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# The Variable Hierarchy for the Games $\mu$ -Calculus

Walid Belkhir and Luigi Santocanale

Laboratoire d'Informatique Fondamentale de Marseille  
Université de Provence

**Abstract.** Parity games are combinatorial representations of closed Boolean  $\mu$ -terms. By adding to them draw positions, they have been organized by Arnold and one of the authors [1,2] into a  $\mu$ -calculus [3]. As done by Berwanger et al. [4,5] for the propositional modal  $\mu$ -calculus, it is possible to classify parity games into levels of hierarchy according to the number of fixed-point variables. We ask whether this hierarchy collapses w.r.t. the standard interpretation of the games  $\mu$ -calculus into the class of all complete lattices. We answer this question negatively by providing, for each  $n \geq 1$ , a parity game  $G_n$  with these properties: it unravels to a  $\mu$ -term built up with  $n$  fixed-point variables, it is semantically equivalent to no game with strictly less than  $n - 2$  fixed-point variables.

## 1 Introduction

Recent work by Berwanger et al. [4,5,6,7] proves that the expressive power of the modal  $\mu$ -calculus [8] increases with the number of fixed point variables. By introducing the variable hierarchy and showing that it does not collapse, they manage to separate the  $\mu$ -calculus from dynamic game logic [9]. Their work solves a longstanding open problem but may also be appreciated for its new insights and the new research paths it discloses within the theory of fixed-points in computer science [3,10]. We already engaged in following one of these paths [11], we deal here with a logical problem. The variable hierarchy may be defined for every  $\mu$ -calculus and even for iteration theories, since just one fixed-point operator is sufficient to define it. In our opinion, asking whether the variable hierarchy for a  $\mu$ -calculus is strict is as fundamental as considering the alternation-depth hierarchy. In this paper we ask this question for the games  $\mu$ -calculus.

Parity games are combinatorial representations of closed Boolean  $\mu$ -terms. By adding to them draw positions (or free variables), A. Arnold and L. Santocanale [1,2] have structured parity games into *the games  $\mu$ -calculus*. In other words, the authors defined substitution, least and greatest fixed-point operators, as usual for  $\mu$ -calculi [3]. The interpretation of the games  $\mu$ -calculus is over the class of all complete lattices.<sup>1</sup> Together with its standard preorder, such a calculus may also be understood as providing a concrete description of free  $\mu$ -lattices [12].

Let us recall the background of the games  $\mu$ -calculus. The interaction between two players in a game is a standard model of the possible interactions between a system

<sup>1</sup> The interpretation in the class of distributive lattices makes the calculus trivial, since every  $\mu$ -term is equivalent to a term with no application of fixed-point operators.

and its potentially adverse environment. Researchers from different communities are still working on this model despite its introduction dates back at least fifteen years [13,14,15]. It was proposed in [16] to develop a theory of communication grounded on similar game theoretic ideas and, moreover, on algebraic concepts such as “free lattice” [17] and “free bicomplete category” [18]. A first work pursued this idea using tools of categorical logic [19]. The proposal was further developed in [12] where cycles were added to lattice terms to enrich the model with possibly infinite behaviors. As a result, lattice terms were replaced by parity games, and one of the most interesting tool from the logics of programs was introduced into the semantics of programming languages. Given two parity games  $G, H$  the witness that the relation  $G \leq H$  holds in every complete lattice interpretation is a winning strategy for a prescribed player, Mediator, in a game  $\langle G, H \rangle$ . A game  $G$  may also be considered as modelling a synchronous communication channel available to two users. Then, a winning strategy for Mediator in  $\langle G, H \rangle$  witnesses the existence of an asynchronous protocol allowing one user of  $G$  to communicate with the other user on  $H$  ensuring absence of deadlocks.

At present, a major interest of this  $\mu$ -calculus is opinion its contribution to the theory of fixed-point logics. The idea that winning strategies for Mediator in the game  $\langle G, H \rangle$  are sort of proofs was formalized in [20]. More interestingly, proof theoretic ideas and tools – the cut elimination procedure and  $\eta$ -expansion, in their game theoretic disguise – have proved quite powerful to solve deep problems arising from fixed-point theory. These are the alternation-depth hierarchy problem [21] and the status of the ambiguous classes [1]. In [2] the authors were able to partially export these ideas to the modal  $\mu$ -calculus. We show here that similar tools success in establishing the strictness of the variable hierarchy.

While dealing with the variable hierarchy problem for the games  $\mu$ -calculus, we shall refer to two digraph complexity measures, the *entanglement* and the *feedback*. The feedback of a vertex  $v$  of a tree with back edges is the number of ancestors of  $v$  that are the target of a back edge whose source is a descendant of  $v$ . The feedback of a tree with back edges is the maximum feedback of its vertices. The entanglement of a digraph  $G$ , denoted  $\mathcal{E}(G)$ , may be defined as follows: *it is the minimum feedback of its finite unravellings into a tree with back edges*. These measures are tied to the logic as follows. A  $\mu$ -term may be represented as a tree with back-edges, the feedback of which corresponds to the minimum number of fixed point variables needed in the  $\mu$ -term, up to  $\alpha$ -conversion. Also, one may consider terms of a vectorial  $\mu$ -calculus, i.e. systems of equations, and these roughly speaking are graphs. The step that constructs a canonical solution of a system of equations by means of  $\mu$ -terms amounts to the construction of a finite unravelling of the graph. In view of these considerations, asking whether a parity game  $G$  is semantically equivalent to a  $\mu$ -term with at most  $n$ -variables amounts to asking whether  $G$  belongs to the level  $\mathcal{L}_n$  defined as follows:

$$\mathcal{L}_n = \{ G \in \mathcal{G} \mid G \sim H \text{ with } H \in \mathcal{G} \text{ and } \mathcal{E}(H) \leq n \}. \quad (1)$$

Here  $\mathcal{G}$  is the collection of parity games with draw positions and  $\sim$  denotes the semantic equivalence. In this paper we ask whether the variable hierarchy, made up of the levels  $\mathcal{L}_n$ , collapses: is there a constant  $k \geq 0$ , such that for all  $n \geq k$ , we have  $\mathcal{L}_k = \mathcal{L}_n$ ? We answer this question negatively, there is no such constant. We shall construct, for each

$n \geq 1$ , a parity game  $G_n$  with two properties: (i)  $G_n$  unravels to a tree with back edges of feedback  $n$ , showing that  $G_n$  belongs to  $\mathcal{L}_n$ , (ii)  $G_n$  is semantically equivalent to no game in  $\mathcal{L}_{n-3}$ . Thus, we prove that the inclusions  $\mathcal{L}_{n-3} \subseteq \mathcal{L}_n$ ,  $n \geq 3$ , are strict.

We combine in this work ideas of [4,5] with the ideas of [21] on the alternation depth of the games  $\mu$ -calculus. On the one hand, our games  $G_n$  mimic the  $n$ -cliques of [4,5] that are shapes for hard  $\mu$ -formulae built up with  $n$  fixed point variables. On the other hand, we adapt the notion of synchronizing game [21] to the context of the variable hierarchy. A strongly synchronizing game  $G$  imposes strong conditions on the structure of a game  $H$  that is semantically equivalent to  $G$ : every move (edge) in  $G$  can be simulated by a non empty finite sequence of moves (a path) of  $H$ ; if two paths simulating distinct edges do intersect, then the edges do intersect as well. We formalize such situation within the notion of  $\star$ -weak simulation. The main result is that if there is a  $\star$ -weak simulation of  $G$  by  $H$ , then  $\mathcal{E}(G) - 2 \leq \mathcal{E}(H)$ . The latter statement holds in the general context of digraphs, not just for the games  $\mu$ -calculus, and might be of general use.

