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Soliton solutions to systems of coupled Schrödinger equations of Hamiltonian type

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1 Introduction

A major role in quantum physics is played by the nonlinear Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta_x \psi + V(x)\psi - \bar{f}(x, \psi), \quad (1.1)$$

where m and \hbar are positive constants, the wave $\psi : \mathbb{R}^+ \times \mathbb{R}^N \rightarrow \mathbb{C}$, $N \geq 3$, V is a potential which is bounded below, and $\bar{f} = f(x, |\psi|)\psi$ is a nonlinear function, for instance in the classical cubic approximation $\bar{f} = |\psi|^2\psi$. One of the questions to which huge attention has been given during the last twenty years is the existence of standing waves (see (1.2) below) for small values of \hbar , which appear due to the geometry of the potential.

This paper is devoted to the corresponding question of existence of solutions of some *systems* of Schrödinger equations. Systems of nonlinear Schrödinger type have been widely used in the applied sciences but mathematical study of standing wave solutions was undertaken only very recently, prompted in particular by the discovery of the importance of these systems as models in nonlinear optics (see for instance [3], [7]) and in the study of Bose-Einstein condensates (see [26]). As in the large majority of other papers on the subject we consider here systems of two equations.

So we suppose ψ is a vector function, $\psi = (\psi_1, \psi_2)$, and satisfies a system of equations like (1.1), with $\bar{f} = (\bar{f}_1, \bar{f}_2)$ and $\bar{f}_k = \sum_j f_{kj}(x, |\psi_1|, |\psi_2|)\psi_j$.

We will be interested in soliton (standing wave) solutions of these systems, that is, solutions in the form

$$\psi_j(t, x) = e^{i\kappa\hbar^{-1}t} u_j(x). \quad (1.2)$$

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Substituting (1.2) into (1.1) and setting $b(x) = V(x) - \kappa$ leads to the system of real elliptic partial differential equations (we write $u = u_1, v = u_2$)

$$(S_{\hbar}) \quad \begin{cases} -\hbar^2 \Delta u + b(x)u = f_1(x, u, v) & \text{in } \mathbb{R}^N, \\ -\hbar^2 \Delta v + b(x)v = f_2(x, u, v) & \text{in } \mathbb{R}^N. \end{cases}$$

We suppose we are in the physical situation when this system is in variational form, that is, when it is the Euler-Lagrange system of some energy functional. This happens when f_1, f_2 are the derivatives of a given function $H(x, u, v)$. There are two types of such systems, Lagrangian - when $f_1 = H_u, f_2 = H_v$, and Hamiltonian - when $f_1 = H_v, f_2 = H_u$. Hamiltonian systems are considerably more difficult to study, since the energy functional is then strongly indefinite, that is, its leading part is respectively coercive and anti-coercive on infinitely dimensional subspaces of the energy space. The present article is devoted to this case. It is our goal to get a general existence result for small \hbar in the case of a superlinear and subcritical Hamiltonian system.

As in many applications, we consider trapping (or "well"-type) potentials, the standard example being $b(x) \sim |x - x_0|^2$ in a neighbourhood of some $x_0 \in \mathbb{R}^N$. A particular case of our result will be the existence of standing waves thanks to a global well structure of b , that is,

$$0 = \inf_{x \in \mathbb{R}^N} b(x) < \liminf_{|x| \rightarrow \infty} b(x). \quad (1.3)$$

Notice that $\inf_{x \in \mathbb{R}^N} b(x) = 0$ is not a restriction, since we can always achieve this through the choice of κ in (1.2).

Unfortunately, as of today PDE theory lacks the means to tackle the existence question under hypothesis (1.3) only, even in the scalar case. However, it turns that we can show that (S_{\hbar}) has a solution provided the constant \hbar is *sufficiently small*. Note that in practice \hbar , the Planck constant, is a very small quantity, so it makes sense to study problem (S_{\hbar}) at the limit $\hbar \rightarrow 0$.

Here are the precise statements. We assume $H(x, u, v)$ is differentiable and strictly convex in $(u, v) \in \mathbb{R}^2$ for all $x \in \mathbb{R}^N$, $H(x, 0, 0) = 0$ and

(H1) there exist constants $p, q, \alpha_k, \beta_k > 1$, such that

$$\frac{1}{p+1} + \frac{1}{q+1} > \frac{N-2}{N}, \quad \frac{\alpha_k}{p+1} + \frac{\beta_k}{q+1} = 1, \quad (1.4)$$

and for some $c_0, d_0 > 0, C_k \geq 0, D_k \geq 0$ we have for $x \in \mathbb{R}^N, (u, v) \in \mathbb{R}^2$

$$\begin{aligned} c_0 |u|^q &\leq |H_u(x, u, v)| \leq C_0 |u|^q + \sum_{k=1}^m C_k |u|^{\alpha_k - 1} |v|^{\beta_k}, \\ d_0 |v|^p &\leq |H_v(x, u, v)| \leq D_0 |v|^p + \sum_{k=1}^m D_k |u|^{\alpha_k} |v|^{\beta_k - 1}. \end{aligned}$$

(H2) There exists $\alpha > 2$ such that for all $x \in \mathbb{R}^N$ and $(u, v) \in \mathbb{R}^2 \setminus \{(0, 0)\}$

$$uH_u(x, u, v) + vH_v(x, u, v) \geq \alpha H(x, u, v) > 0.$$

A typical example of a function satisfying these hypotheses is $H(x, u, v) = a_0(x)|u|^{p+1} + \sum_1^n a_i(x)|u|^{\alpha_i}|v|^{\beta_i} + a_{n+1}(x)|v|^{q+1}$, under (1.4).

We suppose that the continuous potential $b(x)$ satisfies $b \geq 0$ in \mathbb{R}^N and

(b1) there exists $x_0 \in \mathbb{R}^N$ (say $x_0 = 0$) such that $b(x_0) = 0$;

(b2) there exists $A > 0$ such that the level set $G_A = \{x \in \mathbb{R}^N : b(x) < A\}$ has finite Lebesgue measure.

Note that the conditions (b1)-(b2) include (1.3) as a particular case. We shall also suppose that $b(x)$ is bounded. This condition is made for simplicity, since it is irrelevant to the goal of our paper, which is to use the well geometry of the potential. Actually it is even easier to consider potentials which are large at infinity (then there is no restriction on \hbar), since the energy space embeds compactly into Lebesgue spaces, see for instance Theorem 4 in [24].

Note also that (H1) means the problem is superlinear and subcritical, in other words, the couple (p, q) is under the critical hyperbola (given by (1.4)). In particular, one of the nonlinearities in (S_\hbar) can have growth larger than the exponent $(N+2)/(N-2)$, provided the growth of the other is smaller enough to compensate (note that when $p = q$ (1.4) reduces to $p < (N+2)/(N-2)$). In this case the functional associated to (S_\hbar) is not defined for $u, v \in H^1(\mathbb{R}^N)$. It is nowadays well-known that (1.4) is the right notion of criticality for a Hamiltonian system with power-growth nonlinearity, see [5], [13], [22], [23].

