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A new multidimensional Schur-Cohn type stability criterion

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Abstract—In this paper a new multidimensional BIBO stability algorithm is proposed. The algorithm is based on a necessary and sufficient condition for BIBO stability of n-dimensional filters. The criterion involves the use of the functional Schur coefficients, recently introduced by the authors. This new criterion only needs a unique condition to be checked, as an alternative to the set of $N-1$ conditions of the Jury-Anderson or DeCarlo Strintzis stability test. A new procedure involving the Modified Multidimensional Jury Table for testing this criterion is proposed. The procedure is illustrated by a two-dimensional example.

I. INTRODUCTION

In the last years a big amount of research was dedicated to developing techniques for the multidimensional systems. The interest in multidimensional systems is due to their increasing number of applications in many fields (such as digital filtering, image processing, video processing, seismic data processing, biomedical signals processing, control, etc).

One of the many problems that arise naturally in control theory is that of BIBO stability of systems. A system is called BIBO stable (Bounded Input Bounded Output) if for a bounded sequence of input a bounded sequence is always outputted. The BIBO stability of multidimensional filters with multivariable rational transfer function avoid on non-essential second kind singularities is assured if all the zeros of the N -variable denominator P lie outside the closed unit polydisc $\bar{\mathbb{D}}^N$. Testing this condition in the multidimensional case problem is still difficult, as there is no root factorization for multivariable polynomials. Several n-dimensional stability tests were developed in [2], [5], [6], [8], [9], [18]. The 2-dimensional case was considered in [7], [10], [11], [3].

In the one-dimensional case one of the methods used to verify that a one variable polynomial has no zeros inside the closed unit disc is the Schur-Cohn SC criterion (see e.g. [13]). It involves the use of "Schur parameters", also known as the "reflection coefficients". Using an extension of the reflection coefficients to the two-dimensional case [12], a sufficient but not necessary condition for stability of two-dimensional systems was obtained in [1].

The purpose of this paper is to present a new criterion of stability for multidimensional systems. This new criterion is obtained using the functional Schur coefficients, recently introduced by the authors, who are a natural extension to

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the multidimensional case of the Schur parameters. Their use leads to a multidimensional extension of the classical SC algorithm.

The paper is organized as follows: Section II gives a global overview of the one-dimensional case and connections between two different aspects of the classical Schur algorithm. In Section III the slice technique method is presented, the functional Schur coefficients are introduced and the multidimensional stability criterion is formulated. In section IV a multidimensional Jury-Table form procedure for testing the stability in the n-dimensional case is proposed. A 2-D result obtained by Siljak is generalized for the n-D case and the Modified Multidimensional Jury Table is obtained. To illustrate this procedure the Jury table form for the two-dimensional SC algorithm is given in Section V, and an example is provided.

II. THE ONE DIMENSIONAL SCHUR-COHN CRITERION

In [16] the necessary and sufficient condition of stability was obtained by making use of the analytic aspect of the Schur algorithm. We shall briefly recall in this section the mathematical background of the Schur algorithm, and make the connection between the analytic approach and the algebraic approach, which leads to the connection between the Schur coefficients and the leading principal minors in the Schur Cohn matrix.

A. Analytic approach

In this subsection a sequence of Schur parameters is associated to a complex function F . General considerations are made, and the reader will see later on the connection between F and the transfer function of the filter which is tested for stability.

Consider the transform Φ that maps a complex function F analytic around the origin to the function $\Phi(F)$ defined by $\Phi(F) = 0$ if F is a constant function of modulus equal to one, and

$$\Phi(F)(z) = \begin{cases} \frac{F(z)-F(0)}{z(1-F(0)F(z))} & z \neq 0 \\ F'(0)(1-|F(0)|^2)^{-1} & z = 0 \end{cases} \quad (1)$$

otherwise (see [4]).

