The Variable Hierarchy for the Games mu-Calculus

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Abstract

Parity games are combinatorial representations of closed Boolean μ-terms. By adding to them draw positions, they have been organized by Arnold and one of the authors [3, 24] into a μ-calculus [2] whose standard interpretation is over the class of all complete lattices. As done by Berwanger et al. [7, 8] for the propositional modal μ-calculus, it is possible to classify parity games into levels of a hierarchy according to the number of fixed-point variables. We ask whether this hierarchy collapses w.r.t. the standard interpretation. We answer this question negatively by providing, for each $n \geq 1$, a parity game G_n with these properties: it unravels to a μ-term built up with n fixed-point variables, it is semantically equivalent to no game with strictly less than $n - 2$ fixed-point variables.

1 Introduction

Recent work by Berwanger et al. [5, 6, 7, 8] proves that the expressive power of the modal μ-calculus [18] increases with the number of fixed point variables. By introducing the *variable hierarchy* and showing that it does not collapse, they manage to separate the μ-calculus from dynamic game logic [20]. Their work, solving a long-standing open problem, may also be appreciated for the new research paths[1]disclosed to the theory of fixed-points [2, 11]. The variable hierarchy may be defined for every μ-calculus and for iteration theories as well, since one fixed-point operator is enough to define it. Thus, the question whether the variable hierarchy for a μ-calculus is strict is at least as fundamental as considering its alternation-depth hierarchy. In this paper we answer the question for the *games μ-calculus over complete lattices*.

Parity games are combinatorial representations of closed positive Boolean μ-terms. By adding to them draw positions (or free variables), A. Arnold and L. Santocanale [3, 24] have structured parity games into *the games μ-calculus*. In other words, the authors defined substitution, least and greatest fixed-point operators, as usual for μ-calculi [2]. By Tarski's theorem [25] positive Boolean μ-terms have a natural interpretation in an arbitrary complete lattice. Such interpretation transfers to a standard interpretation of this μ-calculus over the class of all complete lattices.[2] The calculus, together with its canonical preorder, may also be understood as a concrete description of the theory of

[1]We already pursued one of these paths in [4]. We deal here with a problem of a more logical nature.

[2]The interpretation in the class of distributive lattices makes the calculus trivial, since every μ-term is equivalent to a term with no application of fixed-point operators.

binary infs and sups, and of least and greatest fixed point over complete lattices, what we called free μ-lattices in [23].

Let us recall the background of the games μ-calculus. The interaction between two players in a game is a standard model of the possible interactions between a system and its potentially adverse environment. Researchers from different communities are still working on this model despite its introduction dates back at least fifteen years [1, 10, 19] or more [9, 15]. It was proposed in [17] to develop a theory of communication grounded on similar game theoretic ideas and, moreover, on algebraic concepts such as "free lattice" [14] and "free bicomplete category" [16]. A first work pursued this idea using tools of categorical logic [13]. The proposal was further developed in [23] where cycles were added to lattice terms to enrich the model with possibly infinite behaviors. As a result, lattice terms were replaced by positive Boolean μ-terms and their combinatorial representation, parity games. The latter, one of the subtlest tool from the logics of programs, was introduced into the semantics of computation. Given two parity games G, H the witness that the relation $G \leq H$ holds in every complete lattice interpretation is a winning strategy for a prescribed player, Mediator, in a game $\langle G, H \rangle$. A game G may also be considered as modelling a synchronous communication channel available to two users. Then, a winning strategy for Mediator in $\langle G, H \rangle$ witnesses the existence of an asynchronous protocol allowing one user of G to communicate with the other user on H ensuring absence of deadlocks.

Apart from its primary goal, that of describing complete lattices, a major interest of this μ-calculus stems from its neat proof-theory, a peculiarity within the theory of fixed-point logics. The idea that winning strategies for Mediator in the game $\langle G, H \rangle$ are sort of circular proofs was formalized in [22]. More interestingly, proof theoretic ideas and tools – the cut elimination procedure and η-expansion, in their game theoretic disguise – have proved quite powerful to solve deep problems arising from fixed-point theory. These are the alternation-depth hierarchy problem [21] and the status of the ambiguous classes [3]. In [24] the authors were able to partially export these ideas to the modal μ-calculus. We show here that similar tools success in establishing the strictness of the variable hierarchy.

While dealing with the variable hierarchy problem for the games μ-calculus, we shall refer to two digraph complexity measures, the *entanglement* and the *feedback*. The feedback of a vertex v of a tree with back edges is the number of ancestors of v that are the target of a back edge whose source is a descendant of v. The feedback of a tree with back edges is the maximum feedback of its vertices. The entanglement of a digraph G, denoted $\mathcal{E}(G)$, may be defined as follows: *it is the minimum feedback of its finite unravellings into a tree with back edges*. These measures are tied to the logic as follows. A μ-term may be represented as a tree with back-edges, the feedback of which corresponds to the minimum number of fixed point variables needed in the μ-term, up to α-conversion. Also, one may consider terms of a vectorial μ-calculus, i.e. systems of equations, and these roughly speaking are graphs. The step that constructs a canonical solution of a system of equations by means of μ-terms amounts to the construction of a finite unravelling of the graph. In view of these considerations, asking whether a parity game G is semantically equivalent to a μ-term with at most n-variables amounts to

asking whether G belongs to the level $\mathcal{L}_n$ defined as follows:

$$\mathcal{L}_n = \{\, G \in \mathcal{G} \mid G \sim H \text{ for some } H \in \mathcal{G} \text{ s.t. } \mathcal{E}(H) \leq n \,\} \tag{1}$$

Here $\mathcal{G}$ is the collection of parity games with draw positions and $\sim$ denotes the semantic equivalence over complete lattices. In this paper we ask whether the variable hierarchy, made up of the levels $\mathcal{L}_n$, collapses: is there a constant $k \geq 0$, such that for all $n \geq k$, we have $\mathcal{L}_k = \mathcal{L}_n$? We answer this question negatively, there is no such constant. We shall construct, for each $n \geq 1$, a parity game G_n with two properties: (i) G_n unravels to a tree with back edges of feedback n, showing that G_n belongs to $\mathcal{L}_n$, (ii) G_n is semantically equivalent to no game in $\mathcal{L}_{n-3}$. Thus, we prove that the inclusions $\mathcal{L}_{n-3} \subseteq \mathcal{L}_n$, $n \geq 3$, are strict.

The games G_n mimic the n-cliques of [7, 8] that are shapes for hard μ-formulae built up with n fixed point variables. This is only the starting point and, to carry on, we strengthen the notion of *synchronizing game*[3] from [21] to the context of the variable hierarchy. By playing with the η-expansion – i.e. the copycat strategy – and the cut-elimination – i.e. composition of strategies – we prove that the syntactical structure of a game H, which is semantically equivalent to a *strongly synchronizing* game G, resembles that of G: every move (edge) in G can be simulated by a non empty finite sequence of moves (a path) of H; if two paths simulating distinct edges do intersect, then the edges do intersect as well. We formalize such situation within the notion of $\star$*-weak simulation*. The main result is that if there is a $\star$-weak simulation of G by H, then $\mathcal{E}(G) - 2 \leq \mathcal{E}(H)$. The latter statement holds in the general context of digraphs, not just for the games μ-calculus, and might be of general use.

We pinpoint next some aspects and open problems arising from the present work. By combining the result on $\star$-weak simulations with the existence of strongly synchronizing games $G_n \in \mathcal{L}_n$, we have been able to prove that the inclusions $\mathcal{L}_{n-3} \subseteq \mathcal{L}_n$ are strict. Yet we do not know whether $\mathcal{L}_{n-1} \subsetneq \mathcal{L}_n$ and, at present, it is not clear that our methods can be improved to establish the strictness of these inclusions. We remark by the way that we are exhibited with another difference with the alternation hierarchy for which its infinity implies that the inclusions between consecutive classes are strict. Also, the reader will notice that the number of free variables in the games G_n increases with n. He might ask whether hard games can be constructed using a fixed number of free variables. Here the question is positively answered: most of the reasoning depends on free variables forming an antichain so that we can exploit the fact that a countable number of free variables (i.e. generators) can be simulated within the free lattice on three generators [14, §1.6]. Finally, the collection of parallel results on the modal μ-calculus and the games μ-calculus – compare for example [12, 21] – calls for the problem of relating these results by interpreting a μ-calculus into the another one. While translations are a classical topic in logic, we are not aware of results in this direction for μ-calculi.

The paper is organized as follows. Section 2 introduces the necessary background on the algebra of parity games, their organization into a μ-calculus, their canonical preorder. In section 3, we firstly recall the definition of entanglement; then we define

[3] A synchronizing game has the property that there exists just one winning strategy for Mediator in $\langle G, G \rangle$, the copycat strategy.

the $\star$-weak simulation between graphs that allows to compare their entanglements. In section 4, we define strongly synchronizing games and we shall prove their *hardness* w.r.t the variable hierarchy, in particular every equivalent game to a strongly synchronizing one is related with it by a $\star$-weak simulation. In section 5, we construct strongly synchronizing games of arbitrary entanglement. We sum up the discussion in our main result, Theorem 5.2.