Combining the result on  $\star$ -weak simulations with the existence of strongly synchronizing games  $G_n \in \mathcal{L}_n$ , we have been able to prove that the inclusions  $\mathcal{L}_{n-3} \subseteq \mathcal{L}_n$  are strict. We do not yet know whether  $\mathcal{L}_{n-1} \subsetneq \mathcal{L}_n$ . To prove the latter strict inclusions, we might try to reduce the constant 2 in the  $\star$ -weak simulation Theorem 14. We remark on the way that we are exhibited with another difference with the alternation hierarchy for which the infiniteness of the hierarchy implies that the inclusions between consecutive classes are strict. We pinpoint next some more problems arising from the present work. The reader will notice that the number of free variables in the games  $G_n$  increases with  $n$ . He might therefore ask whether hard games can be constructed using a fixed number of free variables. The question might be answered by exploiting the fact that a countable number of generators (i.e. free variables) can be simulated within the free lattice on three generators [17, §1.6]. Such kind of simulation calls for a more general problem, that of interpreting some  $\mu$ -calculus into another one. We do not know about existing results in this direction. Yet we are appealed by such a general question since translations might be of help in relating analogous and parallel results proved so far for different  $\mu$ -calculi.

The paper is organized as follows. Section 2 introduces the necessary background on the algebra of parity games and their organization into a  $\mu$ -calculus. In section 3, we firstly recall the definition of entanglement; then we define the  $\star$ -weak simulation between graphs that allows to compare their entanglements. In section 4, we define strongly synchronizing games and we shall prove their *hardness* w.r.t the variable hierarchy, in particular every equivalent game to a strongly synchronizing one is related with it by a  $\star$ -weak simulation. In section 5, we construct strongly synchronizing games of arbitrary entanglement. We sum up the discussion in our main result, Theorem 19.

**Notation, preliminary definitions and elementary facts.** If  $G$  is a graph, then a path in  $G$  is a sequence of the form  $\pi = g_0 g_1 \dots g_n$  such that  $(g_i, g_{i+1}) \in E_G$  for  $0 \leq i < n$ . A path is *simple* if  $g_i \neq g_j$  for  $i, j \in \{0, \dots, n\}$  and  $i \neq j$ . The integer  $n$  is the length of  $\pi$ ,  $g_0$  is the source of  $\pi$ , noted  $\delta_0 \pi = g_0$ , and  $g_n$  is the target of  $\pi$ , noted  $\delta_1 \pi = g_n$ . We denote by  $\Pi^+(G)$  the set of simple non empty (i.e. of length greater than 0) paths in  $G$ . A pointed digraph  $\langle V, E, v_0 \rangle$  of root  $v_0$ , is a *tree* if for each  $v \in V$  there exists a

unique path from  $v_0$  to  $v$ . A *tree with back-edges* is a tuple  $\mathcal{T} = \langle V, T, v_0, B \rangle$  such that  $\langle V, T, v_0 \rangle$  is a tree, and  $B \subseteq V \times V$  is a second set of edges such that if  $(x, y) \in B$  then  $y$  is an ancestor of  $x$  in the tree  $\langle V, T, v_0 \rangle$ . We shall refer to edges in  $T$  as tree edges and to edges in  $B$  as back edges. We say that  $r \in V$  is a return of  $\mathcal{T}$  if there exists  $x \in V$  such that  $(x, r) \in B$ . The *feedback of a vertex*  $v$  is the number of returns  $r$  on the path from  $v_0$  to  $v$  such that, for some descendant  $x$  of  $v$ ,  $(x, r) \in B$ . The *feedback of a tree with back edges* is the minimum feedback of its vertices. We shall say that a pointed directed graph  $(V, E, v_0)$  is a tree with back edges if there is a partition of  $E$  into two disjoint subsets  $T, B$  such that  $\langle V, T, v_0, B \rangle$  is a tree with back edges.

If  $\mathcal{T}$  is a tree with back edges, then a path in  $\mathcal{T}$  can be factored as  $\pi = \pi_1 * \dots * \pi_n * \tau$ , where each factor  $\pi_i$  is a sequence of tree edges followed by a back edge, and  $\tau$  does not contain back edges. Such factorization is uniquely determined by the occurrences of back edges in  $\pi$ . For  $i > 0$ , let  $r_i$  be the return at the end of the factor  $\pi_i$ . Let also  $r_0$  be the source of  $\pi$ . Let the *b-length* of  $\pi$  be the number of back edges in  $\pi$ . i.e.  $r_i = \delta_1 \pi_i$ . For the *b-length* of

**Lemma 1.** *If  $\pi$  is a simple path of b-length  $n$ , then  $r_n$  is the vertex closest to the root visited by  $\pi$ . Hence, if a simple path  $\pi$  lies in the subtree of its source, then it is a tree path.*

We shall deal with trees with back-edges to which a given graph unravels.

**Definition 2.** *A cover or unravelling of a (finite) directed graph  $H$  is a (finite) graph  $K$  together with a surjective graph morphism  $\rho : K \rightarrow H$  such that for each  $v \in V_K$ , the correspondence sending  $k$  to  $\rho(k)$  restricts to a bijection from  $\{k \in V_K \mid (v, k) \in E_K\}$  to  $\{h \in V_H \mid (\rho(v), h) \in E_H\}$ .*

The notion of cover of pointed digraphs is obtained from the previous by replacing the surjectivity constraint by the condition that  $\rho$  preserves the root of the pointed digraphs.

## 2 The Games $\mu$ -Calculus

In this section we recall the definition of parity games with draws and how they can be structured as a  $\mu$ -calculus. We shall skip the most of the details and focus only on the syntactical preorder relation  $\leq$  between  $\mu$ -terms that characterizes the semantical order relation.

A *parity game with draws* is a tuple  $G = \langle Pos_E^G, Pos_A^G, Pos_D^G, M^G, \rho^G \rangle$  where:

- $Pos_E^G, Pos_A^G, Pos_D^G$  are finite pairwise disjoint sets of positions (Eva's positions, Adam's positions, and draw positions),
- $M^G$ , the set of moves, is a subset of  $(Pos_E^G \cup Pos_A^G) \times (Pos_E^G \cup Pos_A^G \cup Pos_D^G)$ ,
- $\rho^G$  is a mapping from  $(Pos_E^G \cup Pos_A^G)$  to  $\mathbb{N}$ .

Whenever an initial position is specified, these data define a game between player Eva and player Adam. The outcome of a finite play is determined according to the normal play condition: a player who cannot move loses. It can also be a draw, if a position in  $Pos_D^G$  is reached.<sup>2</sup> The outcome of an infinite play  $\{(g_k, g_{k+1}) \in M^G\}_{k \geq 0}$  is

<sup>2</sup> Observe that there are no possible moves from a position in  $Pos_D^G$ .

determined by means of the rank function  $\rho^G$  as follows: it is a win for Eva iff the maximum of the set  $\{i \in \mathbb{N} \mid \exists \text{ infinitely many } k \text{ s.t. } \rho^G(g_k) = i\}$  is even. To simplify the notation, we shall use  $Pos_{E,A}^G$  for the set  $Pos_E^G \cup Pos_A^G$  and use similar notations such as  $Pos_{E,D}^G$ , etc. We let  $Max^G = \max \rho^G(Pos_{E,A}^G)$  if the set  $Pos_{E,A}^G$  is not empty, and  $Max^G = -1$  otherwise.