To a function F , one can associate the sequence of functions $(F_k)_{k=0,1,\dots}$ using the following recursion:

$$F_0 = F, \quad F_k = \Phi(F_{k-1}) \quad (k \geq 1). \quad (2)$$

The functions F_k are called *the Schur iterates* of F and the parameters:

$$\gamma_k = F_k(0) \quad (k \geq 0) \quad (3)$$

are called *the Schur coefficients* of the function F . They characterize the function in the sense that a different sequence of Schur coefficients is associated to each function.

Consider now the case when $F = P/Q$, where P and Q are two polynomials in one variable. A simple computation shows that $F_k = P_k/Q_k$ for $k \geq 1$, where (P_k, Q_k) are defined by:

$$P_k(z) = \frac{1}{z}(P_{k-1}(z) - \gamma_{k-1}Q_{k-1}(z)) \quad (k \geq 1) \quad (4)$$

$$Q_k(z) = Q_{k-1}(z) - \bar{\gamma}_{k-1}P_{k-1}(z) \quad (k \geq 1) \quad (5)$$

or, in equivalent form:

$$\begin{bmatrix} P_k(\lambda) \\ Q_k(\lambda) \end{bmatrix} = \begin{bmatrix} z^{-1} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -\gamma_{k-1} \\ -\bar{\gamma}_{k-1} & 1 \end{bmatrix} \begin{bmatrix} P_{k-1}(\lambda) \\ Q_{k-1}(\lambda) \end{bmatrix}$$

where

$$\gamma_k = P_k(0)/Q_k(0) \quad (k \geq 0) \quad (6)$$

Consider now a polynomial P of degree n , with the root factorization $P(z) = \prod_{j=1}^n (z - \alpha_j)$. Let P^T be the transpose of P defined by $P^T(z) = z^n \overline{P(1/\bar{z})}$. It is obvious that if $F = P/P^T$, then:

$$F(z) = \prod_{j=1}^n \frac{z - \alpha_j}{1 - \bar{\alpha}_j z}. \quad (7)$$

In digital filtering F is called an all pass filter if all the roots α_i of P are in \mathbb{D} . This happens if and only if:

$$|\gamma_k| < 1 \quad (k = 0, \dots, n-1). \quad (8)$$

(see [4] for more details)

To see if a polynomial P has no zeros outside the closed unit circle, one has to compute the Schur coefficients for $F := \frac{P}{P^T}$ and verify if condition (8) holds. In order to check if P does not have any roots inside the unit disc the function $F := \frac{P^T}{P}$ is to be used.

B. Algebraic approach

Another way to compute the Schur parameters involves the use of the *the Schur-Cohn matrix*, this procedure being more appropriate for implementation purposes. We briefly present here the context, and connections with the previous subsection will be made.

The SC matrix associated to a complex polynomial

$$P(z) = \sum_{i=0}^n p_i z^i \quad (9)$$

is the matrix $D_P = (d_{ij})_{1 \leq i, j \leq n}$, where:

$$d_{ij} = \sum_{k=1}^i (p_{n-i+k} \bar{p}_{n-j+k} - \bar{p}_{i-k} p_{j-k}), \quad (i \leq j) \quad (10)$$

The classical SC criterion states that the number of zeros of $P(z)$ inside the unit circle is equal to the number of positive eigenvalues of D_P ; the number of zeros outside the unit circle is equal to the number of negative eigenvalues of D_P (reciprocal zeros with respect to the unit circle and

zeros on the circle are not considered), and the number of reciprocal zeros with respect to the unit circle and zeros on the circle is the nullity of D_P (see, e.g. [3]).

A consequence of this criterion is that a polynomial P has all its zeros inside the unit circle if and only if the matrix D_P is positive definite. Moreover, it is easy to see that $D_{P^T} = -D_P$. Therefore, P has all its zeros outside the unit circle if and only if the matrix D_{P^T} is positive definite. The matrix positivity is assured if all its principal leading minors are positive.

