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Boundary stabilization of the wave equations by means of a rotated multiplier method.

PIERRE CORNILLEAU*, JEAN-PIERRE LOHÉAC*[†], AXEL OSSES[‡].

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Abstract

The rotated multiplier method is performed in the case of the boundary stabilization of the wave equation by means of a (linear or non-linear) Neumann feedback. This method leads to new geometrical cases concerning the "active" part of the boundary where the feedback is applied. Due to mixed boundary conditions, these cases generate singularities. Under a simple geometrical condition concerning the orientation of the boundary, we obtain a stabilization result in both cases.

Introduction

In this paper we are concerned with the stabilization of the wave equation in a multi-dimensional body by using a feedback law given by some part of the boundary of the spacial domain and some function defined on this part. The problem can be written as follows

$$\begin{cases} u'' - \Delta u = 0 & \text{in } \Omega \times \mathbb{R}_+^*, \\ u = 0 & \text{on } \partial\Omega_D \times \mathbb{R}_+^*, \\ \partial_\nu u = F & \text{on } \partial\Omega_N \times \mathbb{R}_+^*, \\ u(0) = u_0 & \text{in } \Omega, \\ u'(0) = u_1 & \text{in } \Omega, \end{cases}$$

where we denote by u' , u'' , Δu and $\partial_\nu u$ the first time-derivative of u , the second time-derivative of the scalar function u , the standard Laplacian of u and the normal outward derivative of u on $\partial\Omega$, respectively; $(\partial\Omega_D, \partial\Omega_N)$ is a partition of $\partial\Omega$ and F is the feedback function which may depend on the state (u, u') , the position x and time t .

Our purpose here is to choose the feedback law, that is to say the feedback function F and the "active" part of the boundary, $\partial\Omega_N$, so that for every initial data, the energy function

$$E(u, t) = \frac{1}{2} \int_{\Omega} (|u'(t)|^2 + |\nabla u(t)|^2) dx,$$

is decreasing with respect to time t , and vanishes as $t \rightarrow \infty$.

Formally, we can write the time derivative of E as follows

$$E'(u, t) = \int_{\partial\Omega_N} F u' d\sigma,$$

and E is non-increasing if $F u' \leq 0$ on $\partial\Omega_N$.

Thanks to the multiplier method introduced by L.F. Ho [12] in the framework of Hilbert Uniqueness

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Method [11], it can be shown that the energy function is uniformly decreasing as time t tends to ∞ , under suitable assumptions.

In this method, one chooses some vector field $x \mapsto m(x) = x - x_0$ and

$$\partial\Omega_N = \{x \in \partial\Omega / m(x) \cdot \nu(x) > 0\}, \quad F = -(m \cdot \nu)u',$$

where ν is the normal unit vector pointing outward of Ω .

This method has been performed by many authors, especially by Komornik and Zuazua [10]. Following [7], Bey, Lohéac and Moussaoui obtained in [4] similar results under weaker geometrical conditions concerning the interface $\Gamma = \overline{\partial\Omega_N} \cap \overline{\partial\Omega_D}$. We here extend the above result for rotated multipliers introduced by Osses [15, 16].

1 Notations and main results

Let Ω be a bounded open connected set of \mathbb{R}^n ($n \geq 2$) such that

$$\partial\Omega \text{ is } \mathcal{C}^2 \text{ in the sense of Nečas [14].} \quad (1)$$

Let x_0 be a fixed point in \mathbb{R}^n . We denote by I the $n \times n$ identity matrix, by A a real $n \times n$ skew-symmetric matrix and by d a positive real number such that $d^2 + \|A\|^2 = 1$. We now define the following vector function:

$$\forall x \in \mathbb{R}^n, \quad m(x) = (dI + A)(x - x_0).$$

We consider a partition $(\partial\Omega_N, \partial\Omega_D)$ of $\partial\Omega$ such that

$$\left\{ \begin{array}{l} \Gamma = \overline{\partial\Omega_D} \cap \overline{\partial\Omega_N} \text{ is a } \mathcal{C}^3\text{-manifold of dimension } n-2, \\ m \cdot \nu = 0 \text{ on } \Gamma, \\ \partial\Omega \cap \omega \text{ is a } \mathcal{C}^3\text{-manifold of dimension } n-1, \\ \mathcal{H}^{n-1}(\partial\Omega_D) > 0, \end{array} \right. \quad (2)$$

where ω is a suitable neighborhood of Γ and \mathcal{H}^{n-1} denotes the usual $n-1$ -dimensional Hausdorff measure. Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be a measurable function such that

$$g \text{ is non-decreasing, and } \exists K > 0; \quad |g(s)| \leq K|s| \text{ a.e.} \quad (3)$$

Let us now consider the following wave problem

$$(S) \left\{ \begin{array}{ll} u'' - \Delta u = 0 & \text{in } \Omega \times \mathbb{R}_+^*, \\ u = 0 & \text{on } \partial\Omega_D \times \mathbb{R}_+^*, \\ \partial_\nu u = -(m \cdot \nu)g(u') & \text{on } \partial\Omega_N \times \mathbb{R}_+^*, \\ u(0) = u_0 & \text{in } \Omega, \\ u'(0) = u_1 & \text{in } \Omega, \end{array} \right.$$

for some initial data

$$(u_0, u_1) \in H_D^1(\Omega) \times L^2(\Omega) := H$$

with $H_D^1(\Omega) = \{v \in H^1(\Omega); v = 0 \text{ on } \partial\Omega_D\}$.

This problem is well-posed in this space. Indeed, following Komornik [9], we define the non-linear operator \mathcal{A} on H by

$$\begin{aligned} \mathcal{A}(u, v) &= (-v, -\Delta u), \\ D(\mathcal{A}) &= \{(u, v) \in H_D^1(\Omega) \times H_D^1(\Omega); \Delta u \in L^2(\Omega) \text{ and } \partial_\nu u = -(m \cdot \nu)g(v) \text{ on } \partial\Omega_N\}, \end{aligned}$$

so that (S) can be written in the form

$$\left\{ \begin{array}{l} (u, v)' + \mathcal{A}(u, v) = 0, \\ (u, v)(0) = (u_0, u_1). \end{array} \right.$$

It is classical that \mathcal{A} is a maximal-monotone operator on H and that $D(\mathcal{A})$ is dense in H for the usual norm. Following Brézis [1], we can deduce that for any initial data $(u_0, v_0) \in D(\mathcal{A})$ there is a unique

strong solution (u, v) such that $u \in W^{1,\infty}(\mathbb{R}; H_D^1(\Omega))$ and $\Delta u \in L^\infty(\mathbb{R}_+; L^2(\Omega))$. Moreover, for two initial data, the corresponding solutions satisfy:

$$\forall t \geq 0, \quad \|(u^1(t), v^1(t)) - (u^2(t), v^2(t))\|_H \leq \|(u_0^1, v_0^1) - (u_0^2, v_0^2)\|_H.$$

Using the density of $D(\mathcal{A})$, one can extend the map:

$$\begin{aligned} (u_0, v_0) &\longmapsto (u(t), v(t)) \\ D(\mathcal{A}) &\longrightarrow H \end{aligned}$$

to a strongly continuous semi-group of contractions $(S(t))_{t \geq 0}$ and define for $(u_0, v_0) \in H$ the weak solution $(u(t), u'(t)) = S(t)(u_0, u_1)$ with the regularity $u \in C(\mathbb{R}_+; H_D^1(\Omega)) \cap C^1(\mathbb{R}_+; L^2(\Omega))$. We hence define the energy function of solutions by:

$$E(u, 0) = \frac{1}{2} \int_{\Omega} (|u_1|^2 + |\nabla u_0|^2) dx, \quad E(u, t) = \frac{1}{2} \int_{\Omega} (|u'(t)|^2 + |\nabla u(t)|^2) dx \quad \text{if } t > 0.$$

In order to get stabilization results, we need further assumptions concerning the feedback function g

$$\exists p \geq 1, \exists k > 0, \quad |g(s)| \geq k \min\{|s|, |s|^p\}, \quad \text{a.e.} \quad (4)$$

Concerning the boundary we assume

$$\partial\Omega_N \subset \{x \in \partial\Omega, m(x) \cdot \nu(x) \geq 0\}, \quad \partial\Omega_D \subset \{x \in \partial\Omega, m(x) \cdot \nu(x) \leq 0\}, \quad (5)$$

and the additional geometric assumption

$$m \cdot \tau \leq 0 \quad \text{on } \Gamma, \quad (6)$$

where $\tau(x)$ is the normal unit vector pointing outward of $\partial\Omega_N$ at a point $x \in \Gamma$ when considering $\partial\Omega_N$ as a sub-manifold of $\partial\Omega$.