To obtain a  $\mu$ -calculus, as defined [3, §2], we label draw positions with variables of a countable set  $X$ . If  $\lambda^G : Pos_D^G \rightarrow X$  is such a labelling and  $p_*^G \in Pos_{E,A,D}^G$  is a specified initial position, then we refer to the tuple  $\langle G, p_*^G, \lambda^G \rangle$  as a labeled parity game. We denote by  $G_g$  the game that differs from  $G$  only on the starting position, i.e.  $p_*^{G_g} = g$ , we shall write  $(G, g)$  to mean that the play has reached position  $g$ . We let  $\hat{x}$  be the game with just one final draw position of zero priority and labeled with variable  $x$ . With  $\mathcal{G}$  we shall denote the collection of all labeled parity games; as no confusion will arise, we will call a labeled parity game with simply “game”.

As a  $\mu$ -calculus, formal composition and fixed-point operations may be defined on  $\mathcal{G}$ ; moreover,  $\mathcal{G}$  has meet and join operations. The reader is invited to see [1, §1, §2] for the definitions of these operations. We believe that the algebraic nature of parity games is better understood by defining their semantics. To this goal, let us define the predecessor game  $G^-$ , for  $G$  a game such that  $Max^G \neq -1$ , i.e. there is at least one position in  $Pos_{E,A}^G$ . Let  $Top^G = \{g \in Pos_{E,A}^G \mid \rho^G(g) = Max^G\}$ , then  $G^-$  is defined as follows:

- $Pos_E^{G^-} = Pos_E^G - Top^G$ ,  $Pos_A^{G^-} = Pos_A^G - Top^G$ ,  $Pos_D^{G^-} = Pos_D^G \cup Top^G$ ,
- $M^{G^-} = M^G - (Top^G \times Pos_{E,A,D}^G)$ ,
- $\rho^{G^-}$  is the restriction of  $\rho^G$  to  $Pos_{E,A}^{G^-}$ .

Given a complete lattice  $L$ , the interpretation of a parity game  $G$  in  $L$  is a monotone mapping of the form  $\|G\| : L^{P_D^G} \rightarrow L^{P_{E,A}^G}$ , where  $L^X$  is the  $X$ -fold product lattice of  $L$  with itself. The interpretation of a parity game is defined inductively. If  $P_{E,A}^G = \emptyset$ , then  $L^{P_{E,A}^G} = L^\emptyset = 1$ , the complete lattices with just one element, and there is just one possible definition of the mapping  $\|G\|$ . Otherwise, if  $Max^G$  is odd, then  $\|G\|$  is the parameterized least fixed-point of the monotone mapping  $L^{P_{E,A}^G} \times L^{P_D^G} \rightarrow L^{P_{E,A}^G}$  defined by the system of equations:

$$x_g = \begin{cases} \bigvee \{ x_{g'} \mid (g, g') \in M^G \} & \text{if } g \in Pos_E^G \cap Top^G, \\ \bigwedge \{ x_{g'} \mid (g, g') \in M^G \} & \text{if } g \in Pos_A^G \cap Top^G, \\ \|G_g^-\|(X_{Top^G}, X_{Pos_D^G}) & \text{otherwise.} \end{cases}$$

If  $Max^G$  is even, then  $\|G\|$  is the parameterized greatest fixed-point of this mapping.

**The preorder on  $\mathcal{G}$ .** In order to describe a preorder on the class  $\mathcal{G}$ , we shall define a new game  $\langle G, H \rangle$  for a pair of games  $G$  and  $H$  in  $\mathcal{G}$ . This is not a pointed parity game with draws as defined in the previous section; to emphasize this fact, the two players will be named Mediator and Opponent instead of Eva and Adam.

**Definition 3.** *The game  $\langle G, H \rangle$  is defined as follows:*

- The set of Mediator's positions is  $Pos_A^G \times Pos_{E,D}^H \cup Pos_{A,D}^G \times Pos_E^H \cup \mathcal{L}(M)$ , and the set of Opponent's positions is  $Pos_E^G \times Pos_{E,A,D}^H \cup Pos_{E,A,D}^G \times Pos_A^H \cup \mathcal{L}(O)$ , where  $\mathcal{L}(M), \mathcal{L}(O) \subseteq Pos_D^G \times Pos_D^H$  are the losing positions for Mediator and Opponents respectively. They are defined as follows. If  $(g, h) \in Pos_D^G \times Pos_D^H$ , then: if  $\lambda^G(g) = \lambda^H(h)$ , then the position  $(g, h)$  belongs to Opponents, and there is no move from this position, hence this is a winning position for Mediator. If  $\lambda^G(g) \neq \lambda^H(h)$ , then the position  $(g, h)$  belongs to Mediator and there is no move from this position. The latter is a win for Opponents.
- Moves of  $\langle G, H \rangle$  are either left moves  $(g, h) \rightarrow (g', h)$ , where  $(g, g') \in M^G$ , or right moves  $(g, h) \rightarrow (g, h')$ , where  $(h, h') \in M^H$ ; however the Opponents can play only with Eva on  $G$  or with Adam on  $H$ .
- A finite play is a loss for the player who can not move. An infinite play  $\gamma$  is a win for Mediator if and only if either its left projection  $\pi_G(\gamma)$  is a win for Adam, or its right projection  $\pi_H(\gamma)$  is a win for Eva.

**Definition 4.** If  $G$  and  $H$  belong to  $\mathcal{G}$ , then we declare that  $G \leq H$  if and only if Mediator has a winning strategy in the game  $\langle G, H \rangle$  starting from position  $(p_\star^G, p_\star^H)$ .

The following is the reason to consider such a syntactic relation:

**Proposition 5.** The relation  $\leq$  is sound and complete with respect to the interpretation in any complete lattice, i.e.  $G \leq H$  if and only if  $\|G\| \leq \|H\|$  holds in every complete lattice.

In the sequel, we shall write  $G \sim H$  to mean that  $G \leq H$  and  $H \leq G$ . For other properties of the relation  $\leq$ , see for example Proposition 2.5 of [1]. One can prove that  $G \leq G$ , by exhibiting the *copycat* strategy in the game  $\langle G, G \rangle$ : from a position  $(g, g)$ , it is Opponent's turn to move either on the left or on the right board. When they stop moving, Mediator will have the ability to copy all the moves played by the Opponents so far from the other board until the play reaches the position  $(g', g')$ . There it was also proved that if  $G \leq H$  and  $H \leq K$  then  $G \leq K$ , by describing a game  $\langle G, H, K \rangle$  with the following properties: (1) given two winning strategies  $R$  on  $\langle G, H \rangle$ , and  $S$  on  $\langle H, K \rangle$  there is a winning strategy  $R \parallel S$  on  $\langle G, H, K \rangle$ , that is the composition of the strategies  $R$  and  $S$ , (2) given a winning strategy  $T$  on  $\langle G, H, K \rangle$ , there exists a winning strategy  $T \setminus_H$  on  $\langle G, K \rangle$ .