Notation, preliminary definitions and elementary facts. If G is a graph, then a path in G is a sequence of the form $\pi = g_0 g_1 \dots g_n$ such that $(g_i, g_{i+1}) \in E_G$ for $0 \leq i < n$. A path is *simple* if $g_i \neq g_j$ for $i, j \in \{ 0, \dots, n \}$ and $i \neq j$. The integer n is the length of π, g_0 is the source of π, noted $\delta_0 \pi = g_0$, and g_n is the target of π, noted $\delta_1 \pi = g_n$. We denote by $\mathrm{II}^+(G)$ the set of simple non empty (i.e. of length greater than 0) paths in G. A pointed digraph $\langle V, E, v_0 \rangle$ of root v_0, is a *tree* if for each $v \in V$ there exists a unique path from v_0 to v. A *tree with back-edges* is a tuple $\mathcal{T} = \langle V, T, v_0, B \rangle$ such that $\langle V, T, v_0 \rangle$ is a tree, and $B \subseteq V \times V$ is a second set of edges such that if $(x, y) \in B$ then y is an ancestor of x in the tree $\langle V, T, v_0 \rangle$. We shall refer to edges in T as tree edges and to edges in B as back edges. We say that $r \in V$ is a return of $\mathcal{T}$ if there exists $x \in V$ such that $(x, r) \in B$. The *feedback of a vertex* v is the number of returns r on the path from v_0 to v such that, for some descendant x of v, $(x, r) \in B$. The *feedback of a tree with back edges* is the maximum feedback of its vertices. We shall say that a pointed directed graph (V, E, v_0) is a tree with back edges if there is a partition of E into two disjoint subsets T, B such that $\langle V, T, v_0, B \rangle$ is a tree with back edges.

If $\mathcal{T}$ is a tree with back edges, then a path in $\mathcal{T}$ can be factored as $\pi = \pi_1 * \dots * \pi_n * \tau$, where each factor π_i is a sequence of tree edges followed by a back edge, and τ does not contain back edges. Such factorization is uniquely determined by the occurrences of back edges in π. For $i > 0$, let r_i be the return at the end of the factor π_i. Let also r_0 be the source of π. Let the b-length of π be the number of back edges in π. i.e. $r_i = \delta_1 \pi_i$.

Lemma 1.1. *If π is a simple path of b-length n, then r_n is the vertex closest to the root visited by π. Hence, if a simple path π lies in the subtree of its source, then it is a tree path.*

We shall deal with trees with back-edges to which a given graph unravels.

Definition 1.2. A *cover* or *unravelling* of a (finite) directed graph H is a (finite) graph K together with a surjective graph morphism $\rho : K \longrightarrow H$ such that for each $v \in V_K$, the correspondence sending k to $\rho(k)$ restricts to a bijection from $\{ k \in V_K \mid (v, k) \in E_K \}$ to $\{ h \in V_H \mid (\rho(v), h) \in E_H \}$.

The notion of cover of pointed digraphs is obtained from the previous by replacing the surjectivity constraint by the condition that ρ preserves the root of the pointed digraphs.

2 The Games μ-Calculus

In this section we recall the defintion of parity games with draws and how they can be structured as a μ-calculus. We shall skip the most of the details and focus only on the syntactical preoder relation $\leq$ between μ-terms that characterizes the semantical order relation.

A *parity game with draws* is a tuple $G = \langle Pos_E^G, Pos_A^G, Pos_D^G, M^G, \rho^G \rangle$ where:

- $Pos_E^G, Pos_A^G, Pos_D^G$ are finite pairwise disjoint sets of positions (Eva's positions, Adam's positions, and draw positions),
- M^G, the set of moves, is a subset of $(Pos_E^G \cup Pos_A^G) \times (Pos_E^G \cup Pos_A^G \cup Pos_D^G)$,
- ρ^G is a mapping from $(Pos_E^G \cup Pos_A^G)$ to $\mathbb{N}$.

Whenever an initial position is specified, these data define a game between player Eva and player Adam. The outcome of a finite play is determined according to the normal play condition: a player who cannot move loses. It can also be a draw, if a position in Pos_D^G is reached.[4] The outcome of an infinite play $\{\ (g_k, g_{k+1}) \in M^G\ \}_{k \geq 0}$ is determined by means of the rank function ρ^G as follows: it is a win for Eva iff the maximum of the set $\{\ i \in \mathbb{N} \mid \exists$ infinitely many k s.t. $\rho^G(g_k) = i\ \}$ is even. To simplify the notation, we shall use $Pos_{E,A}^G$ for the set $Pos_E^G \cup Pos_A^G$ and use similar notations such as $Pos_{E,D}^G$, etc. We let $Max^G = \max \rho^G(Pos_{E,A}^G)$ if the set $Pos_{E,A}^G$ is not empty, and $Max^G = -1$ otherwise.

To obtain a μ-calculus, as defined [2, §2], we label draw positions with variables of a countable set X. If $\lambda^G : Pos_D^G \longrightarrow X$ is such a labelling and $p_\star^G \in Pos_{E,A,D}^G$ is a specified initial position, then we refer to the tuple $\langle G, p_\star^G, \lambda^G \rangle$ as a labeled parity game. We denote by (G, g) the game that differs from G only on the starting position, i.e. $p_\star^{(G,g)} = g$, and similarly we write (G, g) to mean that the play has reached position g. We let $\hat{x}$ be the game with just one final draw position of zero priority and labeled with variable x. With $\mathcal{G}$ we shall denote the collection of all labeled parity games; as no confusion will arise, we will call a labeled parity game with simply "game".

As a μ-calculus, formal composition and fixed-point operations may be defined on $\mathcal{G}$; moreover, $\mathcal{G}$ has meet and join operations.When defining these operations on games we shall always assume that the sets of positions of distinct games are pairwise disjoint.

Meets and Joins. For any finite set I, $\bigwedge_I$ is the game defined by letting $Pos_E = \emptyset$, $Pos_A = \{\ p_0\ \}$, $Pos_D = I$, $M = \{\ (p_0, i) \mid i \in I\ \}$ (where $p_0 \notin I$), $\rho(p_0) = 0$. The game $\bigvee_I$ is defined similarly, exchanging Pos_E and Pos_A.

Composition Operation. Given two games G and H and a mapping $\psi : P_D^G \longrightarrow P_{E,A,D}^H$, the game $K = G \circ_\psi H$ is defined as follows:

- $Pos_E^K = Pos_E^G \cup Pos_E^H$,
- $Pos_A^K = Pos_A^G \cup Pos_A^H$,
- $Pos_D^K = Pos_D^H$,

[4] Observe that there are no possible moves from a position in Pos_D^G.

- $M^K = (M^G \cap (Pos^G_{E,A} \times Pos^G_{E,A})) \ \cup \ M^H$
  $\cup \ \{ \ (p, \psi(p')) \mid (p,p') \in M^G \cap (Pos^G_{E,A} \times Pos^G_D) \ \}$.
- ρ^K is such that its restrictions to the positions of G and H are respectively equal to ρ^G and ρ^H.

Sum Operation. Given a finite collection of parity games G_i, $i \in I$, their sum $H = \sum_{i \in I} G_i$ is defined in the obvious way:

- $P^H_Z = \bigcup_{i \in I} P^{G_i}_Z$, for $Z \in \{ E, A, D \}$,
- $M^H = \bigcup_{i \in I} M^{G_i}$,
- ρ^H is such that its restriction to the positions of each G_i is equal to ρ^G_i.

Fixed-Point Operations. If G is a game, a system on G is a tuple $S = \langle E, A, M \rangle$ where:

- E and A are pairwise disjoint subsets of Pos^G_D,
- $M \subseteq (E \cup A) \times Pos^G_{E,A,D}$.

Given a system S and $\theta \in \{ \mu, \nu \}$, we define the parity game $\theta_S.G$:

- $Pos^{\theta_S.G}_E = Pos^G_E \cup E$,
- $Pos^{\theta_S.G}_A = Pos^G_A \cup A$,
- $Pos^{\theta_S.G}_D = Pos^G_D - (E \cup A)$,
- $M^{\theta_S.G} = M^G \cup M$,
- $\rho^{\theta_S.G}$ is the extension of ρ^G to $E \cup A$ such that:
  - if $\theta = \mu$, then $\rho^{\theta_S.G}$ takes on $E \cup A$ the constant value Max^G if this number is odd or $Max^G + 1$ if Max^G is even,
  - if $\theta = \nu$, then $\rho^{\theta_S.G}$ takes on $E \cup A$ the constant value Max^G if this number is even or $Max^G + 1$ if Max^G is odd.

Semantics of $\mathcal{G}$. The algebraic nature of parity games is better understood by defining their semantics. To this goal, let us define the **predecessor game** G^-, for G a game such that $Max^G \neq -1$, i.e. there is at least one position in $Pos^G_{E,A}$. Let $Top^G = \{ g \in Pos^G_{E,A} \mid \rho^G(g) = Max^G \}$, then G^- is defined as follows:

- $Pos^{G^-}_E = Pos^G_E - Top^G$, $Pos^{G^-}_A = Pos^G_A - Top^G$, $Pos^{G^-}_D = Pos^G_D \cup Top^G$,
- $M^{G^-} = M^G - (Top^G \times Pos^G_{E,A,D})$,
- ρ^{G^-} is the restriction of ρ^G to $Pos^{G^-}_{E,A}$.

