The robust shortest path and the single-source shortest path problems: interval data

Martha Salazar-Neumann

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Résumé

Cet article traite de deux problèmes combinatoires robustes. Le premier concerne une version robuste du plus court chemin entre deux nøeuds préfixés 1 et m (trouver un plus court chemin robuste). Le second porte sur les chemins robustes entre un nøeud fixé et tous les autres. Dans ces deux problèmes, les coûts (ou longueurs) des arcs sont modélisés par des intervalles. On énonce des conditions suffisantes permettant de déterminer des arcs dits dits forts ou faibles appartenant ou non au plus court chemin robuste.

Mots-clefs : Plus court chemin robuste, modélisation de l'incertitude

Abstract

We consider two robust deviation problems; the robust deviation version of the problem to find a path of minimum weight connecting two specified nodes 1 and m, (*the robust deviation shortest path problem*) and the robust version of the problem to find shortest paths from a fixed node 1 and all the nodes of the graph (*the robust deviation single-source shortest path problem*). We consider both of them on finite directed graphs where arcs lengths are interval numbers. We give sufficient conditions for an arc (k, r) to be always (for all realization of data) on an optimal solution (strictly strong arcs and T-strictly strong arcs resp.) and for an arc to be never on an optimal solution (non weak arcs and non T-weak arcs resp.). These entities can be used to preprocess a given graph with interval data prior to the solution of the robust problem. In [3] was shown that such preprocessing procedure is quite effective in reducing the time to compute the solution of the robust deviation path problem. Based on these results we present polynomial time recognition algorithms.

Key words : Minimax regret optimization; Robust shortest path problem; uncertainty modeling; arc problem

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1 Introduction

The first problem that we consider is the robust deviation version of the problem to find a path of minimum weight connecting two specified nodes 1 and m,*robust deviation shortest path problem* on finite directed graphs under arc length uncertainties. We model data uncertainty by treating the arc lengths as nonnegative interval numbers, where each arc length can take any value in its interval.

In Zielinski [8] this problem is shown to be NP-hard and to remain NP-hard even when a graph is restricted to be directed acyclic planar and regular of degree three.

In Averbakh et al. [1] authors prove that this problem is strictly NP-hard for acyclic directed graphs. Then in solving this problem, reducing the solution space becomes an important issue. In [3] authors show that for each optimal solution p of the robust deviation shortest path problem there exists a scenario s for which p is an optimal solution of the deterministic problem corresponding to the scenario s.

They show also that knowing those arcs which are never on shortest paths we can preprocess a graph prior to solution of the robust path problem. They study the arc problems on acyclic directed graphs and present a procedure to check whether an arc incident at nodes 1 or m is weak or not. Then authors give a necessary condition for an arc to be non-weak give a procedure to determine if an intermediate arc is non-weak and propose a mixed integer programming approach with preprocessing for the problem. First, the mixed integer program is preprocessed by removing some arcs which will never be in an optimal path, and then it is solved by standard software. They show by computational results that preprocessing of graphs is an efficient method in solving the robust deviation path problem.

Montemani et al. [4], provide a branch and bound algorithm to solve the robust deviation path problem but they do not implement the preprocessing proposed in [3] because in practice it can be used only for acyclic layered graphs with small width.

Another way to reduce the solution space, is by detecting the arcs that are always on a shortest path from 1 to m. In Karasan et al. [3] authors study the robust deviation shortest path problem with interval data, unfortunately, no algorithm to detect the arcs that are always on the solution is given.

In another well-known robust combinatorial problem, the robust deviation spanning tree problem with interval data, Yaman et al. [7] present a mixed integer programming formulation with preprocessing for the problem. The preprocessing removes the non-weak edges and sets the variables corresponding to the strong edges equal to 1. They show by computational results that knowing non-weak and strong edges helps shorten significantly the computation of the robust deviation solution. In Salazar-Neumann [6] characterizations of strictly strong and non-weak edges for the robust deviation spanning tree problem with compact and convex uncertainty are given, and in the case of polyhedral uncertainty we give polynomial time recognition algorithms.

In this work we study the arc problem on a larger class of graphs, finite directed graphs. We give sufficient conditions for an arc to be always or never on a shortest path from 1 to m (strictly strong arcs and non weak arcs respectively). We present polynomial time algorithms to find some strictly strong arcs and some non weak arcs.

The second problem that we consider is the robust version of the problem to find shortest paths from a fixed node 1 to all the remaining nodes of G, the *single-source shortest path problem*. We give sufficient conditions for an arc (k, r) to be always (for all realization of data) on a shortest path from 1 to r (T-strictly strong arcs) and a characterization of the arcs that are never on a shortest path from 1 to another node of G (non T-weak arcs).

Finally, we present polynomial time algorithms to find some T-strictly strong arcs and all the non T-weak arcs.