To see the connection between the principal leading minors of the SC matrix and the Schur coefficients, denote by T_P the infinite Toeplitz matrix associated to P :

$$T_P = \begin{bmatrix} p_0 & 0 & 0 & \dots \\ p_1 & p_0 & 0 & \ddots \\ p_2 & p_1 & p_0 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{bmatrix}. \quad (11)$$

Then one can consider the SC matrix associated to a pair (P, Q) of polynomials, defined by:

$$\Delta(P, Q) = T_Q T_Q^H - T_P T_P^H. \quad (12)$$

Note that if $Q = P^T$ then all the entries of the matrix $\Delta(P, P^T)$ are zero outside the $n \times n$ leading principal submatrix of $\Delta(P, P^T)$.

Denote by $\Delta_k(P, Q)$ the $k \times k$ leading principal submatrix of $\Delta(P, Q)$. The following relation holds for $k \geq 1$:

$$\det \Delta_k(P, Q) = |Q_{k-1}(0)|^2 - |P_{k-1}(0)|^2 = (1 - |\gamma_{k-1}|^2) |Q_{k-1}(0)|^2 \quad (13)$$

where P_{k-1} and Q_{k-1} are given by the Schur recursion (4), (5).

In particular, the positivity condition for the leading principal minors:

$$\det \Delta_k(P, P^T) > 0 \quad (k = 1, 2, \dots, n). \quad (14)$$

holds if and only if the Schur coefficients of the function $F := \frac{P}{P^T}$ given by (6) satisfy:

$$|\gamma_k| < 1 \quad (k = 0, \dots, n-1). \quad (15)$$

It is easy to see that $\Delta_n(P^T, P)$ is the (complex) conjugate of the SC matrix associated to P^T by (10). Therefore we have that P has all his roots outside the unit circle iff $\Delta_n(P^T, P) > 0$. Moreover, $\Delta_n(P, P^T) = -\Delta_n(P^T, P)$. Thus the polynomial P has all his roots inside the unit circle iff $\Delta_n(P, P^T)$ is positive definite.

One of the methods used to check the positivity of the SC matrix, and therefore the stability of a polynomial P is the Jury Table JT (see [11]). The JT which gives a way of computing the principal leading minors in the SC matrix associated to P . We recall in the following the construction of the JT for a polynomial P as in (9).

Jury Table (I)

1. For $i = 0, \dots, n$ let $b_i^0 = p_i$.
2. For $k = 1, \dots, n$ let m be equal to 0 if $k = 1, 2$ and $m = 1$ if $k > 2$. Then construct the k^{th} row of the table with the entries b_i^k for $i = 0, \dots, n - k - 1$ defined by:

$$b_i^k = \left(\frac{1}{b_0^{k-1}} \right)^m \begin{vmatrix} b_0^{k-1} & b_{n-k+1-i}^{k-1} \\ \bar{b}_{n-k+1}^{k-1} & \bar{b}_i^{k-1} \end{vmatrix} \quad (16)$$

3. $P(z) \neq 0$ for all $|z| \leq 1$ if and only if $b_0^k > 0$ for all $k = 1, \dots, n$

In this form of the table P has all the roots outside the unit circle if and only if all the entries of the first column of the table are positive. In order to test if the roots of P are inside the unit circle only a modification of in the initialization of the table is necessary, in order to construct the Jury Table for the polynomial transpose P^T :

Jury Table (II)

1. For $i = 0, \dots, n$ let $b_i^0 = \bar{p}_{n-i}$.
2. For $k = 1, \dots, n$ let m be equal to 0 if $k = 1, 2$ and $m = 1$ if $k > 2$. Then construct the k^{th} row of the table with the entries b_i^k for $i = 0, \dots, n - k - 1$ defined by:

$$b_i^k = \left(\frac{1}{b_0^{k-1}} \right)^m \begin{vmatrix} b_0^{k-1} & b_{n-k+1-i}^{k-1} \\ \bar{b}_{n-k+1}^{k-1} & \bar{b}_i^{k-1} \end{vmatrix} \quad (17)$$

3. $P(z) \neq 0$ for all $|z| \geq 1$ if and only if $b_0^k > 0$ for all $k = 1, \dots, n$.