The game  $\langle G, H, K \rangle$  is the fundamental tool that will allow us to deduce the desired structural properties of games  $H$  which are equivalent to a specified game  $G$ , by considering the game  $\langle G, H, G \rangle$ , section 4. The game  $\langle G, H, K \rangle$  is obtained by gluing the games  $\langle G, H \rangle$  and  $\langle H, K \rangle$  on the central board  $H$  as follows.

**Definition 6.** Positions of the game  $\langle G, H, K \rangle$  are triples  $(g, h, k) \in Pos_{A,E,D}^G \times Pos_{A,E,D}^H \times Pos_{A,E,D}^K$  such that

- the set of Mediator's positions is

$$Pos_A^G \times Pos_{A,E,D}^H \times Pos_{E,D}^K \cup Pos_{A,D}^G \times Pos_{A,E,D}^H \times Pos_E^K \cup \mathcal{L}(M)$$

and the set of Opponent's positions is

$$Pos_E^G \times Pos_{A,E,D}^H \times Pos_{E,A,D}^K \cup Pos_{E,A,D}^G \times Pos_{A,E,D}^H \times Pos_A^K \cup \mathcal{L}(O),$$

where  $\mathcal{L}(M), \mathcal{L}(O) \subseteq Pos_D^G \times Pos_{A,E,D}^H \times Pos_D^K$  are positions of Meditor and Opponents, respectively, defined as follows. Whenever  $(g, h, k) \in Pos_D^G \times Pos_{A,E,D}^H \times Pos_D^K$ , then if  $h \in Pos_{E,A}^H$ , then the position  $(g, h, k)$  belongs to Meditor; otherwise, i.e.  $h \in Pos_D^H$ , then the final position  $(g, h, k)$  belongs to Opponents if and only if  $\lambda^G(g) = \lambda^H(h) = \lambda^K(k)$ .

- Moves of  $\langle G, H, K \rangle$  are either left moves  $(g, h, k) \rightarrow (g', h, k)$  where  $(g, g') \in M^G$  or central moves  $(g, h, k) \rightarrow (g, h', k)$ , where  $(h, h') \in M^H$ , or right moves  $(g, h, k) \rightarrow (g, h, k')$ , where  $(k, k') \in M^K$ ; however the Opponents can play only with Eva on  $G$  or with Adam on  $K$ .
- As usual, a finite play is a loss for the player who cannot move. An infinite play  $\gamma$  is a win for Meditors if and only if  $\pi_G(\gamma)$  is a win for Adam on  $G$ , or  $\pi_K(\gamma)$  is a win for Eva on  $K$ .

### 3 Entanglement and $\star$ -Weak Simulations

Let us recall the main tool which measures the combinatorial essence of the variable hierarchy level on directed graphs. This is the *entanglement* of a digraph  $G$  and might already be defined as *the minimum feedback of the finite unravelings of  $G$  into a tree with back edges*. The entanglement of  $G$  may also be characterized by means of a special Robber and Cops game  $\mathcal{E}(G, k)$ ,  $k = 0, \dots, |V_G|$ . This game, defined in [7], is played by Thief against Cops, a team<sup>3</sup> of  $k$  cops, as follows.

**Definition 7.** *The entanglement game  $\mathcal{E}(G, k)$  of a digraph  $G$  is defined by:*

- Its positions are of the form  $(v, C, P)$ , where  $v \in V_G$ ,  $C \subseteq V_G$  and  $|C| \leq k$ ,  $P \in \{\text{Cops}, \text{Thief}\}$ .
- Initially Thief chooses  $v_0 \in V$  and moves to  $(v_0, \emptyset, \text{Cops})$ .
- Cops can move from  $(v, C, \text{Cops})$  to  $(v, C', \text{Thief})$  where  $C'$  can be
  1.  $C$ : Cops skip,
  2.  $C \cup \{v\}$ : Cops add a new Cop on the current position,
  3.  $(C \setminus \{x\}) \cup \{v\}$ : Cops move a placed Cop to the current position.
- Thief can move from  $(v, C, \text{Thief})$  to  $(v', C, \text{Cops})$  if  $(v, v') \in E_G$  and  $v' \notin C$ .

Every finite play is a win for Cops, and every infinite play is a win for Thief.

The following will constitute our working definition of entanglement:  $\mathcal{E}(G)$ , *the entanglement of  $G$ , is the minimum  $k \in \{0, \dots, |V_G|\}$  such that Cops have a winning strategy in  $\mathcal{E}(G, k)$* . The following proposition provides a useful variant of entanglement games.

**Proposition 8.** *Let  $\tilde{\mathcal{E}}(G, k)$  be the game played as the game  $\mathcal{E}(G, k)$  apart that Cops is allowed to retire a number of cops placed on the graph. That is, Cops moves are of the form*

<sup>3</sup> We shall use the singular to emphasize that Cops constitute a team.



- $(g, C, Cops) \rightarrow (g, C', Thief)$  (*generalized skip move*),
- $(g, C, Cops) \rightarrow (g, C' \cup \{g\}, Thief)$  (*generalized replace move*),

where in both cases  $C' \subseteq C$ . Then *Cops* has a winning strategy in  $\mathcal{E}(G, k)$  if and only if he has a winning strategy in  $\tilde{\mathcal{E}}(G, k)$ .

**$\star$ -Weak Simulations.** We define next a relation between graphs, called  $\star$ -weak simulation, that we shall use to compare their entanglements. Intuitively, there is a  $\star$ -weak simulation of a graph  $G$  by  $H$  if every edge of  $G$  is simulated by a non empty finite path of  $H$ . Moreover, two edges  $e_1, e_2$  of  $G$  not sharing a common endpoint, are simulated by paths  $\pi_1, \pi_2$  that do not intersect. These simulations arise when considering games  $H$  which are semantically equivalent to some strongly synchronizing game  $G$ , as defined in Section 4.

**Definition 9.** A weak simulation  $(R, \varsigma)$  of  $G$  by  $H$  is a binary relation  $R \subseteq V_G \times V_H$  that comes with a partial function  $\varsigma : V_G \times V_G \times V_H \longrightarrow \Pi^+(H)$ , such that:

- $R$  is surjective, i.e. for every  $g \in V_G$  there exists  $h \in V_H$  such that  $gRh$ ,
- $R$  is functional, i.e. if  $g_iRh$  for  $i = 1, 2$ , then  $g_1 = g_2$ ,
- if  $gRh$  and  $g \rightarrow g'$ , then  $\varsigma(g, g', h)$  is defined and  $h' = \delta_1\varsigma(g, g', h)$  is such that  $g'Rh'$ .

Now we want to study conditions under which existence of a weak simulation of  $G$  by  $H$  implies that  $\mathcal{E}(G)$  is some lower bound of  $\mathcal{E}(H)$ . To this goal, we abuse of notation and write  $h \in \varsigma(g, g', h_0)$  if  $\varsigma(g, g', h_0) = h_0h_1 \dots h_n$  and, for some  $i \in \{0, \dots, n\}$ , we have  $h = h_i$ . If  $G = (V_G, E_G)$  is a directed graph then its undirected version  $S(G) = (V_G, E_{S(G)})$  is the undirected graph such that  $\{g, g'\} \in E_{S(G)}$  iff  $(g, g') \in E_G$  or  $(g', g) \in E_G$ . Thus we say that  $G$  has *girth at least  $k$*  if the shortest cycle in  $S(G)$  has length at least  $k$ ,  $G$  does not contain loops, and  $(g, g') \in E_G$  implies  $(g', g) \notin E_G$ .