Given a complete lattice L, the interpretation of a parity game G in L is a monotone mapping of the form $\|G\| : L^{P^G_D} \longrightarrow L^{P^G_{E,A}}$. Here L^X is the X-fold product lattice of L with itself so that, for $x \in X$, $\mathtt{pr}_x : L^X \longrightarrow L$ will denote the projection onto the x-coordinate. The interpretation of a parity game is defined inductively. If $P^G_{E,A} = \emptyset$, then $L^{P^G_{E,A}} = L^\emptyset = 1$, the complete lattices with just one element, and there is just one possible definition of the mapping $\|G\|$. Otherwise, if Max^G is odd, then $\|G\|$ is the parameterized least fixed-point of the monotone mapping $L^{P^G_{E,A}} \times L^{P^G_D} \longrightarrow L^{P^G_{E,A}}$ defined by the system of equations:

$$x_g = \begin{cases} \bigvee \{\ x_{g'} \mid (g,g') \in M^G\ \}\,, & \text{if } g \in Pos^G_E \cap Top^G, \\ \bigwedge \{\ x_{g'} \mid (g,g') \in M^G\ \}\,, & \text{if } g \in Pos^G_A \cap Top^G, \\ \mathtt{pr}_g \circ \|G^-\|(X_{Top^G}, X_{Pos^G_D})\,, & \text{otherwise.} \end{cases}$$

If Max^G is even, then $\|G\|$ is the parameterized greatest fixed-point of this mapping.

The preorder on $\mathcal{G}$. In order to describe a preorder on the class $\mathcal{G}$, we shall define a new game $\langle G, H\rangle$ for a pair of games G and H in $\mathcal{G}$. This is not a pointed parity game with draws as defined in the previous section; to emphasize this fact, the two players will be named Mediator and Opponents instead of Eva and Adam.

Definition 2.1. The game $\langle G, H\rangle$ is defined as follows:

- The set of Mediator's positions is $Pos^G_A \times Pos^H_{E,D} \cup Pos^G_{A,D} \times Pos^H_E \cup \mathcal{L}(M)$, and the set of Opponents' positions is $Pos^G_E \times Pos^H_{E,A,D} \cup Pos^G_{E,A,D} \times Pos^H_A \cup \mathcal{L}(O)$, where $\mathcal{L}(M), \mathcal{L}(O) \subseteq Pos^G_D \times Pos^H_D$ are the losing positions for Mediator and Opponents respectively. They are defined as follows. If $(g,h) \in Pos^G_D \times Pos^H_D$, then: if $\lambda^G(g) = \lambda^H(h)$, then the position (g,h) belongs to Opponents, and there is no move from this position, hence this is a winning position for Mediator. If $\lambda^G(g) \neq \lambda^H(h)$, then the position (g,h) belongs to Mediator and there is no move from this position. The latter is a win for Opponents.

- Moves of $\langle G, H\rangle$ are either left moves $(g,h) \to (g',h)$, where $(g,g') \in M^G$, or right moves $(g,h) \to (g,h')$, where $(h,h') \in M^H$; however the Opponents can play only with Eva on G or with $Adam$ on H.

- A finite play is a loss for the player who can not move. An infinite play γ is a win for Mediator if and only if its left projection $\pi_G(\gamma)$ is a win for Adam, or its right projection $\pi_H(\gamma)$ is a win for Eva.

Definition 2.2. If G and H belong to $\mathcal{G}$, then we declare that $G \leq H$ if and only if Mediator has a winning strategy in the game $\langle G, H\rangle$ starting from position $(p^G_\star, p^H_\star)$.

The following is the reason to consider such a syntactic relation:

Theorem 2.3 (See [23]). *The relation $\leq$ is sound and complete with respect to the interpretation in any complete lattice, i.e. $G \leq H$ if and only if $\|G\| \leq \|H\|$ holds in every complete lattice.*

In the sequel, we shall write $G \sim H$ to mean that $G \leq H$ and $H \leq G$. For other properties of the relation $\leq$, see for example Proposition 2.5 of [3]. One can prove that $G \leq G$, by exibing the *copycat* strategy in the game $\langle G, G\rangle$: from a position (g, g), it is Opponents' turn to move either on the left or on the right board. When they stop moving, Mediator will have the ability to copy all the moves played by the Opponents so far from the other board until the play reaches the position (g', g'). There it was also proved that if $G \leq H$ and $H \leq K$ then $G \leq K$, by describing a game $\langle G, H, K\rangle$ with the following properties: (1) given two winning strategies R on $\langle G, H\rangle$, and S on $\langle H, K\rangle$ there is a winning strategy $R\|S$ on $\langle G, H, K\rangle$, that is the composition of the strategies R and S, (2) given a winning strategy T on $\langle G, H, K\rangle$, there exists a winning strategy $T_{\backslash H}$ on $\langle G, K\rangle$.

The game $\langle G, H, K\rangle$ is the fundamental tool that will allow us to deduce the desired structural properties of games H which are equivalent to a specified game G, by considering the game $\langle G, H, G\rangle$, section 4. The game $\langle G, H, K\rangle$ is obtained by gluing the games $\langle G, H\rangle$ and $\langle H, K\rangle$ on the central board H as follows.

Definition 2.4. Positions of the game $\langle G, H, K\rangle$ are triples $(g, h, k) \in Pos^G_{A,E,D} \times Pos^H_{A,E,D} \times Pos^K_{A,E,D}$ such that

- the set of Mediator's positions is

  $$Pos^G_A \times Pos^H_{A,E,D} \times Pos^K_{E,D} \ \cup \ Pos^G_{A,D} \times Pos^H_{A,E,D} \times Pos^K_E \ \cup \ \mathcal{L}(M)\,,$$

  and the set of Opponents' positions is

  $$Pos^G_E \times Pos^H_{A,E,D} \times Pos^K_{E,A,D} \ \cup \ Pos^G_{E,A,D} \times Pos^H_{A,E,D} \times Pos^K_A \cup \ \mathcal{L}(O)\,,$$

  where $\mathcal{L}(M), \mathcal{L}(O) \subseteq Pos^G_D \times Pos^H_{A,E,D} \times Pos^K_D$ are positions of Mediator and Opponents, respectively, defined as follows. Whenever $(g, h, k) \in Pos^G_D \times Pos^H_{A,E,D} \times Pos^K_D$, then if $h \in Pos^H_{E,A}$, then the position (g, h, k) belongs to Mediator, otherwise, i.e. $h \in Pos^H_D$, then the final position (g, h, k) belongs to Opponents if and only if $\lambda^G(g) = \lambda^H(h) = \lambda^K(k)$.

- Moves of $\langle G, H, K\rangle$ are either left moves $(g, h, k) \to (g', h, k)$ where $(g, g') \in M^G$ or central moves $(g, h, k) \to (g, h', k)$, where $(h, h') \in M^H$, or right moves $(g, h, k) \to (g, h, k')$, where $(k, k') \in M^K$; however the Opponents can play only with Eva on G or with Adam on K.

- As usual, a finite play is a loss for the player who cannot move. An infinite play γ is a win for Mediators if and only if $\pi_G(\gamma)$ is a win for Adam on G, or $\pi_K(\gamma)$ is a win for Eva on K.

3 Entanglement and $\star$-Weak Simulations

Let us recall the main tool which measures the combinatorial essence of the variable hierarchy level on directed graphs. This is the *entanglement* of a digraph G and might already be defined as *the minimum feedback of the finite unravelings of G into a tree with back edges.* The entanglement of G may also be characterized by means of a special Robber and Cops game $\mathcal{E}(G,k)$, $k = 0, \ldots, |V_G|$. This game, defined in [6], is played by Thief against Cops, a team[5] of k cops, as follows.

Definition 3.1. The entanglement game $\mathcal{E}(G,k)$ of a digraph G is defined by:

- Its positions are of the form (v, C, P), where $v \in V_G$, $C \subseteq V_G$ and $|C| \leq k$, $P \in \{Cops, Thief\}$.
- Initially Thief chooses $v_0 \in V$ and moves to $(v_0, \emptyset, Cops)$.
- Cops can move from $(v, C, Cops)$ to $(v, C', Thief)$ where C' can be
  - C : Cops skip,
  - $C \cup \{ v \}$: Cops add a new Cop on the current position,
  - $(C \setminus \{ x \}) \cup \{ v \}$: Cops move a placed Cop to the current position.
- Thief can move from $(v, C, Thief)$ to $(v', C, Cops)$ if $(v, v') \in E_G$ and $v' \notin C$.

Every finite play is a win for Cops, and every infinite play is a win for Thief.

The following will constitute our working definition of entanglement: *$\mathcal{E}(G)$, the entanglement of G, is the minimum $k \in \{ 0, \ldots, |V_G| \}$ such that Cops have a winning strategy in $\mathcal{E}(G,k)$.* The following proposition provides a useful variant of entanglement games.

Proposition 3.2. *Let $\widetilde{\mathcal{E}}(G,k)$ be the game played as the game $\mathcal{E}(G,k)$ apart that Cops is allowed to retire a number of cops placed on the graph. That is, Cops moves are of the form*

- $(g, C, Cops) \to (g, C', Thief)$ *(generalized skip move),*
- $(g, C, Cops) \to (g, C' \cup \{ g \}, Thief)$ *(generalized replace move),*

where in both cases $C' \subseteq C$. Then Cops has a winning strategy in $\mathcal{E}(G,k)$ if and only if he has a winning strategy in $\widetilde{\mathcal{E}}(G,k)$.

$\star$-Weak Simulations. We define next a relation between graphs, called *$\star$-weak simulation*, to be used to compare their entanglements. Intuitively, there is a $\star$-weak simulation of a graph G by H if every edge of G is simulated by a non empty finite path of H. Moreover, two edges e_1, e_2 of G not sharing a common endpoint, are simulated by paths π_1, π_2 that do not intersect. These simulations arise when considering games which are semantically equivalent to strongly synchronizing games, as defined in Section 4.