In [4] authors mention two important applications of this problem ; *"the transportation problems that are usually represented as a weighted digraph, where each arc is associated with a road and costs represent travel times. A shortest path problem has to be solved every time the quickest way to go from one place to another has to be calculated. A similar problem arises in telecomunications when a packet has to be sent from a source node to a destination node on a network. Also in this case, where the network is usually modeled as a weighted digraph and costs are associated with transmission delays, a shortest path problem is faced. In reality it is not easy to estimate arc costs exactly. To overcome this problem, the interval data model is an alternative".*

2 Notation and definitions

Most of the following definitions can be found in [2]. Let G be a finite directed graph, we denote by $V(G)$ and $A(G)$, the set of nodes and arcs of G respectively, we suppose that $\mid V(G) \mid= m$, $\mid A(G) \mid= n$. We consider two special nodes of G, the origin node 1 and the destination node m. We denote by (i, j) the arc from node i to node j. Let A' be a nonempty subset of $A(G)$, the subgraph of G with edge set $A(G) \setminus A'$ is written simply as $G - A'$; it is the subgraph obtained from G by deleting the arcs in A'. If $A' = \{(i, j)\}$ we write $G - (i, j)$ instead of $G - \{(i, j)\}$.

Consider the following deterministic problem called shortest path problem on digraphs : Given G and nonnegative length l_{ij} associated with each arc $(i, j) \in A$, we want to find a shortest directed path connecting two specified nodes 1 and m, where the length of a path is the sum of the lengths of its arcs. An efficient $O(\mid V \mid^2)$ algorithm to solve this problem was given by Dijkstra. It finds not only a shortest (1,m)-path, but shortest paths from 1 to all nodes of G.

We will use some notations given in [3]. Let S be the set of scenarios for the lengths of the arcs of G and let D be the set of our input data. We denote by l_{ij}^s the nonnegative length of the arc (i, j) in the scenario s and we will assume that D is a Cartesian product of intervals, that is to say, each l_{ij}^s can take an arbitrary value in the interval $[\underline{l}_{ij}, \overline{l}_{ij}]$. We denote by $\underline{s}$ the scenario for which for all $(i, j) \in A$, $l_{ij}^{\underline{s}} = \underline{l}_{ij}$ and by $\overline{s}$ the scenario for which for all $(i, j) \in A$, $l_{ij}^{\overline{s}} = \overline{l}_{ij}$.

We denote by $P_1(G)$ the set of all the 1-paths of G, by $P_1^*(G, s)$ the set of all shortest 1-paths of G under the scenario s and by $P_{1j}(G)$ the set of all (1,j)-paths of G. For a path $p \in P_{1j}(G)$ we denote by $A(p)$ the set of arcs of p.

To evaluate a path $p \in P_1(G)$ under the scenario s we use a function $f : P_1(G) \times D \rightarrow \mathbb{R} : (p, l^s) \rightarrow f(p, l^s)$. In this work we assume that this function is the sum of the lengths of the arcs of p under the scenario s.

$$f(p, l^s) = \sum_{(i,j)\in A(p)} l_{ij}^s.$$

By simplicity we will denote by l_p^s the length of path p in the scenario s. In the case where we want to find a shortest directed path connecting two specified nodes 1 and m. We define ;

$$f(p_s^*, l^s) = \min_{p\in P_{1m}(G)} f(p, l^s)$$

The first problem that we will consider in this work is the robust version of the shortest path from 1 to m problem on digraphs under arc length uncertainties. The following definition can be found in the book [5] .

ROBUST DEVIATION : The robust deviation path p_d is such that

$$\max_{s\in S}(f(p_d, l^s) - f(p_s^*, l^s)) = \min_{p\in P_{1m}(G)} \max_{s\in S}(f(p, l^s) - f(p_s^*, l^s))$$

where $p_s^* \in P_{1m}^*(G, s)$.

A worst deviation scenario for a path $p \in P_{1m}(G)$ is a scenario s in which $f(p, l^s) - f(p_s^*, l^s)$, the difference between the cost of the (1,m)-path p and the cost of a shortest path from 1 to m in this scenario is maximum.

The following three definitions can be found in [3] ;

Definition A path is said to be a *weak path* if it is a shortest path from 1 to m for at least one realization of arc lengths.

Definition An arc is a *weak arc* if it lies on some weak path.

Definition An arc is a *strong arc* if it lies on a shortest path from 1 to m for all scenario.

Now we introduce the following definition ;

Definition An arc is a *strictly strong arc* if it lies on all shortest path from 1 to m for all scenario.

The second problem that we will consider is the robust version of the problem to find shortest paths from a fixed node 1 to all the remaining nodes of G, the *robust single-source shortest path problem*. We consider this problem on digraphs under nonnegative arc length uncertainties. We call a feasible solution of the deterministic problem a *spanning 1-tree*, an optimal solution a *minimum spanning 1-tree* of G and a shortest path from 1 to another node of G a *shortest 1-path*.

We denote by $T^1(G)$ the set of all the spanning 1-trees of G, by $T_1^*(G,s)$ the set of all minimum spanning 1-trees of G under the scenario s. For a spanning 1-tree $T \in T^1(G)$ we denote by $A(T)$ the set of arcs of T.

To evaluate a spanning 1-tree $T \in T^1(G)$ under the scenario s we use a function $F : T^1(G) \times D \to \mathbb{R} : (T, l^s) \to F(T, l^s)$. Where

$$F(T, l^s) = \sum_{(i,j)\in A(T)} l_{ij}^s.$$

By simplicity we will denote by l_T^s the length of a 1-tree T in the scenario s. We define ;

$$F(T_s^*, l^s) = \min_{T\in T^1(G)} F(T, l^s)$$

ROBUST DEVIATION : The robust deviation spanning 1-tree T_d is such that

$$\max_{s\in S}(F(T_d, l^s) - F(T_s^*, l^s)) = \min_{T\in T^1(G)} \max_{s\in S}(F(T, l^s) - F(T_s^*, l^s))$$

where $T_s^* \in T_1^*(G, s)$.

