



HAL
open science

Boundary stabilization of the wave equations by means of a rotated multiplier method.

Pierre Cornilleau, Jean-Pierre Lohéac, Axel Osses

► **To cite this version:**

Pierre Cornilleau, Jean-Pierre Lohéac, Axel Osses. Boundary stabilization of the wave equations by means of a rotated multiplier method.. 2007. hal-00178179v2

HAL Id: hal-00178179

<https://hal.science/hal-00178179v2>

Preprint submitted on 12 Oct 2007 (v2), last revised 8 Oct 2011 (v4)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Boundary stabilization of the wave equations by means of a rotated multiplier method.

Pierre Cornilleau, Jean-Pierre Lohéac and Axel Osses

Ecole Centrale de Lyon and Universidad de Chile

October 10, 2007

Abstract

The rotated multipliers method is performed in the case of the boundary stabilization by means of a (linear or non-linear) Neumann feedback. This method leads to new geometrical cases concerning the "active" part of the boundary where the feedback is applied. Due to mixed boundary conditions, these cases generate singularities. Under a simple geometrical condition concerning the orientation of boundary, we obtain a stabilization result in both cases.

1 Introduction

Let Ω be a bounded open connected set of \mathbb{R}^n ($n \geq 2$) such that, in the sense of Nečas ([9]), its boundary $\partial\Omega$ is of class \mathcal{C}^2 if $n \geq 3$ or a curvilinear polygon of class \mathcal{C}^2 if $n = 2$. We define a partition of $\partial\Omega$ in the following way. Let x_0 be a fixed point in \mathbb{R}^n . Let us denote by I the $n \times n$ identity matrix, by A a real $n \times n$ skew-symmetric matrix and by d a positive real number such that: $d^2 + \|A\|^2 = 1$. We now define the following vector function:

$$\forall x \in \mathbb{R}^n, \quad m(x) = (dI + A)(x - x_0). \quad (1)$$

Denoting by $\nu(x)$ the normal unit vector pointing outward of Ω at a point $x \in \partial\Omega$, we set:

$$\partial\Omega_N = \{x \in \partial\Omega, m(x) \cdot \nu(x) > 0\}, \quad \partial\Omega_D = \{x \in \partial\Omega, m(x) \cdot \nu(x) < 0\}, \quad (2)$$

and we suppose:

$$\left| \begin{array}{l} \text{meas}(\partial\Omega_D) \neq 0, \text{meas}(\partial\Omega_N) \neq 0, \\ \Gamma = \overline{\partial\Omega_D} \cap \overline{\partial\Omega_N} \text{ is a } \mathcal{C}^3\text{-manifold of dimension } n - 2, \end{array} \right. \quad (3)$$

and, if $n = 2$:

$$\text{each component of } \partial\Omega \setminus \Gamma \text{ is a } \mathcal{C}^2\text{-manifold of dimension } 1 \quad (4)$$

or, if $n \geq 3$, for a suitable neighbourhood ω of Γ :

$$\partial\Omega \cap \omega \text{ is a } \mathcal{C}^3\text{-manifold of dimension } n - 1. \quad (5)$$

Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous non-decreasing function such that:

$$\exists K > 0; 0 \leq g(s)s \leq Ks^2 \text{ a.e. } . \quad (6)$$

Let us now consider the following wave problem:

$$(S) \begin{cases} u'' - \Delta u = 0 & \text{in } \Omega \times \mathbb{R}_+^*, \\ u = 0 & \text{on } \partial\Omega_D \times \mathbb{R}_+^*, \\ \partial_\nu u = -(m \cdot \nu)g(u') & \text{on } \partial\Omega_N \times \mathbb{R}_+^*, \\ u(0) = u^0 & \text{in } \Omega, \\ u'(0) = u^1 & \text{in } \Omega, \end{cases}$$

where u' (resp. u'') is the first (resp. second) time-derivative of u , $\partial_\nu u = \nabla u \cdot \nu$ is the normal outward derivative of u on $\partial\Omega$, and initial data satisfy:

$$(u_0, u_1) \in H_D^1(\Omega) \times L^2(\Omega) := H$$

with $H_D^1(\Omega) = \{v \in H^1(\Omega); v = 0 \text{ on } \partial\Omega_D\}$.