The following theorem contains our main result.

Theorem 1.1 *If $f_1 = H_v$, $f_2 = H_u$, and (H1)-(H2), (b1)-(b2) are satisfied then (S_\hbar) has a nontrivial solution for small \hbar .*

We now quote previous works related to this result. For the scalar case we refer to [11], [9], [2], [25], [4], and to the references in these papers. Some types of Lagrangian systems with well potentials were studied in [1], [17], [18]. Existence results for radially invariant Hamiltonian systems in \mathbb{R}^N were established in [10] and [24]. A result similar to Theorem 1.1 can be found in [20] (see also [21]) in the particular case when $H = F(u) + G(v)$, that is, the right-hand side of the system is independent of x and has no cross-terms in u, v . This restrictive hypothesis is due to the method used in these papers, which in particular makes use of the choice $\inf_{\mathbb{R}^N} b > 0$, as opposed to (b1). In this case it is possible to use an approach similar to the one in [9]. Finally,

in the recent paper [8] a fairly general result was proved on system (S_{\hbar}) , but under the hypothesis that both p, q are smaller than (or in some cases equal to) the scalar exponent $(N + 2)/(N - 2)$. Using a different approach, here we extend the existence result from [8] to the whole subcritical range for a system, under the hypothesis that the nonlinearity H be convex. Of course the model and most often used Hamiltonians (see above) are convex.

Our work is inspired by [25], where the particular case of Theorem 1.1 when we have one scalar equation was proved. The method in [25] extends readily to Lagrangian systems, since then the functional has the same structure as the scalar one, but the situation turns out to be considerably more involved for Hamiltonian systems. We have used a dual variational structure, relying on the Legendre–Fenchel transformation, which allows us to transform the problem into a new one, to which the Mountain Pass Theorem applies. However, then one of the key points - that the generalized mountain pass value tends to zero as $\hbar \rightarrow 0$ - turns out to be rather delicate to prove. We have found a way to deal with this problem which uses Fourier analysis, a tool that is seldom encountered in this branch of the calculus of variations. Our method will hopefully be useful in other situations as well.

In the next section we describe the variational setting. The proof of Theorem 1.1 is to be found in sections 3. The fact that the mountain pass value (and hence the norms and the energy of the solutions we find) tends to zero as $\hbar \rightarrow 0$ is proved in Section 4.

2 The dual variational formulation

This section has a preliminary character. We recall here some facts which permit to us to set up the variational framework for solving system (S_{\hbar}) .

Lemma 2.1 *Let V be bounded and nonnegative function satisfying (b1) and (b2). Then, for every $g \in L^s(\mathbb{R}^N)$, $1 < s < \infty$, and $\hbar > 0$, the problem*

$$-\Delta u + V(\hbar x)u = g \quad \text{in } \mathbb{R}^N$$

possesses a unique solution $u \in W^{2,s}(\mathbb{R}^N)$. In addition, there exists a constant $K > 0$ (which may depend of \hbar) such that

$$\|u\|_{W^{2,s}(\mathbb{R}^N)} \leq K \|g\|_{L^s(\mathbb{R}^N)}$$

Proof: Denote $V_{\hbar}(x) = V(\hbar x)$. For $s \in (1, \infty)$, consider the operator $R_s : W^{2,s}(\mathbb{R}^N) \rightarrow L^s(\mathbb{R})$ defined by

$$R_s u = (-\Delta + V_{\hbar} I)u \quad \text{for } u \in W^{2,s}(\mathbb{R}^N).$$

It follows for instance from Theorem 1 of [19] that

(i) $\text{Ker}(R_s - \lambda I) = \text{Ker}(R_2 - \lambda I)$, for every $s \in (1, \infty)$.

(ii) $L^s(\mathbb{R}^N) = \text{Ker}(R_s - \lambda I) \oplus \text{Im}(R_s - \lambda I)$.

Since $V_{\hbar} \in L^\infty(\mathbb{R}^N)$, it is known (see for example Lemma 3.10 in [27]) that the spectrum set $\sigma(R_2) \subset [\Lambda, \infty)$ and $\Lambda_{\hbar} \in \sigma(R_2)$, where

$$\Lambda_{\hbar} = \inf \left\{ \int (|\nabla u|^2 + V_{\hbar}(x)u^2) \mid u \in H^1(\mathbb{R}^N), \int u^2 = 1 \right\}.$$

It follows from Lemma 1 in [25] that $\Lambda_{\hbar} > 0$. Therefore $0 \in \rho(R_2)$. Consequently $\text{Ker}(R_s) = \text{Ker}(R_2) = \{0\}$ and

$$L^s(\mathbb{R}^N) = \text{Ker}(R_s) + \text{Im}(R_s) = \text{Im}(R_s).$$

Thus, $R_s : W^{2,s}(\mathbb{R}^N) \subset L^s(\mathbb{R}^N) \rightarrow L^s(\mathbb{R}^N)$ is a isomorphism. Note that R_s is continuous thanks to the immersion $W^{2,s}(\mathbb{R}^N) \subset L^s(\mathbb{R}^N)$. So, there exists a positive constant C such that for all $u \in L^s(\mathbb{R}^N)$

$$\|R_s^{-1}u\|_{W^{2,s}(\mathbb{R}^N)} \leq C\|u\|_{L^s(\mathbb{R}^N)}. \quad \square$$

Given $p, q > 1$ such that $\frac{1}{p+1} + \frac{1}{q+1} > \frac{N-2}{N}$, we define the operators

$$\tilde{R}_{\hbar} : L^{\frac{p+1}{p}}(\mathbb{R}^N) \rightarrow W^{2,\frac{p+1}{p}}(\mathbb{R}^N), \quad \tilde{S}_{\hbar} : L^{\frac{q+1}{q}}(\mathbb{R}^N) \rightarrow W^{2,\frac{q+1}{q}}(\mathbb{R}^N),$$

by

$$R_{\hbar} = S_{\hbar} = (-\Delta + b_{\hbar}I)^{-1},$$

where $b_{\hbar}(x) = b(\hbar x)$. It follows from Lemma 2.1 that the operators R_{\hbar} and S_{\hbar} are well defined and continuous. Since $1/(q+1) > p/(p+1) - 2/N$ holds, we have the continuous Sobolev embeddings

$$i_1 : W^{2,\frac{p+1}{p}}(\mathbb{R}^N) \rightarrow L^{q+1}(\mathbb{R}^N), \quad i_2 : W^{2,\frac{q+1}{q}}(\mathbb{R}^N) \rightarrow L^{p+1}(\mathbb{R}^N),$$

consequently $R_{\hbar} \doteq i_1 \circ \tilde{R}_{\hbar}$, $S_{\hbar} \doteq i_2 \circ \tilde{S}_{\hbar}$ are linear continuous operators.