A worst deviation scenario for a 1-tree $T \in T^1(G)$ is a scenario s in which $F(T, l^s) - F(T_s^*, l^s)$, the difference between the cost of the spanning 1-tree T and the cost of a minimum spanning 1-tree in this scenario is maximum.

In order to deal with the robust single-source shortest path problem we introduce the following concepts ;

Definition A path $p \in P_1(G)$ is said to be a *T-weak path* if p is a shortest 1-path for at least one realization of arc lengths.

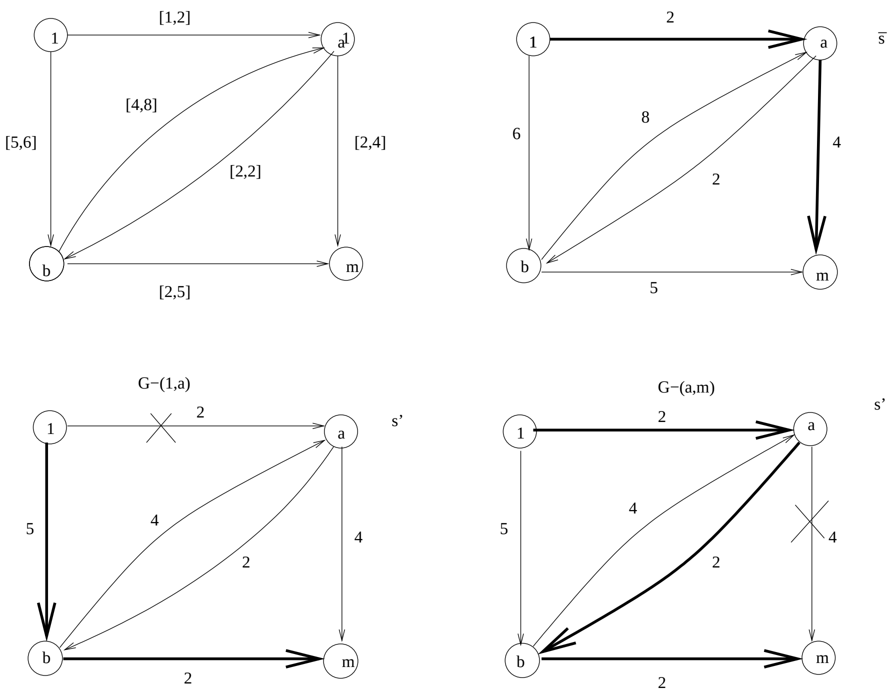

FIG. 1 – Strictly strong arcs algorithm

Definition An arc is a *T-weak arc* if it lies on some T-weak path.

Definition An arc (i, j) is a *T-strictly strong arc* if it lies on all shortest path from 1 to j for all scenario.

Remark We observe that an arc is a non T-weak arc, if and only if it never lies on a minimum spanning 1-tree and an arc is a T-strictly strong arc if and only if it lies on all minimum spanning 1-tree for all scenario.

3 The robust shortest path problem : strictly strong and non-weak arcs

The following proposition give a characterization of the strictly strong arcs.

Proposition 1 Let (k, r) be such that on the graph $G - (k, r)$ the node m is reachable from the node 1. For each $s \in S$ let $p_s^* \in P_{1m}^*(G, s)$. The arc (k, r) is a strictly strong arc of G if and only if for all $q \in P_{1m}(G - (k, r))$ and for all $s \in S$, $l_q^s > l_{p_s^*}^s$

Proof : Let (k, r) be such that on the graph $G - (k, r)$ the node m is reachable from the node 1. If (k, r) is a strictly strong arc of G and $q \in P_{1m}(G - (k, r))$, then q is not a shortest path of G from 1 to m for each scenario s, and consequently for all $s \in S$,

$l_q^s > l_{p_s^*}^s$. Now, if for all $q \in P_{1m}(G-(k,r))$ and for all $s \in S$, $l_q^s > l_{p_s^*}^s$, then q is never a shortest path of G from 1 to m for each scenario s, so for all $s \in S$ a shortest path of G from 1 to m under the scenario s contains the arc (k,r), and (k,r) is then an strictly strong arc.

Remark Let (k,r) be such that there exist $p \in P_{1m}(G)$ such that $(k,r) \in A(p)$ and on the graph $G-(k,r)$ the node m is not reachable from the node 1, then the arc (k,r) is a strictly strong arc.

We can observe that in the last characterization of the strictly strong arcs, the number of extreme scenarios can be exponential, then in the next theorem we will give a sufficient condition easy to test for an arc to be strictly strong. We will use the last proposition on the proof.