Remark 1 *It is worth observing that it is not necessary to assume that*

$$\mathcal{H}^{n-1}(\{x \in \partial\Omega_N; m(x) \cdot \nu(x) > 0\}) > 0.$$

to get stabilization. In fact, our choices of m imply such properties (see examples in Section 4) whether the energy tends to zero.

Following [12], Rellich type relation [17] is useful in control of the wave problem. Then, Komornik and Zuazua [10] have shown how these relations can also help us to stabilize the wave problem, but failed to generalize it in higher dimension than 3. The key-problem was to show the existence of a decomposition in regular and singular part ([6, 8]) which can apply to stabilization problems or control problems in any dimension. The first results towards this direction are due to Moussaoui [13], and Bey-Lohéac-Moussaoui [4] who also have established a Rellich type relation in any dimension. In this new case of Neumann feedback introduced by Osses [15, 16], our goal is then to generalize those Rellich's relations to get stabilization results about (S) . This will lead us to get a stabilization result under (5), (6).

As well as in [9], we shall prove here two results of uniform boundary stabilization.

Exponential boundary stabilization

We here consider the case when $p = 1$ in (4). This is satisfied when g is linear:

$$\exists \alpha > 0 : \quad \forall s \in \mathbb{R}, \quad g(s) = \alpha s.$$

In these cases, the energy function is exponentially decreasing.

Theorem 1 *Assume that geometrical conditions (2), (5) hold and that the feedback function g satisfies (3) and (4) with $p = 1$.*

Then under the further geometrical assumption (6), there exist $C > 0$ and $T > 0$ (independent of d) such that for all initial data in H , the energy of the solution u of satisfies

$$\forall t > \frac{T}{d}, \quad E(u, t) \leq E(u, 0) \exp\left(1 - \frac{d}{C}t\right).$$

The above constants C and T depend only on the geometry.

Rational boundary stabilization

We here consider the general case and we get rational boundary stabilization.

Theorem 2 *Assume that geometrical conditions (2), (5) hold and that the feedback function g satisfies (3) and (4) with $p > 1$.*

Then under the further geometrical assumption (6), there exist $C > 0$ and $T > 0$ (independent of d) such that for all initial data in H , the energy of the solution u of satisfies

$$\forall t > \frac{T}{d}, \quad E(u, t) \leq C t^{2/(p-1)}.$$

where C depends on the initial energy $E(u, 0)$.

Remark 2 *Taking advantage of the works of Banasiak-Roach [2] who generalized Grisvard's results [6] in the piecewise regular case, we will see that Theorems 1 and 2 remain true in the bi-dimensional case when assumption (1) is replaced by following one*

$$\begin{aligned} \partial\Omega \text{ is a curvilinear polygon of class } \mathcal{C}^2, \\ \text{each component of } \partial\Omega \setminus \Gamma \text{ is a } \mathcal{C}^2\text{-manifold of dimension 1,} \end{aligned} \quad (7)$$

and when condition (6) is replaced by

$$\forall x \in \Gamma, \quad 0 \leq \varpi_x \leq \pi \quad \text{and, if } \varpi_x = \pi, \quad m(x) \cdot \tau(x) \leq 0. \quad (8)$$

where ϖ_x is the angle at the boundary in the point x .

These two results are obtained by estimating some integral of the energy function as well as in [9]. This specific estimates are obtained thanks to an adapted Rellich relation.

Hence, this paper is composed of two sections. In the first one we build convenient Rellich relations and in the second one we use it to prove Theorems 1 and 2.

2 Rellich's relation

2.1 A regular case

We can easily build a Rellich relation corresponding to the above vector field m when considered functions are smooth enough.

Proposition 3 *Assume that Ω is an open set of \mathbb{R}^n with boundary of class \mathcal{C}^2 in the sense of Nečas. If u belongs to $H^2(\Omega)$ then*

$$2 \int_{\Omega} \Delta u (m \cdot \nabla u) dx = d(n-2) \int_{\Omega} |\nabla u|^2 dx + \int_{\partial\Omega} (2\partial_{\nu} u (m \cdot \nabla u) - (m \cdot \nu) |\nabla u|^2) d\sigma.$$

Proof. Using Green-Riemann identity we get

$$2 \int_{\Omega_{\varepsilon}} \Delta u (m \cdot \nabla u) dx = \int_{\partial\Omega_{\varepsilon}} 2\partial_{\nu} u (m \cdot \nabla u) d\sigma - 2 \int_{\Omega_{\varepsilon}} \nabla u \cdot \nabla (m \cdot \nabla u) dx.$$

So, observing that

$$\nabla u \cdot \nabla (m \cdot \nabla u) = (D^2 u)(m, \nabla u) + (d\nabla u + A\nabla u) \cdot \nabla u = \frac{1}{2} m \cdot \nabla (|\nabla u|^2) + d|\nabla u|^2 + A\nabla u \cdot \nabla u$$

($D^2 u$ is the Hessian matrix of u) and since A is skew-symmetric, we get

$$2 \int_{\Omega_{\varepsilon}} \Delta u (m \cdot \nabla u) dx = \int_{\partial\Omega_{\varepsilon}} 2\partial_{\nu} u (m \cdot \nabla u) d\sigma - 2d \int_{\Omega_{\varepsilon}} |\nabla u|^2 dx - \int_{\Omega_{\varepsilon}} m \cdot \nabla (|\nabla u|^2) dx.$$

With another use of Green-Riemann formula, we obtain the required result for $\text{div}(m) = nd$. ■

We will now try to extend this result to the case of an element u belonging less regular when Ω is smooth enough.

2.2 Bi-dimensional case

We begin by the plane case. It is the simplest case from the point of view of singularity theory and its understanding dates from Shamir [18].

Theorem 4 *Assume $n = 2$. Under geometrical conditions (2) and (7), let $u \in H^1(\Omega)$ such that*

$$\Delta u \in L^2(\Omega), \quad u|_{\partial\Omega_D} \in H^{3/2}(\partial\Omega_D), \quad \partial_\nu u|_{\partial\Omega_N} \in H^{1/2}(\partial\Omega_N). \quad (9)$$

Then $2\partial_\nu u(m \cdot \nabla u) - (m \cdot \nu)|\nabla u|^2 \in L^1(\partial\Omega)$ and there exist some coefficients $(c_x)_{x \in \Gamma}$ such that

$$2 \int_{\Omega} \Delta u(m \cdot \nabla u) dx = \int_{\partial\Omega} (2\partial_\nu u(m \cdot \nabla u) - (m \cdot \nu)|\nabla u|^2) d\sigma + \frac{\pi}{4} \sum_{x \in \Gamma/\varpi_x = \pi} c_x^2(m \cdot \tau)(x).$$

Proof. We first begin by some general considerations which will be used in the general case too. It is a classical result that $u \in H^2(\omega)$ for every open domain ω such that $\omega \Subset \overline{\Omega} \setminus \Gamma$. For sake of completeness, let us recall the proof.

A trace result shows that there exists $u_R \in H^2(\omega)$ such that $u_R = u$ on $\partial\Omega_D$ and $\partial_\nu u_R = \partial_\nu u$ on $\partial\Omega_N$. Hence, setting $f = \Delta u_R - \Delta u \in L^2(\Omega)$, $u_S = u - u_R$ satisfies

$$\begin{cases} -\Delta u_S = f & \text{in } \Omega, \\ u_S = 0 & \text{on } \partial\Omega_D, \\ \partial_\nu u_S = 0 & \text{on } \partial\Omega_N. \end{cases} \quad (10)$$

Now, if $\omega \Subset \Omega \setminus \Gamma \cup \partial\Omega_D$ and ξ is a cut-off function such that $\xi = 1$ on ω and $\text{supp}(\xi) \subset \Omega$, then, for a suitable $g \in L^2(\Omega)$, $u_\omega = u_S \xi$ satisfies the Dirichlet problem

$$\begin{cases} \Delta u_\omega = g & \text{on } \Omega, \\ u_\omega = 0 & \text{on } \partial\Omega, \end{cases}$$

and using classical method of difference quotients ([6]), one can now conclude that $u_\omega \in H^2(\Omega)$, hence $u_S \in H^2(\omega)$. Else, if $\omega \Subset \Omega \setminus \Gamma \cup \partial\Omega_N$, and ξ is a cut-off function such that $\xi = 1$ on ω and $\text{supp}(\xi) \subset \Omega$, then, for a suitable $g \in L^2(\Omega)$, $u_\omega = u_S \xi$ satisfies the Neumann problem

$$\begin{cases} -\Delta u_\omega + u_\omega = g & \text{on } \Omega, \\ \partial_\nu u_\omega = 0 & \text{on } \partial\Omega, \end{cases}$$

and, using similar argument, one gets $u_S \in H^2(\omega)$.

