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HÖLDER CONTINUITY OF SOLUTIONS OF SECOND-ORDER NON-LINEAR ELLIPTIC INTEGRO-DIFFERENTIAL EQUATIONS

GUY BARLES, EMMANUEL CHASSEIGNE, AND CYRIL IMBERT

ABSTRACT. This paper is concerned with the Hölder regularity of viscosity solutions of second-order, fully non-linear elliptic integro-differential equations. Our results rely on two key ingredients: first we assume that, at each point of the domain, either the equation is strictly elliptic in the classical fully non-linear sense, or (and this is the most original part of our work) the equation is strictly elliptic in a non-local non-linear sense we make precise. Next we impose some regularity and growth conditions on the equation. These results are concerned with a large class of integro-differential operators whose singular measures depend on x and also a large class of equations, including Bellman-Isaacs Equations.

Keywords: Hölder regularity, integro-differential equations, Lévy operators, general non-local operators, viscosity solutions

Mathematics Subject Classification: 35D99, 35J60, 35B05, 47G20

INTRODUCTION

The aim of this paper is to show that viscosity solutions of fully non-linear elliptic integro-differential equations are Hölder continuous under general suitable strict ellipticity and regularity/growth conditions on the equations. We also obtain explicit $C^{0,\alpha}$ estimates.

To be more specific, we describe the general framework we consider. We are interested in equations of the type

$$(1) \quad F(x, u, Du, D^2u, \mathcal{I}[x, u]) = 0 \quad \text{in } \Omega$$

where Ω is a domain of \mathbb{R}^N (not necessarily bounded) and $\mathcal{I}[x, u]$ is an integro-differential operator. The function u is real-valued and Du, D^2u stand respectively for its gradient and Hessian matrix. The non-linearity F is a (continuous) function which is degenerate elliptic: in this context, this means that F is non-increasing with respect to (wrt for short) its last two variables (see below for a precise statement and [4] for further details). The integro-differential operators we will consider in the present paper are defined as follows

$$(2) \quad \mathcal{I}[x, u] = \int_{\mathbb{R}^N} (u(x+z) - u(x) - Du(x) \cdot z \mathbf{1}_B(z)) \mu_x(dz)$$

where $\mathbf{1}_B$ denotes the indicator function of the unit ball B and $\{\mu_x\}_{x \in \mathbb{R}^N}$ is a family of Lévy measures, *i.e.* non-negative singular measures on \mathbb{R}^N such that

$$(3) \quad \int_{\mathbb{R}^N} \min(|z|^2, 1) \mu_x(dz) < +\infty.$$

We point out that the solution u has to be given in the whole space \mathbb{R}^N even if (1) is satisfied on a domain Ω ; see Section 1 for further details. Lévy-Itô operators are important “special cases” of operators of the form (2). They are defined as follows

$$(4) \quad \mathcal{I}_{LI}[x, u] = \int_{\mathbb{R}^N} (u(x+j(x, z)) - u(x) - Du(x) \cdot j(x, z) \mathbf{1}_B(z)) \mu(dz)$$

where μ is a Lévy measure and $j(x, z)$ is the size of the jumps at x . In order that the operator is well-defined, one assumes

$$|j(x, z)| \leq C_0 |z| \quad \text{for some constant } C_0 \text{ and for any } x \in \Omega, z \in \mathbb{R}^N.$$

A model example for a Lévy measure μ and jump function j is the ones associated with fractional Laplacian: $\mu(dz) = dz/|z|^{N+\alpha}$ with $0 < \alpha < 2$ and $j(x, z) = z$.

Many papers deal with Hölder estimates for fully non-linear elliptic equations. There are two kinds of approaches: for uniformly elliptic equations, one can use the powerful approach by Harnack inequalities which leads also to further regularity results; we refer the reader to Cabré and Caffarelli [7] or Trudinger [14, 15] and references therein for results in this direction. A simpler method, more closely related to classical viscosity solutions theory, was introduced by Ishii and Lions in [11]. Besides of its simplicity, it has the further advantage to provide results under weaker ellipticity assumptions and even for some degenerate equations; it was next used in [1], [9] where further regularity results are also proved and later in [5, 3]. As far as integro-differential elliptic equations are concerned, many papers were published in potential theory and equations are linear in most of these papers; moreover they rely on probabilist techniques. See for instance [6]. One of the first paper about Hölder estimates for integro-differential equations with PDE techniques is probably [13]. The author mainly deals with linear equations where singular measures μ_x have a very general x -dependence, improving the previous literature. He is also able to deal with quite particular non-linear equations; more precisely, he treats the case of $F(\mathcal{I}[x, u], \mathcal{J}[x, u]) = 0$ for “strictly elliptic” functions F .

In the present paper, we deal with fully non-linear elliptic equations and we obtain local $C^{0,\alpha}$ regularity and estimates for a quite general class of integro-differential operators, namely operators of the form (2) satisfying proper assumptions (see (6)-(8)). Even if important operators (4) can be seen as special cases of the general one, we will give specific results with specific assumptions.

Let us mention that, on the one hand, we cannot treat all the examples given in [13] (in particular the examples of the Section 3.4 of [13]) and, on the other hand, we can treat examples out of the scope of [13]; indeed, we only assume that the measure μ_x is bounded at infinity (uniformly with respect to x), while Condition (2.2) of [13] requires a (small) power of $|z|$ to be integrable at infinity. Moreover, we can handle much more general non-linear equations, in particular the important Bellman-Isaacs equations and we can also identify the critical Hölder exponent α of the solution. To be more precise, we are able to prove that the solution is α -Hölder continuous for any $\alpha < \beta$ (and even $\alpha = \beta$ under stronger assumptions in the case $\beta < 1$) where β characterizes the singularity of the measure associated with the integral operator.

In order to be able to treat a large class of non-linear elliptic equations, we decided to present the main results by assuming that the non-linearity F in (1) satisfies a proper ellipticity-growth condition (see (H) in Section 2). Freely speaking, this structure condition ensures that either the equation is strictly elliptic in the classical fully non-linear sense or its is strictly elliptic in a non-linear and non-local sense. Since this condition is rather technical, a whole section is devoted to comments and examples (see Section 3).

The techniques we used in the present paper do not seem to yield Lipschitz regularity when $\beta \geq 1$ and we would like to investigate this question in a future work. Let us mention that we proved [2] in quite a general framework that there exists a viscosity solution of the Dirichlet problem without loss of boundary condition. The local Hölder estimates we obtain in the present paper applies to the Dirichlet problem too and we will address naturally in a future work the question of boundary estimates. We would like also to point out that the techniques we develop here can be readily applied to parabolic integro-differential equations. Finally, another possible interesting application of these results is the study of the ergodicity of non-local equations.

The paper is organized as follows. In Section 1, we recall the definition of a viscosity solution of (1). In Section 2, we state the two main results; the first one deals with general x -dependent Lévy measure and the second one deals with integro-differential operators under the Lévy-Itô form. In Section 3, we make comments on the main structure assumption on the non-linearity F and we give examples of direct applications of our results. Section 4 is devoted to proofs of the main results.

Notation. The scalar product in \mathbb{R}^N is denoted by $x \cdot y$. A ball centered at x of radius r is denoted $B(x, r)$. If $x = 0$, we simply write B_r and if $r = 1$, we even write B . \mathcal{S}^{N-1} denotes the unit sphere of \mathbb{R}^N .

The transpose of a matrix A is denoted A^* and $\|A\|$ stands for the usual norm of A , namely $\|A\| := \max_{|z|=1} |Az|$. \mathbb{S}^N is the space of $N \times N$, real symmetric matrices. We recall that $X \geq Y$ if all the eigenvalues of $X - Y$ are non-negative. If $X \in \mathbb{S}^N$ and $\varepsilon \in (0, 1)$ is such that all the eigenvalues of X are strictly less than 1 (resp. strictly greater than -1), we set $X^\varepsilon = (I - \varepsilon X)^{-1}X$ (resp. $X_\varepsilon = (I + \varepsilon X)^{-1}X$). These matrices are obtained from X by applying a sup-convolution procedure (resp. an inf-convolution procedure), namely, for any $\xi \in \mathbb{R}^N$

$$X^\varepsilon \xi \cdot \xi = \sup_{\zeta \in \mathbb{R}^N} \left\{ X\zeta \cdot \zeta - \frac{|\xi - \zeta|^2}{\varepsilon} \right\}, \quad X_\varepsilon \xi \cdot \xi = \inf_{\zeta \in \mathbb{R}^N} \left\{ X\zeta \cdot \zeta + \frac{|\xi - \zeta|^2}{\varepsilon} \right\}.$$

1. VISCOSITY SOLUTIONS FOR PIDE

In this section, we recall the notion of degenerate ellipticity for non-linear non-local equations and the definition of viscosity solutions for such equations.

1.1. Degenerate ellipticity. Throughout the paper, the domain Ω is an open subset of \mathbb{R}^N and the non-linearity F is a continuous function. We also assume that (1) is degenerate elliptic. In this framework, this means that we make the following

ASSUMPTION (E). For any $x \in \mathbb{R}^N$, $u \in \mathbb{R}$, $p \in \mathbb{R}^N$, $X, Y \in \mathbb{S}^N$, $l_1, l_2 \in \mathbb{R}$

$$F(x, u, p, X, l_1) \leq F(x, u, p, Y, l_2) \quad \text{if } X \geq Y, \quad l_1 \geq l_2.$$

1.2. Non-Local operators. In order to define viscosity solutions for (1), we introduce two associated operators $\mathcal{I}^{1,\delta}$ and $\mathcal{I}^{2,\delta}$

$$\begin{aligned} \mathcal{I}^{1,\delta}[x, p, u] &= \int_{|z| < \delta} [u(x+z) - u(x) - p \cdot z \mathbf{1}_B(z)] \mu_x(dz), \\ \mathcal{I}^{2,\delta}[x, p, u] &= \int_{|z| \geq \delta} [u(x+z) - u(x) - p \cdot z \mathbf{1}_B(z)] \mu_x(dz). \end{aligned}$$

In the case of Lévy-Itô operators (4), $\mathcal{I}^{1,\delta}$ and $\mathcal{I}^{2,\delta}$ are defined as follows

$$\begin{aligned} \mathcal{I}^{1,\delta}[x, p, u] &= \int_{|z| < \delta} [u(x+j(x,z)) - u(x) - p \cdot j(x,z) \mathbf{1}_B(z)] \mu(dz), \\ \mathcal{I}^{2,\delta}[x, p, u] &= \int_{|z| \geq \delta} [u(x+j(x,z)) - u(x) - p \cdot j(x,z) \mathbf{1}_B(z)] \mu(dz). \end{aligned}$$

1.3. Definition. We now recall the definition of a viscosity solutions for (1). We assume that we are given a function u defined on the whole space \mathbb{R}^N .