This problem is well-posed in this space. Indeed, following Komornik([6]), we define the non-linear operator \mathcal{A} on H by $\mathcal{A}(u, v) = (-v, -\Delta u)$ on the domain:

$$D(\mathcal{A}) = \{(u, v) \in H_D^1(\Omega) \times H_D^1(\Omega); \Delta u \in L^2(\Omega) \text{ and } \partial_\nu u = -(m \cdot \nu)g(v) \text{ on } \partial\Omega_N\},$$

so that (S) can be written in the form:

$$\begin{cases} (u, v)' + \mathcal{A}(u, v) = 0; \\ (u, v)(0) = (u_0, u_1). \end{cases}$$

It is classical that \mathcal{A} is a maximal-monotone operator on H and that $D(\mathcal{A})$ is

dense in H for the usual norm. Using the works of Brézis, we can deduce that for any initial data $(u^0, v^0) \in D(\mathcal{A})$ there is a unique strong solution (u, v) such that $u \in W^{1,\infty}(\mathbb{R}_+; H_D^1(\Omega))$ and $\Delta u \in L^\infty(\mathbb{R}_+; L^2(\Omega))$. Moreover, for two initial datum, the corresponding solutions satisfy:

$$\forall t \geq 0, \|(u_1(t), v_1(t)) - (u_2(t), v_2(t))\|_H \leq \|(u_1^0, v_1^0) - (u_2^0, v_2^0)\|_H .$$

Using the density of $D(\mathcal{A})$, one can extend the map: $(u_0, u_1) \in D(\mathcal{A}) \mapsto (u(t), u'(t)) \in H$ to a strongly continuous semi-group of contractions $(S(t))_{t \geq 0}$ and define for $(u^0, v^0) \in H$ the weak solution $(u(t), v(t)) = S(t)(u^0, v^0)$ with the regularity $u \in C(\mathbb{R}_+; H_D^1(\Omega)) \cap C^1(\mathbb{R}_+; L^2(\Omega))$. We hence define the energy function of solutions by:

$$\begin{cases} E(u, 0) = \frac{1}{2} \int_\Omega (|u_1|^2 + |\nabla u_0|^2) dx, \\ E(u, t) = \frac{1}{2} \int_\Omega (|u'(t)|^2 + |\nabla u(t)|^2) dx \quad \text{if } t > 0. \end{cases}$$

Following Grisvard([5]), Rellich type relation([12]) is useful in control of the wave problem. Then, Komornik-Zuazua([7]) have shown how these relations can also help us to stabilize the wave problem, but failed to generalize it in higher dimension than 3. The key-problem was to show the existence of a Grisvard-like decomposition([4]) which can apply to stabilization problems or control problems in any dimension. The first results towards this direction are due to Moussaoui([8]), and Bey-Lohéac-Moussaoui([3])who also have established a Rellich type relation in any dimension. In this new case of Neumann feedback introduced by Osses([10],[11]), our goal is then to generalize those Rellich's relations to get stabilization results about (S).

Taking advantage of the works of Banasiak-Roach([1]) who generalized Grisvard's results([4]), we will see that in the case of the plane we have such a relation if the boundary is only piecewise differentiable, provided a condition on the angle ω_x of Γ at the point x :

$$\forall x \in \Gamma, 0 \leq \omega_x \leq \pi \text{ and if } \omega_x = \pi, m.\tau(x) \leq 0 \quad (7)$$

where $\tau(x)$ is the normal unit vector pointing outward of $\partial\Omega_N$ at a point $x \in \Gamma$ when considering $\partial\Omega_N$ as a sub-manifold of $\partial\Omega$.

We will then show how to generalize the proof of Bey-Lohéac-Moussaoui([3]) to get a Rellich inequality in dimension $n \geq 3$ under the following simple geometrical condition:

$$m.\tau \leq 0 \text{ on } \Gamma. \quad (8)$$

2 Rellich's relations

2.1 Plane case

Theorem 1 *Assume $n = 2$. Under the hypotheses (1)– (4), and assuming that $\forall x \in \Gamma, 0 \leq \omega_x \leq \pi$, let $(u, v) \in D(A)$. Then $2\partial_\nu u(m.\nabla u) - (m.\nu)|\nabla u|^2 \in L^1(\partial\Omega)$ and there exists some coefficients $(c_x)_{x \in \Gamma}$ such that:*

$$2 \int_{\Omega} \Delta u(m.\nabla u) dx = \int_{\partial\Omega} (2\partial_\nu u(m.\nabla u) - (m.\nu)|\nabla u|^2) d\sigma + \frac{\pi}{4} \sum_{x; \omega_x = \pi} c_x^2(m.\tau)(x).$$

Proof. We first begin by some general considerations which will be used in the general case two. Let $(u, v) \in D(A)$. It is a classical result that $u \in H^2(\omega)$ for every open domain ω such that $\omega \Subset \bar{\Omega} \setminus \Gamma$. For sake of completeness, let us recall the proof.