Let $\Omega_\varepsilon = \{x \in \Omega; d(x, \Gamma) > \varepsilon\}$. By compactness of Ω_ε , we get $u \in H^2(\Omega_\varepsilon)$. An application of Proposition 3 to our particular situation gives us the following relation

$$2 \int_{\Omega_\varepsilon} \Delta u(m \cdot \nabla u) dx = \int_{\partial\Omega_\varepsilon} (2\partial_\nu u(m \cdot \nabla u) - (m \cdot \nu)|\nabla u|^2) d\sigma$$

and we will try to let $\varepsilon \rightarrow 0$. Using derivative with respect to ν and τ , we get

$$2 \int_{\Omega_\varepsilon} \Delta u(m \cdot \nabla u) dx = \int_{\partial\Omega_\varepsilon} (m \cdot \nu) ((\partial_\nu u)^2 - (\partial_\tau u)^2) d\sigma + 2 \int_{\partial\Omega_\varepsilon} (m \cdot \tau) (\partial_\nu u)(\partial_\tau u) d\sigma.$$

First, since $\Delta u \in L^2(\Omega)$ and $u \in H^1(\Omega)$, Lebesgue's dominated convergence theorem immediately gives

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega_\varepsilon} \Delta u(m \cdot \nabla u) dx = \int_{\Omega} \Delta u(m \cdot \nabla u) dx$$

Now, we work on boundary terms. We shall introduce the following partition of $\partial\Omega_\varepsilon$: let $\widetilde{\partial\Omega_\varepsilon} = \partial\Omega_\varepsilon \cap \partial\Omega$, $\partial\Omega_\varepsilon^* = \partial\Omega_\varepsilon \cap \Omega$. and a decomposition result.

We use a decomposition result due to Banasiak and Roach [2]: every variational solution of (10) can be split as a sum of singular functions. There exists some coefficients $(c_x)_{x \in \Gamma}$ and $u_R \in H^2(\Omega)$ such that

$$u = u_R + \sum_{x \in \Gamma} c_x U_S^x =: u_R + u_S \quad (11)$$

where U_S^x are singular functions which, in some neighborhood of $x \in \Gamma$, are defined in local polar coordinates by:

$$U_S^x(r, \theta) = \rho(r)r^{\frac{\pi}{2\varpi_x}} \sin\left(\frac{\pi}{2\varpi_x}\theta\right).$$

with ρ some cut-off function (see Fig. 1).

Using the density of $C^1(\overline{\Omega})$ in $H^2(\Omega)$, we will be able to assume that $u_R \in C^1(\overline{\Omega})$. Let us look at boundary terms on $\partial\overline{\Omega}_\varepsilon$ first. We claim first that for some constant $C > 0$,

$$|m \cdot \nu| \leq Cd(\cdot, \Gamma).$$

In fact, if $x \in \Omega$ and $x_1 \in \Gamma$ which satisfies $|x - x_1| = d(x, \Gamma)$, one gets:

$$m \cdot \nu(x) = m(x) \cdot (\nu(x) - \nu(x_1)) + (m(x) - m(x_1)) \cdot \nu(x_1) \quad (\text{observe that } m \cdot \nu(x_1) = 0).$$

Hence, using the fact that ν is a piecewise C^1 function (see Fig. 2), we get

$$|m \cdot \nu(x)| \leq (\|m\|_\infty \|\nu'\|_\infty + 1)d(x, \Gamma).$$

Now, working in local coordinates, one gets

$$d(x, \Gamma)|\nabla u|^2 \in L^\infty(\partial\Omega).$$

Hence Lebesgue theorem implies

$$\lim_{\varepsilon \rightarrow 0} \int_{\partial\overline{\Omega}_\varepsilon} (m \cdot \nu) ((\partial_\nu u)^2 - (\partial_\tau u)^2) d\sigma = \int_{\partial\Omega} (m \cdot \nu) ((\partial_\nu u)^2 - (\partial_\tau u)^2) d\sigma.$$

On the other hand, assumptions (9) give $\partial_\nu u \in H^{1/2}(\partial\Omega_N)$, $\partial_\tau u \in H^{-1/2}(\partial\Omega_N)$ and $\partial_\nu u \in H^{-1/2}(\partial\Omega_D)$, $\partial_\tau u \in H^{1/2}(\partial\Omega_D)$. Hence we get

$$\int_{\partial\overline{\Omega}_\varepsilon} (m \cdot \tau)(\partial_\nu u)(\partial_\tau u) d\sigma \rightarrow \int_{\partial\Omega} (m \cdot \tau)(\partial_\nu u)(\partial_\tau u) d\sigma.$$

Now, we have to consider the boundary term on $\partial\Omega_\varepsilon^*$, $I_\varepsilon(\nabla u)$. This quadratic form in ∇u can be decomposed using (11) in

$$I_\varepsilon(\nabla u_R) + 2J_\varepsilon(\nabla u_R, \nabla u_S) + I_\varepsilon(\nabla u_S),$$

where J_ε is the corresponding bilinear form.

Concerning $I_\varepsilon(\nabla u_R)$, regularity of m gives the estimate

$$|I_\varepsilon(\nabla u_R)| \leq C \int_{\partial\Omega_\varepsilon^*} |\nabla u_R|^2 d\sigma$$

This term is then an $O(\varepsilon)$ since ∇u_R is bounded on Ω .

For the term $I_\varepsilon(\nabla u_S)$, we first note that, adjusting the cut-off functions, the supports of u_S^x and u_S^y are disjoint. Hence, using decomposition (11), we can write

$$I_\varepsilon(\nabla u_S) = \sum_{x \in \Gamma} c_x^2 \int_{C_\varepsilon(x)} (2\partial_\nu u_S^x(m \cdot \nabla u_S^x) - (m \cdot \nu)|\nabla u_S^x|^2) d\sigma.$$

If $\varpi_x < \pi$, one has

$$2\partial_\nu u_S^x(m \cdot \nabla u_S^x) - (m \cdot \nu)|\nabla u_S^x|^2 = O(\varepsilon^{\frac{\pi}{\varpi_x} - 2}) \text{ on } C_\varepsilon(x).$$

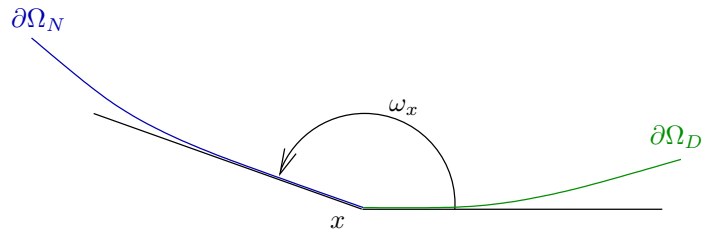
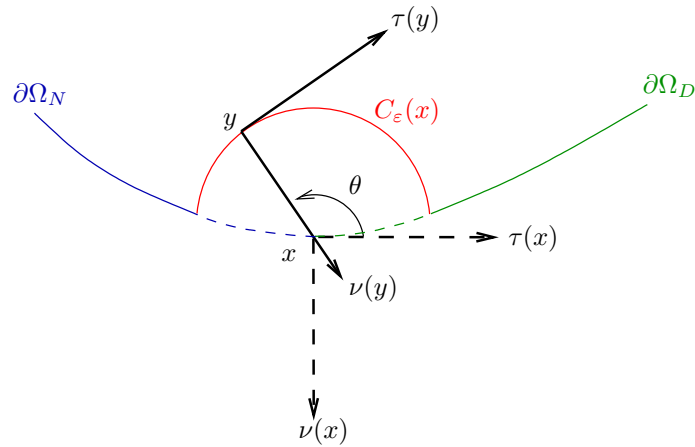
Hence, after integrating on $C_\varepsilon(x)$, we get $\lim_{\varepsilon \rightarrow 0} I_1^x(\varepsilon) = 0$.