So we can define the linear operator

$$T_{\hbar} : L^{\frac{q+1}{q}}(\mathbb{R}^N) \times L^{\frac{p+1}{p}}(\mathbb{R}^N) \rightarrow L^{q+1}(\mathbb{R}^N) \times L^{p+1}(\mathbb{R}^N), \quad T_{\hbar} := \begin{pmatrix} 0 & R_{\hbar} \\ S_{\hbar} & 0 \end{pmatrix},$$

that is, for all $f, \phi \in L^{\frac{q+1}{q}}(\mathbb{R}^N)$, $g, \varphi \in L^{\frac{p+1}{p}}(\mathbb{R}^N)$,

$$\langle T_{\hbar}w, \eta \rangle = \phi R_{\hbar}g + \varphi S_{\hbar}f, \quad \forall \eta = (\phi, \varphi), \quad \forall w = (f, g).$$

Let $X = L^{\frac{q+1}{q}}(\mathbb{R}^N) \times L^{\frac{p+1}{p}}(\mathbb{R}^N)$ be the Banach space endowed with the norm

$$\|w\| = \sqrt{\|f\|_{\frac{q+1}{q}}^2 + \|g\|_{\frac{p+1}{p}}^2}; \quad w = (f, g) \in X,$$

from now on $\|\cdot\|_s$ and $\int h dx$ will denote the L^s -norm in \mathbb{R}^N and $\int_{\mathbb{R}^N} h(x) dx$, respectively.

The dual functional $\Psi^h : X \rightarrow \mathbb{R}$ is defined by

$$\Psi^h(w) = \int H^*(x, w) dx - \frac{1}{2} \int \langle T_h w, w \rangle dx, \quad w \in X,$$

where H^* is the Legendre-Fenchel transform of H , that is, for all $x \in \mathbb{R}$ and $w = (w_1, w_2) \in \mathbb{R}^2$,

$$H^*(x, w) = \sup_{t \in \mathbb{R}^2} \{w_1 t_1 + w_2 t_2 - H(x, t)\}.$$

Lemma 2.2 *The functional Ψ^h is well defined and C^1 on X^* . Its Fréchet derivative is given by*

$$(\Psi^h)'(w)\eta = \int H_w^*(x, w)\eta dx - \int \langle T_h w, \eta \rangle dx, \quad \forall \eta \in X.$$

If $w = (f, g)$ is a critical point of Ψ^h , then $(u, v) = T_h w$ is a solution of the system (obtained by (S_h) through the change $x \rightarrow hx$)

$$\begin{cases} -\Delta u + b(hx)u = H_v(hx, u, v) & \text{in } \mathbb{R}^N, \\ -\Delta v + b(hx)v = H_u(hx, u, v) & \text{in } \mathbb{R}^N. \end{cases} \quad (S_h')$$

Proof: The proof of this lemma is known, for instance we can employ the arguments given in [6] (see Lemma 4.3 there, and also [14]). Let us sketch it for completeness.

The derivative of the second term in Ψ_h is simple to get, by the relation

$$\int \langle \eta, T_h w \rangle dx = \int \langle w, T_h \eta \rangle dx, \quad \forall \eta, w \in X.$$

Consider the functional

$$\mathcal{H}(z) = \int H(x, z) dx, \quad \mathcal{H} : X^* = L^{q+1}(\mathbb{R}^N) \times L^{p+1}(\mathbb{R}^N) \rightarrow \mathbb{R},$$

where $z = (u, v)$. From the hypotheses on H it follows that \mathcal{H} is well-defined on X^* and is a C^1 -functional. The Legendre-Fenchel transform of \mathcal{H} is given by

$$\mathcal{H}^*(w) = \int H^*(x, w) dx, \quad \mathcal{H}^* : X \rightarrow \mathbb{R}.$$

Since H is strictly convex the gradient $H_z : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a homeomorphism. Thus, \mathcal{H}' is a bijection from X^* to X , which is continuous and bounded. Furthermore, \mathcal{H}^* is Gâteaux differentiable, $(\mathcal{H}^*)'(w) = (\mathcal{H}')^{-1}(w)$ for every $w \in X$ (this is a characterization of the Legendre-Fenchel transform), and

$$(\mathcal{H}^*)'(w)\eta = \int H_w^*(x, w)\eta \, dx, \quad \forall \eta, w \in X.$$

Thus, $(\mathcal{H}^*)' : X \rightarrow X^*$ is continuous and bounded, which implies that \mathcal{H}^* is Fréchet differentiable. Now, if w is a critical point of Ψ^h , it follows that $z = (u, v) = T_h w$ is a solution of (S'_h) . In fact, we have

$$(\mathcal{H}^*)'(w) - T_h w = 0 \quad \text{in } X^*,$$

that is

$$(\mathcal{H}')^{-1}(w) - z = 0 \quad \text{in } X^*.$$

As a result,

$$T_h^{-1}z - (\mathcal{H}')(z) = 0 \quad \text{in } W^{2, \frac{p+1}{p}} \times W^{2, \frac{q+1}{q}},$$

because T_h^{-1} is an isomorphism between $W^{2, \frac{p+1}{p}} \times W^{2, \frac{q+1}{q}}$ and $L^{\frac{p+1}{p}} \times L^{\frac{q+1}{q}}$. Thus, $(u, v) = z = T_h w$ is a solution of system (S'_h) . \square

We say that $w = (f, g)$ is the dual solution associated to (u, v) . By making the change of variable $x \mapsto \hbar^{-1}x$ in \mathbb{R}^N , system (S'_h) becomes

$$\begin{cases} -\hbar^2 \Delta u + b(x)u = H_v(x, u, v) & \text{in } \mathbb{R}^N, \\ -\hbar^2 \Delta v + b(x)v = H_u(x, u, v) & \text{in } \mathbb{R}^N. \end{cases} \quad (S_h)$$

3 Proof of Theorem 1.1

We start with the following simple fact.

Lemma 3.1 *The functional Ψ^h has a “mountain pass geometry” on the space X , in the sense that there exist $\rho, \alpha > 0$ and $w \in X$ such that $\Psi^h|_{\partial B_\rho} \geq \alpha$, $\Psi^h(w) < 0$ and $\|w\| > \rho$.*

Proof: It is easy to see that (H1) and (H2) imply that there exist positive constants $c_1 - c_4$ such that

$$c_1|f|^{q+1} + c_2|g|^{p+1} \leq H(x, w) \leq c_3|f|^{q+1} + c_4|g|^{p+1}, \quad w = (f, g).$$

From properties of Legendre-Fenchel transformations, we have

$$d_1|f|^{\frac{q+1}{q}} + d_2|g|^{\frac{p+1}{q}} \leq H^*(x, w) \leq d_3|f|^{\frac{q+1}{q}} + d_4|g|^{\frac{p+1}{p}}, \quad (3.5)$$

for some positive constants $d_1 - d_4$.