In Section IV a modified Jury Table for the multidimensional case will be obtained.

III. FUNCTIONAL SCHUR COEFFICIENTS AND MULTIDIMENSIONAL SCHUR-COHN CRITERION

In [16] the authors obtained a multidimensional extension of the analytic approach of the SC criterion. To each function F a multidimensional analogous of the Schur parameters sequence is associated, and a multidimensional SC criterion is obtained.

The multidimensional extension of the analytic context is based on the slice functions, first introduced by Rudin [15]. They were used in extending to the 2-D or n-D case several results well known in the 1-D case. We present here briefly the "slice" method.

Denote by \mathbb{D} the set $\{z \in \mathbb{C} : |z| < 1\}$, and by \mathbb{T} the set $\{z \in \mathbb{C} : |z| = 1\}$.

For each point $w = (w_1, \dots, w_N)$ on the polytorus \mathbb{T}^N let \tilde{D}_w be the one-dimensional disc that "slices" \mathbb{D}^N through the origin and through w :

$$\tilde{D}_w = \{\lambda w = (\lambda w_1, \dots, \lambda w_N) : \lambda \in \mathbb{D}\}. \quad (18)$$

It is obvious that if u and w are in \mathbb{T}^N such that there is $z \in \mathbb{T}$ with $w = zu$ then $\tilde{D}_w = \tilde{D}_u$. To avoid considering

redundant slices, one has to do a "normalization" on one coordinate, say for now the last one:

$$D_v = \{\lambda(v_1, \dots, v_{N-1}, 1) : \lambda \in \mathbb{D}\} \quad (v \in \mathbb{T}^{N-1}) \quad (19)$$

In the following we introduce the definition of the slice of a multivariable polynomial.

Let $P(z_1, z_2, \dots, z_N)$ be a polynomial in N variables of degree n :

$$P(\mathbf{z}) = \sum_{|\alpha| \leq n} p_\alpha \mathbf{z}^\alpha$$

where $\mathbf{z} = (z_1, z_2, \dots, z_N) \in \mathbb{C}^N$, $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{N}^N$ and $\mathbf{z}^\alpha = z_1^{\alpha_1} \dots z_N^{\alpha_N}$ whose degree is $|\alpha| = \alpha_1 + \dots + \alpha_N$.

For each $v \in \mathbb{T}^{N-1}$ consider the restriction of the multivariable polynomial P to the one-dimensional disc D_v , which can be regarded as a one variable polynomial:

$$P_v(\lambda) = P(\lambda v) \quad (\lambda \in \mathbb{D}). \quad (20)$$

P_v is called the slice of P through v [15].

The authors proved recently [16] the following result: the N -variable polynomial P has no zeros inside the unit polydisc is equivalent with the fact that the one variable polynomial $P_v(\lambda)$ has no zeros inside the unit polydisc, for each $v \in \mathbb{T}^{N-1}$. This was obtained by the means of the Schur coefficients sequence associated to the n variable polynomial P , which is an extension of the Schur parameters sequence in the multidimensional case. In the following we define the functional Schur coefficients sequence for an analytic function F in several variables.

