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# Compact Perturbations and Factorizations of Closed Range Operators

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**Abstract:** In this paper it is shown that if  $A$  et  $B$  are closed range operators in a Hilbert space for which the equation  $B = XA$  has at least a solution, then the compactness of  $A - B$  is equivalent to the existence of a solution  $X$  such that  $X - I$  is compact. This result has several consequences on the description of the compact perturbations of particular classes of operators.

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Let  $H$  be an infinitely dimension separable complex Hilbert space and  $B(H)$  the Banach algebra of bounded linear operators on  $H$ . Several papers deal with the study of perturbations  $B = A + K$  of isometry  $A$  in  $B(H)$ , when  $K$  runs over certain classes of compact operators, such as rank one operators (in [5], [6]), or finite rank operators (see [1], [2], [3]). A first question is to characterize the situations when the perturbation  $B$  is still an isometry, or at least a contraction. For  $K$  a rank one operator, it is shown in [6] that  $B = A + K$  is an isometry (resp. contraction) if and only if  $K = (\alpha - 1)h \otimes A^*h$ , where  $h$  is an unitary vector in  $H$  and  $|\alpha| = 1$  (resp.  $|\alpha| \leq 1$ ).

This condition is equivalent to the existence of a factorization  $B = XA$  where for some unitary (respectively contraction)  $X$  such that  $X - I$  is of rank 1 (more precisely  $X = I + (\alpha - 1)h \otimes h$ ). In [8] we show that this is still true for  $K$  of finite rank (resp. an arbitrary compact), with an factor  $X$  such that  $X - I$  is of finite rank (resp. compact).

It is clear that factorizations of type  $B = XA$  always exist for any isometry  $A$  and any contraction  $B$  (see [4]), so the fact that  $A - B$  is also compact (or finite rank) is equivalent to the existence of a factor  $X$  such that  $X - I$  is of the same type as  $A - B$ .

The aim of this article is to show that this equivalence still holds for any closed range operator  $A$  and any operator  $B$  such that  $\text{Im } B^* \subset \text{Im } A^*$  (i.e. for which there exists at least one factorization  $B = XA$ ).

We remind the Douglas factorization criterion [4], which we use in a dual form:

**Proposition 1** *Let  $A$  and  $B$  in  $B(H)$ . The following affirmations are equivalent:*

- 1)  $\text{Im } B^* \subset \text{Im } A^*$ ;
- 2) *There is  $\lambda > 0$  such that  $B^*B \leq \lambda^2 A^*A$ ;*
- 3) *There is an operator  $X$  in  $B(H)$  such that  $B = XA$ .*

*In this case, there is a unique operator  $X_0$  verifying the following additional conditions:  $\overline{\text{Im } X_0} = \overline{\text{Im } B}$  and  $\overline{\text{Im } X_0^*} \subset \overline{\text{Im } A}$ .*

The unique factor  $X_0$  is called *the reduced solution* of the equation  $B = XA$ , and also satisfies

$$\|X_0\| = \inf_{\lambda > 0} \{ \lambda \in \mathbb{R}_+ : B^*B \leq \lambda^2 A^*A \} \quad \text{and} \quad \text{Ker } X_0 \cap \text{Im } A = A(\overline{\text{Im } A^* \cap \text{Ker } B}).$$

If in addition  $\text{Im } A$  is closed then  $\text{Ker } X_0 = \overline{A(\text{Im } A^* \cap \text{Ker } B)} \oplus (\text{Im } A)^\perp$ .

For a partial isometry  $W$  in  $B(H)$  we denote by  $s_i(W) := W^*W$  the initial support of  $W$  (the orthogonal projection on the orthogonal of  $\text{Ker } (W)$ ) and by  $s_f(W) := WW^*$  the final support of  $W$  (the orthogonal projection on  $\text{Im } (W)$ ). We also note by  $i(W) = s_i(W)H$  the initial subspace and by  $f(W) = s_f(W)H$  the final subspace of  $W$ .