Theorem 1 Let $G(V,A)$ be a finite directed graph, with origin node 1 and destination node m. We suppose that each node $j \in V(G)$ is reachable from the node 1. Let p be the shortest path from 1 to m found by Dijkstra algorithm under the scenario $\overline{s}$. Let $(k,r) \in A(p)$, and let s' the scenario such that, $l_{ij}^{s'} = \overline{l}_{ij}$ if $(i,j) \in A(p)$, and $l_{ij}^{s'} = \underline{l}_{ij}$ if $(i,j) \notin A(p)$. If on the graph $G-(k,r)$ the node m is reachable from the node 1, let $q' \in P_{1m}^*(G-(k,r),s')$ be a shortest path from 1 to m under the scenario s'. If $l_{q'}^{s'} > l_p^{s'}$ then the arc (k,r) is a strictly strong arc.

Proof : Let p be the shortest path from 1 to m found by Dijkstra algorithm under the scenario $\overline{s}$, let $(k,r) \in A(p)$. Now consider the scenario s' such that $l_{ij}^{s'} = \overline{l}_{ij}$ if $(i,j) \in A(p)$, and $l_{ij}^{s'} = \underline{l}_{ij}$ if $(i,j) \notin A(p)$. Let q' a shortest path of $G-(k,r)$ from 1 to m under the scenario s'. If $l_{q'}^{s'} > l_p^{s'}$ then for all $q \in P_{1m}(G-(k,r))$ $l_q^{s'} \geq l_{q'}^{s'} > l_p^{s'}$ and then for all $q \in P_{1m}(G-(k,r))$

$$\sum_{(i,j)\in q\setminus p} \underline{l}_{ij} = \sum_{(i,j)\in q\setminus p} l_{ij}^{s'} > \sum_{(i,j)\in p\setminus q} l_{ij}^{s'} = \sum_{(i,j)\in p\setminus q} \overline{l}_{ij}$$

and then for all $s \in S$

$$\sum_{(i,j)\in q\setminus p} l_{ij}^{s} \geq \sum_{(i,j)\in q\setminus p} \underline{l}_{ij} > \sum_{(i,j)\in p\setminus q} \overline{l}_{ij} \geq \sum_{(i,j)\in p\setminus q} l_{ij}^{s}$$

this implies that for all $s \in S$ and for all $q \in P_{1m}(G-(k,r))$, we have $l_q^s > l_p^s$ and if $p_s^* \in P_{1m}^*(G,s)$, $l_q^s > l_p^s \geq l_{p_s^*}^s$. By proposition 1, (k,r) is a strictly strong arc.

Theorem 1 allow us to derive a polynomial time algorithm to find some strictly strong arcs. An algorithm to find some non-weak arcs is presented in the following section.

Algorithm to find some strictly strong arcs

1. Apply Dijkstra algorithm to G under the scenario $\overline{s}$ to obtain p, a shortest path from 1 to m.

2. Choose an arc $(k, r) \in A(p)$.
3. Construct the scenario s' for which $l_{ij}^{s'} = \bar{l}_{ij}$ if $(i, j) \in A(p)$, and $l_{ij}^{s'} = \underline{l}_{ij}$ if $(i, j) \notin A(p)$.
4. Apply Dijkstra to $G - (k, r)$ under the scenario s'. If there exists a shortest path q' from 1 to m, go to step 5. Otherwise (k, r) is strictly strong.
5. Compute $l_{q'}^{s'}$ and $l_p^{s'}$.
6. If $l_{q'}^{s'} > l_p^{s'}$ then the arc (k, r) is a strictly strong arc.
7. Go to step 2. Choosing another arc in $A(p)$ and continue.

Example 1 To clarify the above procedure, we apply it on the first graph given in Figure 1. We set the lengths of all the arcs to their upper bounds and we find a shortest path p from the node 1 to the node m, we represent such path with bold lines. We construct the scenario s' setting the arcs $(1, a)$ and (a, m) to their upper bounds and the lengths of the remaining arcs to their lower bounds. We delete the arc $(1, a) \in A(p)$ and we apply Dijkstra to $G - (1, a)$ under the scenario s'. As there exists a shortest path q' from 1 to m, we compute $l_{q'}^{s'} = 7$ and $l_p^{s'} = 6$. As $l_{q'}^{s'} > l_p^{s'}$ then the arc $(1, a)$ is a strictly strong arc. Then we take the second arc (a, m) of p, we delete the arc $(1, a) \in A(p)$ and we apply Dijkstra to $G - (a, m)$ under the scenario s'. As there exists a shortest path q' from 1 to m, we compute $l_{q'}^{s'} = 6$ and $l_p^{s'} = 6$. As $l_{q'}^{s'} = l_p^{s'}$ we have not a conclusion for the arc (a, m).

4 The robust single-source shortest path problem : T-strictly strong and non T-weak arcs

The following proposition give a characterization of the T-strictly strong arcs.

Proposition 2 Let $(k, r) \in A(G)$ be such that on the graph $G - (k, r)$ the node r is reachable from the node 1. For each $s \in S$ let $p_s^* \in P_{1r}^*(G, s)$. The arc (k, r) is a T-strictly strong arc of G if and only if for all $q \in P_{1r}(G - (k, r))$ and for all $s \in S$, $l_q^s > l_{p_s^*}^s$.