By using the Hölder inequality and the boundedness of R_{\hbar} and S_{\hbar} , for all $w = (f, g) \in X$ we easily get

$$\begin{aligned} \int \langle w, T_{\hbar} w \rangle &\leq C(\|f\|_{\frac{q+1}{q}} \|g\|_{\frac{p+1}{p}} + \|g\|_{\frac{p+1}{p}} \|f\|_{\frac{q+1}{q}}) \\ &\leq C(\|g\|_{\frac{p+1}{p}}^2 + \|f\|_{\frac{q+1}{q}}^2) = C\|w\|_X^2, \end{aligned} \quad (3.6)$$

Then, from (3.5) and (3.6) we get

$$\Psi^{\hbar}(w) \geq C(\|f\|_{\frac{q+1}{q}}^{\frac{q+1}{q}} + \|g\|_{\frac{p+1}{p}}^{\frac{p+1}{p}}) - C(\|f\|_{\frac{q+1}{q}}^2 + \|g\|_{\frac{p+1}{p}}^2).$$

Thus, since $(p+1)/p < 2$ and $(q+1)/q < 2$, for each $\hbar > 0$ there exist constants $\rho, \alpha > 0$ such that $\Psi^{\hbar}|_{\partial B_{\rho}} \geq \alpha$.

Now, we claim we can find $w \in X$ such that $\Psi^{\hbar}(w) < 0$ and $\|w\| > \rho$. In fact, there exists $w^+ = (f^+, g^+) \in X$ such that $\int \langle T_{\hbar} w^+, w^+ \rangle > 0$ (indeed, it is sufficient to take $f^+ = g^+ \in C_c^{\infty}(\mathbb{R}^N)$). By using (3.5) we obtain, for all $t > 0$,

$$\Psi^{\hbar}(tw^+) \leq Ct^{\frac{q+1}{q}} \int |f|^{\frac{q+1}{q}} + Ct^{\frac{p+1}{p}} \int |g|^{\frac{p+1}{p}} - \frac{t^2}{2} \int \langle T_{\hbar} w^+, w^+ \rangle,$$

for some positive constant C . Since $\frac{p+1}{p}, \frac{q+1}{q} < 2$, the claim follows for $t > 0$ sufficiently large. \square

Set

$$\Gamma_{\hbar} \doteq \{\gamma \in C([0, 1], X) : \gamma(0) = 0, \Psi^{\hbar}(\gamma(1)) < 0\}$$

and

$$c_{\hbar} = \inf_{\gamma \in \Gamma_{\hbar}} \max_{t \in [0, 1]} \Psi^{\hbar}(\gamma(t)).$$

Standard critical point theory implies that for each $\hbar > 0$ we can find a sequence $\{w_n^{\hbar}\}_{n=1}^{\infty} \subset X$ such that

$$\Psi^{\hbar}(w_n^{\hbar}) \rightarrow c_{\hbar} \quad \text{and} \quad (\Psi^{\hbar})'(w_n^{\hbar}) \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (3.7)$$

Our goal will be to show that for sufficiently small values of \hbar each of these sequences possesses an accumulation point, which is nontrivial solution of (S_{\hbar}) .

Lemma 3.2 *For $\hbar > 0$ fixed, the sequence $w_n^{\hbar} = (f_n^{\hbar}, g_n^{\hbar})$ is bounded in X .*

Proof: From properties of the Legendre-Fenchel transform and (H2) we have

$$H^*(x, w_n^{\hbar}) \geq \left(1 - \frac{1}{\alpha}\right) H_f^*(x, w_n^{\hbar}) f_n^{\hbar} + \left(1 - \frac{1}{\alpha}\right) H_g^*(x, w_n^{\hbar}) g_n^{\hbar}. \quad (3.8)$$

Now

$$\begin{aligned} \int H^*(x, w_n^{\hbar}) &= \frac{1}{2} \int \langle T_{\hbar} w_n^{\hbar}, w_n^{\hbar} \rangle + \Psi^{\hbar}(w_n^{\hbar}) \\ &= \Psi^{\hbar}(w_n^{\hbar}) - \frac{1}{2} \langle (\Psi^{\hbar})'(w_n^{\hbar}), w_n^{\hbar} \rangle + \frac{1}{2} \int H_w^*(x, w_n^{\hbar}) w_n^{\hbar}. \end{aligned}$$

Setting $\lambda = \frac{\alpha}{2(\alpha-1)} < 1$, from (3.7) and (3.8) we obtain

$$(1 - \lambda) \int H^*(x, w_n^{\hbar}) \leq c_{\hbar} + o_n(1) \|w_n^{\hbar}\|_X, \quad (3.9)$$

where $o_n(1)$ is a quantity which tends to zero as $n \rightarrow \infty$. By combining (3.5) and (3.9) we get for some $k, K > 0$

$$k \|w_n^{\hbar}\|_X^{\gamma} \leq \|f_n^{\hbar}\|_{L^{\frac{q+1}{q}}}^{\frac{q+1}{q}} + \|g_n^{\hbar}\|_{L^{\frac{p+1}{p}}}^{\frac{p+1}{p}} \leq K c_{\hbar} + o_n(1) \|w_n^{\hbar}\|_X, \quad (3.10)$$

with $\gamma = \min\{1 + 1/p, 1 + 1/q\} > 1$. This trivially implies that $\{w_n^{\hbar}\}$ is bounded in X . \square

With the help of Lemma 3.2 for each $\hbar > 0$ we can extract a subsequence of $\{w_n^{\hbar}\}$ which converges weakly in X to a function $w^{\hbar} = (f^{\hbar}, g^{\hbar})$. We affirm that w is a critical point of Ψ^{\hbar} . First, the sequence $z_n = T_{\hbar} w_n^{\hbar}$ is clearly bounded in X^* , since T_{\hbar} is bounded. Another way of writing (3.7) is

$$T_{\hbar}^{-1} z_n - (\mathcal{H}') (z_n) = o_n(1)$$

(see the proof of Lemma 2.2). Since up to a subsequence we have $z_n \rightharpoonup z$ in $W^{2, \frac{p+1}{p}} \times W^{2, \frac{q+1}{q}}$ we see that the limit function z is a weak solution of (S_{\hbar}) . This implies that $T_{\hbar} z \in X$ and $w = T_{\hbar} z$ is a critical point of Ψ^{\hbar} .

It remains to show that w^{\hbar} is not identically zero. We claim that for small \hbar this is the case. The proof of this claim will be carried out through several steps. First, let u_n^{\hbar} and v_n^{\hbar} be the functions given by

$$u_n^{\hbar} = R_{\hbar} g_n^{\hbar} \in W^{2, \frac{p+1}{p}}(\mathbb{R}^N) \quad \text{and} \quad v_n^{\hbar} = S_{\hbar} f_n^{\hbar} \in W^{2, \frac{q+1}{q}}(\mathbb{R}^N), \quad (3.11)$$

that is,

$$-\Delta u_n^{\hbar} + b(\hbar x) u_n^{\hbar} = g_n^{\hbar} \quad \text{and} \quad -\Delta v_n^{\hbar} + b(\hbar x) v_n^{\hbar} = f_n^{\hbar}, \quad x \in \mathbb{R}^N. \quad (3.12)$$

Next, we note that (1.4) permits to us to choose s, t such that $0 < s, t < 2$, $s + t = 2$ and

$$t < \frac{N}{2}, \quad 2 - t < \frac{N}{2}, \quad \frac{N(p-1)}{2(p+1)} < t < \frac{4(q+1) - N(q-1)}{2(q+1)}. \quad (3.13)$$

Then $p+1 < \frac{2N}{N-2t}$ and $q+1 < \frac{2N}{N-2s}$, which implies

$$W^{2, \frac{p+1}{p}} \hookrightarrow H^s \hookrightarrow L^{q+1} \quad \text{and} \quad W^{2, \frac{q+1}{q}} \hookrightarrow H^t \hookrightarrow L^{p+1},$$

where H^s, H^t are the usual fractional Sobolev spaces over \mathbb{R}^N .