Definition 3.1: Let $F(\mathbf{z})$ be an analytic function in the open unit polydisc \mathbb{D}^N , and let $v = (v_1, \dots, v_{N-1}) \in \mathbb{T}^{N-1}$. For each point $(v_1, \dots, v_{N-1}, 1)$ on the polytorus \mathbb{T}^N consider the restriction of F to the one-dimensional disc D_v , which can be regarded as a one variable function (the slice of F through v):

$$F_v(\lambda) = F(\lambda v_1, \dots, \lambda v_{N-1}, \lambda) \quad (\lambda \in \mathbb{D}), \quad (21)$$

For each v on \mathbb{T}^{N-1} define, by the Schur recursion (2), the sequence $(F_{v,k})_{k \geq 0}$:

$$F_{v,0} = F_v \quad F_{v,k} = \Phi(F_{v,k-1}) \quad (k \geq 1).$$

The functions $\gamma_k : \mathbb{T}^{N-1} \rightarrow \mathbb{C}$ defined in [16] by

$$\gamma_k(v) = F_{v,k}(0) \quad (v \in \mathbb{T}^{N-1}) \quad (22)$$

are called the functional Schur coefficients of the function F .

The following theorem is established in [16]

Theorem 3.2: N -D Schur-Cohn criterion

The following statements are equivalent:

A) P has no zeros in the closed unit polydisc \mathbb{D}^N :

$$P(z_1, z_2, \dots, z_N) \neq 0 \quad |z_i| \leq 1, \quad i = 1 \dots N \quad (23)$$

B) $P_v(\lambda) = P(\lambda v_1, \dots, \lambda v_{N-1}, \lambda)$ has no zeros in \mathbb{D} for $v \in \mathbb{T}^{N-1}$;

C) $|\gamma_k(v)| < 1$ for all $v \in \mathbb{T}^{N-1}$ and $0 \leq k \leq n-1$, were $\gamma_k(v)$ are the functional Schur coefficients associated to $F = \frac{(P_v)^T}{P_v}$

Summing up, in order to associate the multidimensional analogous of the one-dimensional Schur parameters sequence case to a multivariable polynomial P , the functional Schur coefficients for the function

$$F = \frac{(P_v)^T}{P_v}$$

are required.

IV. MULTIDIMENSIONAL JURY TABLE

When testing the stability condition C) one can use again the Jury table form, by using positivity in (13) instead of (8). In the following we give the construction of the Jury Table in the multidimensional case. Let P be a N -variable polynomial:

$$P(\mathbf{z}) = \sum_{|\alpha| \leq n} p_\alpha \mathbf{z}^\alpha \quad (\mathbf{z} = (z_1, \dots, z_N) \in \mathbb{T}^N) \quad (24)$$

Let $v = (v_1, \dots, v_{N-1}) \in \mathbb{T}^{N-1}$. Then the slice pf P through v is:

$$P_v(\lambda) = \sum_{k=1}^n c_k(v) \lambda^k \quad (\lambda \in \mathbb{D}), \quad (25)$$

where the coefficients c_k are polynomials in v given by:

$$c_i(v) = \sum_{|\alpha|=i} p_\alpha v_1^{\alpha_1} v_2^{\alpha_2} \dots v_{N-1}^{\alpha_{N-1}} \quad (0 \leq i \leq n). \quad (26)$$

The Multidimensional Jury Table MJT

1. For $i = 0, \dots, n$ let $b_i^0(v) = c_i(v)$.
2. For $k = 1, \dots, n$ let m be equal to 0 if $k = 1, 2$ and $m = 1$ if $k > 2$. Then construct the k^{th} row of the table with the entries b_i^k for $i = 0, \dots, n-k-1$ defined by:

$$b_i^k(v) = \left(\frac{1}{b_0^{k-1}(v)} \right)^m \left| \begin{array}{cc} b_0^{k-1}(v) & b_{n-k+1-i}^{k-1}(v) \\ \hline b_{n-k+1}^{k-1}(v) & \bar{b}_i^{k-1}(v) \end{array} \right| \quad (27)$$

3. $P(z) \neq 0$ for all $|z| \leq 1$ if and only if $b_0^k(v) > 0$ for all $k = 1, \dots, n$ and $v \in \mathbb{T}^{N-1}$.

In the following we give the generalization of a result of Siljak [17] for the multidimensional case, and as a consequence we provide with the Multidimensional Modified Jury Table MMJT.