**Definition 10.** We say that a weak simulation  $(R, \varsigma)$  of  $G$  by  $H$  is a  $\star$ -weak simulation (or that it has the  $\star$ -property) if  $G$  has girth at least 4, and if  $(g, g'), (\tilde{g}, \tilde{g}')$  are distinct edges of  $G$  and  $h \in \varsigma(g, g', h_0), \varsigma(\tilde{g}, \tilde{g}', \tilde{h}_0)$ , then  $|\{g, g', \tilde{g}, \tilde{g}'\}| = 3$ .

We explain next this property. Given  $(R, \varsigma)$ , consider the set

$$C(h) = \{(g, g') \in E_G \mid \exists h_0 \text{ s.t. } h \in \varsigma(g, g', h_0)\}.$$

**Lemma 11.** Let  $(R, \varsigma)$  be a  $\star$ -weak simulation of  $G$  by  $H$ . If  $C(h)$  is not empty, then there exists an element  $c(h) \in V_G$  such that for each  $(g, g') \in C(h)$  either  $c(h) = g$  or  $c(h) = g'$ . If moreover  $|C(h)| \geq 2$ , then this element is unique.

That is,  $C(h)$  considered as an undirected graph, is a star. Since  $c(h)$  is unique whenever  $|C(h)| \geq 2$ , then  $c(h)$  is a partial function which is defined for all  $h$  with  $|C(h)| \geq 2$ . This allows to define a partial function  $f : V_H \longrightarrow V_G$ , which is defined for all the  $h$  for which  $C(h) \neq \emptyset$ , as follows:

$$f(h) = \begin{cases} c(h), & |C(h)| \geq 2, \\ g, & \text{if } C(h) = \{(g, g')\} \text{ and } h \text{ has no predecessor,} \\ g', & \text{otherwise, provided that } C(h) = \{(g, g')\}. \end{cases} \quad (2)$$

Let us remark that if  $h \in \varsigma(g, g', h_0)$ , then  $f(h) \in \{g, g'\}$ . If  $gRh$  and  $h$  has no predecessor, then  $f(h) = g$ . Also, if  $h'$  is the target of  $\varsigma(g, g', h_0)$  and  $g'$  has a successor, then  $f(h') = g'$ .

**Lemma 12.** *If  $(R, \varsigma)$  is a  $\star$ -weak simulation of  $G$  by  $H$  and  $\rho : K \rightarrow H$  is an unravelling of  $H$ , then there exists a  $\star$ -weak simulation  $(\tilde{R}, \tilde{\varsigma})$  of  $G$  by  $K$ .*

Let us now recall that if  $H$  is a tree with back edges, rooted at  $h_0$ , of feedback  $k$ , then Cops has a *canonical winning strategy* in the game  $\mathcal{E}(H, k)$  from position  $(h_0, C, Cops)$ . Every time a return is visited, a cop is dropped on such a return. If a cop has to be replaced in order to occupy such a return, then the cop which is closest to the root is chosen.

*Remark 13.* Let us remark that, by using the canonical strategy, (i) every path chosen by Thief in  $H$  is a tree path, (ii) if the position in  $\mathcal{E}(H, k)$  is of the form  $(h, C, Thief)$ , and  $h' \neq h$  is in the subtree of  $h$ , then the unique tree path from  $h$  to  $h'$  does contain no cops, apart possibly for the vertex  $h$ . Finally, a vertex  $h \in V_H$  determines a position  $(h, C_H(h), Thief)$  in the game  $\mathcal{E}(H, k)$  that has been reached from the initial position  $(h_0, \emptyset, Cops)$  and where Cops have been playing according the canonical strategy.  $C_H(h)$  is determined as the set of returns  $r$  of  $H$  on the tree path from  $h_0$  to  $h$  such that the tree path from  $r$  to  $h$  contains at most  $k$  returns.

The following Theorem establishes the desired connection between  $\star$ -weak simulations and entanglement.

**Theorem 14.** *If  $(R, \varsigma)$  is a  $\star$ -weak simulation of  $G$  by  $H$ , then  $\mathcal{E}(G) \leq \mathcal{E}(H) + 2$ .*

*Proof.* Let  $k = \mathcal{E}(H)$ . We shall define first a strategy for Cops in the game  $\tilde{\mathcal{E}}(G, k+2)$ . In a second time, we shall prove that this strategy is a *winning* strategy for Cops.

Let us consider Thief's first move in  $\tilde{\mathcal{E}}(G, k+2)$ . This move picks  $g \in G$  leading to the position  $(g, \emptyset, Cops)$  of  $\tilde{\mathcal{E}}(G, k+2)$ . Cops answers by occupying the current position, i.e. he moves to  $(g, \{g\}, Thief)$ . After this move, Cops also chooses a tree with back edges of feedback  $k$  to which  $H$  unravel,  $\pi : \mathcal{T}(H) \rightarrow H$ , such that the root  $h_0$  of  $\mathcal{T}(H)$  satisfies  $gR\pi(h_0)$ . WE can also suppose that  $h_0$  is not a return, thus it has no predecessor. According to Lemma 12 we can lift the  $\star$ -weak simulation  $(R, \varsigma)$  to a  $\star$ -weak simulation  $(\tilde{R}, \tilde{\varsigma})$  of  $G$  by  $\mathcal{T}(H)$ . In other words, we can suppose from now on that  $H$  itself is a tree with back edges of feedback  $k$  rooted at  $h_0$  and, moreover, that  $gRh_0$ .

From this point on, Cops uses a memory to choose how to place cops in the game  $\tilde{\mathcal{E}}(G, k+2)$ . To each Thief's position  $(g, C_G, Thief)$  in  $\tilde{\mathcal{E}}(G, k+2)$  we associate a data structure (the memory) consisting of a triple  $M(g, C, Thief) = (p, c, h)$ , where  $c, h \in V_H$  and  $p \in V_H \cup \{\perp\}$  (we assume that  $\perp \notin V_H$ ). Moreover  $c$  is an ancestor of  $h$  in the tree and, whenever  $p \neq \perp$ ,  $p$  is an ancestor of  $c$  as well.

Intuitively, we are matching the play in  $\tilde{\mathcal{E}}(G, k+2)$  with a play in  $\mathcal{E}(H, k)$ , started at the root  $h_0$  and played by Cops according to the canonical strategy. Thus  $c$  is the vertex of  $H$  currently occupied by Thief in the game  $\mathcal{E}(H, k)$ .<sup>4</sup> Instead of recalling all the play

<sup>4</sup> More precisely we are associating to the position  $(g, C_G, Thief)$  of  $\mathcal{E}(G, k+2)$  the position  $(c, C_H, Thief)$  in  $\mathcal{E}(H, k)$ , where  $C_H$  is determined as  $C_H = C_H(c)$  as in Remark 13.

(that is, the history of all the positions played so far), we need to record the last position played in  $\mathcal{E}(H, k)$ : this is  $p$ , which is undefined when the play begins. Cops on  $G$  are positioned on the images of Cops on  $H$  by the function  $f$  defined in (2). Moreover, Cops eagerly occupies the last two vertices visited on  $G$ . Thief's moves on  $G$  are going to be simulated by sequences of Thief's moves on  $H$ , using the  $\star$ -weak simulation  $(R, \varsigma)$ . In order to make this possible, a simulation of the form  $\varsigma(\tilde{g}, g, \tilde{h})$  must be halted before its target  $h$ ; the current position  $c$  is such halt-point. This implies that the simulation of  $g \rightarrow g'$  by  $(R, \varsigma)$  and the sequence of moves in  $H$  matching Thief's move on  $G$  are slightly out of phase. To cope with that, Cops must guess in advance what might happen in the rest of the simulation and this is why he puts cops on the current and previous positions in  $G$ . We also need to record  $h$ , the target of the previous simulation into the memory.