Lemma 3.3 *There exists a constant $\beta > 0$ (independent of \hbar) such that for each $\hbar > 0$ we can find $R = R(\hbar) > 0$, for which*

$$\begin{aligned} \|u_n^\hbar\|_{H^s(\mathbb{R}^N)}^{q+1} &\leq \beta c_\hbar^{\frac{(q+1)p}{p+1}} + \beta \|u_n^\hbar\|_{H^s(B_R)}^{q+1} + o_n(1), \\ \|v_n^\hbar\|_{H^t(\mathbb{R}^N)}^{p+1} &\leq \beta c_\hbar^{\frac{(p+1)q}{q+1}} + \beta \|v_n^\hbar\|_{H^t(B_R)}^{p+1} + o_n(1). \end{aligned}$$

Proof: We shall need some functional analysis. For $s \in (0, 1)$ let $H_{b(\hbar x)}^s$ be the space of the functions u such that

$$b^{\frac{1}{2}}(\hbar x)u \in L^2(\mathbb{R}^N) \quad \text{and} \quad \frac{|u(x) - u(y)|}{|x - y|^{s + \frac{N}{2}}} \in L^2(\mathbb{R}^N \times \mathbb{R}^N).$$

One can also define $H_{b(\hbar x)}^s$ by interpolation between the spaces

$$L_{b(\hbar x)}^2 = \{u : \int b(\hbar x)u^2 < \infty\} \quad \text{and} \quad H_{b(\hbar x)}^1 = \{u \in L_{b(\hbar x)}^2 : \int |\nabla u|^2 < \infty\}.$$

Since $b \in L^\infty(\mathbb{R}^N)$, the inclusion $H^s \subset H_{b(\hbar x)}^s$ holds. On the other hand it is standard to check that $H_{b(\hbar x)}^s(\mathbb{R}^N)$ is embedded into $H^s(B_R)$, for any $s > 0$ and any ball B_R . Once more through Lemma 1 in [25] (see also the argument used in the proof of this lemma) we can prove that $H^s(\mathbb{R}^N) = H_{b(\hbar x)}^s(\mathbb{R}^N)$ under hypotheses (b1) and (b2).

Define $L = -\Delta + b(\hbar x) : H^2 \subset L^2 \rightarrow L^2$ (L is a positive operator) and $A^s := (\sqrt{L})^s$, so that $A^s : H_{b(\hbar x)}^s \rightarrow L^2$ is a isomorphism between $H_{b(\hbar x)}^s$ and L^2 . This is standard functional analysis, for details and references see [10], pages 224-226, where the case $b \equiv 1$ was considered. We observe that $\|u\|_{H_{b(\hbar x)}^s} = \|A^s u\|_{L^2}$. Then the weak formulation of the first equation in (3.12) is

$$\int A^s u_n^\hbar A^t \varphi = \int g_n^\hbar \varphi, \quad \forall \varphi \in H^t. \quad (3.14)$$

Putting $\varphi = A^{-t}A^s u_n^{\hbar}$ into (3.14), we obtain

$$\|u\|_{H_{b(\hbar x)}^s}^2 = \int |A^s u_n^{\hbar}|^2 = \int g_n^{\hbar} A^{-t} A^s u_n^{\hbar}.$$

So there exists a positive constant not dependent on \hbar such that for all $R > 0$

$$\begin{aligned} \|u_n^{\hbar}\|_{H_{b(\hbar x)}^s(\mathbb{R}^N)}^2 &\leq \|g_n^{\hbar}\|_{L^{\frac{p+1}{p}}} \|A^{-t}A^s u_n^{\hbar}\|_{L^{p+1}} \\ &\leq C \|g_n^{\hbar}\|_{L^{\frac{p+1}{p}}} \|u_n^{\hbar}\|_{H^s} \\ &= C \|g_n^{\hbar}\|_{L^{\frac{p+1}{p}}} [\|u_n^{\hbar}\|_{H^s(B_R)} + \|u_n^{\hbar}\|_{H^s(\mathbb{R}^N \setminus B_R)}]. \end{aligned} \quad (3.15)$$

On the other hand, hypotheses (b1) and (b2) imply that we can find $c > 0$ such that for any $\hbar > 0$ there exists $R = R(\hbar)$ for which

$$\|w\|_{H_{b(\hbar x)}^s(\mathbb{R}^N)} \geq c \|w\|_{H^s(\mathbb{R}^N \setminus B_R)}, \quad \forall w \in H_{b(\hbar x)}^s(\mathbb{R}^N) = H^s(\mathbb{R}^N).$$

This inequality (particularly easy to check under (1.3)) follows from Lemma 3 in [25] where the case $s = 1$ was studied, and from an interpolation argument.

Since $x^2 \leq a + bx, x \geq 0$ implies $x \leq C(b + \sqrt{a})$ we get from (3.15)

$$\|u_n^{\hbar}\|_{H^s(\mathbb{R}^N \setminus B_R)} \leq C \|g_n^{\hbar}\|_{L^{\frac{p+1}{p}}(\mathbb{R}^N)} + C \|u_n^{\hbar}\|_{H^s(B_R)}, \quad (3.16)$$

for some positive constant C not dependent on \hbar . Similarly,

$$\|v_n^{\hbar}\|_{H^t(\mathbb{R}^N \setminus B_R)} \leq C \|f_n^{\hbar}\|_{L^{\frac{q+1}{q}}(\mathbb{R}^N)} + C \|v_n^{\hbar}\|_{H^t(B_R)}. \quad (3.17)$$

Recall we already proved (Lemma 3.2 and (3.10)) that there exists a positive constant C not dependent on \hbar for which

$$\|g_n^{\hbar}\|_{L^{\frac{p+1}{p}}} \leq C c_{\hbar}^{\frac{p}{p+1}} + o_n(1) \quad \text{and} \quad \|f_n^{\hbar}\|_{L^{\frac{q+1}{q}}} \leq C c_{\hbar}^{\frac{q}{q+1}} + o_n(1). \quad (3.18)$$

By combining these with (3.16) and (3.17) we get the lemma. \square

The final and basic ingredient of the proof of Theorem 1.1 is contained in the following lemma.

Lemma 3.4 *We have*

$$\lim_{\hbar \rightarrow 0} c_{\hbar} = 0. \quad (3.19)$$

The proof of this lemma will be given in the next section. We shall now proceed to the proof of Theorem 1.1.