Consider again a N -variable polynomial as in (24), and associate to $P_v(\lambda)$ given by (25) the SC matrix

$$D_P(v) = (d_{ij}(v))_{1 \leq i, j \leq n} \quad (28)$$

where d_{ij} are given by:

$$d_{ij}(v) = \sum_{k=1}^i (c_{n-i+k}(v) \bar{c}_{n-j+k}(v) - \bar{c}_{i-k}(v) c_{j-k}(v)) \quad (29)$$

for $i \leq j$. We give the following theorem [5]:

Theorem 4.1: The positivity of the SC matrix :

$$D_P(v) > 0, \quad \forall v \in \mathbb{T}^{N-1} \quad (30)$$

is equivalent with the following two conditions:

$$D_P(1) > 0 \quad (31)$$

$$\det D_P(v) > 0 \quad \forall v \in \mathbb{T}^{N-1} \quad (32)$$

Proof: The direct implication is obvious. Assume that (31) and (32) are satisfied. The eigenvalues $r_i(v)$, $i = 1 \dots n$ of the Hermitian matrix $D_P(v)$ are real and continuous. From (32) we have:

$$\prod_{i=1}^n r_i(v) > 0, \quad \forall v \in \mathbb{T}^{N-1} \quad (33)$$

Suppose that there is a $v_0 \in \mathbb{T}^{N-1}$ and a $i_0 \in \{1, \dots, n\}$ such that $r_{i_0}(v_0) < 0$. From (31) we have that $r_{i_0}(1) > 0$. But since \mathbb{T}^{N-1} is a convex set, this necessarily implies that $r_{i_0}(v_1) = 0$ for some $v_1 \in \mathbb{T}^N$, which contradicts (33). Consequently (31) and (32) imply that $r_i(v) > 0$, for all $v \in \mathbb{T}^{N-1}$ and for all $i = 1 \dots n$. Therefore the Schur Cohn matrix $D_P(v)$ is positively defined for all $v \in \mathbb{T}^{N-1}$. ■

As a consequence, using the SC matrix associated to P_v^T , we obtain the following:

Corollary 4.2: Simplified N-D Schur-Cohn Criterion

The following assertions are equivalent:

- A) P has no zeros inside the closed unit polydisc \mathbb{D}^N :

$$P(z_1, z_2, \dots, z_N) \neq 0, \quad |z_i| \leq 1, \quad i = 1 \dots N$$

- B) $P_v(\lambda) = P(\lambda v_1, \dots, \lambda v_{N-1}, \lambda \neq 0)$, $|\lambda| \leq 1$, $v \in \mathbb{T}^{N-1}$;

C) The SC matrix associated to P_v^T given by (28) is positive definite:

$$D_{P^T}(v) > 0, \quad \forall v \in \mathbb{T}^{N-1}$$

- D) D1. $D_{P^T}(v_0) > 0$, for an arbitrary $v_0 \in \mathbb{T}^{N-1}$

$$D2. \det D_{P^T}(v) > 0, \quad \forall v \in \mathbb{T}^{N-1}$$

Using the last assertion, instead of requiring all the principal leading minors of the SC matrix to be positive for all $v \in \mathbb{T}^{N-1}$, all that is required is the positivity of the last principal minor and the positivity of the SC matrix in an arbitrary point on the \mathbb{T}^{N-1} . This leads to a simplification of the MJT table, as only the last entry on the first column table needs to be checked for positivity plus the positivity at an arbitrary point on the unit polytorus of the SC matrix:

The Modified Multidimensional Jury Table MMJT

1. Associate to $P_v^T(\lambda)$ given by (25) the SC matrix given by (28).

2. $P(z) \neq 0$ for all $|z| \leq 1$ if and only if

- a) $\Delta_n(v_0) > 0$, where v_0 is an arbitrary point on \mathbb{T}^{N-1} .