Proof : Let (k, r) be such that on the graph $G - (k, r)$ the node r is reachable from the node 1. If (k, r) is a T-strictly strong arc of G and $q \in P_{1r}(G - (k, r))$, then q is not a shortest path of G from 1 to r for each scenario s, and then for all $s \in S$, $l_q^s > l_{p_s^*}^s$. Now if for all $q \in P_{1r}(G - (k, r))$ and for all $s \in S$, $l_q^s > l_{p_s^*}^s$ then q is never a shortest path of G from 1 to r for each scenario s, so for all $s \in S$ a shortest path of G from 1 to r under the scenario s contains the arc (k, r), then (k, r) is an T-strictly strong arc.

Remark Let (k, r) be such that there exist $p \in P_{1r}(G)$ such that $(k, r) \in A(p)$ and on the graph $G - (k, r)$ the node r is not reachable from the node 1, then the arc (k, r) is a T-strictly strong arc.

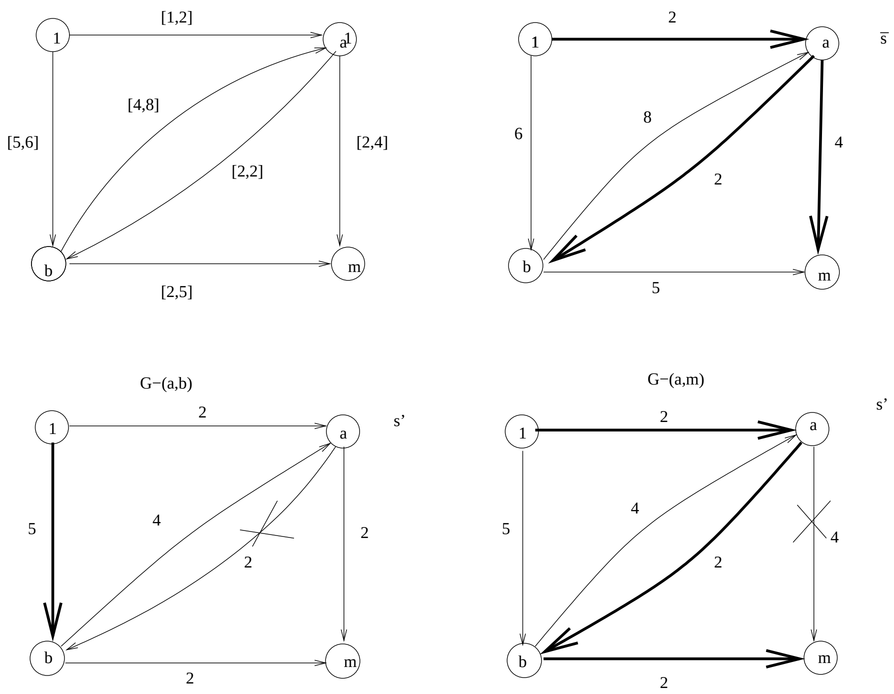

FIG. 2 – T-strictly strong arcs algorithm

On the last proposition, the number of extreme scenarios can be very large, and in that case we can apply the next theorem that give a sufficient condition easy to test for an arc to be T-strictly strong. The proof of next theorem uses the proposition 2.

Theorem 2 Let $G(V, A)$ be a finite directed graph, with origin node 1. Let T the directed tree found by Dijkstra algorithm under the scenario $\overline{s}$, and let $(k, r) \in A(T)$. Let $p \in P_{1r}(G)$ such that $A(p) \subset A(T)$. Let s' be the scenario for which $l_{ij}^{s'} = \bar{l}_{ij}$ if $(i, j) \in A(p)$, and $l_{ij}^{s'} = \underline{l}_{ij}$ if $(i, j) \notin A(p)$. If on the graph $G - (k, r)$ the node r is reachable from the node 1, let $q' \in P_{1r}^{*}(G - (k, r), s')$ be a shortest path from 1 to r under the scenario s'. If $l_{q'}^{s'} > l_{p}^{s'}$ then the arc (k, r) is a T-strictly strong arc.

Proof : Let T be the directed tree found by Dijkstra algorithm under the scenario $\overline{s}$, let $(k, r) \in A(T)$, and let $p \in P_{1r}(G)$ such that $A(p) \subset A(T)$. We observe that p is a shortest path of G from 1 to r under the scenario $\overline{s}$.

Now consider the scenario s' such that $l_{ij}^{s'} = \bar{l}_{ij}$ if $(i, j) \in A(p)$, and $l_{ij}^{s'} = \underline{l}_{ij}$ if $(i, j) \notin A(p)$. Let q' a shortest path of $G - (k, r)$ from 1 to r under the scenario s'. If $l_{q'}^{s'} > l_{p}^{s'}$ then for all $q \in P_{1r}(G - (k, r))$ $l_{q}^{s'} \geq l_{q'}^{s'} > l_{p}^{s'}$ and then for all $q \in P_{1r}(G - (k, r))$

$$\sum_{(i,j) \in q \backslash p} \underline{l}_{ij} = \sum_{(i,j) \in q \backslash p} l_{ij}^{s'} > \sum_{(i,j) \in p \backslash q} l_{ij}^{s'} = \sum_{(i,j) \in p \backslash q} \bar{l}_{ij}$$