The next lemma shows the behaviour of the polar decompositions of  $A$  and  $B$  and the reduced solution  $X_0$  in Douglas criterion, when the operator  $A$  has a closed range and  $A - B$  is compact.

**Lemma 2** *Let  $A$  a closed range operator in  $B(H)$  and  $B$  in  $B(H)$  such that  $A - B$  is compact and  $\text{Im } A^* \supset \text{Im } B^*$ . If  $A = V|A|$  and  $B = W|B|$  are the polar decompositions of  $A$  et  $B$ , then:*

1)  $s_i(V) - s_i(W)$  is a finite rank projection;

2) The operators  $V - W$  and  $s_f(V) - s_f(W)$  are compact in  $B(H)$ ; 3) If  $X_0$  is the reduced solution of the equation  $B = XA$  then  $X_0 - s_f(V)$  is compact (i.e. the restriction of  $X_0 - I$  to  $\text{Im } A$  is a compact operator from  $\text{Im } A$  to  $H$ ).

**Proof.** Note first that  $A - B$  is compact and the operator  $A|_{\text{Im } A^*} : \text{Im } A^* \rightarrow H$  is semi-Fredholm, then the operator  $B|_{\text{Im } A^*} : \text{Im } A^* \rightarrow H$  is also semi-Fredholm, so in particular  $B$  has closed range.

1) Obviously  $s_i(V) - s_i(W)$  is the orthogonal projection on  $K_0 = \text{Im } A^* \cap \text{Ker } B$ . To see that the dimension of  $K_0$  is finite, let  $M$  be the closed unit ball of  $K_0$ . The operator  $A$  is bounded from below on  $\text{Im } A^*$  so there exists  $\delta > 0$  such that  $\delta M \subset AM = (A - B)M$ . But this last set is compact, therefore the unit ball  $M$  is necessarily compact, so  $K_0$  is finite dimensional.

2) Let  $A'$  and  $B'$  in  $B(H)$  the inverses of  $|A| : \text{Im } A^* \mapsto \text{Im } A^*$  and  $|B| : \text{Im } B^* \mapsto \text{Im } B^*$ , extended with 0 on  $\text{Ker } A$  and  $\text{Ker } B$  respectively. We have:  $|A|A' = A'|A| = s_i(V)$  and  $|B|B' = B'|B| = s_i(W)$ . As  $s_i(V)s_i(W) = s_i(W)$ , we have:

$$A' - B' = A's_i(V) - s_i(W)B' = A'(|B| - |A|)B' + A'(s_i(V) - s_i(W))$$

so  $A' - B'$  is compact. But then

$$V - W = Vs_i(V) - Ws_i(W) = V|A|A' - W|B|B' = AA' - BB' = A(A' - B') + (A - B)B'.$$

which shows that  $V - W$  is compact. Finally,

$$s_f(V) - s_f(W) = VV^* - WW^* = (V - W)V^* + W(V^* - W^*)$$

is also a compact operator.

3) If  $X_0$  is the reduced solution of  $B = XA$ , then

$$X_0 - s_f(V) = X_0s_f(V) - s_f(V) = (X_0 - 1)V s_i(V)V^* = (B - A)A'V^*$$

so  $X_0 - s_f(V)$  is compact and the proof is complete.  $\square$

The previous lemma shows that the compactness of  $X_0 - I$  holds only on the range of  $A$ . We now show that  $X_0$  can be extended on the orthogonal of  $\text{Im } A$  while preserving this compactness condition. We do this by the means of a characterisation (shown in [8]), of couples of closed subspaces of  $H$  that are ranges of isometries with compact difference, stated here in a slightly generalized form:

**Proposition 3** Let  $H_1$  and  $H_2$  two closed subspaces of  $H$  of same dimension and  $P_1, P_2$  the corresponding orthogonal projections. The following affirmations are equivalent:

- 1) There exist two partial isometries  $V_1$  and  $V_2$  with the same initial support, such that  $H_i = V_i H$  for  $i = 1, 2$  and  $V_1 - V_2$  is compact;
  - 2)  $P_1 - P_2$  is compact and  $\dim(H_1 \cap H_2^\perp) = \dim(H_2 \cap H_1^\perp)$ .
- Moreover,  $V_1$  and  $V_2$  can be chosen isometric iff  $\dim H_1 = \dim H_2 = \aleph_0$ .