Proof of Theorem 1.1. Since $H^*(w)$ is a convex function on \mathbb{R}^2 we have $\langle \nabla H^*(w), w \rangle \geq H^*(w)$ for all $w \in \mathbb{R}^2$. Hence

$$\begin{aligned} c_h &= \lim_{n \rightarrow \infty} [\Psi^h(w_n^h) - \langle (\Psi^h)'(w_n^h), w_n^h \rangle] \\ &\leq \frac{1}{2} \lim_{n \rightarrow \infty} \int \langle T w_n^h, w_n^h \rangle = \frac{1}{2} \lim_{n \rightarrow \infty} \int f_n^h R_h g_n^h + g_n^h S_h f_n^h \\ &\leq \limsup_{n \rightarrow \infty} \left(\|f_n^h\|_{L^{\frac{q+1}{q}}} \|R_h g_n^h\|_{L^{q+1}} + \|g_n^h\|_{L^{\frac{p+1}{p}}} \|S_h f_n^h\|_{L^{p+1}} \right). \end{aligned}$$

By the Hölder inequality for each $\varepsilon > 0$ there exists $C = C(\varepsilon) > 0$ such that

$$c_h \leq \varepsilon \limsup_{n \rightarrow \infty} \left(\|f_n^h\|_{L^{\frac{q+1}{q}}}^{\frac{q+1}{q}} + \|g_n^h\|_{L^{\frac{p+1}{p}}}^{\frac{p+1}{p}} \right) + C \limsup_{n \rightarrow \infty} \left(\|R_h g_n^h\|_{L^{q+1}}^{q+1} + \|S_h f_n^h\|_{L^{p+1}}^{p+1} \right),$$

so by using (3.18) and by choosing ε sufficiently small we get by the Sobolev embedding and the boundedness of R_h, S_h that

$$\begin{aligned} c_h &\leq C \limsup_{n \rightarrow \infty} \left(\|R_h g_n^h\|_{L^{q+1}}^{q+1} + \|S_h f_n^h\|_{L^{p+1}}^{p+1} \right) \\ &\leq C \limsup_{n \rightarrow \infty} \|u_n^h\|_{H^s(\mathbb{R}^N)}^{q+1} + C \limsup_{n \rightarrow \infty} \|v_n^h\|_{H^t(\mathbb{R}^N)}^{p+1}. \end{aligned}$$

Therefore, by the previous lemma,

$$c_h \leq \beta \left(c_h^{\frac{(q+1)p}{p+1}} + c_h^{\frac{(p+1)q}{q+1}} \right) + C \limsup_{n \rightarrow \infty} \|u_n^h\|_{H^s(B_R)}^{q+1} + C \limsup_{n \rightarrow \infty} \|v_n^h\|_{H^t(B_R)}^{p+1}.$$

Note the embeddings $W^{2, \frac{p+1}{p}} \hookrightarrow H^s$, $W^{2, \frac{q+1}{q}} \hookrightarrow H^t$ are compact on bounded domains, so $\{u_n^h\}, \{v_n^h\}$ converge strongly on B_R as $n \rightarrow \infty$. Hence for the limit functions u^h, v^h we get

$$\|u^h\|_{H^s(B_R)}^{q+1} + \|v^h\|_{H^t(B_R)}^{p+1} \geq \left[1 - C^{-1} \beta \left(c_h^{\frac{pq-1}{p+1}} + c_h^{\frac{pq-1}{q+1}} \right) \right],$$

and the last quantity is strictly positive for small h (since $c_h \rightarrow 0$), which means that the limit functions are not identically zero. Note that of course $(f^h, g^h) = (0, 0)$ if and only if $(u^h, v^h) = (0, 0)$. \square

4 Proof of Lemma 3.4.

We start by observing that

$$c_h \leq \inf_{w \in X \setminus \{0\}} \sup_{t \geq 0} \Psi^h(tw) = \inf_{w \in E} \max_{t \geq 0} \Psi^h(tw),$$

where we have set $E = \{w \in X \mid \int \langle Tw, w \rangle > 0\}$. An explicit computation (see Appendix I) shows that for any $w \in E$ we have

$$\max_{t \geq 0} \Psi^{\hbar}(tw) \leq \text{const.} \left(\frac{\|w\|^2}{\int \langle T_{\hbar}w, w \rangle} \right)^{\gamma}, \quad (4.20)$$

where

$$\gamma = \max \left\{ \frac{\frac{p+1}{p}}{2 - \frac{p+1}{p}}, \frac{\frac{q+1}{q}}{2 - \frac{q+1}{q}} \right\}.$$

So to prove Lemma 3.4 it will be enough to establish the following claim.

$$\inf_{w \in E} \frac{\|w\|^2}{\int \langle T_{\hbar}w, w \rangle} \rightarrow 0, \quad \text{as } \hbar \rightarrow 0. \quad (4.21)$$

In order to verify this, we observe that

$$\inf_{w \in E} \frac{\|w\|^2}{\int \langle T_{\hbar}w, w \rangle} = \inf_{w \in E: \|w\|=1} \frac{1}{\int \langle T_{\hbar}w, w \rangle}.$$

Thus, (4.21) is equivalent to the following result.

Lemma 4.1 *We have*

$$\sup_{w \in E: \|w\|=1} \int \langle T_{\hbar}w, w \rangle dx \rightarrow +\infty \quad \text{as } \hbar \rightarrow 0. \quad (4.22)$$

To facilitate the task of the reader, we first describe the idea behind the proof of (4.22). The point is that if p, q are under the critical hyperbola and s, t are chosen as in (3.13), then it is possible to find (explicitly) a function $g \in L^{\frac{p+1}{p}}(\mathbb{R}^N)$ such that if u satisfies

$$-\Delta u(x) = g(x), \quad x \in \mathbb{R}^N,$$

then u does not belong to the fractional Sobolev space $H^s(\mathbb{R}^N)$, and respectively a function $f \in L^{\frac{q+1}{q}}(\mathbb{R}^N)$ such that the solution of $-\Delta v = f$ is not in H^t . We recall that a function w is in $H^s(\mathbb{R}^N)$ if and only if $w \in L^2(\mathbb{R}^N)$ and its Fourier transform $\widehat{w}(\xi)$ is such that $|\xi|^s \widehat{w}(\xi) \in L^2(\mathbb{R}^N)$. Then, assuming (4.22) does not hold we show we can perturb and cut off the functions f, g , to construct a sequence $w_{\hbar} = (f_{\hbar}, g_{\hbar})$ such that $\|w_{\hbar}\| = 1$ and we can control the corresponding $R_{\hbar}g_{\hbar}, S_{\hbar}f_{\hbar}$ in a way which yields a contradiction for small \hbar .

Proof of Lemma 4.1. Let us suppose (4.22) does not hold, that is, there exists $C_0 > 0$ such that

$$\int \langle T_h w, w \rangle dx \leq C_0, \quad \text{for each } w \in E \text{ with } \|w\| = 1.$$

We start by giving some results from the theory of Fourier transforms, which we shall use. The first theorem is a standard fact from the theory of Fourier transforms of distributions.