and then for all $s \in S$

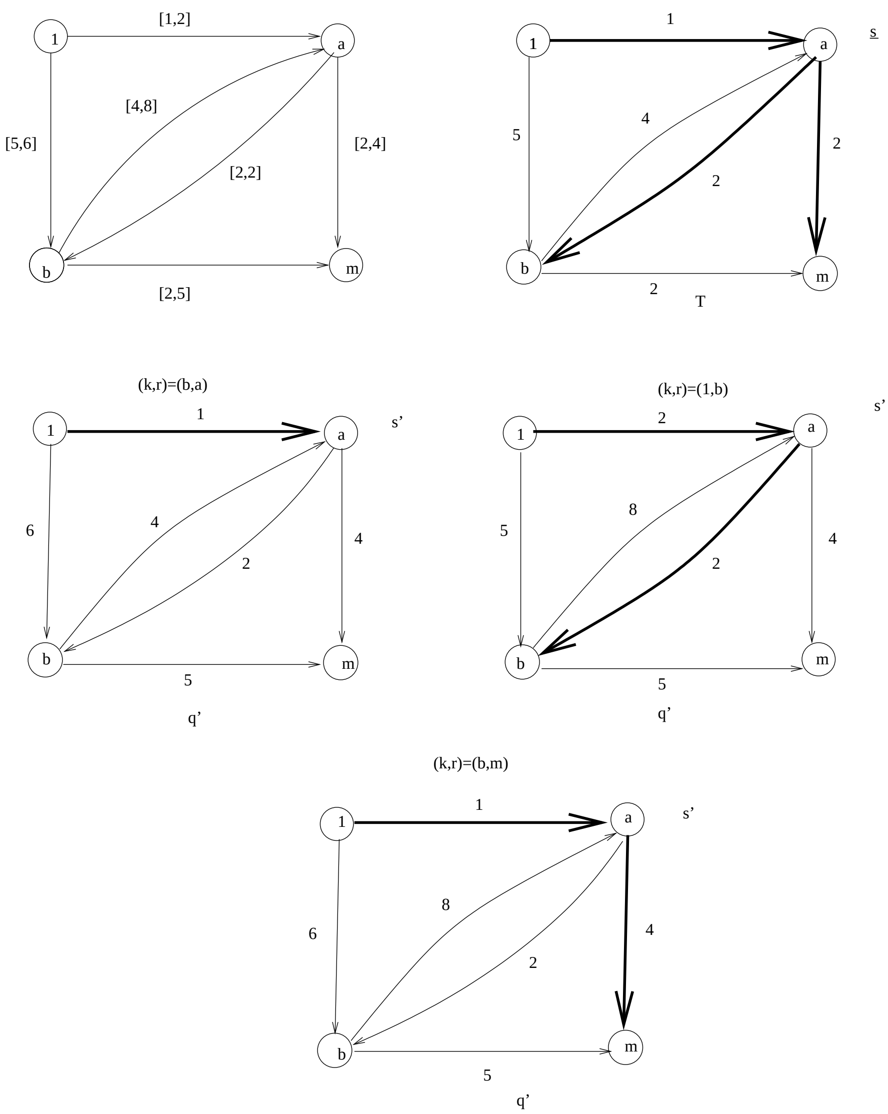

FIG. 3 – Non T-weak arcs algorithm

$$\sum_{(i,j)\in q\setminus p} l_{ij}^{s} \geq \sum_{(i,j)\in q\setminus p} \underline{l}_{ij} > \sum_{(i,j)\in p\setminus q} \bar{l}_{ij} \geq \sum_{(i,j)\in p\setminus q} l_{ij}^{s}$$

this implies that for all $s \in S$ and for all $q \in P_{1r}(G-(k,r))$, if $p_s^* \in P_{1r}^*(G,s)$, $l_q^s > l_p^s \geq l_{p_s^*}^s$ and then by proposition 2, (k,r) is a T-strictly strong arc.

The following proposition give a characterization of the non T-weak arcs.

Proposition 3 For each $s \in S$ let $p_s^* \in P_{1k}^*(G,s)$. The arc (k,r) is a non T-weak arc of G if and only if for each $s \in S$ there exist a path $q \in P_{1r}(G)$, such that $l_q^s < l_{p_s^*}^s + \underline{l}_{kr}$.

Proof : Let (k,r) be a non T-weak arc of G, then for all $p_s^* \in P_{1k}^*(G,s)$, $p_s^* + (k,r)$ is never a shortest path of G from 1 to r, so for all $s \in S$ there exists a path $q \in P_{1r}(G)$ such that $l_q^s < l_{p_s^*}^s + \underline{l}_{kr}$. Now if for all $s \in S$, exists a path $q \in P_{1r}(G)$ such that the inequality ; $l_q^s < l_{p_s^*}^s + \underline{l}_{kr}$ holds, then for all $p \in P_{1k}(G)$ $l_q^s < l_{p_s^*}^s + \underline{l}_{kr} \leq l_p^s + l_{kr}^s$ then all path of G from 1 to r that use the arc (k,r) is never a shortest path from 1 to r for the scenario $s \in S$, then the arc (k,r) is a non T-weak arc.

The following theorem give another characterization easy to test for an arc to be a non T-weak arcs.