If $\varpi_x = \pi$, we will need the following identity

$$2\partial_\nu u_S^x(m \cdot \nabla u_S^x) - (m \cdot \nu)|\nabla u_S^x|^2 = \frac{1}{4\varepsilon} m \cdot \tau(x) \text{ on } C_\varepsilon(x).$$

If we observe that $C_\varepsilon(x)$ behaves as a half-circle when $\varepsilon \rightarrow 0$, an integration gives

$$\lim_{\varepsilon \rightarrow 0} \int_{C_\varepsilon(x)} (2(\nu \cdot \nabla u_S^x)(m \cdot \nabla u_S^x) - (m \cdot \nu)|\nabla u_S^x|^2) d\sigma = \frac{\pi}{4} m(x) \cdot \tau(x).$$

Figure 1: Shape of the boundary near an angular point x .Figure 2: Unit vectors $\nu(x)$, $\tau(x)$, $\nu(y)$ and $\tau(y)$ when $\partial\Omega$ is regular at x .

Finally, the bilinear term $J_\varepsilon(\nabla u_R, \nabla u_S)$ can be written entirely

$$\int_{\partial\Omega_\varepsilon^*} \partial_\nu u_R (m \cdot \nabla u_S) d\sigma + \int_{\partial\Omega_\varepsilon^*} \partial_\nu u_S (m \cdot \nabla u_R) d\sigma - \int_{\partial\Omega_\varepsilon^*} (m \cdot \nu) \nabla u_R \cdot \nabla u_S d\sigma.$$

Using the regularity of m and Cauchy-Schwarz inequality, we get an estimate of the form

$$|J_\varepsilon(\nabla u_R, \nabla u_S)| \leq C \left(\int_{\partial\Omega_\varepsilon^*} |\nabla u_R|^2 d\sigma \right)^{1/2} \left(\int_{\partial\Omega_\varepsilon^*} |\nabla u_S|^2 d\sigma \right)^{1/2}$$

We have seen that the first term in this inequality vanishes when $\varepsilon \rightarrow 0$. For the other term, we now observe that, if ε is small enough

$$\partial\Omega_\varepsilon^* = \bigsqcup_{x \in \Gamma} C_\varepsilon(x)$$

where $C_\varepsilon(x)$ is an arc of circle of radius ε centered in x . Then, we may write

$$\int_{\partial\Omega_\varepsilon^*} |\nabla u_S|^2 d\sigma \leq 2 \sum_{x, y \in \Gamma} c_y^2 \int_{C_\varepsilon(x)} |\nabla U_S^y|^2 d\sigma.$$

A similar computation shows that, for $x \in \Gamma$, $\int_{C_\varepsilon(x)} |\nabla U_S^x|^2 d\sigma = O(1)$. Moreover, if $x \neq y$, U_S^y is bounded near x , we get $\int_{C_\varepsilon(x)} |\nabla U_S^y|^2 d\sigma = O(\varepsilon)$. This completes the proof. ■

Remark 3 *The assumption $\mathcal{H}^1(\partial\Omega_D) > 0$ is not necessary in the above proof. We will now see why we need this assumption on the Dirichlet part in higher dimension.*

2.3 General case

We now state the result in general dimension.

Theorem 5 *Assume $n \geq 3$. Under geometrical conditions (1) and (2), let $u \in H^1(\Omega)$ such that*

$$\Delta u \in L^2(\Omega), \quad u|_{\partial\Omega_D} \in H^{3/2}(\partial\Omega_D), \quad \partial_\nu u|_{\partial\Omega_N} \in H^{1/2}(\partial\Omega_N). \quad (12)$$

Then, $2\partial_\nu u(m \cdot \nabla u) - (m \cdot \nu)|\nabla u|^2 \in L^1(\partial\Omega)$ and there exists $\zeta \in H^{1/2}(\Gamma)$ such that:

$$2 \int_{\Omega} \Delta u(m \cdot \nabla u) dx = d(n-2) \int_{\Omega} |\nabla u|^2 dx + \int_{\partial\Omega} (2\partial_\nu u(m \cdot \nabla u) - (m \cdot \nu)|\nabla u|^2) d\sigma + \int_{\Gamma} (m \cdot \tau)|\zeta|^2 d\gamma.$$

Proof. We will essentially follow [4]. As in the plane case, we let $\Omega_\varepsilon = \{x \in \Omega; d(x, \Gamma) > \varepsilon\}$. For any given $\varepsilon > 0$, we have the identity of Proposition 3

$$2 \int_{\Omega_\varepsilon} \Delta u(m \cdot \nabla u) dx = d(n-2) \int_{\Omega_\varepsilon} |\nabla u|^2 dx + \int_{\partial\Omega_\varepsilon} (2\partial_\nu u(m \cdot \nabla u) - (m \cdot \nu)|\nabla u|^2) d\sigma,$$

and we will again analyze the behavior of each term as $\varepsilon \rightarrow 0$.

First, since $\Delta u \in L^2(\Omega)$ and $u \in H^1(\Omega)$, Lebesgue's dominated convergence immediately gives

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega_\varepsilon} \Delta u(m \cdot \nabla u) dx = \int_{\Omega} \Delta u(m \cdot \nabla u) dx, \quad \lim_{\varepsilon \rightarrow 0} \int_{\Omega_\varepsilon} |\nabla u|^2 dx = \int_{\Omega} |\nabla u|^2 dx.$$

Next, we will work on boundary terms.

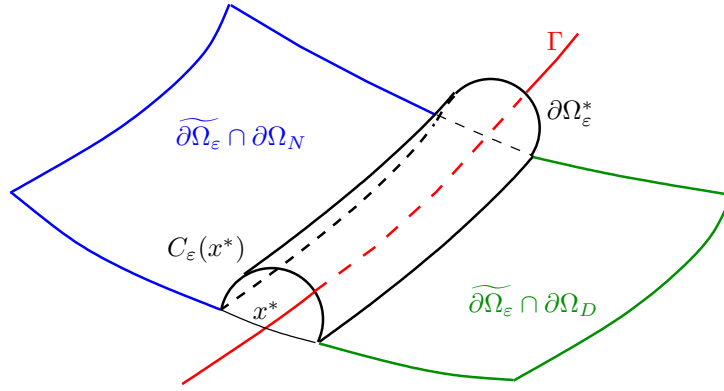


Figure 3: Picture of $\partial\Omega_\varepsilon^*$ and $\widetilde{\partial\Omega_\varepsilon}$

We define again $\widetilde{\partial\Omega_\varepsilon} = \partial\Omega_\varepsilon \cap \partial\Omega$ and $\partial\Omega_\varepsilon^* = \partial\Omega_\varepsilon \cap \Omega$ (see Fig. 3). As well as in the plane case, there exists some constant $C > 0$ such that $|m \cdot \nu| \leq Cd(\cdot, \Gamma)$. Thus, using the fact that

$$d(\cdot, \Gamma)|\nabla u|^2 \in L^1(\partial\Omega)$$

(see [4], Proposition 3), we can use again Lebesgue's theorem to conclude that, as $\varepsilon \rightarrow 0$,

$$\int_{\widetilde{\partial\Omega_\varepsilon}} (m \cdot \nu)|\nabla u|^2 d\sigma \rightarrow \int_{\partial\Omega} (m \cdot \nu)|\nabla u|^2 d\sigma.$$

For the second integral, defining $\nabla_{\partial\Omega}$ as the gradient of derivatives along $\partial\Omega$, we write that

$$\partial_\nu u(m \cdot \nabla u) = (m \cdot \nu)|\partial_\nu u|^2 + \partial_\nu u(m \cdot \nabla_{\partial\Omega} u).$$

The first term is integrable. The second one is, on $\partial\Omega_N$, the product of a $H^{1/2}$ term by a $H^{-1/2}$ one and, on $\partial\Omega_D$, the product of a $H^{-1/2}$ term by a $H^{1/2}$ one. Hence, Lebesgue's theorem gives again, as $\varepsilon \rightarrow 0$,

$$\int_{\widetilde{\partial\Omega_\varepsilon}} \partial_\nu u(m \cdot \nabla u) d\sigma \rightarrow \int_{\partial\Omega} \partial_\nu u(m \cdot \nabla u) d\sigma.$$

Now, we consider the terms on $\partial\Omega_\varepsilon^*$. We assume that $\varepsilon \leq \varepsilon_0$ and we define: $\omega_{\varepsilon_0} := \Omega \setminus \Omega_{\varepsilon_0}$. Now, as well as in the plane case, we can write

$$u = u_R + u_S \quad (13)$$

where u_S is the variational solution of some homogeneous mixed boundary problem and u_R belongs to $H^2(\omega_{\varepsilon_0})$. Working by approximation if necessary, we can suppose that $u_R \in C^1(\overline{\omega_{\varepsilon_0}})$. Considering the same quadratic form as in the bi-dimensional case, this leads to the following splitting of $\int_{\partial\Omega_\varepsilon^*} (2\partial_\nu u(m \cdot \nabla u) - (m \cdot \nu)|\nabla u|^2) d\sigma$:

$$I_\varepsilon(\nabla u_R) + I_\varepsilon(\nabla u_S) + 2J_\varepsilon(\nabla u_R, \nabla u_S).$$

Since $\nabla u_R \in L^\infty(\omega_{\varepsilon_0})$ and $\mathcal{H}^{n-1}(\partial\Omega_\varepsilon^*) \rightarrow 0$ as $\varepsilon \rightarrow 0$, it is obvious that the first term $I_\varepsilon(\nabla u_R)$ tends to zero.

