



**HAL**  
open science

# Asymptotically efficient estimators for nonparametric heteroscedastic regression models

Jean-Yves Brua

► **To cite this version:**

Jean-Yves Brua. Asymptotically efficient estimators for nonparametric heteroscedastic regression models. 2007. hal-00178536v1

**HAL Id: hal-00178536**

**<https://hal.science/hal-00178536v1>**

Preprint submitted on 11 Oct 2007 (v1), last revised 29 Nov 2007 (v2)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Asymptotically efficient estimators for nonparametric heteroscedastic regression models

J.-Y. BRUA\*

*IRMA, Département de Mathématique, Université Louis Pasteur,  
7 rue René Descartes F67084, Strasbourg, Cedex, France*

## Abstract

This paper concerns the estimation of a function at a point in nonparametric heteroscedastic regression models with Gaussian noise or noise having unknown distribution. In those cases an asymptotically efficient kernel estimator is constructed for the minimax absolute error risk.

*MSC:* primary 62G08; secondary 62G20

*Keywords:* Asymptotical efficiency; Kernel estimator; Minimax; Nonparametric regression

---

\**E-mail address:* brua@math.u-strasbg.fr

# 1 Introduction

We consider the problem of estimating a regression function  $S$  at a given point  $z_0 \in ]0; 1[$  under observations

$$y_k = S(x_k) + g(x_k, S)\xi_k, \quad k \in \{1, \dots, n\} \quad (1)$$

where the regressors  $x_k = k/n$  are deterministic,  $\xi_k$  are independent identically distributed random variables which will firstly be assumed Gaussian standard then having unknown density. Notice that the variance of the noises  $g^2$  is unknown and depends on the unknown regression function  $S$  and the regressors  $x_k$ .

Heteroscedastic regression models with this type of scale functionals have been encountered in consumer budget studies utilizing observations on individuals with diverse incomes, in analysis of investment behavior of firms of different sizes and more recently in medical research. For example, Goldfeld and Quandt (1972) considered polynomial regression models such that

$$\begin{aligned} y_k &= \alpha + \beta x_k + u_k, \\ \mathbb{E}(u_k^2) &= a + bx_k + cx_k^2, \end{aligned}$$

which is a particular case of our model (1) if we assume the unknown regression function being like  $S(x) = \alpha + \beta x$  and  $g^2(x, S) = (a - \frac{ac}{\beta^2}) + (b - 2\frac{ac}{\beta})x + \frac{c}{\beta^2}S^2(x)$ . Other heteroscedastic regression models are studied for instance in Efromovich and Pinsker (1996), Galtchouk and Pergamenshchikov (2005) and Efromovich (2007).

The problem of Hölder regression estimation has been studied by several authors. For a regression function belonging to a quasi-Hölder class and estimated at a point with squared error loss, Sacks and Ylvisaker (1981) showed that the linear minimax estimator is a kernel estimator. Donoho and Liu (1991) further found that this estimator is within 17 percent of asymptotically minimax among all procedures and obtained optimal kernels for Hölder classes. For estimating the whole object or its  $k$ th derivative with sup-norm global loss and Hölder class, Korostelev (1993) and Donoho (1994a) proved that a kernel estimator is asymptotically efficient.

This article deals with nonparametric estimation of a regression function belonging to a Hölder ball. We work with the absolute error loss and the corresponding risk. Our aim is to find an efficient estimator, that is to say an estimator which achieves the sharp asymptotic behavior of the minimax risk. To that purpose we use the method developed by Galtchouk and Pergamenshchikov (2006) who introduce the local weak Hölder classes to define the risk of an estimator. So we use the classes  $\mathcal{U}_{z_0, \delta, n}$  which allows an arbitrary large usual Hölder constant but has a weak Hölder condition based on a weak Hölder constant tending to zero (see (2)), then define the risk  $\mathcal{R}_{z_0, \delta, n}(\tilde{S})$  of an estimator  $\tilde{S}$  of  $S(z_0)$  and the minimax risk  $\inf_{\tilde{S}} \mathcal{R}_{z_0, \delta, n}(\tilde{S})$  (see (11)). In these conditions we prove that a kernel estimator is asymptotically efficient, it means that the minimax risk attains the sharp asymptotic constant.

This paper is organized as follows. In section 2 we describe the problem in the case of Gaussian noise with all assumptions needed and define all necessary mathematical objects. Our main results of this problem are written in section 3. The case of unknown noise is related in section 4. Theorems are proved in section 5 and appendix A contains useful results for our proofs.

## 2 Statement of the problem

Consider model (1) where  $g : [0; 1] \times C^1([0; 1], \mathbb{R}) \longrightarrow \mathbb{R}_+^*$  and  $S$  are unknown functions,  $S$  belonging to the class

$$\mathcal{H}(\beta) = \bigcup_{M>0, K>0} \mathcal{H}(M, K, \beta),$$

where  $\beta = 1 + \alpha$  is known,  $\alpha \in ]0; 1[$ ,  $\mathcal{H}(M, K, \beta)$  is the Hölder class defined as

$$\mathcal{H}(M, K, \beta) = \left\{ S \in C^1([0; 1], \mathbb{R}) : \|S'\| \leq M, \sup_{x, y \in [0; 1]} \frac{|S'(y) - S'(x)|}{|x - y|^\alpha} \leq K \right\},$$

with  $\|f\| = \sup_{x \in [0; 1]} |f(x)|$ . We suppose that the noises  $(\xi_k)_{1 \leq k \leq n}$  are independent identically distributed  $\mathcal{N}(0, 1)$ .

As mentioned in the introduction, we will work with a minimax risk taken over the local weak Hölder class at the point  $z_0$  defined as

$$\mathcal{U}_{z_0, \delta, n} = \left\{ S \in \mathcal{H}(\beta) : \|S'\| \leq \delta^{-1} ; \left| \int_{-1}^1 (S(z_0 + hu) - S(z_0)) du \right| \leq \delta h^\beta \right\}, \quad (2)$$

where  $0 < \delta < 1$ ,  $h = h_n = n^{-\frac{1}{2\beta+1}}$ ,  $n \geq 1$ .

Notice that

$$\int_{-1}^1 (S(z_0 + hu) - S(z_0)) du = \int_{-1}^1 \left( \int_{z_0}^{z_0+uh} (S'(t) - S'(z_0)) dt \right) du, \quad (3)$$

so we have for all  $S \in \mathcal{H}(M, K, \beta)$

$$\left| \int_{-1}^1 (S(z_0 + hu) - S(z_0)) du \right| \leq \frac{2K}{\beta(\beta + 1)} h^\beta.$$

That's why the class  $\mathcal{U}_{z_0, \delta, n}$  is called a weak Hölder class.

Furthermore (3) implies that  $\mathcal{H}(\delta^{-1}, \delta, \beta) \subset \mathcal{U}_{z_0, \delta, n}$  for any  $n \geq 1$  and  $0 < \delta < 1$ .

Let us give the assumptions needed. Firstly we suppose that

$$\lim_{n \rightarrow \infty} \sup_{S \in \mathcal{U}_{z_0, \delta, n}} \left| \left( \frac{1}{q_n} \sum_{k=1}^n Q\left(\frac{x_k - z_0}{h}\right) g^2(x_k, S) \right)^{\frac{1}{2}} - g(z_0, S) \right| = 0, \quad (4)$$

with

$$q_n = \sum_{k=1}^n Q\left(\frac{x_k - z_0}{h}\right) \text{ and } Q = \mathbb{I}_{[-1; 1]}.$$

Moreover we assume that there exists  $g_\star > 0$  and  $g^\star < \infty$  such that

$$g_\star \leq \inf_{0 \leq x \leq 1} \inf_{S \in C^1([0; 1], \mathbb{R})} g(x, S) \leq \sup_{0 \leq x \leq 1} \sup_{S \in C^1([0; 1], \mathbb{R})} g(x, S) \leq g^\star \quad (5)$$

and that the function  $g$  is differentiable in the Frechet sense with respect to  $S$  in  $C^1([0; 1], \mathbb{R})$  uniformly over  $x \in [0; 1]$ , i.e. for any  $S, S_0 \in C^1([0; 1], \mathbb{R})$

$$g(x, S) = g(x, S_0) + L_{x, S_0}(S - S_0) + \Gamma_{x, S_0}(S - S_0), \quad (6)$$

where the linear operator  $L_{x, S_0}$  is bounded uniformly over  $x \in [0; 1]$  in  $C^1([0; 1], \mathbb{R})$ , i.e. for any  $S_0 \in C^1([0; 1], \mathbb{R})$  there exists some positive constant  $C_{S_0}$  such that

$$\sup_{x \in [0; 1]} \sup_{S \in C^1([0; 1], \mathbb{R}), \|S\| \neq 0} |L_{x, S_0}(S)| / \|S\| \leq C_{S_0} \quad (7)$$

and the residual term  $\Gamma_{x, S_0}(S)$  satisfies the property

$$\lim_{\|S\| \rightarrow 0} \sup_{x \in [0; 1]} \Gamma_{x, S_0}(S) / \|S\| = 0. \quad (8)$$

**Remark 2.1**

Note that hypothesis (4) is verified when for all  $\varepsilon > 0$ , there exists  $\eta > 0$  such that if  $|x - z_0| \leq \eta$ , then  $\sup_{S \in C^1([0; 1], \mathbb{R})} |g(x, S) - g(z_0, S)| \leq \varepsilon$ .

In particular a function  $g$  satisfies this property if it is uniformly continuous with respect to both variables.

**Remark 2.2**

Let us give a general example of a function  $g$  satisfying hypothesis (4)–(8) above.

Let  $V : \mathbb{R} \rightarrow \mathbb{R}_+$  and  $G : [0; 1] \times \mathbb{R} \rightarrow \mathbb{R}_+$  two differentiable functions such that

$$\begin{aligned} \|V'\|_\infty &< \infty, \\ G_\star &= \inf_{x \in [0; 1], y \in \mathbb{R}} G(x, y) > 0, \\ G'_\star &= \sup_{x \in [0; 1], y \in \mathbb{R}} \left| \frac{\partial G}{\partial y}(x, y) \right| < \infty. \end{aligned}$$

Define

$$g^2(x, S) = G(x, S(x)) + \int_0^1 V(S(t)) dt. \quad (9)$$

The derivative in the Frechet sense of  $g$  is given by

$$L_{x, S}(f) = \frac{1}{2g(x, S)} \frac{\partial G}{\partial y}(x, S(x)) f(x) + \frac{1}{2g(x, S)} \int_0^1 V'(S(t)) f(t) dt,$$

so we have

$$\sup_{x \in [0; 1]} \sup_{S \in C^1([0; 1], \mathbb{R}), \|S\| \neq 0} \frac{|L_{x, S}(f)|}{\|f\|_\infty} \leq \frac{G'_\star + \|V\|_\infty}{2\sqrt{G_\star}}.$$

Writing the Taylor expansion of functions  $y \mapsto G(x, y)$  at the point  $(x, S(x))$  and  $V$  at the point  $S(t)$  to the first order :

$$\begin{aligned} G(x, S(x) + f(x)) &= G(x, S(x)) + \frac{\partial G}{\partial y}(x, S(x)) f(x) + f(x) \varepsilon_{x, S}(f(x)), \\ V(S(t) + f(t)) &= V(S(t)) + V'(S(t)) f(t) + f(t) \tilde{\varepsilon}_{t, S}(f(t)), \end{aligned}$$

we can easily show that

$$\frac{|\Gamma_{x,S}(f)|}{\|f\|_\infty} \leq \frac{G'_* + \|V'\|_\infty}{8G_*^{3/2}} |g^2(x, S+f) - g^2(x, S)| + \frac{1}{2\sqrt{G_*}} \left( |\varepsilon_{x,S}(f(x))| + \int_0^1 |\tilde{\varepsilon}_{t,S}(f(t))| dt \right). \quad (10)$$

Now if we take  $G(x, y) = \alpha_0 + \alpha_1 x + \alpha_2 \sin^2 y$  and  $V(y) = \alpha_3 \sin^2 y$  for all  $(x, y) \in [0, 1] \times \mathbb{R}$ , with  $\alpha_0 > 0$  and  $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}_+$ , then the function  $g$  defined as (9) is uniformly continuous, bounded by  $\sqrt{\alpha_0}$  and  $\sqrt{\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3}$ . Moreover by writing explicitly the functions  $\varepsilon_{x,S}$  and  $\tilde{\varepsilon}_{x,S}$  for this case, we can prove thanks to (10) that  $g$  satisfies hypothesis (8). So we have exhibited an example of function  $g$  which satisfies all assumptions needed.