Clearly, the condition 2) of the previous proposition remains the same when the two subspaces  $H_1$  and  $H_2$  are replaced by their orthogonals. This symmetry is essential in order to prove the main result:

**Theorem 4** Let  $A$  a closed range operator in  $B(H)$  and  $B$  in  $B(H)$  such that  $\text{Im } A^* \supset \text{Im } B^*$ . The following affirmations are equivalent:

- 1)  $A - B$  is compact;
  - 2) There exists an operator  $X$  in  $B(H)$  such that  $X - I$  is compact and  $B = XA$ ;
- If that is the case, one can choose  $X$  such that:

$$\text{Ker } X = A(\text{Im } A^* \cap \text{Ker } B) \quad \text{and} \quad \|X\| = \max\{1, \inf_\lambda \{B^*B \leq \lambda^2 A^*A\}\}.$$

In particular:

- 3) If  $\text{Im } A^* = \text{Im } B^*$  then  $X$  can be chosen invertible.
- 4) If  $B^*B \leq A^*A$  then  $X$  can be chosen contractive.
- 5) If  $A^*A = B^*B$  then  $X$  can be chosen unitary.

**Proof.** The non-trivial part is the fact that 1) implies 2). Let  $A = V|A|$  and  $B = W|B|$  the polar decompositions of  $A$  and  $B$ , and  $X_0$  the reduced solution of the equation  $B = XA$ . Let's first observe that the codimensions of the ranges of  $A$  and  $B$  are either both finite or both infinite. Indeed, if for instance the range codimension of  $A$  is finite, then  $A$  and  $B$  are simultaneously right semi-Fredholm, so the range codimension of  $B$  is finite. It is enough to treat the infinite case, because if the two codimensions are finite,  $X = X_0$  already satisfies the conclusion.

By the Lemma 2 the dimension  $d = \dim(\text{Im } A^* \cap \text{Ker } B)$  is finite, so one can choose a subspace  $K_0$  of  $(\text{Im } B)^\perp$  such that  $\dim K_0 = d$ . Let  $W_0$  an arbitrary partial isometry with  $i(W_0) = \text{Im } A^* \cap \text{Ker } B$  and  $f(W_0) = K_0$  (so  $W_0$  is finite rank), and let  $W' = W + W_0$ . We have then  $i(W') = i(V) = \text{Im } A^*$ ,  $f(W') = \text{Im } B \oplus K_0$ , and moreover  $V - W' = (V - W) - W_0$  is compact (by the lemma 2). However,  $\text{Im } A$  and  $\text{Im } B \oplus K_0$  verify the condition 1) of the Proposition 3, so it is the same for their orthogonals. Thus there exist two partial isometries  $Y$  and  $Z$  such that  $Y - Z$  is compact,  $i(Y) = i(Z)$ ,  $f(Y) = (\text{Im } A)^\perp$  and  $f(Z) = (\text{Im } B)^\perp \ominus K_0 \subset (\text{Im } B)^\perp$ . Let  $U_0$  be the reduced solution of  $Z = UY$  (which is in fact an partial isometry that acts unitarily between  $(\text{Im } A)^\perp$  and  $(\text{Im } B)^\perp \ominus K_0$ ). by the Lemma 2 we know that  $U - s_f(Y)$  is compact.

Let's set  $X = X_0 + U_0$ . Obviously  $B = XA$  and moreover  $X - I = (X_0 - s_f(V)) + (U_0 - s_f(Y))$  is compact, which ends the proof of the implication. Moreover, by the construction of  $U_0$  and the Douglas criterion, we have:

$$\|X\| = \max\{\|U_0\|, \|X_0\|\} = \max\{1, \inf_\lambda \{B^*B \leq \lambda^2 A^*A\}\}.$$

For the supplementary statements:

3) If  $\text{Im } A^* = \text{Im } B^*$  then  $\text{Ker } X_0 = \text{Im } A^* \cap \text{Ker } B = (0)$ , so  $\text{Ker } X = \text{Ker } X_0 = (0)$ . But as  $X$  is Fredholm of zero index, this implies that  $X$  is invertible.