The bilinear term $J_\varepsilon(\nabla u_R, \nabla u_S)$ is, as above,

$$\int_{\partial\Omega_\varepsilon^*} \partial_\nu u_R(m \cdot \nabla u_S) d\sigma + \int_{\partial\Omega_\varepsilon^*} \partial_\nu u_S(m \cdot \nabla u_R) d\sigma - \int_{\partial\Omega_\varepsilon^*} (m \cdot \nu) \nabla u_R \cdot \nabla u_S d\sigma.$$

Using the regularity of m and Cauchy-Schwarz inequality, we get an estimate of the form

$$|J_\varepsilon(\nabla u_R, \nabla u_S)| \leq C \left(\int_{\partial\Omega_\varepsilon^*} |\nabla u_R|^2 d\sigma \right)^{1/2} \left(\int_{\partial\Omega_\varepsilon^*} |\nabla u_S|^2 d\sigma \right)^{1/2} \quad (14)$$

As above, it is clear that the first term tends to zero as $\varepsilon \rightarrow 0$. In order to analyze $I_\varepsilon(\nabla u_S)$ we will need further results.

To begin with, we shall use some definitions. Every $x \in \partial\Omega_\varepsilon^*$ belongs to a unique plane $x^* + \langle \tau^*, \nu^* \rangle$ (setting: $\tau^* = \tau(x^*)$, $\nu^* = \nu(x^*)$) and more precisely to an arc-circle $C_\varepsilon(x^*)$ in this plane of center $x^* \in \Gamma$ and of radius ε (the figure is similar to Fig. 2 in the plane $x^* + \langle \tau^*, \nu^* \rangle$). We define

$$D_\varepsilon(x^*) := \omega_\varepsilon \cap (x^* + \langle \tau^*, \nu^* \rangle).$$

For any $x \in D_\varepsilon(x^*)$, we separate the derivatives of u along the sub-manifold $x - x^* + \Gamma$ with the co-normal derivatives:

$$\nabla u(x) = \nabla_\Gamma u(x) + \nabla_2 u(x), \quad \nabla_\Gamma u(x) \in T_{x^*} \Gamma, \quad \nabla_2 u(x) \in \langle \tau^*, \nu^* \rangle. \quad (15)$$

Using methods of difference quotients (see for instance [4], Theorem 4), one gets $\nabla_\Gamma u \in H^1(\omega_{\varepsilon_0})$ i.e. $\nabla_\Gamma u_S \in H^1(\omega_{\varepsilon_0})$. We will also need the following result concerning the behavior of boundary integrals

Lemma 6 *Let $\varepsilon_0 > 0$. Assume that u is such that $u = 0$ on $\partial\omega_{\varepsilon_0} \cap \partial\Omega_D$,*

$$\forall x^* \in \Gamma, \quad u(x^*, \cdot) \in H^1(D_{\varepsilon_0}(x^*))$$

and

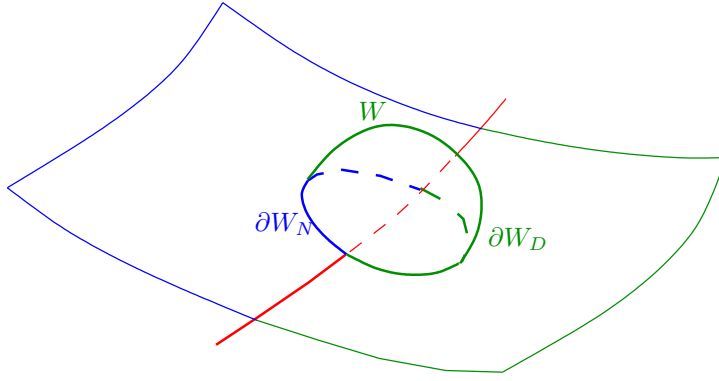
$$(x^* \mapsto \|u(\cdot, \cdot)\|_{H^1(D_{\varepsilon_0}(x^*))}) \in L^2(\Gamma).$$

Then there exists $C > 0$ depending only on Ω such that, for any ε sufficiently small,

$$\int_\Gamma \|u(x^*, \cdot)\|_{L^2(C_\varepsilon(x^*))}^2 d\gamma \leq C\varepsilon \int_\Gamma \|u(x^*, \cdot)\|_{H^1(D_\varepsilon(x^*))}^2 d\gamma(x^*).$$

Proof of Lemma 6. We begin by changing coordinates as well as in [4]. For every $x_0^* \in \Gamma$, there exists $\rho_0 > 0$, a C^2 diffeomorphism Θ from an open neighborhood W of x_0^* to $B(\rho_0) := B_{n-2}(\rho_0) \times B_2(\rho_0)$ (see Fig. 5) such that

$$\begin{aligned} \Theta(x_0^*) &= 0, \\ \Theta(W \cap \Omega) &= \{y \in B(\rho_0); y_n > 0\}, \\ \Theta(W \cap \partial\Omega_D) &= \{y \in B(\rho_0); y_{n-1} > 0, y_n = 0\}, \\ \Theta(W \cap \partial\Omega_N) &= \{y \in B(\rho_0); y_{n-1} < 0, y_n = 0\}, \\ \Theta(W \cap \Gamma) &= \{y \in B(\rho_0); y_{n-1} = 0, y_n = 0\} := \gamma(\rho_0). \end{aligned}$$

Figure 4: The set W .

Reducing ε_0 if necessary, we may assume that $D_{\varepsilon_0}(x_0^*) \subset W$. We then get, writing for $x \in W$, $\Theta(x) = (y', \tilde{y})$ and $v := u \circ \Theta^{-1}$:

$$\int_{W \cap \Gamma} \left(\int_{C_\varepsilon(x^*)} u^2 dl \right) d\gamma(x^*) = \int_{\gamma(\rho_0)} \left(\int_{\Theta(C_\varepsilon(x^*))} v^2 dl(\tilde{y}) \right) dy'.$$

Setting

$$B_2^+(\rho) := \{\tilde{y} = (y_{n-1}, y_n) \in B_2(\rho); y_n > 0\}, \quad C_2^+(\rho) := \{\tilde{y} = (y_{n-1}, y_n) \in \partial B_2(\rho); y_n > 0\},$$

we first observe that we can choose ρ_{x^*} such that $\{y'\} \times B_2^+(\rho) \subset \Theta(D_\varepsilon(x^*))$. Hence if we denote by π_2 the projection on $\{0_{\mathbb{R}^{n-2}}\} \times \mathbb{R}^2$, the change of variables

$$\begin{aligned} \tilde{y} &\longmapsto z = \rho \frac{\tilde{y}}{\|\tilde{y}\|} \\ \pi_2(\Theta(C_\varepsilon(x^*))) &\longrightarrow C_2^+(\rho) \end{aligned}$$

gives the estimate

$$\int_{\Theta(C_\varepsilon(x^*))} v(y', \tilde{y})^2 dl(\tilde{y}) \leq C \int_{C_2^+(\rho)} v(y', z)^2 dl(z) \quad (16)$$

for a constant C depending only on x_0^* .