For any estimator  $\tilde{S}_n(z_0)$  of  $S(z_0)$  we define the following risk

$$\mathcal{R}_{z_0, \delta, n}(\tilde{S}_n) = \sup_{S \in \mathcal{U}_{z_0, \delta, n}} \mathbb{E}_S \varphi_n \frac{|\tilde{S}_n(z_0) - S(z_0)|}{g(z_0, S)}, \quad (11)$$

where  $\mathbb{E}_S$  is the expectation taken with respect to the law  $\mathbb{P}_S$  in (1) and  $\varphi_n = n^{\frac{\beta}{2\beta+1}}$ .

### 3 Lower and upper bounds

In this section we give the lower bound for the minimax risk and show that the kernel estimator  $\hat{S}_n(z_0)$ , defined by

$$\hat{S}_n(z_0) = \frac{1}{q_n} \sum_{k=1}^n Q\left(\frac{x_k - z_0}{h}\right) y_k, \quad (12)$$

is asymptotically efficient as we give the upper bound for its risk.

**Theorem 3.1** For any  $\delta \in ]0; 1[$ ,

$$\liminf_{n \rightarrow \infty} \inf_{\tilde{S}} \mathcal{R}_{z_0, \delta, n}(\tilde{S}) \geq \frac{\mathbb{E}|\xi|}{\sqrt{2}}, \quad \xi \sim \mathcal{N}(0, 1),$$

where the infimum is taken over all estimators  $\tilde{S}$  of  $S(z_0)$ .

**Theorem 3.2** For the estimator  $\hat{S}_n(z_0)$  from (12), the following inequality holds:

$$\limsup_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \mathcal{R}_{z_0, \delta, n}(\hat{S}_n(z_0)) \leq \frac{\mathbb{E}|\xi|}{\sqrt{2}}, \quad \xi \sim \mathcal{N}(0, 1).$$

### 4 Case of unknown noise distribution

In this section we suppose that the  $(\xi_k)$  in model (1) are independent identically distributed with an unknown density  $p$  belonging to

$$\mathcal{P}_{\varepsilon, L} = \left\{ p : \int_{-\infty}^{+\infty} xp(x)dx = 0, \int_{-\infty}^{+\infty} x^2 p(x)dx = 1, \int_{-\infty}^{+\infty} |x|^{2+\varepsilon} p(x)dx \leq L \right\},$$

with  $\varepsilon > 0$  and  $L > 0$  sufficiently large to have the density of the standard Gaussian random variable in  $\mathcal{P}_{\varepsilon, L}$ .

We define the risk corresponding to this case as

$$\tilde{\mathcal{R}}_{z_0, \delta, n}(\tilde{S}_n) = \sup_{p \in \mathcal{P}_{\varepsilon, L}} \sup_{S \in \mathcal{U}_{z_0, \delta, n}} \mathbb{E}_S \varphi_n \frac{|\tilde{S}_n(z_0) - S(z_0)|}{g(z_0, S)}.$$

In the following theorems we give the sharp lower bound for the minimax risk over all estimators and establish the upper bound for the minimax risk for the kernel estimator  $\hat{S}_n(z_0)$  of  $S(z_0)$  defined in (12).

**Theorem 4.1** *For any  $\delta \in ]0; 1[$ , one has:*

$$\liminf_{n \rightarrow \infty} \inf_{\tilde{S}} \tilde{\mathcal{R}}_{z_0, \delta, n}(\tilde{S}) \geq \frac{\mathbb{E}|\eta|}{\sqrt{2}}, \quad \eta \sim \mathcal{N}(0, 1),$$

where the infimum is taken over all estimators  $\tilde{S}$  of  $S(z_0)$ .

**Theorem 4.2** *The kernel estimator (12) is asymptotically efficient. Indeed it satisfies the inequality:*

$$\limsup_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \tilde{\mathcal{R}}_{z_0, \delta, n}(\hat{S}_n(z_0)) \leq \frac{\mathbb{E}|\eta|}{\sqrt{2}}, \quad \eta \sim \mathcal{N}(0, 1).$$

## 5 Proof of the theorems

### 5.1 Proof of theorem 3.1

For all  $\nu \in ]0; \frac{1}{4}[$ , denote

$$S_\nu(x) = \varphi_n^{-1} V_\nu \left( \frac{x - z_0}{h} \right),$$

where the function  $V_\nu$  is defined by:

$$\begin{aligned} V_\nu(x) &= \frac{1}{\nu} \int_{-\infty}^{+\infty} \tilde{Q}_\nu(u) l \left( \frac{u - x}{\nu} \right) du \\ \tilde{Q}_\nu(u) &= \mathbb{I}_{\{|u| \leq 1 - 2\nu\}} + 2\mathbb{I}_{\{1 - 2\nu \leq |u| \leq 1 - \nu\}}, \end{aligned}$$

and  $l$  is a non-negative infinitely differentiable on  $\mathbb{R}$  function such that  $l(z) = 0$  for  $|z| \geq 1$  and  $\int_{-1}^1 l(z) dz = 1$ .

One can easily see that for any  $0 < \nu < \frac{1}{4}$ , we have  $V_\nu(0) = 1$  and  $\int_{-1}^1 V_\nu(x) dx = 2$ .

Let  $b > 0$  and  $\delta \in ]0; 1[$ . Denote

$$S_{\nu, u}(x) = \frac{u}{\varphi_n} V_\nu \left( \frac{x - z_0}{h} \right),$$

where  $x, u \in \mathbb{R}$ .

Thanks to lemma A.1, if  $|u| \leq b$  there exists an integer  $n_{b, \delta} > 0$  such that  $S_{\nu, u} \in \mathcal{U}_{z_0, \delta, n}$  for all

$n \geq n_{b,\delta}$ .

Therefore for  $n \geq n_{b,\delta}$ , one has:

$$\begin{aligned} \mathcal{R}_{z_0,\delta,n}(\tilde{S}) &\geq \sup_{|u| \leq b} \frac{1}{g(z_0, S_{\nu,u})} \mathbb{E}_{S_{\nu,u}} \varphi_n |\tilde{S}(z_0) - S_{\nu,u}(z_0)| \\ &\geq \frac{1}{2b} \int_{-b}^b \frac{1}{g(z_0, S_{\nu,u})} \mathbb{E}_{S_{\nu,u}} v_a \left( \varphi_n(\tilde{S}(z_0) - S_{\nu,u}(z_0)) \right) du := I_n(a, b) \end{aligned}$$

where  $v_a(x) = |x| \wedge a$ ,  $a > 0$ .

Write  $\mathbb{P}_{S_{\nu,u}}$  the law of  $(y_k^{(1)})_{k=1,\dots,n}$ , where  $y_k^{(1)} = S_{\nu,u}(x_k) + g(x_k, S_{\nu,u})\xi_k$ , and  $\mathbb{P}$  the law of  $(y_k^{(0)})_{k=1,\dots,n}$ , where  $y_k^{(0)} = g(x_k, S_{\nu,u})\xi_k$ .

These two measures are equivalent and the corresponding Radon-Nikodym derivative is at the point  $(y_1, \dots, y_n)$ :

$$\begin{aligned} \rho_n(u) &= \frac{d\mathbb{P}_{S_{\nu,u}}}{d\mathbb{P}}(y_1, \dots, y_n) = \exp \left\{ -\frac{1}{2} \sum_{k=1}^n \left( \left( \frac{y_k - S_{\nu,u}(x_k)}{g(x_k, S_{\nu,u})} \right)^2 - \left( \frac{y_k}{g(x_k, S_{\nu,u})} \right)^2 \right) \right\} \\ &= \exp \left( u \varsigma_n \eta_n - \frac{u^2}{2} \varsigma_n^2 \right) \end{aligned}$$

where  $\varsigma_n^2 = \frac{1}{\varphi_n^2} \sum_{k=1}^n \frac{V_\nu^2 \left( \frac{x_k - z_0}{h} \right)}{g^2(x_k, S_{\nu,u})}$  and  $\eta_n = \frac{1}{\varsigma_n \varphi_n} \sum_{k=1}^n \frac{V_\nu \left( \frac{x_k - z_0}{h} \right)}{g^2(x_k, S_{\nu,u})} y_k$ .

Under the law  $\mathbb{P}$ ,  $\eta_n$  is a Gaussian random variable with mean 0 and variance

$$\frac{1}{\varsigma_n^2 \varphi_n^2} \sum_{k=1}^n \frac{V_\nu^2 \left( \frac{x_k - z_0}{h} \right)}{g^4(x_k, S_{\nu,u})} g^2(x_k, S_{\nu,u}) = \frac{1}{\varsigma_n^2 \varphi_n^2} \sum_{k=1}^n \frac{V_\nu^2 \left( \frac{x_k - z_0}{h} \right)}{g^2(x_k, S_{\nu,u})} = 1.$$

We prove in lemma A.2 that

$$\varsigma_n^2 \xrightarrow{n \rightarrow \infty} \int_{-1}^1 \frac{V_\nu^2(z)}{g^2(z_0, 0)} dz := \sigma_\nu^2. \quad (13)$$

So we can rewrite  $\rho_n(u) = \exp \left\{ u \sigma_\nu \eta_n - \frac{u^2 \sigma_\nu^2}{2} + r_n \right\}$ , where

$$r_n = u \eta_n (\varsigma_n - \sigma_\nu) - \frac{u^2}{2} (\varsigma_n^2 - \sigma_\nu^2) \sim \mathcal{N} \left( \frac{u^2}{2} (\sigma_\nu^2 - \varsigma_n^2); u^2 (\varsigma_n - \sigma_\nu)^2 \right).$$

According to Tchebychev's inequality one has for all  $\epsilon > 0$ ,

$$\mathbb{P} \left( \left| r_n + \frac{u^2}{2} (\varsigma_n^2 - \sigma_\nu^2) \right| > \epsilon \right) \leq \frac{u^2 (\varsigma_n - \sigma_\nu)^2}{\epsilon^2} \xrightarrow{n \rightarrow \infty} 0.$$

Thus  $r_n + \frac{u^2}{2} (\varsigma_n^2 - \sigma_\nu^2) \xrightarrow{n \rightarrow \infty} 0$ .