4) If  $A^*A \leq B^*B$  then  $\|X_0\| \leq 1$ , so  $\|X\| \leq 1$ .

5) Finally, if  $A^*A = B^*B$  then  $X_0$  acts unitarily between  $\text{Im } A$  and  $\text{Im } B$ , while  $U_0$  acts unitarily between  $(\text{Im } A)^\perp$  and  $(\text{Im } B)^\perp$ , so  $X$  is unitary, and the proof is complete.  $\square$

This result is not true if the closed range condition on  $A$  is dropped. For example, if  $A$  is any dense range compact in  $H$ ,  $X_0$  an unitary such that  $X_0 - I$  is not compact, and  $B = X_0A$ , then  $A - B$  is trivially compact, but  $X_0$  is the unique solution of the equation  $B = XA$ .

**Corollary 5** *If  $A$  and  $B$  are in  $B(H)$  such that  $A$  is an isometry and  $A - B$  is a compact operator then  $B$  is a contraction (resp. isometry) iff there exists a contraction (resp. unitary)  $X$  in  $B(H)$  such that  $B = XA$  and that  $X - I$  is compact.*

In [8] it is shown that in this last corollary one can replace "compact" by "finite rank". A close look at the proofs shows that the same can be done in Theorem 4, but only under the additional hypothesis that  $|A| - |B|$  is also a finite rank operator.

More precisely, this additional condition does not follow in general from the fact that  $A - B$  has finite rank (as it happens to its analogous condition in the compact case) because if the difference  $R_1 - R_2$  of two positive operators has finite rank, this doesn't necessarily imply that  $R_1^{1/2} - R_2^{1/2}$  is finite rank (in our case  $R_1 = A^*A$  and  $R_2 = B^*B$ ).

Such an example can be found as follows (cf. H. Bercovici, private communication): consider  $0 < R_1 < R_2$  two positive operators such that  $R_2 - R_1$  has rank one and the image of  $R_2 - R_1$  is cyclic for  $R_2$ . Then  $R_1^{1/2} - R_2^{1/2}$  is injective, and thus cannot have finite rank. To see this take  $x$  in  $\text{Ker}(R_1^{1/2} - R_2^{1/2})$  and write

$$R_i^{1/2} = -\frac{1}{\pi} \int_0^\infty \sqrt{t}((t + R_i)^{-1} - t^{-1})dt \quad (i = 1, 2)$$

Then  $x$  lies in  $\text{Ker}((t + R_1)^{-1} - (t + R_2)^{-1})$  for  $t > 0$ , hence in  $\text{Ker}((R_2 - R_1)(t + R_2)^{-1})$  for  $t > 0$ , which implies that  $(t + R_2)^{-1}x$  is orthogonal to the range of  $R_2 - R_1$  for  $t > 0$ . But then  $R_2^n x$  is orthogonal to the range of  $R_2 - R_1$  for  $n \geq 0$  which is itself cyclic for  $R_2$ , so necessarily  $x = 0$ , meaning that  $R_1^{1/2} - R_2^{1/2}$  is not a finite rank operator.

However, in the particular case when  $A$  is an isometry, the fact that  $A - B$  has finite rank implies that  $A^*A - B^*B = I - B^*B$  has finite rank, so  $B^*B$  is diagonalizable and hence  $B^*B - (B^*B)^{1/2}$  has finite rank, therefore  $|A| - |B| = (I - B^*B) + (B^*B - (B^*B)^{1/2})$  has finite rank.

**Corollary 6** *Let  $A$  in  $B(H)$  bounded from below operator. All the perturbations of  $A$  by compact operators are of type  $XA$  with  $X$  in  $B(H)$  such that  $X - I$  is compact.*

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