We will now estimate this latter integral in terms of $\|\nabla_2 v\|_{L^2(\{y'\} \times B_2^+(\rho))}$. If $v_\rho(\tilde{y}) := v(y', \tilde{y})$, one gets $\nabla v_\rho \in L^2(B_2^+(1))$ and

$$\|\nabla v_\rho\|_{L^2(B_2^+(1))} = \|\nabla_2 v\|_{L^2(\{y'\} \times B_2^+(\rho))}, \quad \|v_\rho\|_{L^2(C_2^+(1))} = \rho^{-\frac{1}{2}} \|v\|_{L^2(\{y'\} \times C_2^+(\rho))}.$$

Observing that $v_\rho = 0$ on $B_2^{++}(1) := \{(y_{n-1}, y_n) \in B_2^+(1); y_n > 0\}$, trace theorem and Poincaré's inequality give, for some universal constant $C > 0$, the estimate

$$\int_{C_2^+(\rho)} v^2(y', \tilde{y}) dl(\tilde{y}) \leq C\rho \|\nabla_2 v\|_{L^2(\{y'\} \times B_2^+(\rho))}^2.$$

Hence, thanks to (16), one gets

$$\int_{\Theta(C_\varepsilon(x^*))} v^2(y', \tilde{y}) dl(\tilde{y}) \leq C\rho_{x^*} \|\nabla_2 v\|_{L^2(\{y'\} \times B_2^+(\rho_{x^*}))}^2.$$

Observing that ρ_{x^*} is uniformly $O(\varepsilon)$ on $W \cap \Gamma$ and the diffeomorphism $\Theta(x^*, \cdot)$ (see Fig. 6), we can conclude that, for some constant $C_{x_0^*}$ depending only on x_0^*

$$\int_{\Theta(C_\varepsilon(x^*))} v^2(y', \tilde{y}) dl(\tilde{y}) \leq C_{x_0^*} \varepsilon \|u(x^*, \cdot)\|_{H^1(\Theta^{-1}(\{y'\} \times B_2^+(\rho)))}^2 \leq C_{x_0^*} \varepsilon \|u(x^*, \cdot)\|_{H^1(D_\varepsilon(x^*))}^2.$$

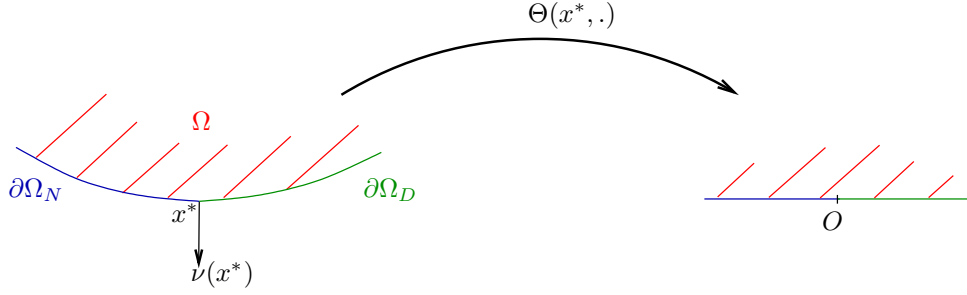


Figure 5: The \mathcal{C}^2 -diffeomorphism $\Theta(x^*, \cdot)$ in the plane $x^* + \langle \tau^*, \nu^* \rangle$.

Hence, after an integration on $W \cap \Gamma$

$$\int_{W \cap \Gamma} \left(\int_{C_\varepsilon(x^*)} u^2 dl \right) d\gamma(x^*) \leq C_{x_0^*} \varepsilon \int_{W \cap \Gamma} \|u(x^*, \cdot)\|_{H^1(D_\varepsilon(x^*))}^2 d\gamma(x^*).$$

We finally complete the proof by using a partition of unity on the open sets $(W_{x_0^*})_{x_0^* \in \Gamma}$.

End of proof of Lemma 6. ■

Let us come back to our problem. Using (15) for u_S , Pythagore's theorem gives

$$\int_{\partial\Omega_\varepsilon^*} |\nabla u_S|^2 d\sigma = \int_{\partial\Omega_\varepsilon^*} |\nabla_\Gamma u_S|^2 d\sigma + \int_{\partial\Omega_\varepsilon^*} |\nabla_2 u_S|^2 d\sigma.$$

Applying Lemma 6 to $\nabla_\Gamma u_S$, we get that the first term vanishes as $\varepsilon \rightarrow 0$. As well as in the bi-dimensional case, we will see that the second term above is bounded, using more information on u_S .

Thanks to [4] (Theorem 4) and Borel-Lebesgue theorem, we may write

$$u_S(x) = \eta(x^*) U_S(x - x^*) := \eta(x^*) U_S^{x^*}(x) \text{ on } \omega_{\varepsilon_0}, \quad (17)$$

with U_S locally diffeomorphic to Shamir's function, and $\eta \in H^{1/2}(\Gamma)$. We then get, thanks to Fubini theorem

$$\int_{\partial\Omega_\varepsilon^*} |\nabla_2 u_S|^2 d\sigma = \int_\Gamma \eta(x^*)^2 \left(\int_{C_\varepsilon(x^*)} |\nabla_2 U_S^{x^*}|^2 dl \right) d\gamma(x^*),$$

and, as well as in the bi-dimensional case, we show that this term is bounded by $O(1) \|\eta\|_{L^2(\Gamma)}^2$. We have now proven that the second term in (14) is bounded, that is

$$J_\varepsilon(\nabla u_R) \rightarrow 0 \text{ as } \varepsilon \rightarrow 0.$$

To treat the last term $I_\varepsilon(\nabla u_S)$, we will use the same tools as above. The splitting (13) for u_S gives us

$$I_\varepsilon(\nabla u_S) = I_\varepsilon(\nabla_2 u_S) + I_\varepsilon(\nabla_\Gamma u_S) + 2J_\varepsilon(\nabla_2 u_S, \nabla_\Gamma u_S).$$

As above, the term $I_\varepsilon(\nabla_\Gamma u_S)$ is estimated by $\int_{\partial\Omega_\varepsilon^*} |\nabla_\Gamma u_S|^2 d\sigma$; it then vanishes for $\varepsilon \rightarrow 0$. The bilinear term is estimated by

$$\left(\int_{\partial\Omega_\varepsilon^*} |\nabla_2 u_S|^2 d\sigma \right)^{1/2} \left(\int_{\partial\Omega_\varepsilon^*} |\nabla_\Gamma u_S|^2 d\sigma \right)^{1/2};$$

it then tends to zero since the first term is bounded and the second one vanishes for $\varepsilon \rightarrow 0$.

For the last term $I_\varepsilon(\nabla_2 u_S)$, we use (17) and Fubini's theorem to write it

$$\int_\Gamma \eta(x^*)^2 \left(\int_{C_\varepsilon(x^*)} 2(\nu \cdot \nabla_2 U_S^{x^*})(m \cdot \nabla_2 U_S^{x^*}) - (m \cdot \nu) |\nabla_2 U_S^{x^*}|^2 \right) dl d\gamma(x^*).$$

We first work in the plane $x^* + \langle \tau^*, -\nu^* \rangle$ and, as above, we get

$$\lim_{\varepsilon \rightarrow 0} \int_{C_\varepsilon(x^*)} (2(\nu \cdot \nabla_2 U_S^{x^*})(m \cdot \nabla_2 U_S^{x^*}) - (m \cdot \nu) |\nabla_2 U_S^{x^*}|^2) dl = \frac{\pi}{4} m(x^*) \cdot \tau(x^*).$$

Moreover, for any $\varepsilon > 0$, this integral term on $C_\varepsilon(x^*)$ is dominated by $\frac{\pi}{2}\|m\|_\infty \in L^1(\Gamma)$; so dominated convergence theorem applies and finally

$$\lim_{\varepsilon \rightarrow 0} I_\varepsilon(\nabla_2 u_S) = \frac{\pi}{4} \int_\Gamma \eta^2 m \cdot \tau d\gamma.$$

The proof is now complete with $\zeta = \frac{\sqrt{\pi}}{2} \eta$. ■

We will now apply Rellich's relation to the stabilization of solutions of (S).

3 Proof of linear and non-linear stabilization

We begin by writing the following consequence of Section 2.