[5] We shall use the singular to emphasize that Cops constitute a team.

Definition 3.3. A *weak simulation* (R, ς) of G by H is a binary relation $R \subseteq V_G \times V_H$ that comes with a partial function $\varsigma : V_G \times V_G \times V_H \longrightarrow \Pi^+(H)$, such that:

- R is surjective, i.e. for every $g \in V_G$ there exists $h \in V_H$ such that gRh,
- R is functional, i.e. if $g_i R h$ for $i = 1, 2$, then $g_1 = g_2$,
- if gRh and $g \to g'$, then $\varsigma(g, g', h)$ is defined and $h' = \delta_1 \varsigma(g, g', h)$ is such that $g'Rh'$.

Now we want to study conditions under which existence of a weak simulation of G by H implies that $\mathcal{E}(G)$ is some lower bound of $\mathcal{E}(H)$. To this goal, we abuse of notation and write $h \in \varsigma(g, g', h_0)$ if $\varsigma(g, g', h_0) = h_0 h_1 \dots h_n$ and, for some $i \in \{ 0, \dots, n \}$, we have $h = h_i$. If $G = (V_G, E_G)$ is a directed graph then its undirected version $S(G) = (V_G, E_{S(G)})$ is the undirected graph such that $\{g, g'\} \in E_{S(G)}$ iff $(g, g') \in E_G$ or $(g', g) \in E_G$. Thus we say that G has *girth at least* k if the shortest cycle in $S(G)$ has length at least k, G does not contain loops, and $(g, g') \in E_G$ implies $(g', g) \notin E_G$.

Definition 3.4. We say that a weak simulation (R, ς) of G by H is a $\star$-weak simulation (or that it has the $\star$-property) if G has girth at least 4, and if $(g, g'), (\tilde{g}, \tilde{g}')$ are distinct edges of G and $h \in \varsigma(g, g', h_0), \varsigma(\tilde{g}, \tilde{g}', \tilde{h}_0)$, then $|\{ g, g', \tilde{g}, \tilde{g}' \}| = 3$.

We explain next this property. Given (R, ς), consider

$$C(h) = \{ (g, g') \in E_G \mid \exists h_0 \text{ s.t. } h \in \varsigma(g, g', h_0) \} .$$

Lemma 3.5. *Let (R, ς) be a $\star$-weak simulation of G by H. If $C(h)$ is not empty, then there exists an element $c(h) \in V_G$ such that for each $(g, g') \in C(h)$ either $c(h) = g$ or $c(h) = g'$. If moreover $|C(h)| \geq 2$, then this element is unique.*

That is, $C(h)$ considered as an undirected graph, is a star. Since $c(h)$ is unique whenever $|C(h)| \geq 2$, then $c(h)$ is a partial function which is defined for all h with $|C(h)| \geq 2$. This allows to define a partial function $f : V_H \longrightarrow V_G$, which is defined for every h for which $C(h) \neq \emptyset$, as follows:

$$f(h) = \begin{cases} c(h), & |C(h)| \geq 2 , \\ g, & \text{if } C(h) = \{ (g, g') \} \text{ and } h \text{ has no predecessor in } H , \\ g', & \text{if } C(h) = \{ (g, g') \} \text{ and } h \text{ has a predecessor in } H . \end{cases} \qquad (2)$$

Let us remark that if $h \in \varsigma(g, g', h_0)$, then $f(h) \in \{ g, g' \}$. If gRh and h has no predecessor, then $f(h) = g$. Also, if h' is the target of $\varsigma(g, g', h_0)$ and g' has a successor, then $f(h') = g'$.

Lemma 3.6. *If (R, ς) is a $\star$-weak simulation of G by H and $\rho : K \longrightarrow H$ is an unravelling of H, then there exists a $\star$-weak simulation $(\tilde{R}, \tilde{\varsigma})$ of G by K.*

Let us now recall that if H is a tree with back edges, rooted at h_0, of feedback k, then Cops has a *canonical winning strategy* in the game $\mathcal{E}(H, k)$ from position $(h_0, C, Cops)$. Every time a return is visited, a cop is dropped on such a return. If a cop has to be replaced in order to occupy such a return, then the cop which is closest to the root is chosen.

Remark 3.7. Let us remark that, by using the canonical strategy, (i) every path chosen by Thief in H is a tree path, (ii) if the position in $\mathcal{E}(H,k)$ is of the form $(h, C, Thief)$, and $h' \neq h$ is in the subtree of h, then the unique tree path from h to h' does contain no cops, apart possibly for the vertex h. Finally, a vertex $h \in V_H$ determines a position $(h, C_H(h), Thief)$ in the game $\mathcal{E}(H,k)$ that has been reached from the initial position $(h_0, \emptyset, Cops)$ and where Cops have been playing according to the canonical strategy. $C_H(h)$ is determined as the set of returns r of H on the tree path from h_0 to h such that the tree path from r to h contains at most k returns.

The following Theorem establishes the desired connection between $\star$-weak simulations and entanglement.

Theorem 3.8. *If (R, ς) is a $\star$-weak simulation of G by H, then $\mathcal{E}(G) \leq \mathcal{E}(H) + 2$.*

Proof. Let $k = \mathcal{E}(H)$. We shall define first a strategy for Cops in the game $\widetilde{\mathcal{E}}(G, k+2)$. In a second time, we shall prove that this strategy is a *winning* strategy for Cops.

Let us consider Thief's first move in $\widetilde{\mathcal{E}}(G, k+2)$. This move picks $g \in G$ leading to the position $(g, \emptyset, Cops)$ of $\widetilde{\mathcal{E}}(G, k+2)$. Cops answers by occupying the current position, i.e. he moves to $(g, \{ g \}, Thief)$. After this move, Cops also chooses a tree with back edges of feedback k to which H unravel, $\pi : \mathcal{T}(H) \longrightarrow H$, such that the root h_0 of $\mathcal{T}(H)$ satisfies $g R \pi(h_0)$. We can also suppose that h_0 is not a return, thus it has no predecessor. According to Lemma 3.6 we can lift the $\star$-weak simulation (R, ς) to a $\star$-weak simulation $(\tilde{R}, \tilde{\varsigma})$ of G by $\mathcal{T}(H)$. In other words, we can suppose from now on that H itself is a tree with back edges of feedback k rooted at h_0 and, moreover, that $g R h_0$.

From this point on, Cops uses a memory to choose how to place cops in the game $\widetilde{\mathcal{E}}(G, k+2)$. To each Thief's position $(g, C_G, Thief)$ in $\widetilde{\mathcal{E}}(G, k+2)$ we associate a data structure (the memory) consisting of a triple $M(g, C, Thief) = (p, c, h)$, where $c, h \in V_H$ and $p \in V_H \cup \{ \perp \}$ (we assume that $\perp \notin V_H$). Moreover c is an ancestor of h in the tree and, whenever $p \neq \perp$, p is an ancestor of c as well.

Intuitively, we are matching the play in $\widetilde{\mathcal{E}}(G, k+2)$ with a play in $\mathcal{E}(H, k)$, started at the root h_0 and played by Cops according to the canonical strategy. Thus c is the vertex of H currently occupied by Thief in the game $\mathcal{E}(H, k)$.[6] Instead of recalling all the play (that is, the history of all the positions played so far), we need to record the last position played in $\mathcal{E}(H, k)$: this is p, which is undefined when the play begins. Cops on G are positioned on the images of Cops on H by the function f defined in (2). Moreover, Cops eagerly occupies the last two vertices visited on G. Thief's moves on G are going to be simulated by sequences of Thief's moves on H, using the $\star$-weak simulation (R, ς). In order to make this possible, a simulation of the form $\varsigma(\tilde{g}, g, \tilde{h})$ must be halted before its target h; the current position c is such halt-point. This implies that the simulation of $g \rightarrow g'$ by (R, ς) and the sequence of moves in H matching Thief's move on G are sligthly out of phase. To cope with that, Cops must guess in advance what might happen in the rest of the simulation and this is why he puts cops on the current and previous positions in G. We also need to record h, the target of the previous simulation into the memory.

[6] More precisely we are associating to the position $(g, C_G, Thief)$ of $\mathcal{E}(G, k+2)$ the position $(c, C_H, Thief)$ in $\mathcal{E}(H, k)$, where C_H is determined as $C_H = C_H(c)$ as in Remark 3.7.

The previous considerations are formalized by requiring the following conditions to hold. To make sense of them, let us say that $f(\{\, p \,\}) = f(p)$ if $p \in V_H$ and that $f(\{\, p \,\}) = \emptyset$ if $p = \bot$. In the last two conditions we require that $p \neq \bot$.

- $C_G = f(C_H(c)) \cup f(\{\, p \,\}) \cup \{\, g \,\}$, (COPS)
- $f(c) = g$, and $f(h') \in f(\{\, p \,\}) \cup \{\, g \,\}$, whenever
  h' lies on the tree path from c to h, (TAIL)
- $f(p) \to g$, $f(p) R \tilde{h}$ for some $\tilde{h} \in V_H$, $c \in \varsigma(f(p), g, \tilde{h})$,
  and h is the target of $\varsigma(f(p), g, \tilde{h})$, (HEAD)
- on the tree path from p to c,
  c is the only vertex s.t. $f(c) = g$. (HALT)

Since h_0 has no predecessors, then $g R h_0$ implies $f(h_0) = g$. Thus, at the beginning, the memory is set to $(\bot, h_0, h_0)$ and conditions (COPS) and (TAIL) hold.