As we have the following limit:  $\frac{u^2}{2} (\varsigma_n^2 - \sigma_\nu^2) \xrightarrow{n \rightarrow \infty} 0$ , we get  $r_n \xrightarrow{n \rightarrow \infty} 0$ .



Denoting  $\psi_{a,n}(\tilde{S}, S_{\nu,u}) = v_a(\varphi_n(\tilde{S}_n(z_0) - S_{\nu,u}(z_0)))$  and  $\mathbb{E}$  the expectation for the probability measure  $\mathbb{P}$ , one has

$$I_n(a, b) \geq \frac{1}{2b} \int_{-b}^b \mathbb{E} \mathbb{I}_{B_d} \frac{\psi_{a,n}(\tilde{S}, S_{\nu,u})}{g(z_0, S_{\nu,u})} \varrho_n(u) du + \delta_n(a, b) := J_n(a, b) + \delta_n(a, b), \quad (14)$$

where

$$\begin{aligned} B_d &= \{|\eta_n| \leq d\} \text{ and } d = \sigma_\nu(b - \sqrt{b}), b > 1, \\ \varrho_n(u) &= \exp\left(u\sigma_\nu\eta_n - \frac{u^2\sigma_\nu^2}{2}\right), \\ \delta_n(a, b) &= \frac{1}{2b} \int_{-b}^b \mathbb{E} \mathbb{I}_{B_d} \frac{\psi_{a,n}(\tilde{S}, S_{\nu,u})}{g(z_0, S_{\nu,u})} \theta_n(u) du, \\ \theta_n(u) &= \rho_n(u) - \varrho_n(u). \end{aligned}$$

Remember that  $\eta_n \sim \mathcal{N}(0, 1)$ , then  $\eta_n \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \eta$  with  $\eta \sim \mathcal{N}(0, 1)$ .

According to Slutsky's lemma

$$u\sigma_\nu\eta_n - \frac{u^2\sigma_\nu^2}{2} + r_n \xrightarrow[n \rightarrow \infty]{\mathcal{L}} u\sigma_\nu\eta - \frac{u^2\sigma_\nu^2}{2}.$$

Then  $\rho_n(u) \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \rho_\infty(u) = \exp\left(u\sigma_\nu\eta - \frac{u^2\sigma_\nu^2}{2}\right)$ .

We can easily show that  $\mathbb{E}\rho_\infty(u) = 1$  and we have also  $\mathbb{E}\rho_n(u) = 1$  because  $\rho_n(u)$  is a density. Hence, using theorem 3.6 from Billingsley (1999),  $\{\rho_n(u), n \geq 1\}$  is uniformly integrable. And since  $\varrho_n(u)$  is bounded on  $B_d$ , we obtain that  $\{\rho_n(u) - \varrho_n(u), n \geq 1\}$  is uniformly integrable as well as  $\{\mathbb{I}_{B_d}\psi_{a,n}(\tilde{S}, S_{\nu,u})\theta_n(u), n \geq 1\}$ .

Write  $\theta_n(u) = \exp\left(u\sigma_\nu\eta_n - \frac{u^2\sigma_\nu^2}{2}\right) (e^{r_n} - 1)$  and notice that  $\exp\left(u\sigma_\nu\eta_n - \frac{u^2\sigma_\nu^2}{2}\right)$  is bounded on  $B_d$  and that  $e^{r_n} - 1 \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0$ . As a consequence one has

$$\frac{\mathbb{I}_{B_d}\psi_{a,n}(\tilde{S}, S_{\nu,u})}{g(z_0, S_{\nu,u})} \theta_n(u) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0.$$

From here it follows that  $\frac{\mathbb{I}_{B_d}\psi_{a,n}(\tilde{S}, S_{\nu,u})}{g(z_0, S_{\nu,u})} \theta_n(u) \xrightarrow[n \rightarrow \infty]{\mathbb{L}^1} 0$  and  $\mathbb{E} \frac{\mathbb{I}_{B_d}\psi_{a,n}(\tilde{S}, S_{\nu,u})}{g(z_0, S_{\nu,u})} \theta_n(u) \xrightarrow[n \rightarrow \infty]{} 0$ .

But there exists a constant  $A > 0$  such that  $\left| \frac{\mathbb{I}_{B_d}\psi_{a,n}(\tilde{S}, S_{\nu,u})}{g(z_0, S_{\nu,u})} \theta_n(u) \right| \leq A\mathbb{E}|e^{r_n} - 1|$ .

And since  $r_n \sim \mathcal{N}\left(\frac{u^2}{2}(\sigma_\nu^2 - \zeta_n^2), u^2(\zeta_n - \sigma_\nu)^2\right)$ , we have

$$\mathbb{E}(e^{r_n}) = \exp\left(\frac{u^2}{2}(\sigma_\nu^2 - \zeta_n^2) + \frac{u^2(\zeta_n - \sigma_\nu)^2}{2}\right) = \exp(u^2\sigma_\nu^2 - u^2\zeta_n\sigma_\nu) \leq \exp(u^2\sigma_\nu^2).$$

Finally bounded convergence yields  $\delta_n(a, b) \xrightarrow[n \rightarrow \infty]{} 0$  in (14).

From now on we are interested in the term  $J_n(a, b)$  in (14): first rewrite  $\varrho_n(u) = \zeta_n e^{-\sigma_\nu^2(u - \tilde{\eta}_n)^2/2}$  with  $\zeta_n = e^{\eta_n^2/2}$  and  $\tilde{\eta}_n = \frac{\eta_n}{\sigma_\nu}$ , then

$$J_n(a, b) = \frac{1}{2b} \int_{-b}^b \mathbb{E} \mathbb{I}_{B_d} \zeta_n \frac{v_a(u - t_n)}{g(z_0, S_{\nu,u})} \exp\left(-\frac{\sigma_\nu^2}{2}(u - \tilde{\eta}_n)^2\right) du,$$

where  $t_n = \varphi_n \tilde{S}_n(z_0)$ .

Hence if  $\xi \sim \mathcal{N}(0, 1)$  and if we denote  $\tilde{\xi} = \frac{\xi}{\sigma_\nu}$ ,  $\zeta = e^{\xi^2/2}$ ,  $\tilde{B}_d = \{|\xi| \leq d\}$  and  $\tilde{\mathbb{E}}$  the expectation for the probability law of  $\xi$ , we get

$$\begin{aligned} J_n(a, b) &= \frac{1}{2b} \int_{-b}^b \tilde{\mathbb{E}} \mathbb{I}_{\tilde{B}_d} \zeta \frac{v_a(u - t_n)}{g(z_0, S_{\nu, u})} \exp\left(-\frac{\sigma_\nu^2}{2}(u - \tilde{\xi})^2\right) du \\ &= \tilde{\mathbb{E}} \mathbb{I}_{\tilde{B}_d} \zeta \frac{1}{2b} \int_{-b}^b \frac{v_a(u - t_n)}{g(z_0, S_{\nu, u})} \exp\left(-\frac{\sigma_\nu^2}{2}(u - \tilde{\xi})^2\right) du \end{aligned}$$

We have the following limit

$$\tilde{\mathbb{E}} \mathbb{I}_{\tilde{B}_d} \zeta \frac{1}{2b} \int_{-b}^b v_a(u - t_n) \exp\left(-\frac{\sigma_\nu^2}{2}(u - \tilde{\xi})^2\right) \left(\frac{1}{g(z_0, S_{\nu, u})} - \frac{1}{g(z_0, 0)}\right) du \xrightarrow{n \rightarrow \infty} 0.$$

Indeed

$$\begin{aligned} &\left| \tilde{\mathbb{E}} \mathbb{I}_{\tilde{B}_d} \zeta \frac{1}{2b} \int_{-b}^b v_a(u - t_n) \exp\left(-\frac{\sigma_\nu^2}{2}(u - \tilde{\xi})^2\right) \left(\frac{1}{g(z_0, S_{\nu, u})} - \frac{1}{g(z_0, 0)}\right) du \right| \\ &\leq \tilde{\mathbb{E}} \mathbb{I}_{\tilde{B}_d} \zeta \frac{1}{2b} \int_{-b}^b v_a(u - t_n) \exp\left(-\frac{\sigma_\nu^2}{2}(u - \tilde{\xi})^2\right) \left| \frac{g(z_0, 0) - g(z_0, S_{\nu, u})}{g(z_0, S_{\nu, u})g(z_0, 0)} \right| du \\ &\leq \tilde{\mathbb{E}} \mathbb{I}_{\tilde{B}_d} \zeta \frac{1}{2b} \int_{-b}^b v_a(u - t_n) \exp\left(-\frac{\sigma_\nu^2}{2}(u - \tilde{\xi})^2\right) \left| \frac{-L_{z_0, 0}(S_{\nu, u}) + \Gamma_{z_0, 0}(S_{\nu, u})}{g_\star^2} \right| du \\ &\leq \tilde{\mathbb{E}} \mathbb{I}_{\tilde{B}_d} \zeta \frac{1}{2b} \int_{-b}^b a \exp\left(-\frac{\sigma_\nu^2}{2}(u - \tilde{\xi})^2\right) \frac{C_0 \|S_{\nu, u}\| + |\Gamma_{z_0, 0}(S_{\nu, u})|}{g_\star^2} du \\ &\leq \tilde{\mathbb{E}} \mathbb{I}_{\tilde{B}_d} \zeta \frac{1}{2b} \int_{-b}^b a \frac{C_0 \|S_{\nu, u}\| + |\Gamma_{z_0, 0}(S_{\nu, u})|}{g_\star^2} du. \end{aligned}$$

This allows us to say that

$$\liminf_{n \rightarrow \infty} J_n(a, b) = \liminf_{n \rightarrow \infty} \tilde{\mathbb{E}} \mathbb{I}_{\tilde{B}_d} \zeta \frac{1}{2b} \int_{-b}^b \frac{v_a(u - t_n)}{g(z_0, 0)} \exp\left(-\frac{\sigma_\nu^2}{2}(u - \tilde{\xi})^2\right) du.$$

But

$$\begin{aligned} &\tilde{\mathbb{E}} \mathbb{I}_{\tilde{B}_d} \zeta \frac{1}{2b} \int_{-b}^b \frac{v_a(u - t_n)}{g(z_0, 0)} \exp\left(-\frac{\sigma_\nu^2}{2}(u - \tilde{\xi})^2\right) du \\ &= \tilde{\mathbb{E}} \mathbb{I}_{\tilde{B}_d} \zeta \frac{1}{2b} \int_{-b-\tilde{\xi}}^{b-\tilde{\xi}} \frac{v_a(t - t_n + \tilde{\xi})}{g(z_0, 0)} \exp\left(-\frac{\sigma_\nu^2}{2}t^2\right) dt \\ &\geq \tilde{\mathbb{E}} \mathbb{I}_{\tilde{B}_d} \zeta \frac{1}{2b} \int_{-\sqrt{b}}^{\sqrt{b}} \frac{v_a(t - t_n + \tilde{\xi})}{g(z_0, 0)} \exp\left(-\frac{\sigma_\nu^2}{2}t^2\right) dt \\ &\geq \tilde{\mathbb{E}} \mathbb{I}_{\tilde{B}_d} \zeta \frac{1}{2b} \int_{-\sqrt{b}}^{\sqrt{b}} \frac{v_a(t)}{g(z_0, 0)} \exp\left(-\frac{\sigma_\nu^2}{2}t^2\right) dt, \end{aligned}$$

this last inequality holds thanks to Anderson's lemma (see Ibragimov and Hasminskii (1981), Chapter II, Lemma 10.1 and Corollary 10.2).