Corollary 7 *Assume that $t \mapsto (u(t), u'(t))$ is a strong solution of (S) and that the geometrical additional assumption (5) if $n \geq 3$ or (6) if $n = 2$ holds, then, for every time t , $u(t)$ satisfies*

$$2 \int_\Omega \Delta u(m \cdot \nabla u) dx \leq d(n-2) \int_\Omega |\nabla u|^2 dx + \int_{\partial\Omega} (2\partial_\nu u(m \cdot \nabla u) - (m \cdot \nu)|\nabla u|^2) d\sigma.$$

Proof. Indeed, under these hypotheses, for each time t , $(u(t), u'(t)) \in D(\mathcal{A})$ so that $u(t)$ satisfies (9) or (12). The corollary is then an application of Theorem 4 or 5. ■

We will be able to prove Theorems 1 and 2 showing that, for $\alpha = \frac{p-1}{2}$, one can apply the following result ([9]):

Proposition 8 *Let $E : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ a non-increasing function such that there exists $\alpha \geq 0$ and $C > 0$ which fulfills*

$$\forall t \geq 0, \int_t^\infty E^{\alpha+1}(s) ds \leq CE(t).$$

Then, setting $T = CE^\alpha(0)$, one gets

$$\begin{aligned} \text{if } \alpha = 0, \quad \forall t \geq T, \quad E(t) &\leq E(0) \exp\left(1 - \frac{t}{T}\right), \\ \text{if } \alpha > 0, \quad \forall t \geq T, \quad E(t) &\leq E(0) \left(\frac{T + \alpha T}{T + \alpha t}\right)^{1/\alpha}. \end{aligned}$$

We come back to our proof now.

Proof. Following [9] and [5], we will prove the estimates for $(u_0, u_1) \in D(\mathcal{A})$ which, using density of the domain, will be sufficient to get the result for all solutions.

Setting $Mu = 2m \cdot \nabla u + d(n-1)u$, we prove the following result.

Lemma 9 *For any $0 \leq S < T < \infty$, one gets*

$$\begin{aligned} 2d \int_S^T E^{\frac{p+1}{2}} dt &\leq - \left[E^{\frac{p-1}{2}} \int_\Omega u' Mu dx \right]_S^T + \frac{p-1}{2} \int_S^T E^{\frac{p-3}{2}} E' \left(\int_\Omega u' Mu dx \right) dt \\ &\quad + \int_S^T E^{\frac{p-1}{2}} \left(\int_{\partial\Omega_N} (m \cdot \nu)((u')^2 - |\nabla u|^2 - g(u')Mu) d\sigma \right) dt. \end{aligned}$$

Proof of Lemma 9. Using the fact that u is solution of (S) and noting that $u''Mu = (u'Mu)' - u'Mu'$, an integration by parts gives:

$$\begin{aligned} 0 &= \int_S^T E^{\frac{p-1}{2}} \int_\Omega (u'' - \Delta u) Mu dx dt \\ &= \left[E^{\frac{p-1}{2}} \int_\Omega u' Mu dx \right]_S^T - \frac{p-1}{2} \int_S^T E^{\frac{p-3}{2}} E' \left(\int_\Omega u' Mu dx \right) dt - \int_S^T E^{\frac{p-1}{2}} \int_\Omega (u'Mu' + \Delta u Mu) dx dt. \end{aligned}$$

Corollary 7 now gives

$$\int_{\Omega} \Delta u M u \, dx \leq d(n-1) \int_{\Omega} \Delta u u \, dx + d(n-2) \int_{\Omega} |\nabla u|^2 \, dx + \int_{\partial\Omega} (2\partial_{\nu} u (m \cdot \nabla u) - (m \cdot \nu) |\nabla u|^2) \, d\sigma.$$

hence, Green-Riemann formula leads to

$$\int_{\Omega} \Delta u M u \, dx \leq -d \int_{\Omega} |\nabla u|^2 \, dx + \int_{\partial\Omega} (\partial_{\nu} u M u - (m \cdot \nu) |\nabla u|^2) \, d\sigma.$$

Using boundary conditions and the fact that $\nabla u = \partial_{\nu} u \nu$ on $\partial\Omega_D$, we then get

$$\int_{\Omega} \Delta u M u \, dx \leq -d \int_{\Omega} |\nabla u|^2 \, dx - \int_{\partial\Omega_N} (m \cdot \nu) (g(u') M u + |\nabla u|^2) \, d\sigma.$$

On the other hand, using $\operatorname{div}(m) = nd$, an other use of Green formula gives us

$$\int_{\Omega} u' M u' \, dx = -d \int_{\Omega} (u')^2 \, dx + \int_{\partial\Omega_N} (m \cdot \nu) |u'|^2 \, d\sigma.$$

End of proof of Lemma 9. ■

Coming back to our problem, Young inequality gives

$$\left| \int_{\Omega} u' M u \, dx \right| \leq CE(t).$$

Lemma 9 shows that

$$\begin{aligned} 2d \int_S^T E^{\frac{p+1}{2}} \, dt &\leq C(E^{\frac{p+1}{2}}(T) + E^{\frac{p+1}{2}}(S)) + C \int_S^T E^{\frac{p-1}{2}} E' \, dt \\ &\quad + \int_S^T E^{\frac{p-1}{2}} \left(\int_{\partial\Omega_N} (m \cdot \nu) ((u')^2 - |\nabla u|^2 - g(u') M u) \, d\sigma \right) dt \end{aligned}$$

For simplicity, let $d\sigma_m = (m \cdot \nu) d\sigma$. If we observe that $E'(t) = - \int_{\partial\Omega_N} g(u') u' d\sigma_m \leq 0$, we get, for a constant $C > 0$ independent of $E(0)$ if $p = 1$,

$$2d \int_S^T E^{\frac{p+1}{2}} \, dt \leq CE(S) + \int_S^T E^{\frac{p-1}{2}} \left(\int_{\partial\Omega_N} (u')^2 - |\nabla u|^2 - g(u') M u d\sigma_m \right) dt.$$

Using the definition of Mu and Young inequality, we get, for any $\varepsilon > 0$,

$$2d \int_S^T E^{\frac{p+1}{2}} \, dt \leq CE(S) + \int_S^T E^{\frac{p-1}{2}} \left(\int_{\partial\Omega_N} \left((u')^2 + \left(\|m\|_{\infty}^2 + \frac{d^2(n-1)^2}{4\varepsilon} \right) g(u')^2 + \varepsilon u^2 \right) d\sigma_m \right) dt.$$

Now, using Poincaré inequality, we can choose $\varepsilon > 0$ such that

$$\varepsilon \int_{\partial\Omega_N} (m \cdot \nu) u^2 \, d\sigma \leq \frac{d}{2} \int_{\Omega} |\nabla u|^2 \, dx \leq dE,$$

so we conclude

$$d \int_S^T E^{\frac{p+1}{2}} \, dt \leq CE(S) + C \int_S^T E^{\frac{p-1}{2}} \left(\int_{\partial\Omega_N} ((u')^2 + g(u')^2) \, d\sigma_m \right) dt.$$

We split $\partial\Omega_N$ to bound the last term of the above estimate

$$\partial\Omega_N^1 = \{x \in \partial\Omega_N; |u'(x)| > 1\}, \quad \partial\Omega_N^2 = \{x \in \partial\Omega_N; |u'(x)| \leq 1\}.$$

Using (3) and (4), we get

$$\int_S^T E^{\frac{p-1}{2}} \left(\int_{\partial\Omega_N^1} ((u')^2 + g(u')^2) d\sigma_m \right) dt \leq C \int_S^T E^{\frac{p-1}{2}} \left(\int_{\partial\Omega_N} u' g(u') d\sigma_m \right) dt \leq CE(S),$$

where C neither depend on $E(0)$ if $p = 1$.

On the other hand, using (3), (4); Jensen inequality and boundedness of m , one successively obtains

$$\int_{\partial\Omega_N^2} ((u')^2 + g(u')^2) d\sigma_m \leq C \int_{\partial\Omega_N^2} (u' g(u'))^{2/(p+1)} d\sigma_m \leq C \left(\int_{\partial\Omega_N^2} u' g(u') d\sigma_m \right)^{\frac{2}{p+1}} \leq C(-E')^{\frac{2}{p+1}}.$$

Hence, using Young inequality again, we get for every $\varepsilon > 0$

$$\int_S^T E^{\frac{p-1}{2}} \left(\int_{\partial\Omega_N^2} ((u')^2 + g(u')^2) d\sigma_m \right) dt \leq \int_S^T (\varepsilon E^{\frac{p+1}{2}} - C(\varepsilon)E') dt \leq \varepsilon \int_S^T E^{\frac{p+1}{2}} dt + C(\varepsilon)E(S),$$

so finally, we get, for some $C(\varepsilon)$ and C independent of $E(0)$ if $p = 1$

$$d \int_S^T E^{\frac{p+1}{2}} dt \leq C(\varepsilon)E(S) + \varepsilon C \int_S^T E^{\frac{p+1}{2}} dt.$$

Choosing now $\varepsilon C \leq \frac{d}{2}$, theorems results from Proposition 8. ■

4 Examples and numerical results

4.1 Examples

We here consider the case when Ω is a plane convex polygonal domain. The normal unit vector pointing outward of Ω is piecewise constant and the nature of boundary conditions involved by the multiplier method can be determined on each edge, independently of other edges.