Consider now a Thief's move of the form $(g, C_G, Thief) \to (g', C_G, Cops)$, where $g' \notin C_G$. If g' has no successor, then Cops simply skips, thus reaching a winning position. Let us assume that g' has a successor, and write $\varsigma(g, g', h) = h h_1 \dots h_n$, $n \geq 1$; observe that $f(h_n) = g'$. If for some $i = 1, \dots, n$ h_i is not in the subtree of c, then the strategy halts, Cops abandons the game and looses. Otherwise, all the path $\pi = c \dots h h_1 \dots h_n$ lies in the subtree of c. By eliminating cycles from π, we obtain a simple path σ, of source c and target h_n, which entirely lies in the subtree of c. By Lemma 1.1, σ is the tree path from c to h_n. An explicit description of σ is as follows: we can write σ as the compose $\sigma_0 \star \sigma_1$, where the target of σ_0 and source of σ_1 is the vertex of $\varsigma(g, g', h)$ which is closest to the root h_0; moreover σ_0 is a prefix of the tree path from c to h, and σ_1 is a postfix of the path $\varsigma(g, g', h)$.

We cut σ as follows: we let c' be the first vertex on this path such that $f(c') = g'$. Thief's move $g \to g'$ on G is therefore simulated by Thief's moves from c to c' on H. This is possible since every vertex lies in the subtree of c and thus it has not yet been explored. Cops consequently occupies the returns on this path, thus modifying C_H to $C'_H = C_H(c') = (C_H \setminus X) \uplus Y$, where Y is a set of at most k vertexes containing the last returns visited on the path from c to c'.

After the simulation on H, Cops moves to $(g', C'_G, Thief)$ in $\widetilde{\mathcal{E}}(G, k+2)$, where $C'_G = f(C'_H) \cup \{\, g, g' \,\}$. Let us verify that this is an allowed move according to the rules of the game. We remark that $f(Y) \subseteq f(\{\, p \,\}) \cup \{\, g, g' \,\}$ and therefore

$$\begin{aligned} C'_G &= f(C_H \setminus X) \cup f(Y) \cup \{\, g, g' \,\} \\ &= (f(C_H \setminus X) \cup (f(Y) \setminus \{\, g' \,\}) \cup \{\, g \,\}) \cup \{\, g' \,\} \\ &= A \cup \{\, g' \,\}, \end{aligned}$$

where $A = f(C_H \setminus X) \cup (f(Y) \setminus \{\, g' \,\}) \cup \{\, g \,\} \subseteq f(C_H) \cup f(\{\, p \,\}) \cup \{\, g \,\} = C_G$. After the simulation Cops also updates the memory to $M(g', C'_G, Thief) = (c, c', h_n)$. Since $f(c) = g$, then condition (COPS) clearly holds. Also, $f(c) = g \to g'$, $g R h$ and h_n is the target of $\varsigma(f(c), g', h)$. We have also that $c' \in \sigma_1$ and hence $c' \in \varsigma(f(c), g', h)$, since otherwise $c' \in \sigma_0$ and $f(c') \in \{\, f(p), g \,\}$, contradicting $f(c') = g'$

and the condition on the girth of G. Thus condition (HEAD) holds as well. Also, condition (HALT) holds, since by construction c' is the first vertex on the tree path from c to h such that $f(c') = g'$. Let us verify that condition (TAIL) holds: by construction $f(c') = g'$, and the path from c' to h_n is a postfix of $\varsigma(g, g', h)$, and hence $f(h') \in \{ g, g' \}$ if h' lies on this tree path.

Let us now prove that the strategy is winning. If Cops never abandons, then an infinite play in $\widetilde{\mathcal{E}}(G, k+2)$ would give rise to an infinite play in $\mathcal{E}(H, k)$, a contradiction. Thus, let us prove that Cops will never abandon. To this goal we need to argue that when Thief plays the move $g \to g'$ on G, then the simulation $\varsigma(g, g', h) = hh_1 \dots h_n$ lies in the subtree of c. If this is not the case, let i be the first index such that h_i is not in the subtree of c. Therefore h_i is a return and, by the assumptions on H and the on canonical strategy, $h_i \in C_H(c)$. Since $h_i \in \varsigma(g, g', h)$, $f(h_i) \in \{ g, g' \}$. Observe, however that we cannot have $f(h_i) = g'$, otherwise $g' \in f(C_H(c)) \subseteq C_G$. We deduce that $f(h_i) = g$ and that $g \in f(C_H) \subseteq C_G$.

Since $C_G \neq \bot$, then $(g, C_G, Thief)$ is not the initial position of the play, so that, if $M(g, C_G, Thief) = (p, c, h)$, then $p \neq \bot$. Let us now consider the last two moves of the play before reaching position $(g, C_G, Thief)$. These are of the form $(f(p), \tilde{C}_G, Thief) \to (g, \tilde{C}_G, Cops) \to (g, C_G, Thief)$, and have been played according to this strategy. Since $g \notin \tilde{C}_G$, it follows that the Cop on h_i has been dropped on H during the previous round of the strategy, simulating the move $f(p) \to g$ on G by the tree path from p to c. This is however in contradiction with condition (HALT), stating that c is the only vertex h on the tree path from p to c such that $f(h) = c$. □

4 Strongly Synchronizing Games

In this section we define *strongly synchronizing* games, a generalization of synchronizing games introduced in [21]. We shall show that, for every game H equivalent to a strongly synchronizing game G, there is a $\star$-weak simulation of G by H.[7]

Let us say that $G \in \mathcal{G}$ is *bipartite* if $M^G \subseteq Pos_E^G \times Pos_{A,D}^G \cup Pos_A^G \times Pos_{E,D}^G$.

Definition 4.1. A game G is *strongly synchronizing* iff its is bipartite, it has girth strictly greater than 4 and, for every pair of positions g, k, the following conditions hold:

1. if $(G, g) \sim (G, k)$ then $g = k$.
2. if $(G, g) \leq (G, k)$ and $(G, k) \not\leq (G, g)$, then $k \in Pos_E^G$ and $(k, g) \in M^G$, or $g \in Pos_A^G$ and $(g, k) \in M^G$.

A consequence of the previous definition is that *the only winning strategy for Mediator in the game $\langle G, G \rangle$ is the copycat strategy.* Thus strongly synchronizing games are synchronizing as defined in [21]. We list next some useful properties of strongly synchronizing games.

Lemma 4.2. *Let G be a strongly synchronizing and let $(g, g'), (\tilde{g}, \tilde{g}') \in M^G$ be distinct.*

[7] In the sequel, we shall not distinguish between a game and its underlying graph.

1. *If* $(G, g) \sim \hat{x}$ *then* $g \in Pos_D^G$ *and* $\lambda(g) = x$.

2. *If* $g, \tilde{g} \in Pos_E^G$ *and, for some game* H *and* $h \in Pos^H$*, we have*

$$(G, g') \leq (H, h) \leq (G, g) \text{ and}$$
$$(G, \tilde{g}') \leq (H, h) \leq (G, \tilde{g}),$$

then $g = \tilde{g}$ *or* $g' = \tilde{g}'$*, and* $|\{ g, g', \tilde{g}, \tilde{g}' \}| = 3$.

3. *If* $g \in Pos_E^G$ *and* $\tilde{g} \in Pos_A^G$ *and, for some* H *and* $h \in Pos^H$*, we have*

$$(G, g') \leq (H, h) \leq (G, g) \text{ and}$$
$$(G, \tilde{g}) \leq (H, h) \leq (G, \tilde{g}'),$$

then $g = \tilde{g}'$ *or* $g' = \tilde{g}$*, and* $|\{ g, g', \tilde{g}, \tilde{g}' \}| = 3$.

We are ready to state the main result of this section.

Proposition 4.3. *Let* G *be a strongly synchronizing game, and let* $H \in \mathcal{G}$ *be such that* $G \leq H \leq G$*, then there is a* $\star$*-weak simulation of* G *by* H.

Proof. Let S, S' be two winning strategies for Mediator in $\langle G, H \rangle$ and $\langle H, G \rangle$, respectively. Let $T = S||S'$ be the composal strategy in $\langle G, H, G \rangle$. Define

$$gRh \text{ iff } (g, h, g) \text{ is a position of } T$$
$$\text{and } g, h \text{ belong to the same player.}$$

We consider first R and prove that it is functional and surjective. If $g_i Rh, i = 1, 2$ then (g_1, h, g_1) and (g_2, h, g_2) are positions of T, hence $(G, g_1) \leq (H, h) \leq (G, g_1)$ and $(G, g_2) \leq (H, h) \leq (G, g_2)$, consequently $(G, g_1) \sim (G, g_2)$ implies $g_1 = g_2$, by definition 4.1. For surjectivity, we can assume that (a) all the positions of G are reachable from the initial position $p_\star^G$, (b) $p_\star^G$ and $p_\star^H$ belong to the same player (by possibly adding to H a new initial position leading to the old one). Since $T_{\backslash H}$ is the copycat strategy, given $g \in Pos_{E,A,D}^G$, from the initial position $(p_\star^G, p_\star^H, p_\star^G)$ of $\langle G, H, G \rangle$, the Opponents have the ability to reach a position of the form (g, h, g). The explicit construction of the function ς will show that h can be chosen to belong to the same player as g.

We construct now the function ς so that (R, ς) is a weak simulation. If gRh and $(g, g') \in M^G$, then we construct $\pi = h, \ldots, h'$ such that $g'Rh'$. Since G is bipartite, then $h \neq h'$ and π is nonempty. We let $\varsigma(g, g', h)$ be a reduction of π to a nonempty simple path.