Definition 1 (Viscosity solutions). *A usc function $u : \mathbb{R}^N \rightarrow \mathbb{R}$ is a subsolution of (1) if for any test function $\phi \in C^2(B(x, \delta))$ such that $u - \phi$ attains a maximum on $B(x, \delta)$ at $x \in \Omega$,*

$$F(x, u(x), D\phi(x), D^2\phi(x), \mathcal{I}^{1,\delta}[x, p, \phi] + \mathcal{I}^{2,\delta}[x, p, u]) \geq 0$$

where $p = D\phi(x)$.

A lsc function $u : \mathbb{R}^N \rightarrow \mathbb{R}$ is a supersolution of (1) if for any test function $\phi \in C^2(\mathbb{R}^N)$ such that $u - \phi$ attains a maximum on $B(x, r)$ at $x \in \Omega$ for some $r > 0$,

$$F(x, u(x), D\phi(x), D^2\phi(x), \mathcal{I}^{1,r}[x, p, \phi] + \mathcal{I}^{2,r}[x, u]) \leq 0.$$

A continuous function $u : \mathbb{R}^N \rightarrow \mathbb{R}$ is a solution of (1) if it is both a sub and a supersolution.

Remark 1. It is possible to construct solutions of (1) in the case where $\Omega = \mathbb{R}^N$. If $\Omega \neq \mathbb{R}^N$, boundary conditions must be imposed. For instance, as far as the Dirichlet problem is concerned, the function u can be prescribed outside of Ω . See [2] for further details.

Remark 2. As remarked in [4], one can choose $r = 0$ in the previous definition, at least in the case of Lévy-Itô operators. See [4] for further details.

2. MAIN RESULTS

This section is devoted to state the two main results of this paper. The first one is concerned with non-local operators of the form (2) (Theorem 1) and the second one permits to handle Lévy-Itô operators (4) (Theorem 2).

The two results rely on a structure condition imposed to the non-linearity F . In order to formulate it, we consider two functions $\Lambda_1, \Lambda_2 : \bar{\Omega} \rightarrow [0, +\infty)$ such that $\Lambda_1(x) + \Lambda_2(x) \geq \Lambda_0 > 0$ on $\bar{\Omega}$.

(H) (*Ellipticity-Growth condition*) For any $R > 0$, there exists constants $k \geq 0$, $\tau, \theta \in (0, 1]$, a locally bounded function $\chi : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$, a modulus of continuity $\omega_F : (0, +\infty) \rightarrow (0, +\infty)$, $\omega_F(0+) = 0$ and a constant $\eta > 0$ such that for any $x, y \in \Omega$ with $|x - y| \leq \eta$, $|u|, |v| \leq R$, $p \in \mathbb{R}^N$, $|q| \leq R$, $l_1 \leq l_2$, and any $L, \bar{\varepsilon}, \varpi > 0$ and $\hat{a} \in \mathcal{S}^{N-1}$, we have

$$\begin{aligned} F(y, u, p, Y, l_2) - F(x, v, p + q, X, l_1) & \leq \Lambda_1(x) \left(\text{Tr}(X - Y) + \frac{\omega_F(|x - y|)}{\bar{\varepsilon}} + |x - y|^\tau |p|^{2+\tau} + |p|^2 + \chi(L, \eta) \right) \\ & \quad + \Lambda_2(x) \left((l_1 - l_2) + \frac{|x - y|^{2\theta}}{\bar{\varepsilon}} + |x - y|^\tau |p|^{k+\tau} + C_F |p|^k + \chi(L, \eta) \right), \end{aligned}$$

if the matrices X, Y satisfy

$$(5) \quad -\frac{4}{\bar{\varepsilon}} I \leq \begin{bmatrix} X & 0 \\ 0 & -Y \end{bmatrix} \leq \frac{2}{\bar{\varepsilon}} \begin{bmatrix} Z & -Z \\ -Z & Z \end{bmatrix} + LI,$$

where I denotes the identity matrix and $Z = I - (1 + \varpi)\hat{a} \otimes \hat{a}$.

In the next subsection, we will make comments on this structure condition and give several examples. The general results are the following.

Theorem 1 (Hölder continuity for general non-local operators). *Assume that the measures μ_x satisfy the following: there exist $\beta \in (0, 2)$, a constant $\tilde{C}_\mu > 0$, a modulus of continuity $\omega_\mu : (0, +\infty) \rightarrow (0, +\infty)$, $\omega_\mu(0+) = 0$ and, for $\eta \in (0, 1)$, a constant $C_\mu(\eta) > 0$ such that for any $x, y \in \Omega$, $d \in \mathcal{S}^{N-1}$, $\eta, \delta \in (0, 1)$,*

$$(6) \quad \int_B |z|^2 \mu_x(dz) + \int_{\mathbb{R}^N \setminus B} \mu_x(dz) \leq \tilde{C}_\mu \quad , \quad \int_{|z| \leq \delta, |d \cdot z| \geq (1-\eta)|z|} |z|^2 \mu_x(dz) \geq C_\mu(\eta) \delta^{2-\beta}$$

$$(7) \quad \int_{B_\delta} |z|^2 |\mu_x - \mu_y|(dz) \leq \omega_\mu(|x - y|) \delta^{2-\beta}$$

$$(8) \quad \begin{cases} \forall \alpha \neq \beta, & \int_{B \setminus B_\delta} |z|^\alpha |\mu_x - \mu_y|(dz) \leq \omega_\mu(|x - y|) \delta^{\alpha-\beta} \\ & \int_{B \setminus B_\delta} |z|^\beta |\mu_x - \mu_y|(dz) \leq \omega_\mu(|x - y|) (\ln \delta)^{-1} \end{cases}$$

with, if $\beta = 1$, $\omega_\mu(\cdot)$ such that $\omega_\mu(r) (\ln r)^{-1} \rightarrow 0$ as $r \rightarrow 0$. Suppose also that the non-linearity F satisfies (H) for some parameters k, τ, θ .

(i) If

$$\theta > \frac{1}{2}(2 - \beta) \quad \text{and} \quad \begin{cases} k = \beta & \text{if } \beta > 1, \\ k < \beta & \text{if } \beta \leq 1, \end{cases}$$

then any bounded continuous viscosity solution u of (1) is locally α -Hölder continuous for α small enough. Precisely, α must satisfy: $\alpha < 1$ if $\beta \geq 1$ and $\alpha < \frac{\beta-k}{1-k}$ if $\beta < 1$. Moreover, the $C^{0,\alpha}$ estimate depends on $\|u\|_\infty$, N (dimension), the constants \tilde{C}_μ , $C_\mu(\eta)$ and the function ω_μ appearing in (6)-(8), on the constants and functions appearing in (H).

(ii) If $\beta < 1$ and if we assume moreover that

- $C_F = 0$ in (H) and $\tau > k(\beta^{-1} - 1)$
- $\omega_\mu(r)$ in (7) and (8) is replaced with $C_\mu r^{\tilde{\theta}}$ for some $\tilde{\theta} \in (0, 1)$,

then u is β -Hölder continuous and its Hölder constant depends on the data as the Hölder estimate obtained in the previous case.

We now turn to Lévy-Itô operators.

Theorem 2 (Hölder continuity with Lévy-Itô operators). *Assume that the function j appearing in the definition of \mathcal{I}_{LI} satisfies : there exist $c_0, C_0 > 0$ such that, for any $x \in \Omega$ and $z \in \mathbb{R}^N$,*

$$(9) \quad \begin{cases} c_0|z| \leq |j(x, z)| \leq C_0|z| \\ |j(x, z) - j(y, z)| \leq C_0|z||x - y|^{\tilde{\theta}} \end{cases}$$

with $\tilde{\theta} \in (0, 1)$. Assume, in addition, that the measure μ satisfies : there exist $\beta \in (0, 2)$, a constant $\tilde{C}_\mu > 0$ and, for any $\eta \in (0, 1)$, a constant $C_\mu(\eta) > 0$ such that for any $x \in \Omega$, $d \in \mathcal{S}^{N-1}$, $\eta, \delta \in (0, 1)$,

$$(10) \quad \int_B |j(x, z)|^2 \mu(dz) + \int_{\mathbb{R}^N \setminus B} \mu(dz) \leq \tilde{C}_\mu \quad , \quad \int_{\mathcal{C}_{\delta, \eta}(x)} |j(x, z)|^2 \mu(dz) \geq C_\mu(\eta) \delta^{2-\beta}$$

where $\mathcal{C}_{\delta, \eta}(x) := \{z : |j(x, z)| \leq \delta, |d \cdot j(x, z)| \geq (1 - \eta)|j(x, z)|\}$, and that, moreover, for any $\alpha \in (0, 2]$ and for δ small enough

$$(11) \quad \begin{cases} \forall \alpha \neq \beta, & \int_{B \setminus B_\delta} |z|^\alpha \mu(dz) \leq \tilde{C}_\mu \delta^{\alpha-\beta} \\ & \int_{B \setminus B_\delta} |z|^\beta \mu(dz) \leq \tilde{C}_\mu (\ln \delta)^{-1} \end{cases}$$

Assume, finally, that the non-linearity F satisfies (H) with parameters k, τ, θ . If

$$\theta, \tilde{\theta} > \frac{1}{2}(2 - \beta) \quad \text{and} \quad \begin{cases} k = \beta & \text{if } \beta > 1, \\ k < \beta & \text{if } \beta \leq 1, \end{cases}$$

then any bounded continuous viscosity solution u of (1) with $\mathcal{I}[x, u]$ replaced with (4) is locally α -Hölder continuous for any $\alpha < \min(1, \beta)$.

If, in addition, $C_F = 0$ in (H) and $\tau > k(1 - \beta)\beta^{-1}$, then u is β -Hölder continuous when $\beta < 1$.

Moreover, the $C^{0, \alpha}$ estimate depends on $\|u\|_\infty$, N and on the constants and functions appearing in (H) and (9)-(11).

Remark 3. It is worth pointing out that (9)-(10)-(11) are the analogues (or the translations) of (6)-(7)-(8) in a context which turns out to be a little bit simpler for the viscosity solutions point of view.

Typically we have in mind the measures μ which satisfy for some $C_\mu^\pm > 0$ and $\beta \in (0, 2)$

$$(12) \quad \frac{C_\mu^-}{|z|^{N+\beta}} dz \leq \mu(dz) \leq \frac{C_\mu^+}{|z|^{N+\beta}} dz.$$

and functions $j(x, z)$ such that $z \mapsto j(x, z)$ has an inverse function $J(x, Z)$ and that there exist $c_0, C_0 > 0$ such that

$$(13) \quad \begin{cases} \forall (x, z) \in B_r(x_0, 0), & c_0|z| \leq |j(x, z)| \leq C_0|z|, \\ \forall (x, Z) \in B_R(x_0, 0), & c_0|Z| \leq |j(x, Z)| \leq C_0|Z| \\ \forall Z \in \mathbb{R}^N, & c_0 \leq |\det D_z J(x_0, Z)| \\ \forall z \in \mathbb{R}^N, & |j(x, z) - j(y, z)| \leq C_0|z||x - y|^{\tilde{\theta}} \end{cases}$$

and these are the properties we will use. We are in such a case if, for instance, for any x , $D_z j(x, z)$ exists for $|z|$ small enough, is continuous in (x, z) and nonsingular for $z = 0$. Such a condition appears in [6].