Theorem 3 Let $G(V,A)$ be a finite directed graph, with origin node 1. We suppose that each node $j \in V(G)$ is reachable from the node 1. Let T the directed tree found by Dijkstra algorithm under the scenario $\underline{s}$ and let $(k,r) \in A \setminus A(T)$. Let $p \in P_{1k}(G)$ such that $A(p) \subset A(T)$. Let s' be the scenario for which, $l_{kr}^{s'} = \underline{l}_{kr}$, $l_{ij}^{s'} = \underline{l}_{ij}$ if $(i,j) \in A(p)$, and $l_{ij}^{s'} = \bar{l}_{ij}$ if $(i,j) \notin A(p)$ and $(i,j) \neq (k,r)$. Let q' be a shortest path of G from 1 to r for the scenario s'. An arc (k,r) is a non T-weak arc if and only if $l_{q'}^{s'} < l_p^{s'} + \underline{l}_{kr}$.

Proof : Let T be the directed tree found by Dijkstra algorithm under the scenario $\underline{s}$, let $(k,r) \in A \setminus A(T)$, and let $p \in P_{1k}(G)$ such that $A(p) \subset A(T)$, then p is a shortest path of G from 1 to k under the scenario $\underline{s}$. Consider the scenario s' described before and let $q' \in P_{1r}^*(G,s')$. If $l_{q'}^{s'} < l_p^{s'} + \underline{l}_{kr}$

$$\sum_{(i,j)\in q'\setminus p} l_{ij}^{s} \leq \sum_{(i,j)\in q'\setminus p} \bar{l}_{ij} = \sum_{(i,j)\in q'\setminus p} l_{ij}^{s'} < \sum_{(i,j)\in p\setminus q'} l_{ij}^{s'} + \underline{l}_{kr}$$

then $l_{q'}^{s} < l_p^{s'} + \underline{l}_{kr}$ but $l_p^{s'} = l_p^{\underline{s}} \leq l_{p_s^*}^{s}$ then for all $s \in S$

$l_{q'}^{s} < l_p^{s'} + \underline{l}_{kr} \leq l_{p_s^*}^{s} + \underline{l}_{kr}$ and by proposition 8 that implies that (k,r) is a not T-weak arc.

Now suppose that the arc (k,r) is a non T-weak arc then by the proposition 8, for each $s \in S$ there exist a path $q \in P_{1r}(G)$, such that $l_q^s < l_{p_s^*}^s + \underline{l}_{kr}$. In particular for the scenario s' there exists a path $q \in P_{1r}(G)$, such that $l_q^{s'} < l_{p_{s'}^*}^{s'} + \underline{l}_{kr}$. Consider the shortest path q'

from 1 to r under the scenario s' ($q' \in P^*_{1r}(G)$) then

$$l^{s'}_{q'} \leq l^{s'}_{q} < l^{s'}_{p^*_{s'}} + \underline{l}_{kr} \leq l^{s'}_{p} + \underline{l}_{kr}$$

and we have the result.

Remark All the non T-weak arcs are non-weak arcs.

The last two theorems allow us to derive polynomial time algorithms to find some T-strictly strong arcs all the non T-weak arcs and some non-weak arcs ;

Algorithm to find some T-strictly strong arcs

1. Apply Dijkstra algorithm to G under the scenario $\overline{s}$ to obtain T.
2. Choose an arc $(k, r) \in A(T)$.
3. We consider the path $p \in P_{1r}(G)$ such that $A(p) \subset A(T)$.
4. Construct the scenario s' for which $l^{s'}_{ij} = \bar{l}_{ij}$ if $(i, j) \in A(p)$, and $l^{s'}_{ij} = \underline{l}_{ij}$ if $(i, j) \notin A(p)$.
5. Apply Dijkstra to $G - (k, r)$ under the scenario s'. If there exists a shortest path q' from 1 to r, go to step 5. Otherwise (k, r) is T-strictly strong.
6. Compute $l^{s'}_{q'}$ and $l^{s'}_{p}$.
7. If $l^{s'}_{q'} > l^{s'}_{p}$ then the arc (k, r) is a T-strictly strong arc.
8. Go to step 2. Choosing another arc in $A(T)$ and continue.

Example 2 To clarify the above procedure, we apply it on the first graph given in Figure 2. We set the lengths of all the arcs to their upper bounds and we find shortest paths from the node 1 to all nodes of G. We represent such tree T with bold lines. We choose the arc (a, b) and we construct the scenario s' setting the arcs $(1, a)$, and (a, b) to their upper bounds and the lengths of the remaining arcs to their lower bounds. We delete the arc $(a, b) \in A(p)$ and we apply Dijkstra to $G - (a, b)$ under the scenario s'. As there exists a shortest path q' from 1 to b, we compute $l^{s'}_{q'} = 5$ and $l^{s'}_{p} = 4$. As $l^{s'}_{q'} > l^{s'}_{p}$ then the arc (a, b) is a T-strictly strong arc. Then we take a second arc (a, m) of T, we construct the scenario s' setting the arcs $(1, a)$, and (a, m) to their upper bounds and the lengths of the remaining arcs to their lower bounds. We delete the arc $(a, m) \in A(T)$ and we apply Dijkstra to $G - (a, m)$ under the scenario s'. As there exists a shortest path q' from 1 to m, we compute $l^{s'}_{q'} = 6$ and $l^{s'}_{p} = 6$. As $l^{s'}_{q'} = l^{s'}_{p}$ then we have not a conclusion for the arc (a, m).