We assume $(g, h) \in (Pos_E^G, Pos_E^H)$, the case $(g, h) \in (Pos_A^G, Pos_A^H)$ is dual. From position (g, h, g) it is Opponent's turn to move on the left, they choose a move $(g, g') \in M^G$. Since G is bipartite, we have either $g' \in Pos_D^G$ or $g' \in Pos_A^G$.

Case (i). If $g' \in Pos_D^G$ then the strategy T suggests playing a finite path on H, $(g', h, g) \to^* (g', h^*, g)$, possibly of zero length, and then it will suggest to play on the external right board. An infinite path played only on H cannot arise, since T is a winning strategy and such an infinite path is not a win for Mediator. Since $T_{\backslash H}$ is

the copycat strategy, T suggests the only move $(g', h^*, g) \rightarrow (g', h^*, g')$. From this position T suggests playing a path on H leading to a final draw position $h_f \in Pos_D^H$ as follows $(g', h^*, g') \rightarrow^* (g', h_f, g')$, such that $\lambda^G(g') = \lambda^H(h_f)$, therefore $g' R h_f$.
Case (ii). If $g' \in Pos_A^G$ then from position (g', h, g) it is Mediator's turn to move. We claim that T will suggest playing a nonempty finite path $(g', h, g) \rightarrow^+ (g', h', g)$ on the central board H, where $h' \in Pos_A^H$, and then suggests the move $(g', h', g) \rightarrow (g', h', g')$. Let $\tilde{h} \in Pos_{A,E,D}^H$ be such that the position $(g', \tilde{h}, g)$ has been reached from (g', h, g), through a (possibly empty) sequence of central moves, by playing with T. Then T cannot suggest a move on the left board $(g', \tilde{h}, g) \rightarrow (g'', \tilde{h}, g)$, since $T_{\setminus H}$ is the copycat strategy. Also, if $\tilde{h} \in Pos_E^H$, T cannot suggest a move on the right board $(g', \tilde{h}, g) \rightarrow (g', \tilde{h}, \tilde{g})$. The reason is that $T = S||S'$, and the position $(\tilde{h}, g)$ of $\langle H, G \rangle$ does not allow a Mediator's move on the right board. Thus a sequence of central moves on H is suggested by T and, as mentioned above, this sequence cannot be infinite. We claim that its endpoint $h' \in Pos_A^H$. We already argued that $h' \notin Pos_E^H$, let us argue that $h' \notin Pos_D^H$. If this were the case, then strategy T suggests the only move $(g', h', g) \rightarrow (g', h_n, g')$, hence $(G, g') \sim (H, h')$. By Lemma 4.2.1, we get $g' \in Pos_D^G$, contradicting $g' \in Pos_A^G$.

This proves that (R, ς) is a weak simulation. We prove next that (R, ς) has the $\star$-property, thus assume that $h^* \in \varsigma(g, g', h_0), \varsigma(\tilde{g}, \tilde{g}', \tilde{h}_0)$. Let us suppose first that $g, \tilde{g} \in Pos_E^H$. By looking at the construction of these paths, we observe that the two sequences of moves

$$(g, h_0, g) \rightarrow (g', h_0, g) \rightarrow^* (g', h^*, g) \rightarrow^* (g', h_n, g) \rightarrow (g', h_n, g') \,,$$
$$(\tilde{g}, \tilde{h}_0, \tilde{g}) \rightarrow (\tilde{g}', \tilde{h}_0, \tilde{g}) \rightarrow^* (\tilde{g}', h^*, \tilde{g}) \rightarrow^* (\tilde{g}', \tilde{h}_m, \tilde{g}) \rightarrow (\tilde{g}', \tilde{h}_m, \tilde{g}') \,,$$

may be played in the game $\langle G, H, G \rangle$, according to the winning strategy $T = S||S'$. We have therefore that $(G, g') \leq (H, h^*) \leq (G, g)$ and $(G, \tilde{g}') \leq (H, h^*) \leq (G, \tilde{g})$.[8] Consequently $|\{ g, g', \tilde{g}, \tilde{g}' \}| = 3$, by Lemma 4.2.2. If $g \in Pos_E^G$ and $\tilde{g} \in Pos_A^G$, a similar argument shows that the positions (g', h^*, g) and $(\tilde{g}, h^*, \tilde{g}')$ may be reached with T and hence $(G, g') \leq (H, h^*) \leq (G, g)$ and $(G, \tilde{g}) \leq (H, h^*) \leq (G, \tilde{g}')$. Lemma 4.2.3 implies then $|\{ g, g', \tilde{g}, \tilde{g}' \}| = 3$. Finally, the cases $(g, \tilde{g}) \in \{ (Pos_A^G, Pos_A^G), (Pos_A^G, Pos_E^G) \}$ are handled by duality. This completes the proof of Proposition 4.3. □

[8] Similar inequalites may be derived even if $h^* \in Pos_D^H$. In this case the moves in the central board may be interleaved with the move on the right board.

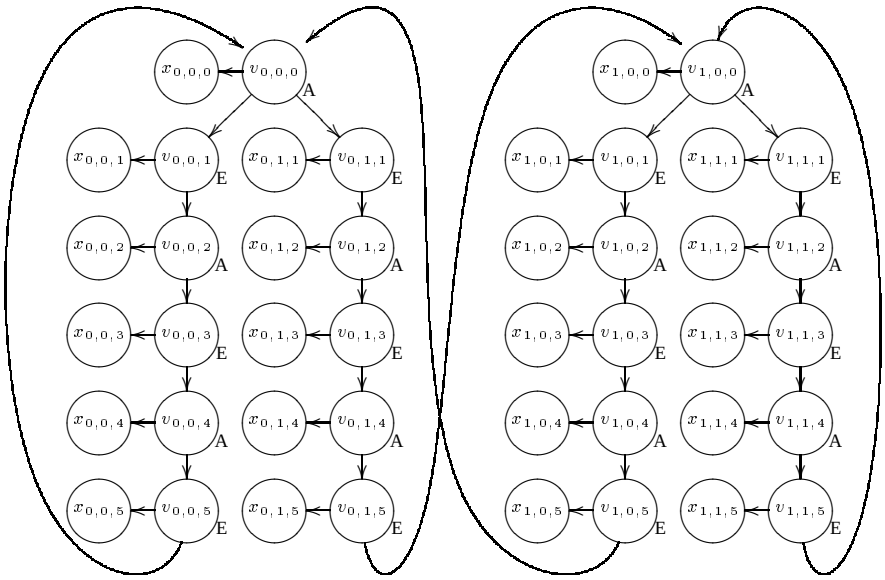

Figure 1: The game G_2

5 Construction of Strongly Synchronizing Games

In this section we complete the hierarchy theorem by constructing, for $n \geq 1$, strongly synchronizing games G_n such that $\mathcal{E}(G_n) = n$. This games mimic the n-cliques already used in [7] to prove that the variable hierarchy for the modal μ-calculus is infinite. The game G_2 appears in Figure 1.

The general definition of the game G_n is as follows. Let $[n]$ denote the set $\{ 0, \ldots, n-1 \}$ and let $I_n = \{ (i,j,k) \in [n] \times [n] \times [6] \mid k = 0 \text{ implies } j = 0 \}$. We define

$$\begin{aligned} Pos_A^{G_n} &= \{ v_{i,j,k} \mid (i,j,k) \in I_n \text{ and } k \bmod 2 = 0 \}, \\ Pos_E^{G_n} &= \{ v_{i,j,k} \mid (i,j,k) \in I_n \text{ and } k \bmod 2 = 1 \}, \\ Pos_D^{G_n} &= \{ w_{i,j,k} \mid (i,j,k) \in I_n \}. \end{aligned}$$

Let $X = \{ x_{i,j,k} \mid i,j \geq 0, k \in [n] \}$ be a countable set of variables, the labelling of draw positions, $\lambda^{G_n} : Pos_D^{G_n} \longrightarrow X$, sends $w_{i,j,k}$ to $x_{i,j,k}$. The moves M^{G_n} either lie on some cycle:

$$\begin{aligned} &v_{i,0,0} \to v_{i,j,1}, && v_{i,j,k} \to v_{i,j,k+1},\ k = 1, \ldots, 4, \\ &v_{i,j,5} \to v_{j,0,0}\ , \end{aligned}$$

or lead to draw positions: $v_{i,j,k} \to w_{i,j,k}$. Finally, the priority function ρ^{G_n} assigns a constant odd priority to all positions. We state next the main facts about the games G_n:

Proposition 5.1. *The games G_n are strongly synchronizing and $\mathcal{E}(G_n) = n$.*

The proof of the statement is omitted for lack of space. We are now ready to state the main achievement of this paper.

Theorem 5.2. *For $n \geq 3$, the inclusions $\mathcal{L}_{n-3} \subseteq \mathcal{L}_n$ are strict. Therefore the variable hierarchy for the games μ-calculus is infinite.*

By the previous Proposition the game $G_n \in \mathcal{L}_n$. Also, since G_n is strongly synchronizing, if $H \sim G_n$, then there exists a $\star$-weak simulation of G_n by H. It follows by Theorem 3.8 that $n - 2 \leq \mathcal{E}(H)$. Therefore $G_n \notin \mathcal{L}_{n-3}$.