- b) $\det \Delta_n(v) > 0$, for all $v \in \mathbb{T}^{N-1}$.

Remark that b) can be written in equivalent form:

- b') $b_0^n(v) > 0$ for all $v \in \mathbb{T}^{N-1}$, where $b_0^n(v)$ is the last entry on the first column in the MJT (given by 27).

In the next section we show how this criterion can be applied to check the stability of the two-dimensional systems.

V. IMPLEMENTATION IN THE TWO DIMENSIONAL CASE

To illustrate this new stability criterion we provide with a 2-dimensional example.

Let $P(z_1, z_2)$ be a two-variable polynomial of degree n . Let $\lambda \in \mathbb{D}$ and consider $P_v(\lambda) = P(\lambda v, \lambda)$ the slice of P through $v \in \mathbb{T}$. Let $\gamma_k(v)$ be the functional Schur coefficients associated to $F = \frac{(P_v)^T}{P_v}$ and let $b_0^k(v)$ be the first entries on each row in MJT (27), for $k = 1, \dots, n$. Denote by $D(v)$ the SC matrix associated to P_v^T . Summing up the previous results, the following assertions are equivalent:

2-D Schur-CohnC criterion:

- A) $P(z_1, z_2) \neq 0 \quad |z_i| \leq 1, i = 1, 2;$
- B) $P_v(\lambda) := P(\lambda v, \lambda) \neq 0$, for all $|v| = 1$ and $|\lambda| \leq 1;$
- C) $|\gamma_k(v)| < 1$ for all $|v| = 1$ and $k = 0 \dots, n - 1;$
- D) $b_0^k(v) > 0$ for all $|v| = 1$ and $k = 1 \dots, n;$
- E) $D(v)$ is positive definite;
- F) $D(v_0) > 0$ for a fixed $|v_0| = 1$ and $b_0^n(v) > 0$ for all $|v| = 1$.

The principal leading minors $b_0^k(v)$ of the Schur Cohn matrix are trigonometric polynomials in v_1 real valued on \mathbb{T}^2 . Several techniques exist for testing their positivity (see e.g. [3]) and will not be discussed here.

Example. Let P be the two variable polynomial (from [3])

$$P(z_1, z_2) = 12 + 10z_1 + 6z_2 + 5z_1z_2 + 2z_1^2 + z_1^2z_2.$$

The slice of P through $v \in \mathbb{T}$ is $P_v(\lambda) = P(\lambda v, \lambda)$,

$$P_v(\lambda) = \lambda^3 v^2 + (2v^2 + 5v)\lambda^2 + (6 + 10v)\lambda + 12.$$

In Table I the MJT (27) is constructed for P . The entries are computed by (27), and represented after simplification on each line with some positive integer common factor. In order to check the condition of stability for P , all entries of the first column of the Table I must be positive for $v \in \mathbb{T}$ (first row of the table is to be ruled out, as it represent the coefficients of $P_v(\lambda)$):

$$\begin{aligned} b_0^1 &= 143 \\ b_0^2 &= 15(235 - 12(v + \bar{v})) \\ b_0^3 &= 677 - 250(v + \bar{v}) + 24(v^2 + \bar{v}^2). \end{aligned}$$

Let $v = e^{it}$, with t real, and set $x = \cos t$. We have:

$$\begin{aligned} b_0^2 &= -24x + 235 \\ b_0^3 &= 96x^2 - 500x + 629 \end{aligned}$$

and it is easy to see that they are strictly positive for $x \in [-1, 1]$ (for instance using [14], [19]). Therefore we have stability for $P(z_1, z_2)$.