The previous considerations are formalized by requiring the following conditions to hold. To make sense of them, let us say that  $f(\{p\}) = f(p)$  if  $p \in V_H$  and that  $f(\{p\}) = \emptyset$  if  $p = \perp$ . In the last two conditions we require that  $p \neq \perp$ .

- $C_G = f(C_H(c)) \cup f(\{p\}) \cup \{g\}$ , (COPS)
- $f(c) = g$ , and  $f(h') \in f(\{p\}) \cup \{g\}$ , (TAIL)  
whenever  $h'$  lies on the tree path from  $c$  to  $h$ .
- $f(p) \rightarrow g, f(p)R\tilde{h}$  for some  $\tilde{h} \in V_H, c \in \varsigma(f(p), g, \tilde{h})$ , (HEAD)  
and  $h$  is the target of  $\varsigma(f(p), g, \tilde{h})$ ,
- on the tree path from  $p$  to  $c$ ,  $c$  is the only vertex s.t.  $f(c) = g$ . (HALT)

Since  $h_0$  has no predecessors, then  $gRh_0$  implies  $f(h_0) = g$ . Thus, at the beginning, the memory is set to  $(\perp, h_0, h_0)$  and conditions (COPS) and (TAIL) hold.

Consider now a Thief's move of the form  $(g, C_G, Thief) \rightarrow (g', C_G, Cops)$ , where  $g' \notin C_G$ . If  $g'$  has no successor, then Cops simply skips, thus reaching a winning position. Let us assume that  $g'$  has a successor, and write  $\varsigma(g, g', h) = hh_1 \dots h_n$ ,  $n \geq 1$ ; observe that  $f(h_n) = g'$ . If for some  $i = 1, \dots, n$   $h_i$  is not in the subtree of  $c$ , then the strategy halts, Cops abandons the game and loses. Otherwise, all the path  $\pi = c \dots hh_1 \dots h_n$  lies in the subtree of  $c$ . By eliminating cycles from  $\pi$ , we obtain a simple path  $\sigma$ , of source  $c$  and target  $h_n$ , which entirely lies in the subtree of  $c$ . By Lemma 1,  $\sigma$  is the tree path from  $c$  to  $h_n$ . An explicit description of  $\sigma$  is as follows: we can write  $\sigma$  as the compose  $\sigma_0 \star \sigma_1$ , where the target of  $\sigma_0$  and source of  $\sigma_1$  is the vertex of  $\varsigma(g, g', h)$  which is closest to the root  $h_0$ ; moreover  $\sigma_0$  is a prefix of the tree path from  $c$  to  $h$ , and  $\sigma_1$  is a postfix of the path  $\varsigma(g, g', h)$ .

We cut  $\sigma$  as follows: we let  $c'$  be the first vertex on this path such that  $f(c') = g'$ . Thief's move  $g \rightarrow g'$  on  $G$  is therefore simulated by Thief's moves from  $c$  to  $c'$  on  $H$ . This is possible since every vertex lies in the subtree of  $c$  and thus it has not yet been explored. Cops consequently occupies the returns on this path, thus modifying  $C_H$  to  $C'_H = C_H(c') = (C_H \setminus X) \uplus Y$ , where  $Y$  is a set of at most  $k$  vertexes containing the last returns visited on the path from  $c$  to  $c'$ .

After the simulation on  $H$ , Cops moves to  $(g', C'_G, Thief)$  in  $\tilde{\mathcal{E}}(G, k + 2)$ , where  $C'_G = f(C'_H) \cup \{g, g'\}$ . Let us verify that this is an allowed move according to the

rules of the game. We remark that  $f(Y) \subseteq f(\{p\}) \cup \{g, g'\}$  and therefore

$$\begin{aligned} C'_G &= f(C_H \setminus X) \cup f(Y) \cup \{g, g'\} \\ &= (f(C_H \setminus X) \cup (f(Y) \setminus \{g'\}) \cup \{g\}) \cup \{g'\} = A \cup \{g'\}, \end{aligned}$$

where  $A = f(C_H \setminus X) \cup (f(Y) \setminus \{g'\}) \cup \{g\} \subseteq f(C_H) \cup f(\{p\}) \cup \{g\} = C_G$ . After the simulation Cops also updates the memory to  $M(g', C'_G, Thief) = (c, c', h_n)$ . Since  $f(c) = g$ , then condition (COPS) clearly holds. Also,  $f(c) = g \rightarrow g'$ ,  $gRh$  and  $h_n$  is the target of  $\varsigma(f(c), g', h)$ . We have also that  $c' \in \sigma_1$  and hence  $c' \in \varsigma(f(c), g', h)$ , since otherwise  $c' \in \sigma_0$  and  $f(c') \in \{f(p), g\}$ , contradicting  $f(c') = g'$  and the condition on the girth of  $G$ . Thus condition (HEAD) holds as well. Also, condition (HALT) holds, since by construction  $c'$  is the first vertex on the tree path from  $c$  to  $h$  such that  $f(c') = g'$ . Let us verify that condition (TAIL) holds: by construction  $f(c') = g'$ , and the path from  $c'$  to  $h_n$  is a postfix of  $\varsigma(g, g', h)$ , and hence  $f(h') \in \{g, g'\}$  if  $h'$  lies on this tree path.

Let us now prove that the strategy is winning: Cops will never abandon. To this goal we need to argue that when Thief plays the move  $g \rightarrow g'$  on  $G$ , then the simulation  $\varsigma(g, g', h) = hh_1 \dots h_n$  lies in the subtree of  $c$ . If this is not the case, let  $i$  be the first index such that  $h_i$  is not in the subtree of  $c$ . Therefore  $h_i$  is a return and, by the assumptions on  $H$  and the on canonical strategy,  $h_i \in C_H(c)$ . Since  $h_i \in \varsigma(g, g', h)$ ,  $f(h_i) \in \{g, g'\}$ . Observe, however that we cannot have  $f(h_i) = g'$ , otherwise  $g' \in f(C_H(c)) \subseteq C_G$ . We deduce that  $f(h_i) = g$  and that  $g \in f(C_H) \subseteq C_G$ .

Since  $C_G \neq \perp$ , then  $(g, C_G, Thief)$  is not the initial position of the play, so that, if  $M(g, C_G, Thief) = (p, c, h)$ , then  $p \neq \perp$ . Let us now consider the last two moves of the play before reaching position  $(g, C_G, Thief)$ . These are of the form  $(f(p), \tilde{C}_G, Thief) \rightarrow (g, \tilde{C}_G, Cops) \rightarrow (g, C_G, Thief)$ , and have been played according to this strategy. Since  $g \notin \tilde{C}_G$ , it follows that the Cop on  $h_i$  has been dropped on  $H$  during the previous round of the strategy, simulating the move  $f(p) \rightarrow g$  on  $G$  by the tree path from  $p$  to  $c$ . This is however in contradiction with condition (HALT), stating that  $c$  is the only vertex  $h$  on the tree path from  $p$  to  $c$  such that  $f(h) = c$ .  $\square$

## 4 Strongly Synchronizing Games

In this section we define *strongly synchronizing* games, a generalization of synchronizing games introduced in [21]. We shall show that, for every game  $H$  equivalent to a strongly synchronizing game  $G$ , there is a  $\star$ -weak simulation of  $G$  by  $H$ .<sup>5</sup>

Let us say that  $G \in \mathcal{G}$  is *bipartite* if  $M^G \subseteq Pos_E^G \times Pos_{A,D}^G \cup Pos_A^G \times Pos_{E,D}^G$ .