Algorithm to find all the non T-weak arcs and some non weak arcs.

1. Apply Dijkstra algorithm to G under the scenario $\underline{s}$ to obtain T.
2. Choose an arc $(k, r) \in A \setminus A(T)$.

3. Consider the path $p \in P_{1k}(G)$ such that $A(p) \subset A(T)$.
4. Construct the scenario s' for which $l^{s'}_{kr} = \underline{l}_{kr}$, $l^{s'}_{ij} = \underline{l}_{ij}$ if $(i,j) \in A(p)$, and $l^{s'}_{ij} = \overline{l}_{ij}$ if $(i,j) \notin A(p)$ and $(i,j) \neq (k,r)$.
5. Apply Dijkstra to G under the scenario s' to obtain q' a shortest path from 1 to r.
6. Compute $l^{s'}_{q'}$ and $l^{s'}_{p}$.
7. If $l^{s'}_{q'} < l^{s'}_{p} + \underline{l}_{kr}$ then the arc (k,r) is a non T-weak arc and then a non weak arc. Otherwise (k,r) is a T-weak arc.

Example 3 To clarify the above procedure, we apply it on the first graph given in Figure 3. We set the lengths of all the arcs to their lower bounds and we find shortest paths from the node 1 to all nodes of G. We represent such tree T with bold lines. We choose the arc $(b,a) \notin A(T)$ and we consider a path $p \in P_{1b}(G)$ such that $A(p) \subset A(T)$. We construct the scenario s' setting the arcs (b,a), and $(1,a)$ to their lower bounds and the lengths of the remaining arcs to their upper bounds. We apply Dijkstra to G under the scenario s' to obtain a shortest path q' from 1 to a, we compute $l^{s'}_{q'} = 1$, $l^{s'}_{p} = 3$ and $\underline{l}_{ba} = 4$. As $l^{s'}_{q'} < l^{s'}_{p} + \underline{l}_{ba}$ then the arc (b,a) is a non T-weak arc. Then we take a second arc $(1,b) \notin A(T)$ and we consider a path $p \in P_{11}(G)$ such that $A(p) \subset A(T)$. We construct the scenario s' setting the arc $(1,b)$, to their lower bound and the lengths of the remaining arcs to their upper bounds. We apply Dijkstra to G under the scenario s' to obtain a shortest path q' from 1 to b, we compute $l^{s'}_{q'} = 4$, $l^{s'}_{p} = 0$ and $\underline{l}_{1b} = 5$. As $l^{s'}_{q'} < l^{s'}_{p} + \underline{l}_{1b}$ then the arc $(1,b)$ is a non T-weak arc. Finally we choose the arc $(b,m) \notin A(T)$ and we consider a path $p \in P_{1b}(G)$ such that $A(p) \subset A(T)$. We construct the scenario s' setting the arcs (b,a), and $(1,a)$ to their lower bounds and the lengths of the remaining arcs to their upper bounds. We apply Dijkstra to G under the scenario s' to obtain a shortest path q' from 1 to m, we compute $l^{s'}_{q'} = 5$, $l^{s'}_{p} = 3$ and $\underline{l}_{bm} = 2$. As $l^{s'}_{q'} = l^{s'}_{p} + \underline{l}_{ba}$ then the arc (b,m) is a T-weak arc.

Conclusions and extensions :

We have given sufficient conditions for an arc (k,r) to be always on an optimal solution of the robust shortest path and for the single-source shortest path problems (strictly strong arcs and T-strictly strong arcs resp.) and for an arc to be never on an optimal solution (non weak arcs and non T-weak arcs resp.). Such conditions are based on the topology of the graph combined with the structure of the data set. Based on these results we have presented polynomial time recognition algorithms that can be used to preprocess a given graph with interval data prior to the solution of the robust problem. In [3] was shown that such type of preprocessing procedures are quite effective in reducing the time to compute the solution of the robust deviation path problem.

Future considerations in this area would involve the question : what are the cases when any shortest path algorithm would provide a robust deviation shortest path ? and the

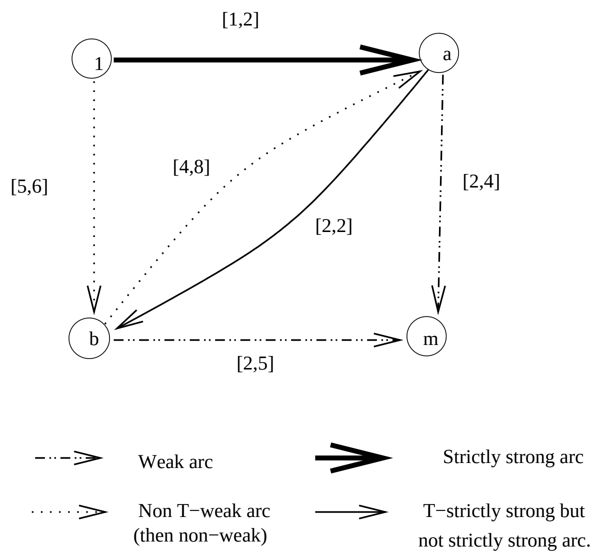

FIG. 4 – Classification of the arcs

investigation of the behavior of the number of arcs strong and non-weak on digraphs of big size.

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