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6 Appendix: complete proofs

6.1 On tree with back edges

Lemma 6.1 (i.e. Lemma 1.1). *If π is a simple path of b-length n, then r_n is the vertex closest to the root visited by π.*

Proof. It is enough to observe that, for each i, r_i is the highest vertex visited by π_i. To this goal, if $\pi_i = d_i * b_i$, where d_i is a tree path and b_i is a back-edge, then either r_i belongs to d_i or it is an ancestor of the source of d_i. The first case is excluded by π_i being simple. □

6.2 A variant of the entanglement game

Proposition 6.2 (i.e. Proposition 3.2). *Let $\widetilde{\mathcal{E}}(G,k)$ be the game played as the game $\mathcal{E}(G,k)$ except that Cops is allowed to retire a number of cops placed on the graph. That is, Cops moves are of the form*

- $(g, C, Cops) \to (g, C', Thief)$ *(generalized skip move),*
- $(g, C, Cops) \to (g, C' \cup \{ g \}, Thief)$ *(generalized replace move),*

where in both cases $C' \subseteq C$. Then Cops has a winning strategy in $\mathcal{E}(G,k)$ if and only of he has a winning strategy in $\widetilde{\mathcal{E}}(G,k)$.

Proof. Since every Cops' move in the game $\mathcal{E}(G,k)$ is a Cops' move in the game $\widetilde{\mathcal{E}}(G,k)$, and since there is no new kind of moves for Thief in the game $\widetilde{\mathcal{E}}(G,k)$, then a Cops' winning strategy in $\mathcal{E}(G,k)$ can be used to let Cops win in $\widetilde{\mathcal{E}}(G,k)$.

On the other direction, a winning strategy for Cops in $\widetilde{\mathcal{E}}(G,k)$ can be mapped to a winning strategy for Cops in $\mathcal{E}(G,k)$ as follows.

Each position (g, C, P) of $\mathcal{E}(G,k)$ is matched by a position (g, C^-, P) of $\widetilde{\mathcal{E}}(G,k)$ such that $C^- \subseteq C$. A Thief's move $(g, C, Thief) \to (g', C, Cops)$ in $\mathcal{E}(G,k)$ can certainly be simulated by the move $(g, C^-, Thief) \to (g', C^-, Cops)$ in $\widetilde{\mathcal{E}}(G,k)$, note that Thief has the ability to perform such a move because since if $g' \in C^-$ then already $g' \in C$.

Assume that the position $(g, C_0, Cops)$ of $\mathcal{E}(G,k)$ is matched by the position $(g, C_0^-, Cops)$ of $\widetilde{\mathcal{E}}(G,k)$. From $(g, C_0^-, Cops)$, Cops' winning strategy may suggest two kinds of moves.

It may suggest a generalized skip $(g, C_0^-, Cops) \to (g, C_1^-, Cops)$ with $C_1^- \subseteq C_0^-$. If this is the case, the Cops just skips on from the related position $(g, C_0, Cops)$.

It may suggest a generalized replace move $(g, C_0^-, Cops) \to (g, C_1^- \cup \{ g \}, Thief)$. If $|C_0| < k$, then the such a move becomes an add move $(g, C_0, Cops) \to (g, C_0 \cup \{ g \}, Thief)$. Otherwise $|C_0| = k$ and $|C_1^-| < k$ – since $g \notin C_1^-$ and $|C_1^- \cup \{ g \}| \leq k$ – and consequently we can pick $x \in C_0 \setminus C_1^-$, this is possible since $C_0 \setminus C_1^-$ is not empty, because $C_1^- \subseteq C_0^- \subseteq C_0$ and $|C_1^-| < |C_0|$. Observe also that $x \neq g$, since this would mean that Thief has been trapped. Therefore the move $(g, C_0^-, Cops) \to (g, C_1^- \cup \{ g \}, Thief)$ is simulated by the replace move $(g, C_0, Cops) \to (g, C_0 \setminus$

$\{ x \} \cup \{ g \}, Thief)$. Moreover the invariant $C_1^- \cup \{ g \} \subseteq C_0 \setminus \{ x \} \cup \{ g \}$ is maintained. □

6.3 On the $\star$ property of weak simulations

Lemma 6.3 (i.e. Lemma 3.5). *Let (R, ς) be a $\star$-weak simulation of G by H. If $C(h)$ is not empty, then there exists an element $c(h) \in V_G$ such that for each $(g, g') \in C(h)$ either $c(h) = g$ or $c(h) = g'$. If moreover $|C(h)| \geq 2$, then this element is unique.*

Proof. Clearly the condition holds if $|C(h)| \leq 2$, by definition 3.4. Let us suppose that $|C(h)| \geq 3$.

Fix two undirected edges $\{ c(h), g_1 \}, \{ c(h), g_2 \}$ in the undirected version of $C(h)$. Consider a third undirected edge $\{ \tilde{g}_1, \tilde{g}_2 \} \in C(h)$, so that $|\{ \tilde{g}_1, \tilde{g}_2 \} \cup \{ c(h), g_1 \}| = 3$, and similarly $|\{ \tilde{g}_1, \tilde{g}_2 \} \cup \{ c(h), g_2 \}| = 3$.[9]If $c(h) \notin \{ \tilde{g}_1, \tilde{g}_2 \}$, then $\{ \tilde{g}_1, \tilde{g}_2 \} = \{ g_1, g_2 \}$, thus creating an undirected 3-cycle and contradicting the condition on the girth of G. □

Lemma 6.4 (i.e. Lemma 3.6). *If (R, ς) is a $\star$-weak simulation of G by H and $\rho : K \longrightarrow H$ is a cover, then there exists a $\star$-weak simulation $(\tilde{R}, \tilde{\varsigma})$ of G by K.*

Proof. We construct the $\star$-weak simulation $(\tilde{R}, \tilde{\varsigma})$, where $\tilde{R} \subseteq V_G \times V_K$, as follows

$$g \tilde{R} k \iff g R \rho(k)$$

We consider first $\tilde{R}$ and we prove it to be surjective and functional. Since for each $g \in V_G$ there exists $h \in V_H$ such that gRh and since ρ is surjective, then there exists $k \in V_K$ such that $h = \rho(k)$, and hence $gR\rho(k)$, thus $g\tilde{R}k$. Therefore $\tilde{R}$ is surjective.
If $g_i \tilde{R} k$, $i = 1, 2$, then $g_i R \rho(k)$. Since R is functional, then $g_1 = g_2$. Therefore $\tilde{R}$ is functional.
We exhibit $\tilde{\varsigma}$ as follows. If $g \tilde{R} k_0$ and $g \to g'$, then, we take $\tilde{\varsigma}(g, g', k_0) = k_0, \ldots, k_n$, such that $\varsigma(g, g', \rho(k_0)) = \rho(k_0), \ldots, \rho(k_n)$. Note that the path $k_0, \ldots, k_n$ is unique. Therefore, $(\tilde{R}, \tilde{\varsigma})$ is a weak simulation.

Finally, whenever $(g, g'), (\tilde{g}, \tilde{g}')$ are distinct edges of G and $k_i \in \tilde{\varsigma}(g, g', k_0) \cap \tilde{\varsigma}(\tilde{g}, \tilde{g}', k_0)$, then $\rho(k_i) \in \varsigma(g, g', \rho(k_0)) \cap \varsigma(\tilde{g}, \tilde{g}', \rho(k_0))$. Since (R, ς) has the $\star$-property, we get $|\{ g, g', \tilde{g}, \tilde{g}' \}| = 3$. It follows that $(\tilde{R}, \tilde{\varsigma})$ has the $\star$-property. □

6.4 Properties of strongly synchronizing games

Lemma 6.5. *If G is strongly synchronizing, then the unique winning strategy in the game $\langle G, G \rangle$ is the copycat strategy.*

Proof. Let us consider a position $g \in Pos_E^G$, and let us analyze the position (g, g) of $\langle G, G \rangle$. Let us suppose that $(g, g') \in M^G$ and consider the possible Mediator's answers to the Opponents' move $(g, g) \to (g', g)$.

[9]Observe that the condition on the cardinality implies that we cannot have $(g_1, g_2), (g_2, g_1) \in C(h)$. Thus, the requirement that G has no directed cycles of length 2 is somewhat superfluous.

Mediator cannot answer $(g', g) \to (g'', g)$, since then the relation $(G, g'') \leq (G, g)$ implies that either $g'' = g$ (hence having a cycle of length 2 in G), or that there is an undirected edge between g'' and g, thus creating a length 3 cycle.

Similarly Mediator cannot answer $(g', g) \to (g', \tilde{g})$ with $g' \neq \tilde{g}$. Again, this would create a length 3 cycle in the undirected version of G. □

Lemma 6.6 (i.e. lemma 4.2). *Let G be a strongly synchronizing and $(g, g'), (\tilde{g}, \tilde{g}') \in M^G$.*

1. *If $(G, g) \sim \hat{x}$ then $g \in Pos_D^G$ and $\lambda(g) = x$.*
2. *If $g, \tilde{g} \in Pos_E^G$ and, for some game H and $h \in Pos^H$, we have*

$$(G, g') \leq (H, h) \leq (G, g) \text{ and } (G, \tilde{g}') \leq (H, h) \leq (G, \tilde{g}) ,$$

   then $g = \tilde{g}$ or $g' = \tilde{g}'$, and $|\{ g, g', \tilde{g}, \tilde{g}' \}| = 3$.