3. COMMENTS AND EXAMPLES

In this section, we make comments on assumptions of the main theorems and give examples of applications. More precisely, we illustrate the different terms appearing in the structure condition (H); we give examples of non-local operators of type (2) and (4); eventually, we give a regularity result that applies to the Bellman-Isaacs equation.

3.1. Non-Linearities. In this subsection, we illustrate the structure condition (H) we used in Theorems 1 and 2. We can see that this structure condition combines two different terms. The first one permits to handle equations that are strictly elliptic in the usual sense. The second one permits to handle non-local equations that are strictly elliptic in a generalized (non-local) sense. Notice that imposing $\Lambda_1(x) + \Lambda_2(x) \geq \Lambda_0 > 0$ means that at each point $x \in \Omega$, the non-linearity is either strictly elliptic in the classical (non-linear) sense or strictly elliptic in the generalized (non-local) sense.

A typical situation is the following: we are given two open subsets $\mathcal{O}_1, \mathcal{O}_2$ of Ω such that $\mathcal{O}_1 \cup \mathcal{O}_2 = \Omega$ and the closure of $(\mathcal{O}_1)^c$ is included in \mathcal{O}_2 ; moreover we know that F satisfies (H) in \mathcal{O}_1 with $\Lambda_1(x) \equiv 1, \Lambda_2(x) \equiv 0$ while F satisfies (H) in \mathcal{O}_2 with $\Lambda_1(x) \equiv 0, \Lambda_2(x) \equiv 1$. Then, if Λ is a continuous function in Ω which is equal to 1 on the closure of $(\mathcal{O}_1)^c$ and which has a support which is included in \mathcal{O}_2 , then it is easy to check that F satisfies (H) with $\Lambda_1(x) \equiv 1 - \Lambda(x), \Lambda_2(x) \equiv \Lambda(x)$.

SECOND-ORDER EQUATIONS. A non-linearity is degenerate elliptic if, for $X \leq Y$,

$$F(x, u, p, Y, l) - F(x, u, p, X, l) \leq 0.$$

We recall that in the linear case, *i.e.* $F = -\text{tr}(AX)$ with A a (constant) symmetric matrix, this is equivalent to $A \geq 0$. In this special case, a strict ellipticity condition reads: $A \geq \Lambda I$. In the non-linear case, it reads

$$F(x, u, p, Y, l) - F(x, u, p, X, l) \leq \Lambda \text{tr}(X - Y)$$

as soon as $X \leq Y$. Hence, the first term of the structure condition is the non-linear analogue of the classical strict ellipticity condition.

We would like next to be able to treat equations with non-constant coefficients: $A = A(x)$. Consider for instance the following special linear case

$$F(x, X) = -\text{tr}(\sigma(x)\sigma^*(x)X)$$

where σ maps Ω into the space of $p \times N$ -matrices for some $p \leq N$. Such a linear operator appears naturally in the probabilistic framework. The second term appearing in the structure condition permits to treat the case of bounded uniformly continuous σ . Indeed, if ω_σ is a modulus of continuity of σ , apply (5) to the vector $z = (z_1, z_2)$ with $z_1 = \sigma(\bar{x})e, z_2 = \sigma(\bar{y})e$ and an arbitrary $e \in \mathcal{S}^{N-1}$, and get

$$\sigma^T(\bar{x})X\sigma(\bar{x})e \cdot e - \sigma^T(\bar{y})Y\sigma(\bar{y})e \cdot e \leq \frac{1}{\varepsilon}\omega_\sigma^2(|\bar{x} - \bar{y}|) + L\|\sigma\|_\infty^2$$

(we used that $Z \leq I$). Therefore, if $a := \sigma\sigma^*$

$$\text{Tr}(a(\bar{x})X) - \text{Tr}(a(\bar{y})Y) \leq \frac{1}{\varepsilon}d\omega_\sigma^2(|\bar{x} - \bar{y}|) + Ld\|\sigma\|_\infty^2.$$

Hence, choose $\omega_F(r) = d\omega_\sigma^2(r)$ and $\chi(L, \eta) = Ld\|\sigma\|_\infty^2$. It is worth pointing out that, in the uniformly elliptic case, a or σ is only required to be continuous; we will see below that a more restrictive Hölder continuity has to be satisfied when the equation is not uniformly elliptic and when the regularity comes from the non-local term.

We now turn to the third term, involving gradients p and $x - y$. Roughly speaking, we are able to treat local terms of the form $c(x)|Du(x)|^l$ with $l = 2$ if c is merely bounded and $l = 2 + \tau$ if c is locally τ -Hölder continuous. We leave details to the reader.

NON-LOCAL EQUATIONS. We now turn to the second terms appearing in (H) but corresponding to a non-local strict ellipticity. More precisely, imposing

$$F(x, u, p, X, l_1) - F(x, u, p, X, l_2) \leq 0$$

as soon as $l_1 \geq l_2$ is the non-local degenerate ellipticity condition. Hence, the term $\Lambda_2(x)(l_1 - l_2)$ can be interpreted as a strict non-local ellipticity condition.

The next terms are interpreted as for local equations. In particular, we see that on regions of Ω where F is only strictly elliptic through the non-local term ($\Lambda_1(x) = 0$), then

- σ has to be Hölder continuous (the same computations as above provide the $(|x - y|^{2\theta})$ -term if $\omega_\sigma^2(r) = Cr^\theta$);

- The (natural) quadratic growth in the gradient which can be handled in the case of uniformly elliptic equations, has to be replaced here by a $|Du|^l$ type growth with $l \leq k$, where k is related, in the regularity results, to the type of non-local uniform ellipticity we have: roughly speaking, for the fractional Laplacian case $(-\Delta)^{\frac{\beta}{2}}$, the natural growth turns out to be $k = \beta$, even if Theorem 1 shows that the case $\beta \leq 1$ is a little bit more particular. We refer the reader to Subsection 3.3 where an example of equation involving a fractional Laplacian is treated in details.

We say more about these assumptions in the next subsections. In particular, we treat equations that are not exactly of the form (1) but can be handled with the same techniques (see Subsection 3.4).

3.2. Singular measures. The model singular measure is the Lévy measure associated with the fractional Laplacian $(-\Delta)^{\frac{\beta}{2}}$. In this case, $\mu_x(z) = \mu(z) = 1/|z|^{N+\beta}$ with $0 < \beta < 2$.

A second simple example of measure μ_x is $c(x, z)\mu(dz)$ with a Lévy measure μ satisfying (10)-(11) and $c(x, z)$ satisfying for any $x, y \in \Omega$ and $z \in B$

$$|c(x, z) - c(y, z)| \leq \omega(|x - y|).$$

One can thus easily check (6), (7), (8) and Theorem 1 applies for suitable non-linearities F . Lévy measures associated with tempered stable Lévy processes satisfy (10)-(11). Indeed, in this case

$$\mu(dz) = \mathbf{1}_{(0, +\infty)}(z) \left(G^+ e^{-\lambda^+ |z|} \frac{dz}{|z|^{N+\alpha}} \right) + \mathbf{1}_{(-\infty, 0)}(z) \left(G^- e^{-\lambda^- |z|} \frac{dz}{|z|^{N+\alpha}} \right).$$

These measures appear in financial modeling, see for instance [8].

3.3. A non-Local equation involving the fractional Laplacian. In order to illustrate further our results, we next consider the following model equation

$$(-\Delta)^{\frac{\beta}{2}} u + b(x)|Du|^{k+\tau} + |Du|^k = 0$$

where $b \in C^{0,\tau}$, $0 < \tau < 1$, $0 < k < 2$. In this case, Condition (H) is satisfied with $\Lambda_1 = 0$, $\Lambda_2(x) = 1 > 0$, $\theta > 0$ is arbitrary and τ, k appear in the equation. It is easy to check that (6) is satisfied with exponent β : first, it is a Lévy measure; if $\mathcal{C}_{\delta, \eta}$ denotes $\{|z| < |\delta|, (d \cdot z) \geq (1 - \eta)|z|\}$, then by homogeneity and symmetry of μ ,

$$\begin{aligned} \int_{\mathcal{C}_{\delta, \eta}} |z|^2 \mu(dz) &= \frac{|\mathcal{C}_{\delta, \eta}|}{|B_\delta|} \int_{B_\delta} |z|^2 \mu(dz) = \frac{|\mathcal{C}_{1, \eta}|}{|B_1|} \int_{B_\delta} |z|^2 \mu(dz) = \frac{|\mathcal{C}_{1, \eta}|}{|B_1|} \int_{B_\delta} |z|^{2-\beta-N} \mu(dz) \\ &= C_\mu(\eta) \delta^{2-\beta}. \end{aligned}$$

Moreover, the other hypotheses on μ (namely (7) and (8)) are automatically satisfied since μ is independent of x (i.e., one can choose $\omega_\mu = 0$). Since $\beta < 2$, we cannot allow here a quadratic growth for the gradient term, indeed Theorem 1 and 2 work only for $k \leq \beta$ in the absence of local ellipticity.

3.4. The Bellman-Isaacs equation. Let us illustrate Theorem 2 by considering an important second-order non-linear elliptic integro-differential equations appearing in the study of stochastic control of processes with jumps, namely the Bellman-Isaacs equation. Let us mention the work of Jakobsen and Karlsen [12] in the evolution case where the authors use completely different techniques.