**Definition 15.** *A game  $G$  is strongly synchronizing iff it is bipartite, it has girth strictly greater than 4 and, for every pair of positions  $g, k$ , the following conditions hold:*

1. *if  $(G, g) \sim (G, k)$  then  $g = k$ .*
2. *if  $(G, g) \leq (G, k)$  and  $(G, k) \not\leq (G, g)$ , then  $k \in Pos_E^G$  and  $(k, g) \in M^G$ , or  $g \in Pos_A^G$  and  $(g, k) \in M^G$ .*

<sup>5</sup> In the sequel, we shall not distinguish between a game and its underlying graph.

A consequence of the previous definition is that *the only winning strategy for Mediator in the game  $\langle G, G \rangle$  is the copycat strategy*. Thus strongly synchronizing games are synchronizing as defined in [21]. We list next some useful properties of strongly synchronizing games.

**Lemma 16.** *Let  $G$  be a strongly synchronizing and let  $(g, g'), (\tilde{g}, \tilde{g}') \in M^G$  be distinct.*

1. *If  $(G, g) \sim \hat{x}$  then  $g \in Pos_D^G$  and  $\lambda(g) = x$ .*
2. *If  $g, \tilde{g} \in Pos_E^G$  and, for some game  $H$  and  $h \in Pos^H$ , we have*

$$(G, g') \leq (H, h) \leq (G, g) \text{ and } (G, \tilde{g}') \leq (H, h) \leq (G, \tilde{g}),$$

*then  $g = \tilde{g}$  or  $g' = \tilde{g}'$ , and  $|\{g, g', \tilde{g}, \tilde{g}'\}| = 3$ .*

3. *If  $g \in Pos_E^G$  and  $\tilde{g} \in Pos_A^G$  and, for some  $H$  and  $h \in Pos^H$ , we have*

$$(G, g') \leq (H, h) \leq (G, g) \text{ and } (G, \tilde{g}) \leq (H, h) \leq (G, \tilde{g}'),$$

*then  $g = \tilde{g}'$  or  $g' = \tilde{g}$ , and  $|\{g, g', \tilde{g}, \tilde{g}'\}| = 3$ .*

We are ready to state the main result of this section.

**Proposition 17.** *Let  $G$  be a strongly synchronizing game, and let  $H \in \mathcal{G}$  be such that  $G \leq H \leq G$ , then there is a  $\star$ -weak simulation of  $G$  by  $H$ .*

*Proof.* Let  $R, S$  be two winning strategies for Mediator in  $\langle G, H \rangle$  and  $\langle H, G \rangle$ , respectively. Let  $T = R || S$  be the composal strategy in  $\langle G, H, G \rangle$ . Define

$$gRh \text{ iff } (g, h, g) \text{ is a position of } T \text{ and } g, h \text{ belong to the same player.}$$

We consider first  $R$  and prove that it is functional and surjective. If  $g_i Rh, i = 1, 2$  then  $(g_1, h, g_1)$  and  $(g_2, h, g_2)$  are positions of  $T$ , hence  $(G, g_1) \leq (H, h) \leq (G, g_1)$  and  $(G, g_2) \leq (H, h) \leq (G, g_2)$ , consequently  $(G, g_1) \sim (G, g_2)$  implies  $g_1 = g_2$ , by definition 15. For surjectivity, we can assume that (a) all the positions of  $G$  are reachable from the initial position  $p_\star^G$ , (b)  $p_\star^G$  and  $p_\star^H$  belong to the same player (by possibly adding to  $H$  a new initial position leading to the old one). Since  $T \setminus H$  is the copycat strategy, given  $g \in Pos_{E,A,D}^G$ , from the initial position  $(p_\star^G, p_\star^H, p_\star^G)$  of  $\langle G, H, G \rangle$ , the Opponents have the ability to reach a position of the form  $(g, h, g)$ . The explicit construction of the function  $\varsigma$  will show that  $h$  can be chosen to belong to the same player as  $g$ .

We construct now the function  $\varsigma$  so that  $(R, \varsigma)$  is a weak simulation. If  $gRh$  and  $(g, g') \in M^G$ , then we construct  $\pi = h, \dots, h'$  such that  $g'Rh'$ . Since  $G$  is bipartite, then  $h \neq h'$  and  $\pi$  is nonempty. We let  $\varsigma(g, g', h)$  be a reduction of  $\pi$  to a nonempty simple path.

We assume  $(g, h) \in (Pos_E^G, Pos_E^H)$ , the case  $(g, h) \in (Pos_A^G, Pos_A^H)$  is dual. From position  $(g, h, g)$  it is Opponent's turn to move on the left, they choose a move  $(g, g') \in M^G$ . Since  $G$  is bipartite, we have either  $g' \in Pos_D^G$  or  $g' \in Pos_A^G$ .

*Case (i).* If  $g' \in Pos_D^G$  then the strategy  $T$  suggests playing a finite path on  $H$ ,  $(g', h, g) \rightarrow^* (g', h^*, g)$ , possibly of zero length, and then it will suggest to play on the external right board. An infinite path played only on  $H$  cannot arise, since  $T$  is a

winning strategy and such an infinite path is not a win for Mediator. Since  $T \setminus_H$  is the copycat strategy,  $T$  suggests the only move  $(g', h^*, g) \rightarrow (g', h^*, g')$ . From this position  $T$  suggests playing a path on  $H$  leading to a final draw position  $h_f \in Pos_D^H$  as follows  $(g', h^*, g') \rightarrow^* (g', h_f, g')$ , such that  $\lambda^G(g') = \lambda^H(h_f)$ , therefore  $g' R h_f$ .

*Case (ii).* If  $g' \in Pos_A^G$  then from position  $(g', h, g)$  it is Mediator's turn to move. We claim that  $T$  will suggest playing a nonempty finite path  $(g', h, g) \rightarrow^+ (g', h', g)$  on the central board  $H$ , where  $h' \in Pos_A^H$ , and then suggests the move  $(g', h', g) \rightarrow (g', h', g')$ . Let  $\tilde{h} \in Pos_{A,E,D}^H$  be such that the position  $(g', \tilde{h}, g)$  has been reached from  $(g', h, g)$ , through a (possibly empty) sequence of central moves, by playing with  $T$ . Then  $T$  cannot suggest a move on the left board  $(g', \tilde{h}, g) \rightarrow (g'', \tilde{h}, g)$ , since  $T \setminus_H$  is the copycat strategy. Also, if  $\tilde{h} \in Pos_E^H$ ,  $T$  cannot suggest a move on the right board  $(g', \tilde{h}, g) \rightarrow (g', \tilde{h}, \tilde{g})$ . The reason is that  $T = R \parallel S$ , and the position  $(\tilde{h}, g)$  of  $\langle H, G \rangle$  does not allow a Mediator's move on the right board. Thus a sequence of central moves on  $H$  is suggested by  $T$  and, as mentioned above, this sequence cannot be infinite. We pretend that its endpoint  $h' \in Pos_A^H$ . We already argued that  $h' \notin Pos_E^H$ , let us argue that  $h' \notin Pos_D^H$ . If this were the case, then strategy  $T$  suggests the only move  $(g', h', g) \rightarrow (g', h_n, g')$ , hence  $(G, g') \sim (H, h')$ . By Lemma 16.1, we get  $g' \in Pos_D^G$ , contradicting  $g' \in Pos_A^G$ .