Now, when applying the MMJT for P , we have to consider the Schur Cohn matrix associated to $P_v^T(\lambda)$:

$$P_v^T(\lambda) = 12\lambda^3 + (6 + 10\bar{v})\lambda^2 + (2\bar{v}^2 + 5\bar{v})\lambda + \bar{v}^2,$$

TABLE I
2-D JURY TABLE FOR P

λ^0	λ^1	λ^2	λ^3
12	$6 + 10v$	$5v + 2v^2$	v^2
143	$70 + 115\bar{v}$	$18\bar{v}^2 + 50\bar{v}$	
$1175 - 60(v + \bar{v})$	$284 + 725v - 84v^2$		
$677 - 250(v + \bar{v}) + 24(v^2 + \bar{v}^2)$			

which is equal to:

$$D(v) = \begin{pmatrix} 143 & 115\bar{v} + 70 & 50\bar{v} + 18\bar{v}^2 \\ 115v + 70 & 100(v + \bar{v}) + 250 & 115\bar{v} + 70 \\ 50v + 18v^2 & 115v + 70 & 143 \end{pmatrix}$$

It is easy to see that $D(1) = \begin{pmatrix} 143 & 185 & 68 \\ 185 & 350 & 185 \\ 68 & 185 & 143 \end{pmatrix}$ is positive definite. Moreover, for $v = e^{it}$, with $t \in \mathbb{R}$, we have positivity for $\det D_n(t) = 1800(8 \cos t - 17)(12 \cos t - 37)$. Condition F) is satisfied, thus $D_n(v)$ is positive definite for all $v \in \mathbb{T}$ and furthermore $P_v^T(\lambda)$ has no zeros outside $\overline{\mathbb{D}^2}$. Therefore $P_v(\lambda)$ has no zeros inside $\overline{\mathbb{D}^2}$, which implies the stability of $P(z_1, z_2)$.

VI. CONCLUSIONS AND FUTURE WORKS

In this contribution a new multidimensional SC type stability criterion was introduced, and a procedure for testing this criterion was given MMJT. The implementation technique in the two-dimensional case was detailed, and illustrated by means of an example. Application in design of filters with guaranteed stability is straightforward.

In order to compare our criterion with others criteria, remind that the existing multidimensional tests of stability relies either on the DeCarlo Strintzis criterion or on the Jury-Anderson criterion:

DeCarlo Strintzis criterion

The condition (23) holds if and only if all the following N conditions are satisfied:

$$P(z_1, 1, 1, \dots, 1) \neq 0 \quad |z_1| \leq 1; \quad (34)$$

$$P(1, z_2, 1, \dots, 1) \neq 0 \quad |z_2| \leq 1; \quad (35)$$

.....

$$P(1, 1, \dots, 1, z_N) \neq 0 \quad |z_N| \leq 1; \quad (36)$$

$$P(z_1, z_2, \dots, z_N) \neq 0 \quad |z_i| = 1, \quad i = 1 \dots N. \quad (37)$$

Jury-Anderson criterion

The condition (23) holds if and only if all the following N conditions are satisfied:

$$P(z_1, 0, 0, \dots, 0) \neq 0 \quad |z_1| \leq 1 \quad (38)$$

$$P(z_1, z_2, 0, \dots, 0) \neq 0 \quad |z_1| = 1, |z_2| \leq 1 \quad (39)$$

.....

$$P(z_1, \dots, z_{N-1}, 0) \neq 0 \quad |z_i| = 1, \quad i = 1 \dots N - 2, \quad |z_{N-1}| \leq 1 \quad (40)$$

$$P(z_1, z_2, \dots, z_N) \neq 0 \quad |z_i| = 1, \quad i = 1 \dots N - 1 \quad |z_N| \leq 1 \quad (41)$$

Both criteria involve checking a set of N conditions. The new multidimensional stability criterion 3.2 proposed in

Section III states that (23) holds if and only if the following unique condition is satisfied:

$$P(\lambda v_1, \dots, \lambda v_{N-1}, \lambda) \neq 0 \quad i = 1 \dots N - 1, |v_i| = 1, \lambda \leq 1.$$

This criterion is based on the use of functional Schur coefficients and future research should be directed to investigate their properties.

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