Eventually using the fact that  $\tilde{\mathbb{E}}\mathbb{I}_{\tilde{B}_d}\zeta = \frac{2\sigma_\nu(b - \sqrt{b})}{\sqrt{2\pi}}$  and  $\sigma_\nu^2 \xrightarrow{\nu \rightarrow 0} \frac{2}{g^2(z_0, 0)}$ , we obtain sequentially:

$$\begin{aligned} \liminf_{n \rightarrow \infty} J_n(a, b) &\geq \frac{\sigma_\nu}{\sqrt{2\pi}} \frac{b - \sqrt{b}}{b} \int_{-\sqrt{b}}^{\sqrt{b}} \frac{v_a(t)}{g(z_0, 0)} \exp\left(-\frac{\sigma_\nu^2}{2} t^2\right) dt, \\ \liminf_{a \rightarrow \infty} \liminf_{n \rightarrow \infty} J_n(a, b) &\geq \frac{\sigma_\nu}{\sqrt{2\pi}} \frac{b - \sqrt{b}}{b} \int_{-\sqrt{b}}^{\sqrt{b}} \frac{|t|}{g(z_0, 0)} \exp\left(-\frac{\sigma_\nu^2}{2} t^2\right) dt, \\ \liminf_{b \rightarrow \infty} \liminf_{a \rightarrow \infty} \liminf_{n \rightarrow \infty} J_n(a, b) &\geq \frac{\sigma_\nu}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{|t|}{g(z_0, 0)} \exp\left(-\frac{\sigma_\nu^2}{2} t^2\right) dt = \frac{\tilde{\mathbb{E}}|\xi|}{\sigma_\nu g(z_0, 0)}, \\ \liminf_{\nu \rightarrow 0} \liminf_{b \rightarrow \infty} \liminf_{a \rightarrow \infty} \liminf_{n \rightarrow \infty} J_n(a, b) &\geq \frac{\tilde{\mathbb{E}}|\xi|}{\sqrt{2}}. \end{aligned}$$

Then the proof of theorem 3.1 is completed.  $\square$

## 5.2 Proof of theorem 3.2

We begin by rewriting the kernel estimator as  $\hat{S}_n(z_0) - S(z_0) = B_n + \frac{1}{\sqrt{q_n}}\zeta_n$  with

$$B_n = \frac{1}{q_n} \sum_{k=1}^n Q\left(\frac{x_k - z_0}{h}\right) (S(x_k) - S(z_0)) \quad (15)$$

$$\zeta_n = \frac{1}{\sqrt{q_n}} \sum_{k=1}^n Q\left(\frac{x_k - z_0}{h}\right) g(x_k, S) \xi_k. \quad (16)$$

First we take a look at the term  $\frac{\zeta_n}{\sqrt{q_n}}$ . By (16),  $\zeta_n$  is a Gaussian random variable  $\mathcal{N}(0, \sigma_n^2(S))$  where  $\sigma_n^2(S) = \frac{1}{q_n} \sum_{k=1}^n Q\left(\frac{x_k - z_0}{h}\right) g^2(x_k, S)$ . Notice that the variance  $\sigma_n^2(S)$  satisfies (see lemma A.3)

$$\sigma_n^2(S) \xrightarrow{n \rightarrow \infty} g^2(z_0, S).$$

If  $\xi \sim \mathcal{N}(0, 1)$ , one has

$$\begin{aligned} \sup_{S \in \mathcal{U}_{z_0, \delta, n}} \frac{1}{g(z_0, S)} \mathbb{E}_S \left| \frac{\varphi_n}{\sqrt{q_n}} \zeta_n \right| &= \frac{\varphi_n}{\sqrt{q_n}} \sup_{S \in \mathcal{U}_{z_0, \delta, n}} \frac{\mathbb{E}_S |\zeta_n|}{g(z_0, S)} \frac{\sigma_n(S)}{\sigma_n(S)} = \frac{\varphi_n}{\sqrt{q_n}} \mathbb{E}|\xi| \sup_{S \in \mathcal{U}_{z_0, \delta, n}} \frac{\sigma_n(S)}{g(z_0, S)} \\ &\leq \frac{\varphi_n}{\sqrt{q_n}} \frac{\mathbb{E}|\xi|}{g_\star} \left( \sup_{S \in \mathcal{U}_{z_0, \delta, n}} |\sigma_n(S) - g(z_0, S)| + g_\star \right). \end{aligned}$$

According to the hypothesis (4) we obtain

$$\limsup_{n \rightarrow \infty} \sup_{S \in \mathcal{U}_{z_0, \delta, n}} \mathbb{E}_S \frac{\varphi_n}{g(z_0, S)} \frac{|\zeta_n|}{\sqrt{q_n}} \leq \frac{\mathbb{E}|\xi|}{\sqrt{2}},$$

since we can easily see that  $\frac{q_n}{\varphi_n^2} = \frac{q_n}{nh} \xrightarrow{n \rightarrow \infty} 2$ .

Denote  $u_k = \frac{x_k - z_0}{h}$ ,  $\Delta u_k = \frac{1}{nh}$  and rewrite (15) as

$$\begin{aligned} B_n &= \frac{1}{q_n} \sum_{k=1}^n Q\left(\frac{x_k - z_0}{h}\right) (S(x_k) - S(z_0)) \\ &= \frac{\varphi_n^2}{q_n} \sum_{k=1}^n Q(u_k) (S(z_0 + hu_k) - S(z_0)) \Delta u_k \\ &= \frac{\varphi_n^2}{q_n} \int_{-1}^1 (S(z_0 + hu) - S(z_0)) du + \frac{\varphi_n^2}{q_n} R_n \end{aligned}$$

with

$$\begin{aligned} R_n &= \sum_{k=1}^n Q(u_k) (S(z_0 + hu_k) - S(z_0)) \Delta u_k - \int_{-1}^1 (S(z_0 + hu) - S(z_0)) du \\ &= \sum_{k=k_*}^{k^*} \int_{u_{k-1}}^{u_k} (S(z_0 + hu_k) - S(z_0 + hu)) du \\ &\quad - \int_{u_{k_*}}^1 (S(z_0 + hu) - S(z_0)) du + \int_{u_{k_*-1}}^{-1} (S(z_0 + hu) - S(z_0)) du, \end{aligned}$$

where  $k^* = [n(z_0 + h)]$  et  $k_* = [n(z_0 - h)] + 1$  and  $[a]$  is the integer part of a number  $a$ .

We can bound  $R_n$  as follows:

$$\begin{aligned} |R_n| &\leq \sum_{k=k_*}^{k^*} \int_{u_{k-1}}^{u_k} h(u_k - u) \delta^{-1} du + \int_{u_{k_*}}^1 h \delta^{-1} u du + \int_{u_{k_*-1}}^{-1} h \delta^{-1} |u| du \\ &\leq h \delta^{-1} \left( \sum_{k=k_*}^{k^*} (u_k - u_{k-1}) \frac{1}{nh} + (1 - u_{k_*}) + 2(-1 - u_{k_*-1}) \right) \\ &\leq h \delta^{-1} \left( \frac{1}{nh} (u_{k^*} - u_{k_*}) + \frac{1}{nh} + \frac{2}{nh} \right) \\ &\leq \frac{6\delta^{-1}}{n}. \end{aligned}$$

Hence

$$\limsup_{n \rightarrow \infty} \sup_{S \in \mathcal{U}_{z_0, \delta, n}} \mathbb{E}_{S \varphi_n} \left| \frac{\varphi_n^2}{q_n} R_n \right| = 0.$$

With regard to the term  $\frac{\varphi_n^2}{q_n} \int_{-1}^1 (S(z_0 + hu) - S(z_0)) du$  one has

$$\left| \frac{\varphi_n^2}{q_n} \int_{-1}^1 (S(z_0 + hu) - S(z_0)) du \right| \leq \frac{\varphi_n^2}{q_n} \delta n^{\frac{-\beta}{2\beta+1}} = \delta \frac{\varphi_n}{q_n}.$$

Then using the definition of  $\mathcal{U}_{z_0, \delta, n}$  we get

$$\limsup_{n \rightarrow \infty} \sup_{S \in \mathcal{U}_{z_0, \delta, n}} \mathbb{E}_S \varphi_n \left| \frac{\varphi_n^2}{q_n} \int_{-1}^1 (S(z_0 + hu) - S(z_0)) du \right| \leq \limsup_{n \rightarrow \infty} \frac{\varphi_n^2}{q_n} \delta \leq \frac{\delta}{2}.$$

Finally limiting  $\delta \rightarrow 0$  yields

$$\limsup_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \mathcal{R}_{z_0, \delta, n}(\hat{S}_n(z_0)) \leq \frac{\mathbb{E}|\xi|}{\sqrt{2}}.$$

□

### 5.3 Proof of theorem 4.1

This is a consequence of the theorem 3.1 which gives the sharp lower bound in the case of Gaussian errors having expectation zero and unknown variance which depends on the design point and the regression function. The corresponding risk  $\mathcal{R}_{z_0, \delta, n}$  is less than our risk  $\tilde{\mathcal{R}}_{z_0, \delta, n}$  because the density of the standard Gaussian random variable belongs to  $\mathcal{P}_{\epsilon, L}$ . The inequality in theorem 4.1 is then proved. □

### 5.4 Proof of theorem 4.2

By writing  $\hat{S}_n(z_0) - S(z_0) = B_n + \zeta_n / \sqrt{q_n}$ , with  $B_n$  and  $\zeta_n$  defined by (15) and (16), we remark that  $B_n$  does not depend on the distributions of the random variables  $\xi_k$ . However with the same model where  $(\xi_k)$  are standard Gaussian random variables, 3.1 shows that for any  $\delta \in ]0; 1[$ ,

$$\limsup_{n \rightarrow \infty} \sup_{S \in \mathcal{U}_{z_0, \delta, n}} \varphi_n |B_n| \leq \delta/2.$$

Hence it suffices to prove that

$$\lim_{n \rightarrow \infty} \sup_{p \in \mathcal{P}_{\epsilon, L}} \sup_{S \in \mathcal{U}_{z_0, \delta, n}} \left| \frac{\mathbb{E}_S |\zeta_n|}{g(z_0, S)} - \mathbb{E}|\eta| \right| = 0,$$

because one would then have

$$\lim_{n \rightarrow \infty} \sup_{p \in \mathcal{P}_{\epsilon, L}} \sup_{S \in \mathcal{U}_{z_0, \delta, n}} \mathbb{E}_S \frac{\varphi_n}{\sqrt{q_n}} \frac{|\zeta_n|}{g(z_0, S)} = \frac{\mathbb{E}|\eta|}{\sqrt{2}},$$

with  $\eta \sim \mathcal{N}(0, 1)$ .