Corollary 1. Consider the following Bellman-Isaacs equation in \mathbb{R}^N

$$(14) \quad cu + \sup_{\lambda \in \Lambda} \inf_{\gamma \in \Gamma} \left\{ -\frac{1}{2} \text{Tr}(\sigma_{\lambda, \gamma}(x) \sigma_{\lambda, \gamma}^*(x) D^2 u) - b_{\lambda, \gamma}(x) \cdot Du - \mathcal{I}_{LI}^{\lambda, \gamma}[x, u] - f_{\lambda, \gamma}(x) \right\} = 0$$

with $c \geq 0$ and where $\mathcal{I}_{LI}^{\lambda, \gamma}[x, u]$ is a family of Lévy-Itô operators associated with a common Lévy measure μ and a family of functions $j_{\lambda, \gamma}(x, z)$. Assume that

- (i) μ verifies (10)-(11),
- (ii) there exists $c_0, C_0 > 0$ and $\tilde{\theta} \in (0, 1)$ such that for any $(\lambda, \gamma) \in \Lambda \times \Gamma$, $j_{\lambda, \gamma}$ satisfies (9),
- (iii) $\sigma_{\lambda, \gamma}$, $b_{\lambda, \gamma}$ and $f_{\lambda, \gamma}$ satisfy for some $\theta \in (0, 1)$ (and some constant $C_F > 0$)

$$\forall \alpha, \beta, \quad \|\sigma_{\lambda, \gamma}\|_{0, \theta} + \|b_{\lambda, \gamma}\|_{0, \theta} + \|f_{\lambda, \gamma}\|_{0, \theta} \leq C_F.$$

If $\theta, \tilde{\theta} > \frac{1}{2}(2 - \beta)$, then any bounded viscosity solution u of (14) is α -Hölder continuous for any $\alpha < 1$ if $\beta \geq 1$ and for $\alpha < \frac{\beta-k}{1-k}$ if $\beta < 1$.

Proof. Remark that (14) is not exactly of the form (1). Nevertheless, the proof of Theorem 2 we present below can be adapted to this framework. It is enough to check that the structure condition are satisfied by the linear equations whose corresponding function F is

$$F_{\lambda,\gamma}(x, u, p, A, l) = \Lambda u - \frac{1}{2} \text{Tr}(\sigma_{\lambda,\gamma}(x) \sigma_{\lambda,\gamma}^*(x) A) - b_{\lambda,\gamma}(x) \cdot p - l - f_{\lambda,\gamma}(x).$$

Indeed, remark that if X denotes (x, u, p, A, l) and $F(X) = \sup_{\lambda} \inf_{\gamma} F_{\lambda,\gamma}(X)$, remark that

$$F(X) - F(Y) \leq \sup_{\lambda,\gamma} (F_{\lambda,\gamma}(X) - F_{\lambda,\gamma}(Y)).$$

Hence, if the constants and functions appearing in Condition (H) do not depend on λ, γ , we can conclude in the case of F too. \square

4. PROOFS OF THEOREMS 1 AND 2

We prove successively Theorems 1 and 2. On one hand, the different proofs are very similar and we will skip details when adapting arguments. On the other hand, we need to use very precisely every parameter. This is the reason why constants are computed from line to line and explicit formulae are given for each of them in order to be able to use them later.

4.1. Proof of Part (i) of Theorem 1.

Proof of Theorem 1. (i). Without loss of generality, we assume here that $C_F = 1$ in the ellipticity-growth condition (H). In order to prove the local Hölder continuity of u , we are going to show that, for any $x_0 \in \Omega$, there exists $L_2 = L_2(x_0)$ such that, for some well chosen $\alpha \in (0, 1)$ and for $L_1 = L_1(x_0) > 0$ large enough, we have

$$M = \sup_{x,y \in \mathbb{R}^N} \{u(x) - u(y) - \phi(x-y) - \Gamma(x)\} \leq 0$$

where $\phi(z) = L_1|z|^\alpha$ and $\Gamma(x) = L_2|x - x_0|^2$. We point out that the role of the Γ -term is to localize around x_0 , while the ϕ -one is concerned with the Hölder continuity. Proving such a result with a suitable control on α, L_1, L_2 clearly implies the desired property.

If $\Omega \neq \mathbb{R}^N$, we first choose L_2 in order that $u(x) - u(y) - \phi(x-y) - \Gamma(x) \leq 0$ if $x \notin \Omega$: to do so, we first choose

$$L_2 \geq \frac{8\|u\|_\infty}{[d(x_0, \partial\Omega)]^2}.$$

If $\Omega = \mathbb{R}^N$, L_2 is arbitrary. Then we argue by contradiction: we assume that $M > 0$ and we are going to get a contradiction for L_1 large enough and for a suitable choice of α .

If the supremum defining M is attained at (\bar{x}, \bar{y}) , we then deduce from $M > 0$ that $\bar{x} \neq \bar{y}$ and

$$(15) \quad |\bar{x} - \bar{y}| \leq \left(\frac{2\|u\|_\infty}{L_1} \right)^{\frac{1}{\alpha}} =: A, \quad |\bar{x} - x_0| < \sqrt{\frac{2\|u\|_\infty}{L_2}} =: R_2 \leq \frac{d(x_0, \partial\Omega)}{2}, \quad u(\bar{x}) > u(\bar{y}).$$

If L_1 is large enough in order to have $A < \frac{d(x_0, \partial\Omega)}{2}$, we also have $\bar{x}, \bar{y} \in \Omega$.

Next, we pick some $\nu_0 \in (0, 1)$ and we define

$$a = \bar{x} - \bar{y}, \quad \varepsilon = |a| \quad \hat{a} = \frac{a}{|a|}, \quad \delta = \nu_0 \varepsilon < \varepsilon.$$

First, ν_0 will be chosen independently of L_1 . But next, δ will be chosen as a power of ε and in this case, so will be ν_0 .

From the study of the maximum point property for (\bar{x}, \bar{y}) , we also get

$$(16) \quad L_1 \varepsilon^\alpha \leq u(\bar{x}) - u(\bar{y}) \leq \omega_u(\varepsilon)$$

where ω_u denotes the modulus of continuity of u on $B(x_0, d(x_0, \partial\Omega)/2)$. We will use this piece of information below (see Step 4). Notice also that if $\chi(x, y)$ denotes $\phi(x - y) + \Gamma(x)$, then $x \mapsto u(x) - \chi(x, \bar{y})$ (resp. $y \mapsto u(y) + \chi(\bar{x}, y)$) attains a global maximum (resp. minimum) at \bar{x} (resp. \bar{y}) with $\chi(\cdot, \bar{y})$ (resp. $-\chi(\bar{x}, \cdot)$) of class C^2 on $B(\bar{x}, \delta)$. In particular, we can use $\chi(\bar{x}, \cdot)$ and $-\chi(\cdot, \bar{y})$ as test-functions in Definition 1 with any $\delta' < \delta$.

The remaining of the proof is divided in four steps. We write down the viscosity inequalities and combine them (Step 1), we get suitable matrices inequalities from non-local Jensen-Ishii's lemma (Step 2), we estimate from above the difference of the non-local terms (Step 3) and we conclude (Step 4).

STEP 1: WRITING DOWN VISCOSITY INEQUALITIES. Let p denote $D\phi(a)$ and q denote $D\Gamma(\bar{x})$. We use Corollary 1 of [4] which, for $\iota > 0$ small enough, provides us with two matrices $X_\iota, Y_\iota \in \mathbb{S}^N$ such that, for any $\delta' \ll 1$

$$\begin{aligned} F(\bar{x}, u(\bar{x}), p + q, X_\iota, \mathcal{I}^{1, \delta'}[\bar{x}, p + q, \chi_\iota(\cdot, \bar{y})]) + \mathcal{I}^{2, \delta'}[\bar{x}, p + q, u] + o_\iota(1) &\leq 0, \\ F(\bar{y}, u(\bar{y}), p, Y_\iota, \mathcal{I}^{1, \delta'}[\bar{y}, p, -\chi_\iota(\bar{x}, \cdot)]) + \mathcal{I}^{2, \delta'}[\bar{y}, p, u] + o_\iota(1) &\geq 0 \end{aligned}$$

(where χ_ι is an approximation of χ by a localized inf-convolution, see [4]) with the following matrix inequality

$$(17) \quad -\frac{1}{\iota}I \leq \begin{bmatrix} X_\iota & 0 \\ 0 & -Y_\iota \end{bmatrix} \leq \begin{bmatrix} Z & -Z \\ -Z & Z \end{bmatrix} + 2L_2 \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} + o_\iota(1),$$

where $Z = D^2\phi(a)$.

Our aim is to first let ι tend to 0 in order to get rid of all the artificial ι dependences in these inequalities: in order to do so, but also in order to apply (H) which requires a two-side bound on the matrices, we are first going to build matrices X, Y such that the above viscosity inequalities still hold if we replace X_ι, Y_ι by X, Y and such that the matrices X, Y satisfy the required inequality in (H).

Then, if we set

$$l_1 := \mathcal{I}^{1, \delta'}[\bar{x}, p + q, \chi_\iota(\cdot, \bar{y})] + \mathcal{I}^{2, \delta'}[\bar{x}, p + q, u], \quad l_2 := \mathcal{I}^{1, \delta'}[\bar{y}, p, -\chi_\iota(\bar{x}, \cdot)] + \mathcal{I}^{2, \delta'}[\bar{y}, p, u],$$

and if we subtract the viscosity inequalities, dropping all the ι dependences, we will have

$$(18) \quad 0 \leq F(\bar{y}, u(\bar{y}), p, Y, l_2) - F(\bar{x}, u(\bar{x}), p + q, X, l_1).$$

In order to get the desired contradiction, the rest of the proof consists in obtaining various estimates, and in particular on the differences $X - Y$ and $\mathcal{I}^{2, \delta'}[\bar{x}, p + q, u] - \mathcal{I}^{2, \delta'}[\bar{y}, p, u]$, in order to apply the ellipticity-growth condition (H) to show that the right-hand side of this inequality is strictly negative. We point out that, since we are going to let first δ' tend to 0, the terms $\mathcal{I}^{1, \delta'}[\bar{x}, p + q, \chi_\iota(\cdot, \bar{y})]$, $\mathcal{I}^{1, \delta'}[\bar{y}, p, -\chi_\iota(\bar{x}, \cdot)]$ create no difficulty because they tend to 0 with δ' .