Using (6), it is obvious that $(m.\nu)g(v) \in H^{\frac{1}{2}}(\partial\Omega)$. A trace result shows that there exists $u_R \in H^2(\Omega)$ such that $u_R = 0$ on $\partial\Omega$ and $\partial_\nu u_R = (m.\nu)g(v)$ on $\partial\Omega$. So, letting $f = \Delta u_R - \Delta u \in L^2(\Omega)$, $u_S = u - u_R$ satisfies:

$$\begin{cases} -\Delta u_S = f & \text{in } \Omega, \\ u_S = 0 & \text{on } \partial\Omega_D, \\ \partial_\nu u_S = 0 & \text{on } \partial\Omega_N. \end{cases}$$

Now, if $\omega \Subset \Omega \setminus \Gamma \cup \partial\Omega_D$, and ξ is a cut-off function so that $\xi = 1$ on ω and $\text{supp}(\xi) \subset \Omega$ then, for a suitable $g \in L^2(\Omega)$, $u_\omega = u_S \xi$ is the solution of the Dirichlet problem:

$$\begin{cases} \Delta u_\omega = g & \text{on } \Omega, \\ u_\omega = 0 & \text{on } \partial\Omega, \end{cases}$$

and using classical method of difference quotients (see [4]), one can now conclude that $u_\omega \in H^2(\Omega)$, so $u_S \in H^2(\omega)$. Else, if $\omega \Subset \Omega \setminus \Gamma \cup \partial\Omega_N$, and ξ is a cut-off function so that $\xi = 1$ on ω and $\text{supp}(\xi) \subset \Omega$ then, for a suitable $g \in L^2(\Omega)$, $u_\omega = u_S \xi$ is the solution of the Neumann problem:

$$\begin{cases} -\Delta u_\omega + u_\omega = g & \text{on } \Omega, \\ \partial_\nu u_\omega = 0 & \text{on } \partial\Omega, \end{cases}$$

and, using the same argument, one has $u_S \in H^2(\omega)$.

Let $\Omega_\varepsilon = \{x \in \Omega; d(x, \Gamma) > \varepsilon\}$. By compactness of Ω_ε , we have $u \in H^2(\Omega_\varepsilon)$, and in this case Rellich's relation is shorter.

Lemma 2 *For any given $\varepsilon > 0$, we have the following identity:*

$$2 \int_{\Omega_\varepsilon} \Delta u(m \cdot \nabla u) dx = d(n-2) \int_{\Omega_\varepsilon} |\nabla u|^2 dx + \int_{\partial\Omega_\varepsilon} (2\partial_\nu u(m \cdot \nabla u) - (m \cdot \nu) |\nabla u|^2) d\sigma.$$

Proof. Using Green-Riemann identity we get:

$$2 \int_{\Omega_\varepsilon} \Delta u(m \cdot \nabla u) dx = \int_{\partial\Omega_\varepsilon} 2\partial_\nu u(m \cdot \nabla u) d\sigma - 2 \int_{\Omega_\varepsilon} \nabla u \cdot (m \cdot \nabla u) dx.$$

So, noting that $\nabla u \cdot (m \cdot \nabla u) = D^2 u(m, \nabla u) + (d\nabla u + A\nabla u) \cdot \nabla u = \frac{1}{2} m \cdot \nabla (|\nabla u|^2) + d|\nabla u|^2 + A\nabla u \cdot \nabla u$ and using the fact that A is skew-symmetric:

$$2 \int_{\Omega_\varepsilon} \Delta u(m \cdot \nabla u) dx = \int_{\partial\Omega_\varepsilon} 2\partial_\nu u(m \cdot \nabla u) d\sigma - 2d \int_{\Omega_\varepsilon} |\nabla u|^2 dx - \int_{\Omega_\varepsilon} m \cdot \nabla (|\nabla u|^2) dx.$$

With another use of Green-Riemann formula, we obtain the desired result for $\text{div}(m) = nd$. ■