Denote  $\tilde{\zeta}_n = \zeta_n / g(z_0, S) = \sum_{k=1}^n u_k$ , where  $u_k = \frac{1}{\sqrt{q_n}} Q\left(\frac{x_k - z_0}{h}\right) \frac{g(x_k, S)}{g(z_0, S)} \xi_k$ , and rewrite

$\frac{g(x_k, S)}{g(z_0, S)} \xi_k = \xi'_k + \xi''_k$ , where

$$\begin{aligned} \xi'_k &= \frac{g(x_k, S)}{g(z_0, S)} \xi_k \mathbb{I}_{|\xi_k| \leq q_n^{1/4}} - \frac{g(x_k, S)}{g(z_0, S)} \mathbb{E} \left( \xi_1 \mathbb{I}_{|\xi_1| \leq q_n^{1/4}} \right), \\ \xi''_k &= \frac{g(x_k, S)}{g(z_0, S)} \xi_k \mathbb{I}_{|\xi_k| > q_n^{1/4}} - \frac{g(x_k, S)}{g(z_0, S)} \mathbb{E} \left( \xi_1 \mathbb{I}_{|\xi_1| > q_n^{1/4}} \right). \end{aligned}$$

Introducing the notations  $u'_k = \frac{1}{\sqrt{q_n}} Q\left(\frac{x_k - z_0}{h}\right) \xi'_k$  and  $u''_k = \frac{1}{\sqrt{q_n}} Q\left(\frac{x_k - z_0}{h}\right) \xi''_k$ , one gets

$$\tilde{\zeta}_n = \tilde{\zeta}'_n + \tilde{\zeta}''_n = \sum_{k=1}^n u'_k + \sum_{k=1}^n u''_k.$$

Moreover,  $(u'_k)_{k \geq 1}$  is a martingale difference and for all  $k \geq 2$ , we have:

$$\begin{aligned} \mathbb{E}_S((u'_k)^2 | \mathcal{F}_{k-1}) &= \frac{1}{q_n} Q\left(\frac{x_k - z_0}{h}\right) \mathbb{E}_S((\xi'_k)^2 | \mathcal{F}_{k-1}) \\ &= \frac{1}{q_n} Q\left(\frac{x_k - z_0}{h}\right) \frac{g^2(x_k, S)}{g^2(z_0, S)} \text{Var}\left(\xi_1 \mathbb{I}_{|\xi_1| \leq q_n^{1/4}}\right). \end{aligned}$$

Write

$$\sum_{i=1}^n \mathbb{E}_S((u'_i)^2 | \mathcal{F}_{i-1}) = \frac{1}{q_n} \sum_{i=1}^n Q\left(\frac{x_i - z_0}{h}\right) \frac{g^2(x_i, S)}{g^2(z_0, S)} \text{Var}\left(\xi_1 \mathbb{I}_{|\xi_1| \leq q_n^{1/4}}\right) = \frac{G_n(S)}{q_n} a_n,$$

where  $G_n(S) = \sum_{i=1}^n Q\left(\frac{x_i - z_0}{h}\right) \frac{g^2(x_i, S)}{g^2(z_0, S)}$  and  $a_n = \text{Var}\left(\xi_1 \mathbb{I}_{|\xi_1| \leq q_n^{1/4}}\right)$ .

Hence, if we denote  $r_n(S) = \frac{G_n(S)}{q_n} a_n$  and  $\tau_n = \inf\left\{k : \sum_{i=1}^k \mathbb{E}_S(u_i'^2 | \mathcal{F}_{i-1}) \geq r_n(S)\right\}$ , we obtain

$$\tau_n = \inf\left\{k : \sum_{i=1}^k Q\left(\frac{x_i - z_0}{h}\right) \geq q_n\right\}.$$

Indeed,

$$\begin{aligned} \sum_{i=1}^k \mathbb{E}_S(u_i'^2 | \mathcal{F}_{i-1}) &\geq \frac{G_n(S)}{q_n} a_n \\ \Leftrightarrow \sum_{i=1}^k Q\left(\frac{x_i - z_0}{h}\right) \frac{g^2(x_i, S)}{g^2(z_0, S)} a_n &\geq G_n(S) a_n \\ \Leftrightarrow \sum_{i=1}^k Q\left(\frac{x_i - z_0}{h}\right) \frac{g^2(x_i, S)}{g^2(z_0, S)} &\geq \sum_{i=1}^n Q\left(\frac{x_i - z_0}{h}\right) \frac{g^2(x_i, S)}{g^2(z_0, S)}. \end{aligned}$$

As we have  $Q\left(\frac{x_k - z_0}{h}\right) = 0$  if  $n \geq k > \tau_n$ , we can write that  $\tilde{\zeta}'_n = \sum_{k=1}^{\tau_n} u'_k$ .

And one has :  $|u'_k| = \frac{1}{\sqrt{q_n}} Q\left(\frac{x_k - z_0}{h}\right) |\xi'_k| \leq \frac{1}{\sqrt{q_n}} \frac{g^*}{g_*} 2q_n^{1/4} = 2 \frac{g^*}{g_*} q_n^{-1/4}$ .

By the definition of the set  $\mathcal{P}_{\epsilon, L}$ , we get :  $\lim_{a \rightarrow +\infty} \sup_{p \in \mathcal{P}_{\epsilon, L}} \int_{-\infty}^{+\infty} x^2 \mathbb{I}_{|x| > a} p(x) dx = 0$ .

Indeed, denoting  $K_p(a) = \int_{-\infty}^{+\infty} x^2 \mathbb{I}_{|x| > a} p(x) dx = \int_{-\infty}^{+\infty} |x|^{2+\epsilon} |x|^{-\epsilon} \mathbb{I}_{|x| > a} p(x) dx$ , and taking into account that if  $|x| > a > 0$ , then  $|x|^{-\epsilon} < a^{-\epsilon}$ , we obtain that  $\sup_{p \in \mathcal{P}_{\epsilon, L}} K_p(a) \leq M a^{-\epsilon}$ . Hence

$$\lim_{a \rightarrow +\infty} \sup_{p \in \mathcal{P}_{\epsilon, L}} K_p(a) = 0.$$

Let us show that  $a_n \rightarrow 1$  uniformly in  $p \in \mathcal{P}_{\epsilon, L}$  and in  $S \in \mathcal{U}_{z_0, \delta, n}$ .  
 Firstly we have:

$$\begin{aligned} |a_n - 1| &= |Var(\xi_1 \mathbb{I}_{|\xi_1| \leq q_n^{1/4}}) - 1| \\ &= |\mathbb{E}(\xi_1^2 \mathbb{I}_{|\xi_1| \leq q_n^{1/4}}) - \mathbb{E}(\xi_1 \mathbb{I}_{|\xi_1| \leq q_n^{1/4}})^2 - 1| \\ &\leq \left| \int_{-q_n^{1/4}}^{q_n^{1/4}} x^2 p(x) dx - 1 \right| + \left| \int_{-q_n^{1/4}}^{q_n^{1/4}} x p(x) dx \right|^2. \end{aligned}$$

The Cauchy-Schwarz inequality brings us:

$$\begin{aligned} \left| \int_{-q_n^{1/4}}^{q_n^{1/4}} x p(x) dx \right|^2 &= \left| \int_{-\infty}^{+\infty} x p(x) \mathbb{I}_{|x| > q_n^{1/4}} dx \right|^2 \\ &\leq \int_{-\infty}^{+\infty} x^2 p(x) \mathbb{I}_{|x| > q_n^{1/4}} dx \int_{-\infty}^{+\infty} p(x) dx \\ &\leq K_p(q_n^{1/4}). \end{aligned}$$

From here it follows that

$$\sup_{p \in \mathcal{P}_{\epsilon, L}} \sup_{S \in \mathcal{U}_{z_0, \delta, n}} |a_n - 1| \leq 2 \sup_{p \in \mathcal{P}_{\epsilon, L}} K_p(q_n^{1/4}),$$

so the left term goes to zero as  $n$  goes to infinity.

Using the assumption (4) and the inequality

$$|r_n(S) - 1| \leq \left| \frac{G_n(S)}{q_n} - 1 \right| + \frac{G_n(S)}{q_n} |a_n - 1|$$

we get on the one hand the convergence of  $r_n(S)$  to 1 uniformly in  $p$  and in  $S$ .

On the other hand one has  $\mathbb{E}(\tilde{\zeta}_n''^2) \rightarrow 0$  uniformly in  $p$  and in  $S$ . Indeed:

$$\begin{aligned}
\mathbb{E}(\tilde{\zeta}_n''^2) &= \mathbb{E} \left( \left( \sum_{k=1}^n u_k'' \right)^2 \right) = \text{Var} \left( \sum_{k=1}^n u_k'' \right) = \sum_{k=1}^n \text{Var}(u_k'') \\
&= \sum_{k=1}^n \text{Var}(\xi_k'') \frac{1}{q_n} Q \left( \frac{x_k - z_0}{h} \right) \\
&= \frac{1}{q_n} \sum_{k=1}^n Q \left( \frac{x_k - z_0}{h} \right) \frac{g^2(x_k, S)}{g^2(z_0, S)} \text{Var} \left( \xi_k \mathbb{I}_{|\xi_k| \geq q_n^{1/4}} \right) \\
&= \frac{1}{q_n} \sum_{k=1}^n Q \left( \frac{x_k - z_0}{h} \right) \frac{g^2(x_k, S)}{g^2(z_0, S)} \text{Var} \left( \xi_1 \mathbb{I}_{|\xi_1| \geq q_n^{1/4}} \right) \\
&= \frac{1}{q_n} \sum_{k=1}^n Q \left( \frac{x_k - z_0}{h} \right) \frac{g^2(x_k, S)}{g^2(z_0, S)} \mathbb{E} \left( \xi_1^2 \mathbb{I}_{|\xi_1| \geq q_n^{1/4}} \right) \\
&= \frac{1}{q_n} \sum_{k=1}^n Q \left( \frac{x_k - z_0}{h} \right) \frac{g^2(x_k, S)}{g^2(z_0, S)} \int_{-\infty}^{+\infty} x^2 p(x) \mathbb{I}_{|x| \geq q_n^{1/4}} dx \\
&= \frac{G_n(S)}{q_n} K_p(q_n^{1/4}).
\end{aligned}$$

As a consequence we have:

$$\sup_{p \in \mathcal{P}_{\epsilon, L}} \sup_{S \in \mathcal{U}_{z_0, \delta, n}} \mathbb{E}_S(\tilde{\zeta}_n''^2) \rightarrow 0.$$

Applying the lemma A.4 shows the convergence in distribution of  $\zeta_n'$  to  $\mathcal{N}(0, 1)$  uniformly in  $p \in \mathcal{P}_{\epsilon, L}$  and in  $S \in \mathcal{U}_{z_0, \delta, n}$  because the function  $\rho$  in the lemma A.4 does not depend on the law of the martingale difference. In fact, if  $\Phi$  denotes the standard Gaussian distribution function, one has:

$$\begin{aligned}
&\sup_{p \in \mathcal{P}_{\epsilon, L}} \sup_{S \in \mathcal{U}_{z_0, \delta, n}} \sup_{x \in \mathbb{R}} \left| \mathbb{P} \left( \sum_{k=1}^{\tau_n} u_k' \leq x \right) - \Phi(x) \right| \\
&\leq \sup_{p \in \mathcal{P}_{\epsilon, L}} \sup_{S \in \mathcal{U}_{z_0, \delta, n}} \sup_{x \in \mathbb{R}} \left| \mathbb{P} \left( \sum_{k=1}^{\tau_n} u_k' \leq x \right) - \Phi(x/\sqrt{r_n(S)}) \right| \\
&+ \sup_{p \in \mathcal{P}_{\epsilon, L}} \sup_{S \in \mathcal{U}_{z_0, \delta, n}} \sup_{x \in \mathbb{R}} \left| \Phi(x) - \Phi(x/\sqrt{r_n(S)}) \right|.
\end{aligned}$$

The second term of the right member of this inequality tends toward zero because  $r_n(S) \rightarrow 1$  uniformly in  $p$  and in  $S$  and because  $\Phi$  is uniformly continuous on  $\mathbb{R}$ .