STEP 2: BUILDING AND ESTIMATING THE MATRICES X, Y . We follow here ideas introduced by Crandall and Ishii [10] to obtain these matrices, by using only the upper bounds on X_ι, Y_ι . We rewrite Inequality (17) as: for any $z_1, z_2 \in \mathbb{R}^N$, we have

$$X_\iota z_1 \cdot z_1 - Y_\iota z_2 \cdot z_2 \leq Z(z_1 - z_2) \cdot (z_1 - z_2) + 2L_2|z_1|^2.$$

We have dropped the $o_\iota(1)$ -term in the right-hand side for the sake of simplicity since it plays no role. In fact, we use the previous matrix inequality on the following form

$$(X_\iota - 2L_2I)z_1 \cdot z_1 - Y_\iota z_2 \cdot z_2 \leq Z(z_1 - z_2) \cdot (z_1 - z_2).$$

Next we have to compute Z , as well as, for the rest of the proof, the derivatives of ϕ . It will be convenient to do the proof for $\phi(x) = \varphi(|x|)$ for a general smooth function $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}$. We thus get for any $b \in \mathbb{R}^N$

$$(19) \quad \begin{aligned} D\phi(b) &= \varphi'(|b|)\hat{b} \\ D^2\phi(b) &= \frac{\varphi'(|b|)}{|b|}P_{b^\perp} + \varphi''(|b|)\hat{b} \otimes \hat{b} \end{aligned}$$

where $\hat{b} = b/|b|$ and $P_{b^\perp} = I - \hat{b} \otimes \hat{b}$ is the projection on the orthogonal space of b . Hence, if $\varphi(r) = L_1 r^\alpha$, we get

$$(20) \quad \begin{aligned} D\phi(b) &= L_1 \alpha |b|^{\alpha-2} b \\ D^2\phi(b) &= L_1 (\alpha |b|^{\alpha-2} I + \alpha(\alpha-2) |b|^{\alpha-4} b \otimes b) = L_1 \alpha |b|^{\alpha-4} (|b|^2 I - (2-\alpha) b \otimes b) \end{aligned}$$

$$(21) \quad D^2\phi(b) \leq L_1 \alpha |b|^{\alpha-2} I.$$

We have

$$Z = \frac{1}{\bar{\varepsilon}} \left(I - (2-\alpha) \hat{a} \otimes \hat{a} \right) \quad \text{with} \quad \bar{\varepsilon} = (L_1 \alpha \varepsilon^{\alpha-2})^{-1}.$$

Now we come back to the X_ℓ, Y_ℓ inequality: we apply to this inequality a sup-convolution in both variables z_1 and z_2 with a parameter which is $\frac{1}{4}\bar{\varepsilon}$. Noticing that this corresponds to an inf-convolution on the $(Y_\ell z_2 \cdot z_2)$ -term, we easily get, with the notation introduced at the end of the Introduction,

$$(X_\ell - 2L_2 I)^{\frac{1}{4}\bar{\varepsilon}} z_1 \cdot z_1 - (Y_\ell)^{\frac{1}{4}\bar{\varepsilon}} z_2 \cdot z_2 \leq Z^{\frac{1}{2}\bar{\varepsilon}} (z_1 - z_2) \cdot (z_1 - z_2).$$

On the other hand, tedious but easy, explicit computations yield $Z^{\frac{1}{2}\bar{\varepsilon}} = 2\bar{\varepsilon}^{-1} (I - (1+\varpi) \hat{a} \otimes \hat{a})$ with

$$\varpi := \frac{1-\alpha}{3-\alpha} > 0.$$

If we set $X = (X_\ell - 2L_2 I)^{\frac{1}{4}\bar{\varepsilon}} + 2L_2 I$, $Y = (Y_\ell)^{\frac{1}{4}\bar{\varepsilon}}$, then X, Y satisfy (5) with $L = 2L_2$ and since $X_\ell \leq X$ and $Y \leq Y_\ell$, the viscosity inequalities still hold for X and Y because F is degenerate elliptic.

From this new form of inequality (17), we can obtain several type of estimates on X and Y : first, choosing $z_2 = -z_1 = \hat{a}$, we get

$$X \hat{a} \cdot \hat{a} - Y \hat{a} \cdot \hat{a} \leq -\frac{8\varpi}{\bar{\varepsilon}} + O(L_2) = -8L_1 \alpha \varpi \varepsilon^{\alpha-2} + O(L_2).$$

Next choosing $z_2 = z_1 = z$ with z being orthogonal to \hat{a} , we have

$$X z \cdot z - Y z \cdot z \leq O(L_2).$$

In particular, this yields

$$(22) \quad \text{Tr}(X - Y) \leq -8L_1 \alpha \varpi \varepsilon^{\alpha-2} + O(L_2).$$

STEP 3: ESTIMATES OF THE NON-LOCAL TERMS. The difference of the non-local terms, denoted by T_{nl} , can be rewritten as follows

$$T_{nl} = \int_{|z| \geq \delta'} [u(\bar{x} + z) - u(\bar{x}) - (p+q) \cdot z \mathbf{1}_B(z)] \mu_{\bar{x}}(dz) - \int_{|z| \geq \delta'} [u(\bar{y} + z) - u(\bar{y}) - p \cdot z \mathbf{1}_B(z)] \mu_{\bar{y}}(dz).$$

In order to estimate it, we part the domain of integration $\{|z| \geq \delta'\}$ into three pieces $\mathbb{R}^N \setminus B$ which leads to the T_1 term below, $\mathcal{C} = \{z \in B_\delta : |z \cdot \hat{a}| \geq (1-\eta)|z|\} \subset B$ which leads to the T_2 term below and $B \setminus \mathcal{C}$ which leads to the T_3 term below.

In order not to add further technicalities, we assume from now on that $\delta' = 0$; the reader can check that if $\delta' > 0$, the estimates we present below remain valid up to $o_{\delta'}(1)$.

Therefore we have to estimate from above $T_{nl} = T_1 + T_2 + T_3 + o_{\delta'}(1)$ with

$$\begin{aligned} T_1 &= \int_{|z| \geq 1} [u(\bar{x} + z) - u(\bar{x})] \mu_{\bar{x}}(dz) - \int_{|z| \geq 1} [u(\bar{y} + z) - u(\bar{y})] \mu_{\bar{y}}(dz) \\ T_2 &= \int_{\mathcal{C}} [u(\bar{x} + z) - u(\bar{x}) - (p+q) \cdot z] \mu_{\bar{x}}(dz) - \int_{\mathcal{C}} [u(\bar{y} + z) - u(\bar{y}) - p \cdot z] \mu_{\bar{y}}(dz) \\ T_3 &= \int_{B \setminus \mathcal{C}} [u(\bar{x} + z) - u(\bar{x}) - (p+q) \cdot z] \mu_{\bar{x}}(dz) - \int_{B \setminus \mathcal{C}} [u(\bar{y} + z) - u(\bar{y}) - p \cdot z] \mu_{\bar{y}}(dz) \end{aligned}$$

Estimate of T_1 . Since u is bounded and so are the μ_x away from the origin, we conclude that T_1 is bounded, uniformly with respect to all the parameters we introduced. More precisely,

$$(23) \quad T_1 \leq C_1$$

where

$$(24) \quad C_1 = 4\|u\|_\infty \sup_{x \in B(x_0, d(x_0, \partial\Omega)/2)} \mu_x(\mathbb{R}^N \setminus B).$$

Estimate of T_2 . We estimate T_2 from above by using the definition of M . Indeed, its definition provides the following key inequality

$$(25) \quad \begin{aligned} u(\bar{x} + d) - u(\bar{x}) - (p + q) \cdot d &\leq u(\bar{y} + d') - u(\bar{y}) - p \cdot d' \\ &+ \left\{ \phi(a + d - d') - \phi(a) - D\phi(a) \cdot (d - d') \right\} \\ &+ \left\{ \Gamma(\bar{x} + d) - \Gamma(\bar{x}) - D\Gamma(\bar{x}) \cdot d \right\}. \end{aligned}$$

We then use (25) with $d = z$ and $d' = 0$ (resp. with $d = 0$ and $d' = z$). We obtain

$$\begin{aligned} T_2 &\leq \int_{\mathcal{C}} [\phi(a + z) - \phi(a) - D\phi(a) \cdot z] \mu_{\bar{x}}(dz) \\ &+ \int_{\mathcal{C}} [\phi(a - z) - \phi(a) + D\phi(a) \cdot z] \mu_{\bar{y}}(dz) + \int_{\mathcal{C}} [\Gamma(\bar{x} + z) - \Gamma(\bar{x}) - D\Gamma(\bar{x}) \cdot z] \mu_{\bar{x}}(dz) \end{aligned}$$

We will now use a second-order Taylor expansion in each integral. First, according to the form of Γ , we have

$$\int_{\mathcal{C}} [\Gamma(\bar{x} + z) - \Gamma(\bar{x}) - D\Gamma(\bar{x}) \cdot z] \mu_{\bar{x}}(dz) = L_2 \int_{\mathcal{C}} |z|^2 \mu_{\bar{x}}(dz) = O(L_2).$$

Next, for the two other terms and for a general $\phi(x) = \varphi(|x|)$, we recall (19) and we finally get

$$T_2 \leq \frac{1}{2} \int_{\mathcal{C}} \sup_{t \in (-1, 1)} \left(\frac{\varphi'(|a + tz|)}{|a + tz|} P_{(a+tz)^\perp} z \cdot z + \varphi''(|a + tz|) (\widehat{a + tz} \cdot z)^2 \right) (\mu_{\bar{x}} + \mu_{\bar{y}})(dz) + O(L_2).$$

and using next that $\varphi(r) = L_1 r^\alpha$, we obtain

$$T_2 \leq \frac{L_1 \alpha}{2} \int_{\mathcal{C}} \sup_{t \in (-1, 1)} |a + tz|^{\alpha-4} (|a + tz|^2 |z|^2 - (2 - \alpha)((a + tz) \cdot z)^2) (\mu_{\bar{x}} + \mu_{\bar{y}})(dz) + O(L_2).$$

Let b denote $a + tz$ and estimate $|b|$ and $b \cdot z$ for $z \in \mathcal{C}$ as follows

$$\begin{aligned} |b| &\leq \varepsilon + t|z| \leq \varepsilon + \delta \leq (1 + \nu_0)\varepsilon \\ |b \cdot z| &= |a \cdot z + t|z|^2| \geq (1 - \eta)\varepsilon|z| - \delta|z| \geq (1 - \eta - \nu_0)\varepsilon|z| \\ |b|^2 |z|^2 - (2 - \alpha)(b \cdot z)^2 &\leq (1 + \nu_0)^2 \varepsilon^2 |z|^2 - (2 - \alpha)(1 - \eta - \nu_0)^2 \varepsilon^2 |z|^2 \\ &\leq ((1 + \nu_0)^2 - (2 - \alpha)(1 - \eta - \nu_0)^2) \varepsilon^2 |z|^2 \end{aligned}$$

where we chose η, ν_0 small enough so that

$$(26) \quad (2 - \alpha)(1 - \eta - \nu_0)^2 - (1 + \nu_0)^2 > 0.$$

Hence

$$T_2 \leq -L_1 C_4 \varepsilon^{\alpha-2} \int_{\mathcal{C}} |z|^2 (\mu_{\bar{x}} + \mu_{\bar{y}})(dz) + O(L_2)$$

with

$$(27) \quad C_4 = \frac{\alpha}{2} \left((2 - \alpha)(1 - \eta - \nu_0)^2 - (1 + \nu_0)^2 \right) (1 + \nu_0)^{\alpha-4}.$$