3. *If $g \in Pos_E^G$ and $\tilde{g} \in Pos_A^G$ and, for some H and $h \in Pos^H$, we have*

$$(G, g') \leq (H, h) \leq (G, g) \text{ and } (G, \tilde{g}) \leq (H, h) \leq (G, \tilde{g}') ,$$

   then $g = \tilde{g}'$ or $g' = \tilde{g}$, and $|\{ g, g', \tilde{g}, \tilde{g}' \}| = 3$.

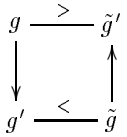
Proof. 1. Let χ_G be the set of free variables of G. First, we have the following claim.

Claim 6.7*: If $(G, g) \sim \hat{x}$, then $x \in \chi_G$.*

Proof. On the one hand, if $x \notin \chi_G$ then $G[x/\top] \sim G[x/\bot]$. One the other hand, $G[x/\top] \sim \hat{x}[x/\top] \sim \top$ and $G[x/\bot] \sim \hat{x}[x/\bot] \sim \bot$, thus $\bot = \top$. This ends the proof of the claim. □

If g has a successor, then the winning strategy in $\langle G, \hat{x}, G \rangle$ will suggest for example to play $(g, p_\star^{\hat{x}}, g) \to (g', p_\star^{\hat{x}}, g) \to (g', p_\star^{\hat{x}}, g')$, for some $(g, g') \in M^G$. Therefore $(G, g) \sim \hat{x} \sim (G, g')$, contradicting the fact that G is strongly synchronizing. Thus g has no successor, and clearly $g \in Pos_D^G$ and $\lambda^G(g) = x$, according to the claim.

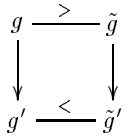
2. We derive first $(G, g') \leq (G, \tilde{g})$ and $(G, \tilde{g}') \leq (G, g)$ and observe that each inequality is strict, because the game is bipartite. Therefore from item 2 of Definition 4.1 we have a diagram of the form



that is we have an undirected edge bewteen g and $\tilde{g}'$, and an undirected edge between g' and $\tilde{g}$.

If $g \neq \tilde{g}$ and $g' \neq \tilde{g}'$, then the above diagram gives rise to an undirected cycle of length 4, which cannot happen.

3. As before, we derive $(G, \tilde{g}) \leq (G, g)$ and $(G, g') \leq (G, \tilde{g}')$ and moreover $(G, \tilde{g}) < (G, g)$ and $(G, g') < (G, \tilde{g}')$, since g and $\tilde{g}$ belong to opposite players. Therefore from item 2 of definition 4.1 we obtain a diagram of the form



If $g \neq \tilde{g}'$ and $g' \neq \tilde{g}$, then the above diagram gives rise to an undirected cycle of length 4, which cannot happen.

□

6.5 The games G_n are strongly synchronizing

It is clear that the game G_n is bipartite and $\mathcal{E}(G_n) = n$, moreover the girth of G_n is 6. To accomplish the proof that G_n is stronlgy synchronizing, we need some intermediary lemmas.

Lemma 6.8. *If $(G_n, w_{i,j,k}) \leq (G_n, g)$ then either $g = w_{i,j,k}$ or $g \in Pos_E^{G_n}$ and $g = v_{i,j,k}$.*

Proof. Case (i). If $g = w_{i',j',k'}$, then surely we need to have $(i, j, k) = (i', j', k')$.

Let therefore $g = v_{i',j',k'}$.

Case (ii). If $g \in Pos_A^{G_n}$ and $(i, j, k) \neq (i', j', k')$, Opponents can choose to move $(w_{i,j,k}, v_{i',j',k'}) \to (w_{i,j,k}, w_{i',j',k'})$, the latter being a lost position for Mediator.

Case (iii). If $g \in Pos_A^{G_n}$ and $(i, j, k) = (i', j', k')$, Opponents can choose to move $(w_{i,j,k}, v_{i,j,k}) \to (w_{i,j,k}, v_{i',j',k'})$ with $(i, j, k) \neq (i', j', k')$. From this position Mediator cannot move $(w_{i,j,k}, v_{i',j',k'}) \to (w_{i,j,k}, w_{i',j',k'})$, nor $(w_{i,j,k}, v_{i',j',k'}) \to (w_{i,j,k}, v_{i'',j''',k''})$, since the girth of G_n being equal to 6 implies that $(i, j, k) \neq (i'', j'', k'')$ and $v_{i'',j'',k''} \in Pos_A^{G_n}$, falling back into case (ii).

Case (iv). If $g \in Pos_E^{G_n}$ and $(i, j, k) \neq (i', j', k')$, then Mediator cannot move $(w_{i,j,k}, v_{i',j',k'}) \to (w_{i,j,k}, w_{i',j',k'})$. He cannot either move $(w_{i,j,k}, v_{i',j',k'}) \to (w_{i,j,k}, v_{i'',j'',k''})$ since $v_{i'',j'',k''} \in Pos_A^{G_n}$, thus falling back either into case (ii), or into case (iii).

Therefore, the only possibility is that $g \in Pos_E^{G_n}$ and $(i, j, k) = (i', j', k')$. □

Dualizing the previous proof we obtain:

Lemma 6.9. *If $(G_n, g) \leq (G_n, w_{i,j,k})$ then either $g = w_{i,j,k}$ or $g \in Pos_A^{G_n}$ and $g = v_{i,j,k}$.*

Lemma 6.10. *If $(G, v_{i,j,k}) \leq (G, v_{i',j',k'})$ and $v_{i,j,k} \neq v_{i',j',k'}$, then either $v_{i,j,k} \in Pos_A^{G_n}$ and $(v_{i,j,k}, v_{i',j',k'}) \in M^{G_n}$, or $v_{i',j',k'} \in Pos_E^{G_n}$ and $(v_{i',j',k'}, v_{i,j,k}) \in M^{G_n}$.*

Proof. Let us suppose that $v_{i,j,k} \in Pos_A^{G_n}$. We remark that $v_{i',j',k'} \notin Pos_D^{G_n}$, and thus we split the proof into two cases.

Case (i). If $v_{i',j',k'} \in Pos_A^{G_n}$, then Opponents can move $(v_{i,j,k}, v_{i',j',k'}) \to (v_{i,j,k}, w_{i',j',k'})$. This is a lost position by Lemma 6.9.

Case (ii). Therefore we have $v_{i',j',k'} \in Pos_E^{G_n}$. Mediator has two kinds of moves. He can choose to move to a "variable", that is, to move $(v_{i,j,k}, v_{i',j',k'}) \to (v_{i,j,k}, w_{i',j',k'})$ or $(v_{i,j,k}, v_{i',j',k'}) \to (w_{i,j,k}, v_{i',j',k'})$. These moves, however, lead to lost positions, by Lemmas 6.8 and 6.9. Therefore, if the position $(v_{i,j,k}, v_{i',j',k'})$ is winning, then he can only move $(v_{i,j,k}, v_{i',j',k'}) \to (v_{i,j,k}, v_{i'',j'',k''})$ or $(v_{i,j,k}, v_{i',j',k'}) \to (v_{i'',j'',k''}, v_{i',j',k'})$. In the first case, if the position $(v_{i,j,k}, v_{i'',j'',k''})$ is winning, then $(i,j,k) = (i'',j'',k'')$ by case (i); hence $(v_{i',j',k'}, v_{i,j,k}) \in M^{G_n}$. In the second case, if Mediator moves to a winning position $(v_{i,j,k}, v_{i',j',k'}) \to (v_{i'',j'',k''}, v_{i',j',k'})$, then $(i',j',k') = (i'',j'',k'')$ by the dual of case (i) and hence $(v_{i,j,k}, v_{i',j',k'}) \in M^{G_n}$. □

Thus we are ready to prove:

Proposition 6.11. *The games G_n are strongly synchronizing.*

Proof. Let us prove first that $(G,g) \sim (G,\tilde{g})$ implies $g = \tilde{g}$. Let us assume that $(G,g) \sim (G,\tilde{g})$, we split the proof that $g = \tilde{g}$ into three cases, according to the color of g.

Case (i). Assume $g \in Pos_D^{G_n}$ and thus let $g = w_{i,j,k}$. If $g \neq \tilde{g}$, then Lemma 6.8 implies that $\tilde{g} = v_{i,j,k}$ with $\tilde{g} \in Pos_E^{G_n}$. Similarly Lemma 6.9 implies that $\tilde{g} = v_{i,j,k}$ with $\tilde{g} \in Pos_A^{G_n}$. Thus we reach a contradiction, and therefore $g = \tilde{g}$.

Case (ii). Let us assume that $g = v_{i,j,k} \in Pos_E^{G_n}$. Then $(G, w_{i,j,k}) < (G,g) \sim (G,\tilde{g})$ and therefore $\tilde{g} = v_{i,j,k}$ by Lemma 6.8.

Case (iii). If $g = v_{i,j,k} \in Pos_A^{G_n}$ then $(G,\tilde{g}) \sim (G,g) < (G, w_{i,j,k})$ and therefore $\tilde{g} = v_{i,j,k}$ by Lemma 6.9.

Let us now prove that $(G,g) \leq (G,\tilde{g})$ and $g \neq \tilde{g}$ implies $\tilde{g} \in Pos_E^{G_n}$ and $(\tilde{g},g) \in M^{G_n}$ or $g \in Pos_E^{G_n}$ and $(g,\tilde{g}) \in M^{G_n}$.

This is the case if $g \in Pos_D^{G_n}$ or $\tilde{g} \in Pos_D^{G_n}$, by Lemmas 6.8 and 6.9. If both $g, \tilde{g} \in Pos_{E,A}^{G_n}$, then the statement follows from Lemma 6.10. □