Along each edge, vector ν is constant and the boundary conditions are defined by the sign of

$$m(x) \cdot \nu(x) = (R_\theta(x - x_0)) \cdot \nu(x) = (x - x_0) \cdot R_{-\theta}(\nu(x)).$$

Hence we build ν , $R_{-\theta}(\nu)$ and we can determine the sign of above coefficient with respect to the position of x_0 . To this end, we construct two straight lines, orthogonal with respect to $R_{-\theta}(\nu)$ so that each of them contains one vertex of the considered edge.

This determines a belt and if x_0 belongs to this belt, we obtained mixed boundary conditions along this edge, if x_0 does not belong to this belt, then we get Dirichlet or Neumann boundary conditions along whole the edge (see Figure 6).

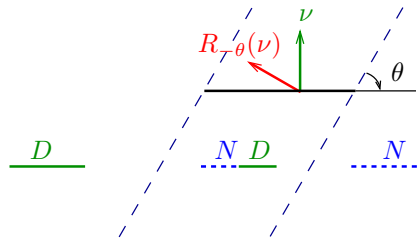


Figure 6: Boundary conditions along some edge depending on the position of x_0 .

Performing this method for a square, $\Omega = (0, 1)^2$, we show in Figure 7 the different cases of boundary conditions depending on the position of x_0 . Three main cases are considered

1. $0 < \theta < \frac{\pi}{4}$: above belts controlling opposite edges have a non-empty intersection, which is a belt of positive thickness,

2. $\theta = \frac{\pi}{4}$: this intersection is a straight line,
3. $\frac{\pi}{4} < \theta < \frac{\pi}{2}$: the intersection is empty.

The case when θ is negative can be easily deduced by symmetry.

In the three above cases, there are four angular sectors (shaded areas in Figure 7) such that if x_0 belongs to one of them, then geometrical condition (6) is satisfied.

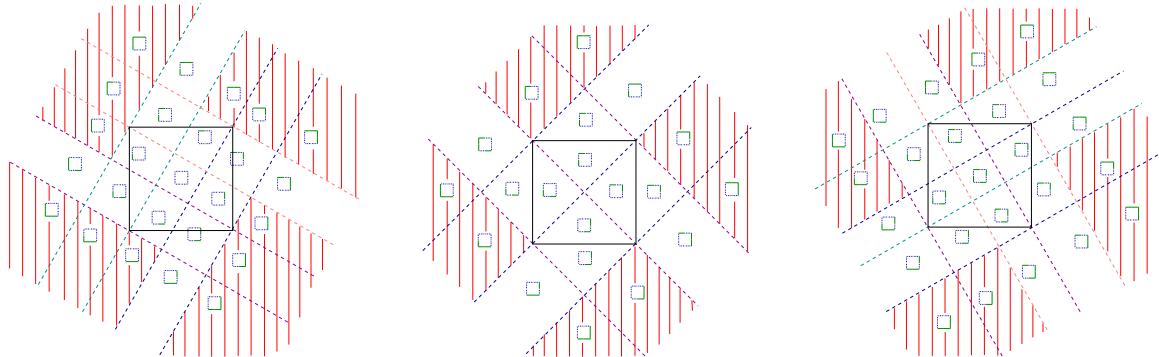


Figure 7: Shape of boundary data with respect to x_0 (from left to right, cases 1,2,3).

4.2 Numerical results

We perform numerical experiments by considering the following case

$$\Omega = (0, 1)^2, \quad \partial\Omega_D = (\{0\} \times [0, \frac{1}{2}]) \cup ([0, 1] \times \{0\}), \quad \partial\Omega_N = \partial\Omega \setminus \partial\Omega_D,$$

and using above vector field

$$m(x) = R_\theta(x - x_0).$$

We only consider the case of a linear feedback. Let us write down the problem.

$$\begin{cases} u'' - \Delta u = 0, & \text{in } \Omega \times \mathbb{R}_+^*, \\ u = 0, & \text{on } \partial\Omega_D \times \mathbb{R}_+^*, \\ \partial_\nu u = -(m \cdot \nu) u', & \text{on } \partial\Omega_N \times \mathbb{R}_+^*, \\ u(0) = u_0, & \text{in } \Omega, \\ u'(0) = u_1, & \text{in } \Omega. \end{cases}$$

We will investigate cases when θ varies in $[0, \arctan(2)]$. A particular case is given in Figure 8.

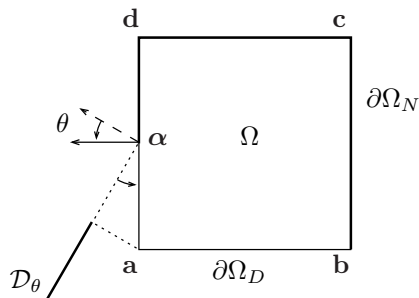


Figure 8: When x_0 belongs to \mathcal{D}_θ , geometrical condition (6) is satisfied at α .

Our aim here is to study numerically the variations of the speed of stabilization with respect to the position of x_0 and the value of θ .

To this end, we have built a finite differences scheme (in space). This leads to a linear second order differential equation

$$U'' + BU' + KU = 0, \quad (18)$$

where B is the feedback matrix and $-K$ is the discretized Laplace operator. This differential equation can be rewritten as follows

$$\begin{pmatrix} V \\ U' \end{pmatrix}' = \begin{pmatrix} 0 & K^{1/2} \\ -K^{1/2} & -B \end{pmatrix} \begin{pmatrix} V \\ U' \end{pmatrix}$$

and the energy function is approximated by $\frac{1}{2}(\langle U, KU \rangle + \|U'\|^2) = \frac{1}{2}(\|V\|^2 + \|U'\|^2)$ with $V = K^{1/2}U$. The decreasing rate is given by the highest eigenvalue of above matrix. Results of our computations are shown in Figure 9 where we built the decreasing rate as a function depending on θ and the position of x_0 represented by the abscissa λ along \mathcal{D}_θ .

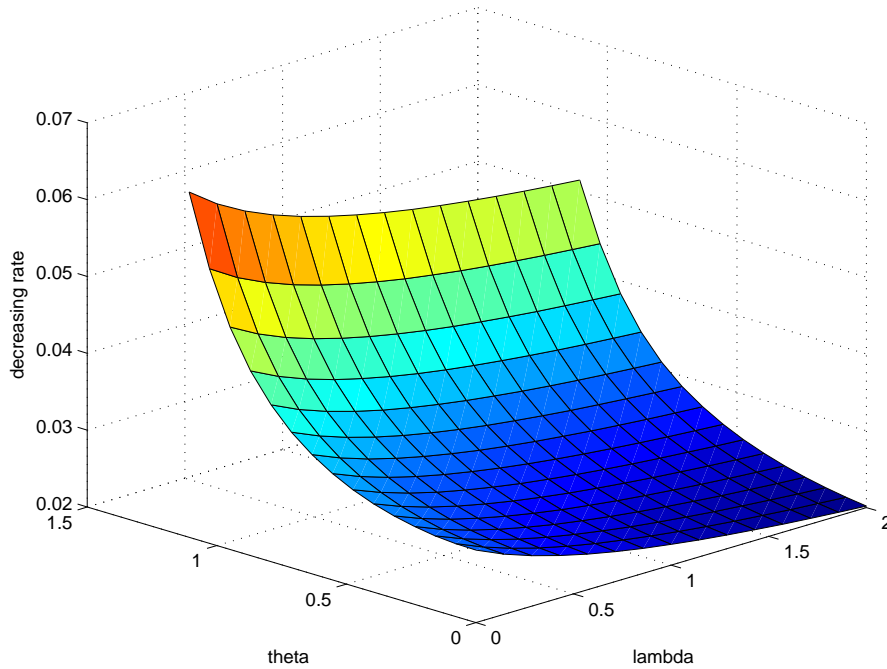


Figure 9: Dependence of the decreasing rate with respect to θ , λ .

It can be observed that in this case, the decreasing rate is increasing with θ and the best position for x_0 is the origin of half-line \mathcal{D}_θ .

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