Using (6), we thus obtain

$$(28) \quad T_2 \leq -L_1 C_4 C_\mu(\eta) \varepsilon^{\alpha-2} \delta^{2-\beta} + O(L_2),$$

and finally,

$$(29) \quad T_2 \leq -L_1 C_5 \varepsilon^{\alpha-\beta} + O(L_2),$$

with

$$(30) \quad C_5 = C_4 C_\mu(\eta) \nu_0^{2-\beta}.$$

Estimate of T_3 . In order to estimate T_3 from above, it is convenient to introduce

$$\begin{aligned} l_1(z) &= u(\bar{x} + z) - u(\bar{x}) - (p + q) \cdot z \\ l_2(z) &= u(\bar{y} + z) - u(\bar{y}) - p \cdot z \end{aligned}$$

and write

$$T_3 = \int_{B \setminus \mathcal{C}} l_1(z) \mu_{\bar{x}}(dz) - \int_{B \setminus \mathcal{C}} l_2(z) \mu_{\bar{y}}(dz).$$

We first remark that (25) with, successively $d = d'$, $d' = 0$ and $d = 0$ yields

$$(31) \quad \begin{cases} l_1(z) - l_2(z) & \leq \left(\Gamma(\bar{x} + z) - \Gamma(\bar{x}) - D\Gamma(\bar{x}) \cdot z \right) \\ l_1(z) & \leq \left(\phi(a + z) - \phi(a) - D\phi(a) \cdot z \right) + \left(\Gamma(\bar{x} + z) - \Gamma(\bar{x}) - D\Gamma(\bar{x}) \cdot z \right) \\ l_2(z) & \geq - \left(\phi(a - z) - \phi(a) + D\phi(a) \cdot z \right). \end{cases}$$

We next consider the signed measure $\mu(dz) = \mu_{\bar{x}}(dz) - \mu_{\bar{y}}(dz)$. It can be represented with two non-negative measures μ^\pm as follows: $\mu = \mu^+ - \mu^-$. We would like next to introduce a measure $\min(\mu_{\bar{x}}, \mu_{\bar{y}})$. To make it precise, let K denote the support of μ^+ and define $\bar{\mu} = \mathbf{1}_K \mu_{\bar{y}} + (1 - \mathbf{1}_K) \mu_{\bar{x}}$. We now rewrite T_3 with these measures and use (31) to get

$$\begin{aligned} T_3 &= \int_{B \setminus \mathcal{C}} (l_1(z) - l_2(z)) \bar{\mu}(dz) + \int_{B \setminus \mathcal{C}} l_1(z) \mu^+(dz) - \int_{B \setminus \mathcal{C}} l_2(z) \mu^-(dz) \\ &\leq \int_{B \setminus \mathcal{C}} [\Gamma(\bar{x} + z) - \Gamma(\bar{x}) - D\Gamma(\bar{x}) \cdot z] \bar{\mu}(dz) + \int_{B \setminus \mathcal{C}} [\phi(a + z) - \phi(a) - D\phi(a) \cdot z] \mu^+(dz) \\ &\quad + \int_{B \setminus \mathcal{C}} [\phi(a - z) - \phi(a) + D\phi(a) \cdot z] \mu^-(dz). \end{aligned}$$

To estimate T_3 , we first remark that

$$\int_{B \setminus \mathcal{C}} [\Gamma(\bar{x} + z) - \Gamma(\bar{x}) - D\Gamma(\bar{x}) \cdot z] \bar{\mu}(dz) = L_2 \int_B |z|^2 \bar{\mu}(dz) = \tilde{C}_\mu L_2.$$

Next, for the two other terms, we split the integration domain into $B \setminus B_\delta$ and $B_\delta \setminus \mathcal{C}$. Using once again a second-order Taylor expansion for ϕ together with (6), it comes

$$\begin{aligned} T_3 &\leq \tilde{C}_\mu L_2 + L_1 \int_{B_\delta \setminus \mathcal{C}} (\varepsilon - \delta)^{\alpha-2} |z|^2 |\mu|(dz) + L_1 \int_{B \setminus B_\delta} [|z|^\alpha + \varepsilon^{\alpha-1} |z|] |\mu|(dz) \\ &\leq \tilde{C}_\mu L_2 + (1 - \nu_0)^{\alpha-2} L_1 \varepsilon^{\alpha-2} \int_{B_\delta \setminus \mathcal{C}} |z|^2 |\mu_{\bar{x}} - \mu_{\bar{y}}|(dz) \\ &\quad + L_1 \int_{B \setminus B_\delta} |z|^\alpha |\mu_{\bar{x}} - \mu_{\bar{y}}|(dz) + L_1 \varepsilon^{\alpha-1} \int_{B \setminus B_\delta} |z| |\mu_{\bar{x}} - \mu_{\bar{y}}|(dz). \end{aligned}$$

where $|\mu| = \mu^+ + \mu^-$ and where we used (21), that $\alpha < 1$ and the α -Hölder continuity of ϕ . We now use (7) and (8). It is convenient to introduce

$$\psi_\gamma(r) = \begin{cases} r^{\gamma-\beta} & \text{if } \gamma \neq \beta \\ \ln r^{-1} & \text{if } \gamma = \beta. \end{cases}$$

We get for $\alpha < \min(1, \beta)$

$$(32) \quad T_3 \leq \tilde{C}_\mu L_2 + 2\nu_0^{\alpha-\beta} L_1 \omega_\mu(\varepsilon) \varepsilon^{\alpha-\beta} + L_1 \omega_\mu(\varepsilon) \varepsilon^{\alpha-1} \psi_1(\nu_0 \varepsilon)$$

for ν_0 small enough. We use here that $\nu_0^{\alpha-\beta}$ controls $(1 - \nu_0)^{\alpha-2} \nu_0^{2-\beta}$ from above.

If now $\alpha = \beta < 1$, we get

$$(33) \quad \begin{aligned} T_3 &\leq \tilde{C}_\mu L_2 + L_1 (\varepsilon - \delta)^{\beta-2} \delta^{2-\beta} \omega_\mu(\varepsilon) + L_1 \omega_\mu(\varepsilon) \ln \delta^{-1} + L_1 \omega_\mu(\varepsilon) \varepsilon^{\beta-1} \delta^{1-\beta} \\ T_3 &\leq \tilde{C}_\mu L_2 + 2L_1 \omega_\mu(\varepsilon) \ln \delta^{-1} + o(L_1 \omega_\mu(\varepsilon) \ln \delta^{-1}). \end{aligned}$$

Final estimate for T_{nl} . Combining (23),(29), (32), we finally get for $\alpha < \min(1, \beta)$ and $\beta \neq 1$

$$(34) \quad T_{nl} \leq -L_1 C_5 \varepsilon^{\alpha-\beta} + o(\varepsilon^{\alpha-\beta}) + O(L_2).$$

If $\alpha < 1$ and $\beta = 1$

$$T_{nl} \leq -L_1 C_5 \varepsilon^{\alpha-1} + 2\nu_0^{\alpha-1} L_1 \omega_\mu(\varepsilon) \varepsilon^{\alpha-1} + L_1 \omega_\mu(\varepsilon) \varepsilon^{\alpha-1} \ln(\nu_0 \varepsilon)^{-1} + O(L_2).$$

We see that if $\omega_\mu(r)$ satisfies $\omega_\mu(r) \ln r^{-1} \rightarrow 0$ as $r \rightarrow 0$, (34) still holds true in this case.

For $\alpha = \beta < 1$, combine (23), (29) and (33) to get

$$(35) \quad T_{nl} \leq L_1 \left(-C_4 C_\mu(\eta) \nu_0^{2-\beta} + \omega_\mu(\varepsilon) \ln \delta^{-1} + o(\omega_\mu(\varepsilon) \ln \delta^{-1}) \right) + O(L_2)$$

where C_4 is given by (27).

STEP 4: CONCLUSION. Here, we pick $\alpha < \min(1, \beta)$ and we deduce from (34) that $T_{nl} \leq 0$ if L_1 is large enough. Using inequality (18) together with (H) with $L = R = O(L_2)$ and Estimates (22) & (34), and recalling that $\bar{\varepsilon} := (L_1 \alpha \varepsilon^{\alpha-2})^{-1}$, we are thus lead to

$$(36) \quad 0 \leq \Lambda_1(\bar{x}) A_1 + \Lambda_2(\bar{x}) A_2$$

with

$$\begin{aligned} A_1 &= L_1 \alpha \varepsilon^{\alpha-2} \left[-8\varpi + \omega_F(\varepsilon) \right] + \varepsilon^\tau (\alpha L_1 \varepsilon^{\alpha-1})^{2+\tau} + (\alpha L_1 \varepsilon^{\alpha-1})^2 + \tilde{\chi}(L_2) \\ A_2 &= \left[-L_1 C_5 \varepsilon^{\alpha-\beta} + o(\varepsilon^{\alpha-\beta}) \right] + \frac{\varepsilon^{2\theta}}{\bar{\varepsilon}} + \varepsilon^\tau (\alpha L_1 \varepsilon^{\alpha-1})^{k+\tau} + (\alpha L_1 \varepsilon^{\alpha-1})^k + \tilde{\chi}(L_2) \end{aligned}$$

where we have gathered in the $\tilde{\chi}(L_2)$ -term the terms that either depend on L_2 or are bounded. We use the assumption $\Lambda_1 + \Lambda_2 \geq \Lambda_0 > 0$ by rewriting (36) as follows

$$0 \leq \Lambda_0 \max(A_1, A_2).$$

To get the desired contradiction and to obtain the $C^{0,\alpha}$ -estimate, it is enough to prove that $A_i < 0$ for $i = 1, 2$ and L_1 large enough, and to control the size of such L_1 .

As far as A_1 is concerned and as soon as $\alpha < 1$, we can ensure that $-8\varpi + \omega_F(\varepsilon) \leq -4\varpi$ if L_1 is large enough; using (15), it is clear that, in order to do it, the size of L_1 depends only on $\|u\|_\infty$, α and $d(x_0, \partial\Omega)$. This yields an estimate of the type

$$A_1 \leq L_1 \varepsilon^{\alpha-2} \left[-4\varpi \alpha + \alpha^{2+\tau} (L_1 \varepsilon^\alpha)^{1+\tau} + \alpha^2 L_1 \varepsilon^\alpha \right] + \tilde{\chi}(L_2).$$

Now there are two ways to estimate $L_1 \varepsilon^\alpha$: either to use the first part of inequality (16) which yields the estimate $L_1 \varepsilon^\alpha \leq 2\|u\|_\infty$, or to use the second part and the estimate of $L_1 \varepsilon^\alpha$ through the modulus of continuity of u . In the sequel, the strategy of the proof consists in proving the result for α small enough by using the first estimate of $L_1 \varepsilon^\alpha$ and then to use this first step (which provides a modulus of continuity of u) to prove it for all α by using the second estimate of $L_1 \varepsilon^\alpha$.