As we have  $\mathbb{E}_S|\tilde{\zeta}_n''| \leq \sqrt{\mathbb{E}_S(\tilde{\zeta}_n''^2)}$ , we come to  $\sup_{p \in \mathcal{P}_{\epsilon, L}} \sup_{S \in \mathcal{U}_{z_0, \delta, n}} \mathbb{E}_S|\tilde{\zeta}_n''| \rightarrow 0$ . By using Markov's inequality, we show that  $(\tilde{\zeta}_n'')$  tends toward 0 in probability uniformly in  $p$  and in  $S$ .

Now we will show that  $\tilde{\zeta}_n = \tilde{\zeta}_n' + \tilde{\zeta}_n''$  converges in distribution to  $\eta \sim \mathcal{N}(0, 1)$  uniformly in  $p$  and in  $S$ . Let  $x \in \mathbb{R}$ ,  $\kappa > 0$  et  $\gamma > 0$ . As  $(\tilde{\zeta}_n')$  converges in distribution to  $\eta$  uniformly in  $p$  and



in  $S$ , there exists an integer  $N_1$  such that for all  $n \geq N_1$ ,

$$\begin{aligned} \sup_{p \in \mathcal{P}_{\epsilon, L}} \sup_{S \in \mathcal{U}_{z_0, \delta, n}} \left| \mathbb{P}_S(\tilde{\zeta}'_n \leq x) - \mathbb{P}(\eta \leq x) \right| &\leq \kappa, \\ \sup_{p \in \mathcal{P}_{\epsilon, L}} \sup_{S \in \mathcal{U}_{z_0, \delta, n}} \left| \mathbb{P}_S(\tilde{\zeta}'_n \leq x - \gamma) - \mathbb{P}(\eta \leq x - \gamma) \right| &\leq \kappa, \\ \sup_{p \in \mathcal{P}_{\epsilon, L}} \sup_{S \in \mathcal{U}_{z_0, \delta, n}} \left| \mathbb{P}_S(\tilde{\zeta}'_n \leq x + \gamma) - \mathbb{P}(\eta \leq x + \gamma) \right| &\leq \kappa. \end{aligned}$$

As  $(\tilde{\zeta}''_n)$  converges in probability to zero uniformly in  $p$  and in  $S$ , there exists an integer  $N_2$  such that for all  $n \geq N_2$ ,

$$\sup_{p \in \mathcal{P}_{\epsilon, L}} \sup_{S \in \mathcal{U}_{z_0, \delta, n}} \mathbb{P}_S(|\tilde{\zeta}''_n| > \gamma) \leq \kappa.$$

From here it follows that, if  $n \geq \max(N_1, N_2)$  we have:

$$\begin{aligned} &\sup_{p \in \mathcal{P}_{\epsilon, L}} \sup_{S \in \mathcal{U}_{z_0, \delta, n}} \left| \mathbb{P}_S(\tilde{\zeta}_n \leq x) - \mathbb{P}(\eta \leq x) \right| \\ &= \sup_{p \in \mathcal{P}_{\epsilon, L}} \sup_{S \in \mathcal{U}_{z_0, \delta, n}} \left| \mathbb{P}_S(\tilde{\zeta}'_n + \tilde{\zeta}''_n \leq x) - \mathbb{P}(\eta \leq x) \right| \\ &\leq \sup_{p \in \mathcal{P}_{\epsilon, L}} \sup_{S \in \mathcal{U}_{z_0, \delta, n}} \left| \mathbb{P}_S(\tilde{\zeta}'_n \leq x) - \mathbb{P}(\eta \leq x) \right| + \sup_{p \in \mathcal{P}_{\epsilon, L}} \sup_{S \in \mathcal{U}_{z_0, \delta, n}} \left| \mathbb{P}_S(\tilde{\zeta}'_n \leq x) - \mathbb{P}_S(\tilde{\zeta}'_n + \tilde{\zeta}''_n \leq x) \right| \\ &\leq \kappa + \sup_{p \in \mathcal{P}_{\epsilon, L}} \sup_{S \in \mathcal{U}_{z_0, \delta, n}} \left| \mathbb{P}_S(\tilde{\zeta}'_n \leq x) - \mathbb{P}_S(\tilde{\zeta}'_n + \tilde{\zeta}''_n \leq x, |\tilde{\zeta}''_n| \leq \gamma) \right| \\ &+ \sup_{p \in \mathcal{P}_{\epsilon, L}} \sup_{S \in \mathcal{U}_{z_0, \delta, n}} \mathbb{P}_S(\tilde{\zeta}'_n + \tilde{\zeta}''_n \leq x, |\tilde{\zeta}''_n| > \gamma) \\ &\leq 2\kappa + \sup_{p \in \mathcal{P}_{\epsilon, L}} \sup_{S \in \mathcal{U}_{z_0, \delta, n}} \left| \mathbb{P}_S(\tilde{\zeta}'_n \leq x) - \mathbb{P}_S(\tilde{\zeta}'_n + \tilde{\zeta}''_n \leq x, |\tilde{\zeta}''_n| \leq \gamma) \right|. \end{aligned}$$

It is easy to see that we can write the following inequalities:

$$\begin{aligned} &\mathbb{P}_S(\tilde{\zeta}'_n \leq x) - \mathbb{P}_S(\tilde{\zeta}'_n \leq x - \gamma) - \mathbb{P}_S(\tilde{\zeta}'_n \leq x - \gamma, |\tilde{\zeta}''_n| > \gamma) \\ &\leq \mathbb{P}_S(\tilde{\zeta}'_n \leq x - \gamma) - \mathbb{P}_S(\tilde{\zeta}'_n \leq x) - \mathbb{P}_S(\tilde{\zeta}'_n \leq x - \gamma, |\tilde{\zeta}''_n| > \gamma) \\ &= \mathbb{P}_S(\tilde{\zeta}'_n \leq x - \gamma, |\tilde{\zeta}''_n| \leq \gamma) - \mathbb{P}_S(\tilde{\zeta}'_n \leq x) \\ &\leq \mathbb{P}_S(\tilde{\zeta}'_n + \tilde{\zeta}''_n \leq x, |\tilde{\zeta}''_n| \leq \gamma) - \mathbb{P}_S(\tilde{\zeta}'_n \leq x) \\ &\leq \mathbb{P}_S(\tilde{\zeta}'_n \leq x + \gamma) - \mathbb{P}_S(\tilde{\zeta}'_n \leq x). \end{aligned}$$

Remark that the lower term in these inequalities is negative while the greater is positive. In order to obtain the convergence to zero of

$$\left| \mathbb{P}_S(\tilde{\zeta}'_n + \tilde{\zeta}''_n \leq x, |\tilde{\zeta}''_n| \leq \gamma) - \mathbb{P}_S(\tilde{\zeta}'_n \leq x) \right|,$$

uniformly in  $p$  and in  $S$  it suffices to prove that the lower and the greater terms in these inequalities converges to zero uniformly in  $p$  and in  $S$ . To that purpose, we have on the one hand for  $n \geq \max(N_1, N_2)$ :

$$\sup_{p \in \mathcal{P}_{\epsilon, L}} \sup_{S \in \mathcal{U}_{z_0, \delta, n}} \mathbb{P}_S(\tilde{\zeta}'_n \leq x - \gamma, |\tilde{\zeta}''_n| > \gamma) \leq \sup_{p \in \mathcal{P}_{\epsilon, L}} \sup_{S \in \mathcal{U}_{z_0, \delta, n}} \mathbb{P}_S(|\tilde{\zeta}''_n| > \gamma) \leq \kappa$$

and

$$\begin{aligned}
& \sup_{p \in \mathcal{P}_{\epsilon, M}} \sup_{S \in \mathcal{U}_{z_0, \delta, n}} \left| \mathbb{P}_S(\tilde{\zeta}'_n \leq x - \gamma) - \mathbb{P}_S(\tilde{\zeta}'_n \leq x) \right| \\
& \leq \sup_{p \in \mathcal{P}_{\epsilon, M}} \sup_{S \in \mathcal{U}_{z_0, \delta, n}} \left| \mathbb{P}_S(\tilde{\zeta}'_n \leq x - \gamma) - \mathbb{P}(\eta \leq x - \gamma) \right| + |\mathbb{P}(\eta \leq x - \gamma) - \mathbb{P}(\eta \leq x)| \\
& + \sup_{p \in \mathcal{P}_{\epsilon, M}} \sup_{S \in \mathcal{U}_{z_0, \delta, n}} \left| \mathbb{P}_S(\tilde{\zeta}'_n \leq x) - \mathbb{P}(\eta \leq x) \right| \\
& \leq 3\kappa,
\end{aligned}$$

and on the other hand for  $n \geq N_1$ :

$$\begin{aligned}
& \sup_{p \in \mathcal{P}_{\epsilon, M}} \sup_{S \in \mathcal{U}_{z_0, \delta, n}} \left| \mathbb{P}_S(\tilde{\zeta}'_n \leq x + \gamma) - \mathbb{P}_S(\tilde{\zeta}'_n \leq x) \right| \\
& \leq \sup_{p \in \mathcal{P}_{\epsilon, M}} \sup_{S \in \mathcal{U}_{z_0, \delta, n}} \left| \mathbb{P}_S(\tilde{\zeta}'_n \leq x + \gamma) - \mathbb{P}(\eta \leq x + \gamma) \right| + |\mathbb{P}(\eta \leq x + \gamma) - \mathbb{P}(\eta \leq x)| \\
& + \sup_{p \in \mathcal{P}_{\epsilon, M}} \sup_{S \in \mathcal{U}_{z_0, \delta, n}} \left| \mathbb{P}_S(\tilde{\zeta}'_n \leq x) - \mathbb{P}(\eta \leq x) \right| \\
& \leq 3\kappa.
\end{aligned}$$

Hence for  $n \geq \max(N_1, N_2)$ , one has:

$$\sup_{p \in \mathcal{P}_{\epsilon, L}} \sup_{S \in \mathcal{U}_{z_0, \delta, n}} \left| \mathbb{P}_S(\tilde{\zeta}'_n + \tilde{\zeta}''_n \leq x, |\tilde{\zeta}''_n| \leq \gamma) - \mathbb{P}_S(\tilde{\zeta}'_n \leq x) \right| \leq 4\epsilon.$$

Eventually for  $n \geq \max(N_1, N_2)$ , we get:

$$\sup_{p \in \mathcal{P}_{\epsilon, L}} \sup_{S \in \mathcal{U}_{z_0, \delta, n}} \left| \mathbb{P}_S(\tilde{\zeta}_n \leq x) - \mathbb{P}(\eta \leq x) \right| \leq 6\kappa.$$

It finishes the proof of the convergence in distribution of  $(\tilde{\zeta}_n)$  to  $\eta \sim \mathcal{N}(0, 1)$  uniformly in  $p$  and in  $S$ .