Theorem 4.2 *Suppose the function u_0 has slow growth, that is, for some $m \in \mathbb{N}$*

$$\int_{\mathbb{R}^N} \frac{|u_0(x)| dx}{(1 + |x|)^m} < \infty. \quad (4.23)$$

Then the Fourier transform \widehat{u} exists and belongs to the class of tempered distributions \mathcal{S}' . In addition if $\phi \in C_c^\infty(\mathbb{R}^N)$ is such that $\phi \equiv 1$ in B_1 , $\phi \equiv 0$ in $\mathbb{R}^N \setminus B_2$ and we set $\phi_n(x) = \phi(x/n)$ then $\widehat{\phi_n u} \rightarrow \widehat{u}$ in \mathcal{S}' .

We shall use the Fourier transform of the function

$$w_0(x) = \frac{1}{1 + |x|^2},$$

and its powers. It is a well-known fact from Fourier analysis that for any $\alpha > 0$ we have

$$\widehat{w_0^\alpha}(\xi) = C(N, \alpha) |\xi|^{\alpha - \frac{N}{2}} K_{\frac{N}{2} - \alpha}(|\xi|), \quad (4.24)$$

(this is for instance formula (3.11) in [15]) ; here $K_\nu(z)$ is the modified Bessel function of the second kind, given by

$$K_\nu(z) = \frac{\Gamma(\nu + \frac{1}{2})}{\sqrt{\pi}} |z|^\nu \int_0^\infty \frac{\cos(t)}{(t^2 + z^2)^{\nu + \frac{1}{2}}} dt = \frac{\Gamma(\nu + \frac{1}{2})}{\sqrt{\pi} |z|^\nu} \int_0^\infty \frac{\cos(sz)}{(1 + s^2)^{\nu + \frac{1}{2}}} ds.$$

Standard analysis shows that $K_\nu(z) > 0$, $K_\nu(z) \in C^\infty(\mathbb{R} \setminus \{0\})$, K decays exponentially as $|z| \rightarrow \infty$, and, most importantly, $K_\nu(z) \sim \text{const.} |z|^{-\nu}$ as $z \rightarrow 0$. Hence

$$\widehat{w_0^\alpha}(\xi) \sim C(N, \alpha) |\xi|^{2\alpha - N} \quad \text{as } |\xi| \rightarrow 0. \quad (4.25)$$

We now fix $p' > p$ and $q' > q$ such that p', q' are still under the critical hyperbola, $\frac{1}{p' + 1} + \frac{1}{q' + 1} > 1 - \frac{2}{N}$. We set

$$\alpha = \frac{Np'}{2(p' + 1)}, \quad \beta = \frac{Nq'}{2(q' + 1)},$$

so that in particular

$$w_0^\alpha \in L^{\frac{p+1}{p}}(\mathbb{R}^N), \quad w_0^\beta \in L^{\frac{q+1}{q}}(\mathbb{R}^N).$$

Let u_0, v_0 be the solutions of

$$-\Delta u_0 = k_1 w_0^\alpha, \quad -\Delta v_0 = k_2 w_0^\beta \quad \text{in } \mathbb{R}^N, \quad (4.26)$$

where $k_1 = k_1(p, q, N) := \|w_0^\alpha\|_{\frac{p+1}{p}}^{-1}$, $k_2 = k_2(p, q, N) := \|w_0^\beta\|_{\frac{q+1}{q}}^{-1}$. By standard PDE theory u_0 and v_0 are functions which belong to some Lebesgue spaces over \mathbb{R}^N (see for instance Theorem 10.2 (i) in [16]), which in particular implies that they have slow growth, as in (4.23) (by the Hölder inequality). Hence Theorem 4.2 applies, and, by taking the Fourier transform on both sides of the equations in (4.26) we get

$$\widehat{u}_0(\xi) = k_1 |\xi|^{-2} \widehat{w}_0^\alpha(\xi), \quad \widehat{v}_0(\xi) = k_2 |\xi|^{-2} \widehat{w}_0^\beta(\xi). \quad (4.27)$$

Note that $\widehat{u}_0, \widehat{v}_0$ are positive.

Lemma 4.3 *We have*

$$\int_{\mathbb{R}^N} |\xi|^2 \widehat{u}_0(\xi) \widehat{v}_0(\xi) d\xi = \infty.$$

Proof. By (4.24) and (4.27) we have

$$|\xi|^2 \widehat{u}_0(\xi) \widehat{v}_0(\xi) \sim C(N, \alpha, \beta) |\xi|^{2(\alpha+\beta-1-N)} \quad \text{as } |\xi| \rightarrow 0.$$

However, by the choice of α and β that we made

$$\begin{aligned} \alpha + \beta - 1 - N &= \frac{N}{2} \left(\frac{p'}{p'+1} + \frac{q'}{q'+1} \right) - 1 - N \\ &= \frac{N}{2} \left(2 - \frac{1}{p'+1} - \frac{1}{q'+1} \right) - 1 - N \\ &< \frac{N}{2} \left(1 + \frac{2}{N} \right) - 1 - N = -\frac{N}{2}, \end{aligned}$$

and the lemma follows. \square

We set $u_n = \phi_n u_0 \in C_c^\infty(\mathbb{R}^N)$, $v_n = \phi_n v_0 \in C_c^\infty(\mathbb{R}^N)$, where ϕ_n is a function as in Theorem 4.2, and

$$\begin{aligned} \widetilde{g}_{n,\hbar} &:= -\Delta u_n + b(\hbar x) u_n, \\ \widetilde{f}_{n,\hbar} &:= -\Delta v_n + b(\hbar x) v_n. \end{aligned}$$

Since u_n, v_n have compact support and $b(0) = 0$ for each fixed n we have

$$\tilde{g}_{n,\hbar} \rightarrow -\Delta u_n \quad \text{in } L^{\frac{p+1}{p}}, \quad \tilde{f}_{n,\hbar} \rightarrow -\Delta v_n \quad \text{in } L^{\frac{q+1}{q}} \quad \text{as } \hbar \rightarrow 0.$$

Clearly

$$-\Delta u_n \rightarrow -\Delta u_0 \quad \text{in } L^{\frac{p+1}{p}}, \quad -\Delta v_n \rightarrow -\Delta v_0 \quad \text{in } L^{\frac{q+1}{q}} \quad \text{as } n \rightarrow \infty,$$

and, recalling that we have taken u_0, v_0 so that $\|\Delta u_0\|_{\frac{p+1}{p}} = \|\Delta u_0\|_{\frac{q+1}{q}} = 1$, we see that we can find n_0 such that for each $n \geq n_0$ there exists \hbar_n for which

$$2 \geq \|\tilde{g}_{n,\hbar}\|_{\frac{p+1}{p}} \geq \frac{1}{2}, \quad 2 \geq \|\tilde{f}_{n,\hbar}\|_{\frac{q+1}{q}} \geq \frac{1}{2}, \quad \text{if } \hbar < \hbar_n.$$

Now set

$$g_{n,\hbar} = \frac{\tilde{g}_{n,\hbar}}{\sqrt{2}\|\tilde{g}_{n,\hbar}\|_{\frac{p+1}{p}}}, \quad f_{n,\hbar} = \frac{\tilde{f}_{n,\hbar}}{\sqrt{2}\|\tilde{f}_{n,\hbar}\|_{\frac{q+1}{q}}},$$

and $w_{n,\hbar} = (f_{n,\hbar}, g_{n,\hbar})$. So $w_{n,\hbar} \in E$ and $\|w_{n,\hbar}\|_X = 1$. By the hypothesis we made

$$\int \langle T_{\hbar} w_{n,\hbar}, w_{n,\hbar} \rangle \leq C_0,$$

for all $n \geq n_0$ and all $\hbar < \hbar_n$.