Using $L_1 \varepsilon^\alpha \leq 2\|u\|_\infty$ in the above inequality yields

$$A_1 \leq L_1 \varepsilon^{\alpha-2} \left[-4\varpi \alpha + \alpha^{2+\tau} (2\|u\|_\infty)^{1+\tau} + 2\alpha^2 \|u\|_\infty \right] + \tilde{\chi}(L_2).$$

For α small enough (depending only on $\|u\|_\infty$ and τ), the bracket is less than $-2\varpi\alpha < 0$ and, recalling (15), it is clear that the right-hand side is (strictly) negative if L_1 is large enough (depending on α , ϖ , $\tilde{\chi}(L_2)$), providing the desired property.

For the A_2 -term, we first write

$$\begin{aligned} A_2 &= -L_1 C_5 \varepsilon^{\alpha-\beta} + o(\varepsilon^{\alpha-\beta}) + \varepsilon^{2\theta} L_1 \varepsilon^{\alpha-2} + \varepsilon^\tau (\alpha L_1 \varepsilon^{\alpha-1})^{k+\tau} + (L_1 \alpha \varepsilon^{\alpha-1})^k + \tilde{\chi}(L_2) \\ &= L_1 \varepsilon^{\alpha-\beta} \left[-C_5 + o_\varepsilon(1) + \varepsilon^{2\theta-2+\beta} + \alpha^{k+\tau} \varepsilon^{\beta-k} (L_1 \varepsilon^\alpha)^{k+\tau-1} + \alpha^k \varepsilon^{\beta-k} (L_1 \varepsilon^\alpha)^{k-1} \right] + \tilde{\chi}(L_2) \\ &= L_1 \varepsilon^{\alpha-\beta} \left[-C_5 + o_\varepsilon(1) + \alpha^{k+\tau} \varepsilon^{\beta-k} (L_1 \varepsilon^\alpha)^{k+\tau-1} + \alpha^k \varepsilon^{\beta-k} (L_1 \varepsilon^\alpha)^{k-1} \right] + \tilde{\chi}(L_2) \end{aligned}$$

where C_5 is given by (30); we used $2\theta + \beta - 2 > 0$.

The key difference with A_1 is the fact that the exponents of the term $L_1 \varepsilon^\alpha$ can be non-positive and we have to argue differently if it is indeed the case.

We have the following cases.

- If $\beta > 1$, since $C_5 \geq \alpha C'_5$ for some constant C'_5 independent of α (at least for $\alpha \leq 1/2$), then one can argue as for A_1 with $k = \beta$ since $k + \tau > 1$ and $k = \beta > 1$ and obtain the $C^{0,\alpha}$ regularity and estimates for α small enough.
- If $\beta \leq 1$, then we cannot use this argument anymore since k must satisfy (at least) $k \leq 1$. In order to conclude, it is enough to ensure

$$(37) \quad \varepsilon^{\beta-k} (L_1 \varepsilon^\alpha)^{k+\tau-1} = o_\varepsilon(1) \quad \text{and} \quad \varepsilon^{\beta-k} (L_1 \varepsilon^\alpha)^{k-1} = o_\varepsilon(1).$$

Writing $\varepsilon^{\beta-k} (L_1 \varepsilon^\alpha)^{k-1} = L_1^{k-1} \varepsilon^{\beta-k+\alpha(k-1)}$, we see that this term is $o_\varepsilon(1)$ if $k < 1$ and $\beta - k + \alpha(k-1) \geq 0$; notice that we do not know how to compare, in full generality, L_1 -terms and ε -terms. The second condition implies that $k < \beta$. In the same way, for the other term, either $k + \tau - 1 \geq 0$ and the condition $\beta > k$ is sufficient or $k + \tau - 1 < 0$ and we are lead to $\alpha \leq (\beta - k)/(1 - k - \tau)$. Gathering all these informations yields the conditions

$$1 \geq \beta > k \quad \text{and} \quad \alpha \leq \frac{\beta - k}{1 - k}.$$

At this point, putting together the informations on A_1 and A_2 , we have shown that u is locally in $C^{0,\bar{\alpha}}$ for $\bar{\alpha}$ small enough (depending only on the data and the L^∞ -norm of u) and we have an estimate of the local $C^{0,\bar{\alpha}}$ -norm of u . In order to conclude the proof, we need to come back to the estimate on A_1 and A_2 (in the case when $\beta > 1$) and to estimate the terms $L_1 \varepsilon^\alpha$ using (16) and the local $C^{0,\bar{\alpha}}$ -modulus of continuity. This easily yields the full result and the proof is complete. \square

4.2. Proof of Part (ii) of Theorem 1.

Proof of Theorem 1. (ii). In order to prove the second part of the theorem, we now choose $\alpha = \beta$ and we only need to adapt the final step (Step 4) in the previous proof. We proceed as before by writing (36) with A_1 unchanged and A_2 given by

$$A_2 = L_1 \left[-C_4 C_\mu(\eta) \nu_0^{2-\beta} + \varepsilon^{\bar{\theta}} \ln \delta^{-1} + o(\varepsilon^{\bar{\theta}} \ln \delta^{-1}) \right] + L_1 \varepsilon^{\beta-2+2\theta} + \varepsilon^\tau (\beta L_1 \varepsilon^{\beta-1})^{k+\tau} + \tilde{\chi}(L_2).$$

Notice that we used the additional assumption about ω_μ . Notice also that at this stage of the proof, L_2 is fixed and L_1 can be chosen large enough in order to control the term $\tilde{\chi}(L_2)$. From (16) and Part (i), we have

$$(38) \quad L_1 \varepsilon^\beta \leq C \varepsilon^{\alpha'}$$

for any $\alpha' < \beta$. Next, the idea is to choose $\nu_0 = \varepsilon^\gamma$ for some $\gamma > 0$. We will see that computations are simplified by choosing directly ν_0 such that $\nu_0^{2-\beta} = \varepsilon^\gamma$ for some $\gamma > 0$ small enough. We will get $A_2 < 0$

for γ small enough and L_1 big enough. We thus get

$$\begin{aligned} A_2 &= L_1 \left[-C_4 C_\mu(\eta) \varepsilon^\gamma + (1 + \gamma(2 - \beta)) \varepsilon^{\tilde{\theta}} \ln \varepsilon^{-1} + o(\varepsilon^{\tilde{\theta}} \ln \varepsilon^{-1}) \right] \\ &\quad + L_1 \left[\varepsilon^{\beta-2+2\theta} + L_1^{k+\tau-1} \varepsilon^{\beta(k+\tau)-k} \right] + \tilde{\chi}(L_2). \end{aligned}$$

By assumption, we have $\beta - 2 + 2\theta > 0$ and $\beta(k + \tau) - k > 0$. Using (38) if $k + \tau - 1 > 0$, we see that if γ is chosen such that

$$\gamma < \min(\tilde{\theta}, \beta - 2 + 2\theta, \beta(k + \tau) - k)$$

then $A_2/L_1 < 0$ for L_1 large enough. Indeed, it is immediate if $k + \tau - 1 \leq 0$ and on the other case, we use (38) with $\alpha = \beta - \eta$ with η small enough. The proof is now complete. \square

4.3. Proof of Theorem 2.

Proof of Theorem 2. This proof follows along the lines of the previous one, the only difference is the way of getting Estimates (34) and (35) for the non-local term in the new framework and under the new assumptions. Let us explain this point.

In order to shed light on the fact that j has to be Hölder continuous wrt x , we let $\omega(r)$ denote $C_0 r^{\tilde{\theta}}$ and we will see at the end of the present proof that ω has to be chosen as a power law. Precisely, in this case,

$$\begin{aligned} T_{nl} &= \int_{|z| \geq \delta'} [u(\bar{x} + j(\bar{x}, z)) - u(\bar{x}) - (p + q) \cdot j(\bar{x}, z) \mathbf{1}_B(z)] \mu(dz) \\ &\quad - \int_{|z| \geq \delta'} [u(\bar{y} + j(\bar{y}, z)) - u(\bar{y}) - p \cdot j(\bar{y}, z) \mathbf{1}_B(z)] \mu(dz). \end{aligned}$$

We then write $T_{nl} = T_1 + T_2 + T_3$ with

$$\begin{aligned} T_1 &= \int_{|z| \geq 1} [u(\bar{x} + j(\bar{x}, z)) - u(\bar{x})] \mu(dz) - \int_{|z| \geq 1} [u(\bar{y} + j(\bar{y}, z)) - u(\bar{y})] \mu(dz) \\ T_2 &= \int_{z \in \mathcal{C}} [u(\bar{x} + j(\bar{x}, z)) - u(\bar{x}) - (p + q) \cdot j(\bar{x}, z)] \mu(dz) \\ &\quad - \int_{z \in \mathcal{C}} [u(\bar{y} + j(\bar{y}, z)) - u(\bar{y}) - p \cdot j(\bar{y}, z)] \mu(dz) \\ T_3 &= \int_{z \in B, z \notin \mathcal{C}} [\dots] \mu_{\bar{x}}(dz) - \int_{z \in B, z \notin \mathcal{C}} [\dots] \mu_{\bar{y}}(dz) \end{aligned}$$

where \mathcal{C} is defined in the following way

$$\mathcal{C} := \left\{ z : \left| j\left(\frac{\bar{x} + \bar{y}}{2}, z\right) \right| \leq \frac{\delta}{2} \text{ and } \left| j\left(\frac{\bar{x} + \bar{y}}{2}, z\right) \cdot \hat{a} \right| \geq \left(1 - \frac{\eta}{2}\right) \left| j\left(\frac{\bar{x} + \bar{y}}{2}, z\right) \right| \right\} = \mathcal{C}_{\delta/2, \eta/2}\left(\frac{\bar{x} + \bar{y}}{2}\right),$$

where the notation $\mathcal{C}_{\delta, \eta}(x)$ is defined in the statement of Theorem 2. Roughly speaking, \mathcal{C} is the analogue of the cone used in the proof of Theorem 1 where we have replaced z by $j(\frac{\bar{x} + \bar{y}}{2}, z)$. Notice that, because of (9), $\mathcal{C} \subset B$ if δ is small enough.

We have chosen such a set \mathcal{C} for the following reason : if L_1 is large enough (or equivalently ε or δ are small enough)

$$(39) \quad \mathcal{C} \subset \mathcal{C}_{\delta, \eta}(\bar{x}) \cap \mathcal{C}_{\delta, \eta}(\bar{y}),$$

which means that both $j(\bar{x}, z)$ and $j(\bar{y}, z)$ are in the ‘‘good’’ cones.