This proves that  $(R, \varsigma)$  is a weak simulation. We prove next that  $(R, \varsigma)$  has the  $\star$ -property, thus assume that  $h^* \in \varsigma(g, g', h_0), \varsigma(\tilde{g}, \tilde{g}', \tilde{h}_0)$ . Let us suppose first that  $g, \tilde{g} \in Pos_E^H$ . By looking at the construction of these paths, we observe that the sequence of moves

$$\begin{aligned} (g, h_0, g) &\rightarrow (g', h_0, g) \rightarrow^* (g', h^*, g) \rightarrow^* (g', h_n, g) \rightarrow (g', h_n, g') \\ (\tilde{g}, \tilde{h}_0, \tilde{g}) &\rightarrow (\tilde{g}', \tilde{h}_0, \tilde{g}) \rightarrow^* (\tilde{g}', h^*, \tilde{g}) \rightarrow^* (\tilde{g}', \tilde{h}_m, \tilde{g}) \rightarrow (\tilde{g}', \tilde{h}_m, \tilde{g}') \end{aligned}$$

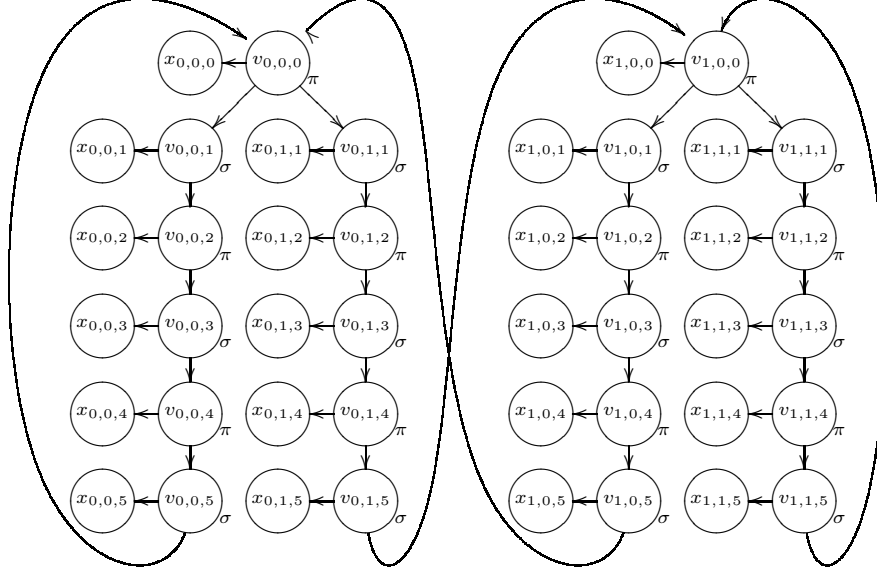
maybe played in the game  $\langle G, H, G \rangle$ , according to the winning strategy  $T = R \parallel S$ . We have therefore that  $(G, g') \leq (H, h^*) \leq (G, g)$  and  $(G, \tilde{g}') \leq (H, h^*) \leq (G, \tilde{g})$ .<sup>6</sup> Consequently  $|\{g, g', \tilde{g}, \tilde{g}'\}| = 3$ , by Lemma 16.2. If  $g \in Pos_E^G$  and  $\tilde{g} \in Pos_A^G$ , a similar argument shows that the positions  $(g', h^*, g)$  and  $(\tilde{g}, h^*, \tilde{g}')$  may be reached with  $T$  and hence  $(G, g') \leq (H, h^*) \leq (G, g)$  and  $(G, \tilde{g}) \leq (H, h^*) \leq (G, \tilde{g}')$ . Lemma 16.3 implies then  $|\{g, g', \tilde{g}, \tilde{g}'\}| = 3$ . Finally, the cases  $(g, \tilde{g}) \in \{(Pos_A^G, Pos_A^G), (Pos_A^G, Pos_E^G)\}$  are handled by duality. This completes the proof of Proposition 17.  $\square$

## 5 Construction of Strongly Synchronizing Games

In this section we complete the hierarchy theorem by constructing, for  $n \geq 1$ , strongly synchronizing games  $G_n$  such that  $\mathcal{E}(G_n) = n$ . This games mimic the  $n$ -cliques already used in [4] to prove that the variable hierarchy for the modal  $\mu$ -calculus is infinite. We

<sup>6</sup> Similar inequalities may be derived even if  $h^* \in Pos_D^H$ . In this case the moves in the central board may be interleaved with the move on the right board.

begin by drawing the game  $G_2$ :



The general definition of the game  $G_n$  is as follows. Let  $[n]$  denote the set  $\{0, \dots, n-1\}$  and let  $I_n = \{(i, j, k) \in [n] \times [n] \times [5] \mid k \bmod 2 = 0 \text{ implies } j = 0\}$ . We define

$$\begin{aligned} Pos_A^{G_n} &= \{v_{i,j,k} \mid (i, j, k) \in I_n \text{ and } k \bmod 2 = 0\}, \\ Pos_E^{G_n} &= \{v_{i,j,k} \mid (i, j, k) \in I_n \text{ and } k \bmod 2 = 1\}, \\ Pos_D^{G_n} &= \{w_{i,j,k} \mid (i, j, k) \in I_n\}. \end{aligned}$$

Let  $X = \{x_{i,j,k} \mid i, j \geq 0, k \in [n]\}$  be a countable set of variables, the labelling of draw positions,  $\lambda^{G_n} : Pos_D^{G_n} \rightarrow X$ , sends  $w_{i,j,k}$  to  $x_{i,j,k}$ . The moves  $M^{G_n}$  either lye on some cycle:

$$v_{i,0,0} \rightarrow v_{i,j,1}, \quad v_{i,j,k} \rightarrow v_{i,j,k+1}, \quad k = 1, \dots, 4, \quad v_{i,j,5} \rightarrow v_{j,0,0},$$

or lead to draw positions:  $v_{i,j,k} \rightarrow w_{i,j,k}$ . Finally, the priority function  $\rho^{G_n}$  assigns a constant odd priority to all positions. We state next the main facts about the games  $G_n$ :

**Proposition 18.** *The games  $G_n$  are strongly synchronizing and  $\mathcal{E}(G_n) = n$ .*

The proof of the statement is omitted for lack of space. We are now ready to state our main achievement of this paper.

**Theorem 19.** *For  $n \geq 3$ , the inclusions  $\mathcal{L}_{n-3} \subseteq \mathcal{L}_n$  are strict. Therefore the variable hierarchy for the games  $\mu$ -calculus is infinite.*

By the previous Proposition the game  $G_n \in \mathcal{L}_n$ . Also, since  $G_n$  is strongly synchronizing, if  $H \sim G_n$ , then there exists a  $\star$ -weak simulation of  $G_n$  by  $H$ . It follows by Theorem 14 that  $n-2 \leq \mathcal{E}(H)$ . Therefore  $G \notin \mathcal{L}_{n-3}$ .

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