Finally let us show that

$$\lim_{n \rightarrow \infty} \sup_{p \in \mathcal{P}_{\epsilon, L}} \sup_{S \in \mathcal{U}_{z_0, \delta, n}} \left| \mathbb{E}_S |\tilde{\zeta}_n| - \mathbb{E} |\eta| \right| = 0.$$

Let  $\kappa > 0$  and  $A > 0$  such that  $\frac{2}{A} \frac{g^{\star 2}}{g_\star^2} \leq \kappa$  and  $\frac{2}{A} \mathbb{E}(\eta^2) \leq \kappa$ .

Denote  $f_A(x) = |x|$  if  $|x| \leq A$  and  $f_A(x) = A$  if  $|x| > A$ .

Remark that  $\sup_{n \in \mathbb{N}^*} \sup_{p \in \mathcal{P}_{\epsilon, L}} \sup_{S \in \mathcal{U}_{z_0, \delta, n}} \mathbb{E}_S \tilde{\zeta}_n^2 \leq \frac{g^{\star 2}}{g_\star^2}$ .

One has:

$$\left| \mathbb{E}_S |\tilde{\zeta}_n| - \mathbb{E} |\eta| \right| \leq \left| \mathbb{E}_S |\tilde{\zeta}_n| - \mathbb{E}_S f_A(\tilde{\zeta}_n) \right| + \left| \mathbb{E}_S f_A(\tilde{\zeta}_n) - \mathbb{E} f_A(\eta) \right| + \left| \mathbb{E} f_A(\eta) - \mathbb{E} |\eta| \right|.$$

However, since  $f_A$  is continuous and bounded on  $\mathbb{R}$  and since  $(\tilde{\zeta}_n)$  converges in distribution to  $\eta$  uniformly in  $p$  and in  $S$ , there exists an integer  $N$  such that if  $n \geq N$ , then:

$$\sup_{p \in \mathcal{P}_{\epsilon, L}} \sup_{S \in \mathcal{U}_{z_0, \delta, n}} \left| \mathbb{E}_S f_A(\tilde{\zeta}_n) - \mathbb{E} f_A(\eta) \right| \leq \kappa.$$

Moreover by using Cauchy-Schwarz and Bienaymé-Tchebychev inequalities, we obtain:

$$\begin{aligned} & \sup_{p \in \mathcal{P}_{\epsilon, L}} \sup_{S \in \mathcal{U}_{z_0, \delta, n}} \left| \mathbb{E}_S |\tilde{\zeta}_n| - \mathbb{E}_S f_A(\tilde{\zeta}_n) \right| = \sup_{p \in \mathcal{P}_{\epsilon, L}} \sup_{S \in \mathcal{U}_{z_0, \delta, n}} \left| \mathbb{E}_S |\tilde{\zeta}_n| \mathbb{I}_{|\tilde{\zeta}_n| > A} - A \mathbb{E}_S \mathbb{I}_{|\tilde{\zeta}_n| > A} \right| \\ & \leq \sup_{p \in \mathcal{P}_{\epsilon, L}} \sup_{S \in \mathcal{U}_{z_0, \delta, n}} \left( \mathbb{E}_S \tilde{\zeta}_n^2 \right)^{1/2} \sqrt{\mathbb{P}_S(|\tilde{\zeta}_n| > A)} + \sup_{p \in \mathcal{P}_{\epsilon, M}} \sup_{S \in \mathcal{U}_{z_0, \delta, n}} A \mathbb{P}_S(|\tilde{\zeta}_n| > A) \\ & \leq \frac{g^*}{g_*} \sup_{p \in \mathcal{P}_{\epsilon, L}} \sup_{S \in \mathcal{U}_{z_0, \delta, n}} \sqrt{\frac{\mathbb{E}_S \tilde{\zeta}_n^2}{A^2}} + A \sup_{p \in \mathcal{P}_{\epsilon, M}} \sup_{S \in \mathcal{U}_{z_0, \delta, n}} \frac{\mathbb{E}_S \tilde{\zeta}_n^2}{A^2} \\ & \leq \frac{2}{A} \frac{g^{*2}}{g_*^2} \\ & \leq \kappa. \end{aligned}$$

In the same way we show that

$$\begin{aligned} |\mathbb{E} f_A(\eta) - \mathbb{E} |\eta|| &= \left| \mathbb{E} A \mathbb{I}_{|\eta| > A} - \mathbb{E} |\eta| \mathbb{I}_{|\eta| > A} \right| \\ &\leq A \mathbb{P}(|\eta| > A) + \mathbb{E} |\eta| \mathbb{I}_{|\eta| > A} \\ &\leq A \frac{\mathbb{E} \eta^2}{A^2} + (\mathbb{E} \eta^2)^{1/2} \sqrt{\mathbb{P}(|\eta| > A)} \\ &\leq \frac{2}{A} \mathbb{E} \eta^2 \\ &\leq \kappa. \end{aligned}$$

Now we have shown that for  $n \geq N$  :

$$\sup_{p \in \mathcal{P}_{\epsilon, L}} \sup_{S \in \mathcal{U}_{z_0, \delta, n}} \left| \mathbb{E}_S |\tilde{\zeta}_n| - \mathbb{E} |\eta| \right| \leq 3\kappa,$$

and that completes the proof of theorem 4.2.

## A Appendix

**Lemma A.1** *The family of functions  $\mathcal{S} = \{S_\nu, \nu \in ]0; \frac{1}{4} [ \}$  satisfies*

$$\mathcal{S} \subset \liminf_{n \rightarrow \infty} \mathcal{U}_{z_0, \delta, n} = \bigcup_{k \geq 1} \bigcap_{n \geq k} \mathcal{U}_{z_0, \delta, n}.$$

PROOF: First remark that

$$\int_{-1}^1 (S_\nu(z_0 + uh) - S_\nu(z_0)) du = \frac{1}{\varphi_n} \int_{-1}^1 (V_\nu(u) - V_\nu(0)) du = \frac{2}{\varphi_n} - \frac{2}{\varphi_n} = 0.$$

Moreover one has

$$|S'_\nu(x)| = \frac{1}{\varphi_n h} \left| V'_\nu \left( \frac{x - z_0}{h} \right) \right| \leq \frac{2n^{\frac{-\beta+1}{2\beta+1}}}{\nu} \int_{-1}^1 |l'(u)| du$$

since

$$\left| V'_\nu \left( \frac{x - z_0}{h} \right) \right| = \left| -\nu^{-2} \int_{-\infty}^{+\infty} \tilde{Q}_\nu(u) l' \left( \frac{u - \frac{x-z_0}{h}}{\nu} \right) du \right| \leq \frac{2}{\nu} \int_{-1}^1 |l'(u)| du.$$

For any fixed  $\delta$  in  $]0; 1[$ , if we choose  $n \geq 1$  such that

$$n^{\frac{-\beta+1}{2\beta+1}} \frac{2}{\nu} \int_{-1}^1 |l'(u)| du \leq \delta^{-1} \quad \text{i.e.} \quad n \geq \left( \frac{2\delta}{\nu} \int_{-1}^1 |l'(u)| du \right)^{\frac{2\beta+1}{\beta-1}},$$

then  $S_\nu \in \mathcal{U}_{z_0, \delta, n}$ .

Therefore we have the desired result.  $\square$

**Lemma A.2** *We have the following limit:*

$$\zeta_n^2 \xrightarrow{n \rightarrow \infty} \int_{-1}^1 \frac{V_\nu^2(z)}{g^2(z_0, 0)} dz.$$

PROOF: For sufficiently large  $n$  we have

$$\zeta_n^2 = \frac{1}{nh} \sum_{k=1}^n \frac{V_\nu^2 \left( \frac{x_k - z_0}{h} \right)}{g^2(x_k, S_{\nu, u})} = \frac{1}{h} \int_{z_0-h}^{z_0+h} \frac{V_\nu^2 \left( \frac{x-z_0}{h} \right)}{g^2(x, S_{\nu, u})} \mu_n(dx) = \int_0^1 \frac{V_\nu^2 \left( \frac{x-z_0}{h} \right)}{g^2(x, S_{\nu, u})} \nu_n(dx)$$

with  $\mu_n = \frac{1}{n} \sum_{k=1}^n \delta_{k/n} =$  and  $\nu_n = \frac{\mathbb{I}_{[z_0-h; z_0+h]}}{h} \mu_n$ .

Using hypothesis (6) and (7) to the function  $g$ , we can write for all  $x \in [0; 1]$

$$\begin{aligned} \left| \frac{1}{g^2(x, S_{\nu, u})} - \frac{1}{g^2(x, 0)} \right| &= \left| \frac{g^2(x, 0) - g^2(x, S_{\nu, u})}{g^2(x, S_{\nu, u}) g^2(x, 0)} \right| \\ &\leq \frac{1}{g_\star^4} \left| 2g(x, 0) L_{x,0}(S_{\nu, u}) + (L_{x,0}(S_{\nu, u}))^2 + (\Gamma_{x,0}(S_{\nu, u}))^2 \right. \\ &\quad \left. + 2g(x, 0) \Gamma_{x,0}(S_{\nu, u}) + 2L_{x,0}(S_{\nu, u}) \Gamma_{x,0}(S_{\nu, u}) \right| \\ &\leq \frac{1}{g_\star^4} \left( 2g^\star C_0 \|S_{\nu, u}\| + C_0^2 \|S_{\nu, u}\|^2 + \frac{|\Gamma_{x,0}(S_{\nu, u})|^2}{\|S_{\nu, u}\|^2} \|S_{\nu, u}\|^2 \right. \\ &\quad \left. + 2g^\star \frac{|\Gamma_{x,0}(S_{\nu, u})|}{\|S_{\nu, u}\|} \|S_{\nu, u}\| + 2C_0 \|S_{\nu, u}\| \frac{|\Gamma_{x,0}(S_{\nu, u})|}{\|S_{\nu, u}\|} \|S_{\nu, u}\| \right). \end{aligned}$$