To check these properties, we write

$$\begin{aligned} |j(\bar{x}, z) \cdot \hat{a}| &\geq \left| j\left(\frac{\bar{x} + \bar{y}}{2}, z\right) \cdot \hat{a} \right| - \left| j(\bar{x}, z) - j\left(\frac{\bar{x} + \bar{y}}{2}, z\right) \right| \geq (1 - \eta/2) \left| j\left(\frac{\bar{x} + \bar{y}}{2}, z\right) \right| - |z| \omega\left(\left|\bar{x} - \frac{\bar{x} + \bar{y}}{2}\right|\right) \\ &\geq (1 - \eta/2) |j(\bar{x}, z)| - (2 - \eta/2) |z| \omega(\varepsilon/2) \geq (1 - \eta/2 - (2 - \eta) C_0^{-1} \omega(\varepsilon/2)) |j(\bar{x}, z)| \\ &\geq (1 - \eta) |j(\bar{x}, z)| \end{aligned}$$

and

$$|j(\bar{x}, z)| \leq |j(\frac{\bar{x} + \bar{y}}{2}, z)| + \omega(\varepsilon/2)|z| \leq \frac{\delta}{2} + \frac{\delta}{2c_0}\omega(\varepsilon/2) \leq \delta$$

for L_1 such that

$$\omega(\varepsilon/2) \leq \min\left(\frac{\eta c_0}{4 - \eta}, c_0\right) = \min(C_0, c_0) = c_0$$

for $\eta < 1$.

Estimate of T_1 . We remark that, thanks to the properties of j and μ , (23) still holds true.

Estimate of T_2 . One can check that (28) and (29) still hold true. Indeed, we use (10)-(11) in order to get (6). More precisely, using (10)-(11) and recalling the computation we made in Subsection 3.3, we obtain

$$\forall \hat{a}, \quad \int_{\mathcal{C}} |z|^2 \mu(dz) \geq C_\mu \delta^{2-\beta}$$

where $C_\mu = C(c_0, C_0, C_\mu^-, \eta, d, \beta)$. Notice that (26) is slightly modified and so is C_4 (and consequently C_5).

Estimate of T_3 . In the case of Lévy-Itô operators, we estimate T_3 as follows. By using (25) with $d = j(\bar{x}, z)$ and $d' = j(\bar{y}, z)$, we get

$$\begin{aligned} T_3 &= \int_{z \in B, z \notin \mathcal{C}} [u(\bar{x} + j(\bar{x}, z)) - u(\bar{x}) - (p + q) \cdot j(\bar{x}, z) \\ &\quad - u(\bar{y} + j(\bar{y}, z)) + u(\bar{y}) + p \cdot j(\bar{y}, z)] \mu(dz) \\ &\leq T_3^1 + T_3^2 \end{aligned}$$

with

$$\begin{aligned} T_3^1 &= \int_{z \in B, z \notin \mathcal{C}} [\Gamma(\bar{x} + j(\bar{x}, z)) - \Gamma(\bar{x}) - q \cdot j(\bar{x}, z)] \mu(dz) \\ T_3^2 &= \int_{z \in B, z \notin \mathcal{C}} [\phi(a + \Delta(z)) - \phi(a) - p \cdot \Delta(z)] \mu(dz) \end{aligned}$$

where $\Delta(z) = j(\bar{x}, z) - j(\bar{y}, z)$. Let us first estimate T_3^1 as follows.

$$T_3^1 \leq \frac{1}{2} \int_B \sup_{t \in (0,1)} \left[D^2 \Gamma(\bar{x} + tj(\bar{x}, z)) j(\bar{x}, z) \cdot j(\bar{x}, z) \right] \mu(dz)$$

and we deduce that

$$(40) \quad T_3^1 \leq C_2 L_2$$

where

$$(41) \quad C_2 = C_0^2 \int_{|z| \leq 1} |z|^2 \mu(dz).$$

We now turn to T_3^2 and we write $T_3^2 = T_3^{2,1} + T_3^{2,2}$ with

$$\begin{aligned} T_3^{2,1} &= \int_{z \in B, z \notin \mathcal{C}, |\Delta(z)| \geq \delta} [\phi(a + \Delta(z)) - \phi(a) - D\phi(a) \cdot \Delta(z)] \mu(dz), \\ T_3^{2,2} &= \int_{z \in B, z \notin \mathcal{C}, |\Delta(z)| \leq \delta} [\phi(a + \Delta(z)) - \phi(a) - D\phi(a) \cdot \Delta(z)] \mu(dz). \end{aligned}$$

We can now estimate $T_3^{2,1}$. In order to clarify computations, we write ω for $\omega(\varepsilon)$ in the following lines. Remark that

$$\delta \leq |\Delta(z)| \leq \omega|z|$$

from which we deduce

$$\begin{aligned} T_3^{2,1} &\leq \int_{\delta\omega^{-1} \leq |z| \leq 1} (|\Delta(z)|^\alpha + L_1\alpha\varepsilon^{\alpha-1}|\Delta(z)|)\mu(dz) \leq \int_{\delta\omega^{-1} \leq |z| \leq 1} (\omega^\alpha|z|^\alpha + L_1\alpha\varepsilon^{\alpha-1}\omega|z|)\mu(dz) \\ &\leq \omega^\alpha\psi_\alpha(\delta\omega^{-1}) + L_1\varepsilon^{\alpha-1}\omega\psi_1(\delta\omega^{-1}) \end{aligned}$$

where

$$(42) \quad \psi_\gamma(r) = \int_{r \leq |z| \leq 1} |z|^\gamma \mu(dz) \leq \begin{cases} \frac{C_\mu^+}{\gamma-\beta} & \text{if } \gamma > \beta \\ C_\mu^+ \ln \frac{1}{r} & \text{if } \gamma = \beta \\ C_\mu^+ \frac{r^{-(\beta-\gamma)}}{\beta-\gamma} & \text{if } \gamma < \beta \end{cases}$$

(we used (10)-(11)). In order to estimate $T_3^{2,2}$, we use a Taylor expansion together with (21) and the fact that $|a + t\Delta(z)| \geq \varepsilon - \delta > 0$. It comes

$$\begin{aligned} T_3^{2,2} &\leq \frac{L_1\alpha}{2} \int_{|z| \leq 1} \sup_{t \in (0,1)} |a + t\Delta(z)|^{\alpha-2} |\Delta(z)|^2 \mu(dz) \\ &\leq \frac{L_1}{2} \int_B |z|^2 \mu(dz) (\varepsilon - \delta)^{\alpha-2} \omega^2 \\ &\leq \frac{L_1}{2} \int_B |z|^2 \mu(dz) (1 - \nu_0)^{\alpha-2} \times \omega^2 \varepsilon^{\alpha-2}. \end{aligned}$$

Gathering the estimates on $T_3^{2,1}$ and $T_3^{2,2}$, we obtain

$$(43) \quad T_3^2 \leq \omega^\alpha\psi_\alpha(\delta\omega^{-1}) + L_1\varepsilon^{\alpha-1}\omega\psi_1(\delta\omega^{-1}) + L_1C_3\omega^2\varepsilon^{\alpha-2}$$

with

$$(44) \quad C_3 = \frac{1}{2} \int_B |z|^2 \mu(dz) (1 - \nu_0)^{\alpha-2}$$

Final estimate of T_{nl} . Gathering estimates (23), (29), (40) and (43), we finally obtain

$$(45) \quad T_{nl} \leq \omega^\alpha\psi_\alpha(\nu_0\omega^{-1}\varepsilon) + L_1\omega\psi_1(\nu_0\omega^{-1}\varepsilon)\varepsilon^{\alpha-1} + L_1C_3\omega^2\varepsilon^{\alpha-2} - L_1C_5\varepsilon^{\alpha-\beta} + O(L_2).$$

Looking at the $\omega^2\varepsilon^{\alpha-2}$ -term, we see that we need to assume that j is Hölder continuous wrt x and we thus replace $\omega(\varepsilon)$ with $C\varepsilon^{\tilde{\theta}}$. Assuming without loss of generality that $C = 1$, we get

$$T_{nl} \leq \varepsilon^{\alpha\tilde{\theta}}\psi_\alpha(\nu_0\varepsilon^{1-\tilde{\theta}}) + L_1\varepsilon^{\alpha+\tilde{\theta}-1}\psi_1(\nu_0\varepsilon^{1-\tilde{\theta}}) + L_1\varepsilon^{\alpha-2+2\tilde{\theta}} - L_1C_5\varepsilon^{\alpha-\beta} + O(L_2).$$

We claim that as in the case of Theorem 1, (34) holds true under the assumptions of Theorem 2. To see this, we write

$$\begin{aligned} \varepsilon^{\alpha\tilde{\theta}}\psi_\alpha(\nu_0\varepsilon^{1-\tilde{\theta}}) &+ L_1\varepsilon^{\alpha+\tilde{\theta}-1}\psi_1(\nu_0\varepsilon^{1-\tilde{\theta}}) + L_1\varepsilon^{\alpha-2+2\tilde{\theta}} \\ &= L_1\varepsilon^{\alpha-\beta} \left[L_1^{-1}\varepsilon^{\alpha\tilde{\theta}-\alpha+\beta}\psi_\alpha(\nu_0\varepsilon^{1-\tilde{\theta}}) + \varepsilon^{\beta+\tilde{\theta}-1}\psi_1(\nu_0\varepsilon^{1-\tilde{\theta}}) + \varepsilon^{\beta-2+2\tilde{\theta}} \right] \\ &= L_1\varepsilon^{\alpha-\beta} \left[o(\varepsilon^{\tilde{\theta}\beta}) + \varepsilon^{\beta+\tilde{\theta}-1}\psi_1(\nu_0\varepsilon^{1-\tilde{\theta}}) + o_\varepsilon(1) \right] \\ &= L_1\varepsilon^{\alpha-\beta} \left[\varepsilon^{\beta+\tilde{\theta}-1}\psi_1(\nu_0\varepsilon^{1-\tilde{\theta}}) + o_\varepsilon(1) \right] \end{aligned}$$

since $\tilde{\theta} > \frac{1}{2}(2 - \beta)$ and $\alpha < \beta$ (we used (42)). We next distinguish cases.

- If $\beta > 1$, then by (42), we get

$$\varepsilon^{\beta+\tilde{\theta}-1}\psi_1(\nu_0\varepsilon^{1-\tilde{\theta}}) = O(\varepsilon^{\tilde{\theta}\beta}).$$

and we conclude in this case.

- If $\beta = 1$,

$$\varepsilon^{\beta+\tilde{\theta}-1}\psi_1(\nu_0\varepsilon^{1-\tilde{\theta}}) = O(\varepsilon^{\tilde{\theta}} \ln \frac{1}{\varepsilon^{1-\tilde{\theta}}}) = o_\varepsilon(1)$$

we we can conclude in this case too.

- Finally, if $\beta < 1$,

$$\varepsilon^{\beta+\tilde{\theta}-1}\psi_1(\nu_0\varepsilon^{1-\tilde{\theta}}) = O(\varepsilon^{\tilde{\theta}+\beta-1}) = o(\varepsilon^{\frac{\beta}{2}})$$

and (34) follows. The proof of Theorem 2 is now complete. \square

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