On the other hand, setting $k_{n,\hbar}^{-1} = 2\|\tilde{g}_{n,\hbar}\|_{L^{\frac{p+1}{p}}} \|\tilde{g}_{n,\hbar}\|_{L^{\frac{p+1}{p}}}$ (by the above $k_{n,\hbar} \in (1/8, 2)$), we have

$$\begin{aligned} \int \langle T_{\hbar} w_{n,\hbar}, w_{n,\hbar} \rangle &= k_{n,\hbar} \int (\tilde{f}_{\hbar,n} R_{\hbar} \tilde{g}_{n,\hbar} + \tilde{g}_{n,\hbar} S_{\hbar} \tilde{f}_{n,\hbar}) \\ &= k_{n,\hbar} \int u_n(-\Delta v_n) + v_n(-\Delta u_n) + 2b(\hbar x) u_n v_n \\ &\geq k_{n,\hbar} \int \widehat{u}_n(-\Delta v_n) + \widehat{v}_n(-\Delta u_n) \\ &= 2k_{n,\hbar} \int |\xi|^2 \widehat{\phi}_n u_0(\xi) \widehat{\phi}_n v_0(\xi) d\xi, \end{aligned}$$

where we used Parseval's identity and the positivity of b, u_n, v_n . Hence

$$\int |\xi|^2 \widehat{\phi}_n u_0(\xi) \widehat{\phi}_n v_0 d\xi \leq 4C_0,$$

and Fatou's lemma gives a contradiction with Lemma 4.3 (note that the definition of the Fourier transform implies $\widehat{\phi}_n u_0(\xi) \rightarrow \widehat{u}_0(\xi)$ for each $\xi \neq 0$).

This completes the proof of Theorem 1. \square

5 Appendix

In this Appendix, we verify the estimate (4.20) used in Lemma 3.4. First, we note that for $w = (f, g)$

$$\Psi^h(tw) = \frac{A}{\alpha}t^\alpha + \frac{B}{\beta}t^\beta - \frac{C}{2}t^2, \quad (5.28)$$

where

$$\alpha = (p+1)/p, \quad \beta = (q+1)/q, \\ A = \int |g|^{\frac{p+1}{p}}, \quad B = \int |f|^{\frac{q+1}{q}}, \quad C = \int \langle T_h w, w \rangle.$$

Denoting the right hand side of (5.28) by $h(t)$, it is easy to check that

$$\max\{\Psi^h(tw), t \geq 0\} = h(\bar{t}),$$

for some $\bar{t} > 0$ if and only if $h'(\bar{t}) = 0$, that is

$$\bar{t}^2 = \frac{A}{C}\bar{t}^\alpha + \frac{B}{C}\bar{t}^\beta.$$

This implies that there exists a positive constant K such that

$$\bar{t} \leq K \left[\left(\frac{A}{C} \right)^{\frac{1}{2-\alpha}} + \left(\frac{B}{C} \right)^{\frac{1}{2-\beta}} \right].$$

Then, for some constant $K' > 0$,

$$\begin{aligned} h(\bar{t}) &= A\left(\frac{1}{\alpha} - \frac{1}{2}\right)\bar{t}^\alpha + B\left(\frac{1}{\beta} - \frac{1}{2}\right)\bar{t}^\beta \\ &\leq K' \left[\frac{A^{\frac{2}{2-\alpha}}}{C^{\frac{\alpha}{2-\alpha}}} + \frac{AB^{\frac{\alpha}{2-\beta}}}{C^{\frac{\alpha}{2-\beta}}} + \frac{BA^{\frac{\beta}{2-\alpha}}}{C^{\frac{\beta}{2-\alpha}}} + \frac{B^{\frac{2}{2-\beta}}}{C^{\frac{\beta}{2-\beta}}} \right] \\ &= K' \left[\frac{A^{\frac{2}{2-\alpha}}}{C^{\frac{\alpha}{2-\alpha}}} + \frac{A}{C^{\frac{\alpha}{2}}} \frac{B^{\frac{\alpha}{2-\beta}}}{C^{\frac{\alpha\beta}{2(2-\beta)}}} + \frac{B}{C^{\frac{\beta}{2}}} \frac{A^{\frac{\beta}{2-\alpha}}}{C^{\frac{\alpha\beta}{2(2-\alpha)}}} + \frac{B^{\frac{2}{2-\beta}}}{C^{\frac{\beta}{2-\beta}}} \right]. \end{aligned}$$

By using the Young inequality, we obtain

$$\begin{aligned} \frac{A}{C^{\frac{\alpha}{2}}} \frac{B^{\frac{\alpha}{2-\beta}}}{C^{\frac{\alpha\beta}{2(2-\beta)}}} + \frac{B}{C^{\frac{\beta}{2}}} \frac{A^{\frac{\beta}{2-\alpha}}}{C^{\frac{\alpha\beta}{2(2-\alpha)}}} &\leq \left(\frac{A}{C^{\frac{\alpha}{2}}} \right)^{\frac{2}{2-\alpha}} + \left(\frac{B^{\frac{\alpha}{2-\beta}}}{C^{\frac{\alpha\beta}{2(2-\beta)}}} \right)^{\frac{2}{\alpha}} \\ &\quad + \left(\frac{B}{C^{\frac{\beta}{2}}} \right)^{\frac{2}{2-\beta}} + \left(\frac{A^{\frac{\beta}{2-\alpha}}}{C^{\frac{\alpha\beta}{2(2-\alpha)}}} \right)^{\frac{2}{\beta}}. \end{aligned}$$

Thus,

$$h(\bar{t}) \leq 2K' \left[\left(\frac{A^{\frac{2}{\alpha}}}{C} \right)^{\frac{\alpha}{2-\alpha}} + \left(\frac{B^{\frac{2}{\beta}}}{C} \right)^{\frac{\beta}{2-\beta}} \right] \leq 2K' \left[\frac{A^{\frac{2}{\alpha}} + B^{\frac{2}{\beta}}}{C} \right]^{\gamma},$$

where $\gamma = \max \left\{ \frac{\alpha}{2-\alpha}, \frac{\beta}{2-\beta} \right\}$.

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