Hence

$$\begin{aligned}
& \left| \int_0^1 \left( \frac{1}{g^2(x, S_{\nu, u})} - \frac{1}{g^2(x, 0)} \right) \nu_n(dx) \right| \\
& \leq \frac{1}{g_*^4} \int_0^1 \nu_n(dx) \left( 2g^* C_0 \| S_{\nu, u} \| + C_0^2 \| S_{\nu, u} \|^2 + \left( \sup_{x \in [0;1]} \frac{|\Gamma_{x,0}(S_{\nu, u})|}{\| S_{\nu, u} \|} \right)^2 \| S_{\nu, u} \|^2 \right. \\
& \quad \left. + 2g^* \left( \sup_{x \in [0;1]} \frac{|\Gamma_{x,0}(S_{\nu, u})|}{\| S_{\nu, u} \|} \right) \| S_{\nu, u} \| + 2C_0 \| S_{\nu, u} \| \left( \sup_{x \in [0;1]} \frac{|\Gamma_{x,0}(S_{\nu, u})|}{\| S_{\nu, u} \|} \right) \| S_{\nu, u} \| \right).
\end{aligned}$$

As  $(\nu_n)$  weakly tends to  $2\delta_{z_0}$  when  $n \rightarrow \infty$ , one has

$$\lim_{n \rightarrow \infty} \int_0^1 \nu_n(dx) = 2 \quad \text{et} \quad \lim_{n \rightarrow \infty} \int_0^1 \left( \frac{1}{g^2(x, 0)} - \frac{1}{g^2(z_0, 0)} \right) \nu_n(dx) = 0.$$

Then taking into account hypothesis (8) and because  $\| S_{\nu, u} \|^2$  tends to 0 as  $n \rightarrow \infty$ , we obtain on the one hand

$$\begin{aligned}
& \int_0^1 \left( \frac{1}{g^2(x, S_{\nu, u})} - \frac{1}{g^2(z_0, 0)} \right) \nu_n(dx) \\
& = \int_0^1 \left( \frac{1}{g^2(x, S_{\nu, u})} - \frac{1}{g^2(x, 0)} \right) \nu_n(dx) + \int_0^1 \left( \frac{1}{g^2(x, 0)} - \frac{1}{g^2(z_0, 0)} \right) \nu_n(dx) \xrightarrow{n \rightarrow \infty} 0.
\end{aligned}$$

On the other hand

$$\int_0^1 \frac{V_\nu^2\left(\frac{x-z_0}{h}\right)}{g^2(z_0, 0)} \nu_n(dx) = \frac{1}{nh} \sum_{k=1}^n \frac{V_\nu^2\left(\frac{x_k-z_0}{h}\right)}{g^2(z_0, 0)} \xrightarrow{n \rightarrow \infty} \int_{-1}^1 \frac{V_\nu^2(y)}{g^2(z_0, 0)} dy.$$

Now, if  $V_\nu^*$  denotes the maximum of  $V_\nu^2$  on  $\mathbb{R}$ , one has

$$\begin{aligned}
& \left| \zeta_n^2 - \int_{-1}^1 \frac{V_\nu^2(z)}{g^2(z_0, 0)} dz \right| = \left| \int_0^1 \frac{V_\nu^2\left(\frac{x-z_0}{h}\right)}{g^2(x, S_{\nu, u})} \nu_n(dx) - \int_{-1}^1 \frac{V_\nu^2(z)}{g^2(z_0, 0)} dz \right| \\
& \leq \left| \int_0^1 \frac{V_\nu^2\left(\frac{x-z_0}{h}\right)}{g^2(x, S_{\nu, u})} \nu_n(dx) - \int_0^1 \frac{V_\nu^2\left(\frac{x-z_0}{h}\right)}{g^2(z_0, 0)} \nu_n(dx) \right| + \left| \int_0^1 \frac{V_\nu^2\left(\frac{x-z_0}{h}\right)}{g^2(z_0, 0)} \nu_n(dx) - \int_{-1}^1 \frac{V_\nu^2(z)}{g^2(z_0, 0)} dz \right| \\
& \leq V_\nu^* \int_0^1 \left| \frac{1}{g^2(x, S_{\nu, u})} - \frac{1}{g^2(z_0, 0)} \right| \nu_n(dx) + \left| \int_0^1 \frac{V_\nu^2\left(\frac{x-z_0}{h}\right)}{g^2(z_0, 0)} \nu_n(dx) - \int_{-1}^1 \frac{V_\nu^2(z)}{g^2(z_0, 0)} dz \right|.
\end{aligned}$$

Let  $n$  goes to  $\infty$  and then we have completed the proof of lemma A.2.  $\square$

**Lemma A.3** *The variance  $\sigma_n^2(S)$  of  $\zeta_n$  satisfies*

$$\sigma_n^2(S) \xrightarrow{n \rightarrow \infty} g^2(z_0, S).$$

PROOF: One has

$$\sum_{k=1}^n Q\left(\frac{x_k - z_0}{h}\right) g^2(x_k, S) = n \int_{z_0-h}^{z_0+h} g^2(x, S) \mu_n(dx)$$

with the measure

$$\mu_n = \frac{1}{n} \sum_{k=1}^n \delta_{k/n}.$$

We know that  $(\mu_n)_{n \geq 1}$  weakly tends to the uniform measure on  $[0; 1]$ .

Moreover for sufficiently large  $n$ ,

$$\frac{1}{nh} \sum_{k=1}^n Q\left(\frac{x_k - z_0}{h}\right) g^2(x_k, S) = \int_{z_0-h}^{z_0+h} g^2(x, S) \nu_n(dx)$$

with  $\nu_n = \frac{\mu_n \mathbb{1}_{[z_0-h; z_0+h]}}{h}$ .

Like this  $(\nu_n)_{n \geq 1}$  weakly tends to  $2\delta_{z_0}$ , the Dirac measure at  $z_0$ , when  $n \rightarrow \infty$ .

Then we can conclude as we remember that  $\frac{q_n}{\varphi_n^2} \xrightarrow{n \rightarrow \infty} 2$  and that  $nh = \varphi_n^2$ . □

**Lemma A.4** (cf. Freedman, 1971, pp. 90-91). Let  $\delta \in ]0; 1[$  and  $r > 0$ . Assume that  $(u_k)_{k \geq 0}$  is a martingale difference with respect to the filtration  $(\mathcal{F}_k)_{k \geq 0}$  such that  $|u_k| \leq \delta$  for all  $k$  and

$$\sum_{k=1}^{\infty} \mathbb{E}(u_k^2 | \mathcal{F}_{k-1}) \geq r. \text{ Define } \tau = \inf \left\{ n : \sum_{k=1}^n \mathbb{E}(u_k^2 | \mathcal{F}_{k-1}) \geq r \right\}.$$

Then there exists a function  $\rho : ]0; +\infty[ \rightarrow [0; 2]$  not depending on the distribution of the martingale difference, such that  $\lim_{x \rightarrow 0} \rho(x) = 0$  and

$$\sup_{x \in \mathbb{R}} \left| \mathbb{P} \left( \sum_{k=1}^{\tau} u_k \leq x \right) - \Phi(x/\sqrt{r}) \right| \leq \rho(\delta/\sqrt{r}),$$

where  $\Phi$  is the standard Gaussian distribution function.

## References

- Barron, A., Birgé, L. and Massart, P. (1999), ‘Risk bounds for model selection via penalization’, *Probab. Theory Related Fields* **113**, 301–413.
- Billingsley, P. (1999), *Convergence of probability measures*, Wiley Series in Probability and Statistics: Probability and Statistics, second edn, John Wiley & Sons Inc., New York. A Wiley-Interscience Publication.
- Chatterjee, S. and Olkin, I. (2006), ‘Nonparametric estimation for quadratic regression’, *Statist. Probab. Lett.* **76**, 1156–1163.
- Dahlhaus, R. (1995), ‘Efficient location and regression estimation for long range dependent regression models’, *Ann. Statist.* **23**, 1029–1047.

- Donoho, D. L. (1994a), ‘Asymptotic minimax risk for sup-norm loss: solution via optimal recovery’, *Probab. Theory Related Fields* **99**(2), 145–170.
- Donoho, D. L. (1994b), ‘Statistical estimation and optimal recovery’, *Ann. Statist.* **22**, 238–270.
- Donoho, D. L., Johnstone, I. M., Kerkyacharian, G. and Picard, D. (1995), ‘Wavelet shrinkage: asymptopia?’, *J. Roy. Statist. Soc. Ser. B* **57**, 301–369.
- Donoho, D. and Liu, R. C. (1991), ‘Geometrizing rates of convergence. III’, *Ann. Statist.* **19**, 668–701.
- Efromovich, S. (1999), *Nonparametric Curve Estimation. Methods, Theory and Applications*, Springer, Berlin, New York.
- Efromovich, S. (2007), ‘Sequential design and estimation in heteroscedastic nonparametric regression’, *Sequential Anal.* **26**(1), 3–25.
- Efromovich, S. and Pinsker, M. (1996), ‘Sharp-optimal and adaptive estimation for heteroscedastic nonparametric regression’, *Statist. Sinica* **6**(4), 925–942.
- Freedman, D. (1971), *Brownian Motion and Diffusion*, Holden Day, San Francisco.
- Galtchouk, L. and Pergamenshchikov, S. (2005), ‘Efficient adaptive nonparametric estimation in heteroscedastic regression models’, *Preprint of the Strasbourg Louis Pasteur University, IRMA* pp. available online at <http://hal.archives-ouvertes.fr/hal-00129707/fr/>.
- Galtchouk, L. and Pergamenshchikov, S. (2006), ‘Asymptotically efficient estimates for nonparametric regression models’, *Statist. Probab. Lett.* **76**, 852–860.
- Goldfeld, S. and Quandt, R. (1972), *Nonlinear Methods in Econometrics*, North-Holland, Amsterdam, London.
- Golubev, G. K. (1992), ‘Asymptotically minimax estimation of a regression function in an additive model’, *Problems Inform. Trans.* **28**, 3–15.
- Golubev, G. K. and Nussbaum, M. (1993), ‘Adaptive spline estimates in a non parametric regression model’, *Theory Probab. Appl.* **37**, 521–529.
- Gunst, R. and Mason, R. (1980), *Regression Analysis and its Applications: A Data oriented Approach*, Marcel Dekker, New York.
- Ibragimov, I. and Hasminskii, R. (1981), *Statistical Estimation: Asymptotic Theory*, Springer, Berlin, New York.
- Kalifa, J. and Mallat, S. (2003), ‘Thresholding estimators for linear inverse problems and deconvolutions’, *Ann. Statist.* **31**, 58–109.
- Korostelev, A. (1993), ‘Exact asymptotically minimax estimator for nonparametric regression in uniform norm’, *Theory Probab. Appl.* **38**, 737–743.

- Nemirovski, A. (2000), Topics in non-parametric statistics, in ‘Lectures on probability theory and statistics (Saint-Flour, 1998)’, Vol. 1738 of *Lecture Notes in Math.*, Springer, Berlin, pp. 85–277.
- Nussbaum, M. (1985), ‘Spline smoothing in regression models and asymptotic efficiency in  $L_2$ ’, *Ann. Statist.* **13**, 984–997.
- Sacks, J. and Strawderman, W. (1982), Improvements on linear minimax estimates, in ‘Statistical decision theory and related topics, III, Vol. 2 (West Lafayette, Ind., 1981)’, Academic Press, New York, pp. 287–304.
- Sacks, J. and Ylvisaker, D. (1981), ‘Asymptotically optimum kernels for density estimation at a point’, *Ann. Statist.* **9**(2), 334–346.