Coming back to our particular situation, we have the following relation:

$$2 \int_{\Omega_\varepsilon} \Delta u(m \cdot \nabla u) dx = \int_{\partial\Omega_\varepsilon} (2\partial_\nu u(m \cdot \nabla u) - (m \cdot \nu) |\nabla u|^2) d\sigma$$

and we will try to let $\varepsilon \rightarrow 0$. Using derivative with respect to ν and τ , we have:

$$2 \int_{\Omega_\varepsilon} \Delta u(m \cdot \nabla u) dx = \int_{\partial\Omega_\varepsilon} (m \cdot \nu) ((\partial_\nu u)^2 - (\partial_\tau u)^2) d\sigma + 2 \int_{\partial\Omega_\varepsilon} (m \cdot \tau) (\partial_\nu u) (\partial_\tau u) d\sigma.$$

Firstly, because $\Delta u \in L^2(\Omega)$ and $u \in H^1(\Omega)$, Lebesgue's dominated convergence theorem immediatly gives:

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega_\varepsilon} \Delta u(m \cdot \nabla u) dx = \int_{\Omega} \Delta u(m \cdot \nabla u) dx$$

Now, we work on boundary terms. We shall need a partition of $\partial\Omega_\varepsilon$: we define $\widetilde{\partial\Omega_\varepsilon} = \partial\Omega_\varepsilon \cap \partial\Omega$, $\partial\Omega_\varepsilon^* = \partial\Omega_\varepsilon \cap \Omega$ and a decomposition result. Banasiak and Roach([1]) have shown that, under the hypothesis (4), every variational solution of a mixed boundary value problem with smooth data does have a Grisvard-like decomposition. Consequently, there exists some coefficients $(c_x)_{x \in \Gamma}$ and $u_R \in H^2(\Omega)$ such that:

$$u = u_R + \sum_{x \in \Gamma} c_x u_S^x \quad (9)$$

where u_S^x singular functions of $x \in \Gamma$ which, in a neighbourhood of x , are defined in local coordinates by:

$$u_S^x(r, \theta) = r^{\frac{\pi}{2\omega_x}} \sin\left(\frac{\pi}{2\omega_x}\theta\right).$$

Using the density of $C^1(\overline{\Omega})$ in $H^2(\Omega)$, we will be able to assume that $u_R \in C^1(\overline{\Omega})$.

Let's look at boundary terms on $\widetilde{\partial\Omega_\varepsilon}$ first. We claim first that for some constant $C > 0$,

$$|m \cdot \nu| \leq Cd(\cdot, \Gamma).$$

In fact, if $x \in \Omega$ and $x_1 \in \Gamma$ which satisfies $|x - x_1| = d(x, \Gamma)$, one has: $m \cdot \nu(x) = m(x) \cdot (\nu(x) - \nu(x_1)) + (m(x) - m(x_1)) \cdot \nu(x_1)$ (note that $m \cdot \nu(x_1) = 0$); so using the fact v is a piecewise C^1 function, we get: $|m \cdot \nu(x)| \leq (\|m\|_\infty \|\nu'\|_\infty + 1)d(x, \Gamma)$. Now, working in local coordinates, one has:

$$d(x, \Gamma)|\nabla u|^2 \in L^\infty(\partial\Omega)$$

so Lebesgue's dominated convergence theorem implies

$$\lim_{\varepsilon \rightarrow 0} \int_{\widetilde{\partial\Omega_\varepsilon}} (m \cdot \nu) ((\partial_\nu u)^2 - (\partial_\tau u)^2) d\sigma = \int_{\partial\Omega} (m \cdot \nu) ((\partial_\nu u)^2 - (\partial_\tau u)^2) d\sigma.$$

On the otherhand, if we note that $|(\partial_\nu u)(\partial_\tau u)| \leq C\|m\|_\infty(d(\cdot, \Gamma)|\nabla u|^2)^{\frac{1}{2}}|v|$ on $\partial\Omega$, it is then obvious that:

$$\int_{\widetilde{\partial\Omega_\varepsilon}} (m \cdot \tau)(\partial_\nu u)(\partial_\tau u) d\sigma \rightarrow \int_{\partial\Omega} (m \cdot \tau)(\partial_\nu u)(\partial_\tau u) d\sigma.$$

Now, we have to consider boundary terms on $\partial\Omega_\varepsilon^*$. We first note that, if ε is sufficiently small: $\partial\Omega_\varepsilon^* = \coprod_{x \in \Gamma} C_\varepsilon(x)$ with $C_\varepsilon(x)$ an arc-circle centered in x and of radius ε . We can then write

$$\int_{\partial\Omega_\varepsilon^*} (2\partial_\nu u(m \cdot \nabla u) - (m \cdot \nu)|\nabla u|^2) d\sigma = \sum_{x \in \Gamma} \int_{C_\varepsilon(x)} (2\partial_\nu u(m \cdot \nabla u) - (m \cdot \nu)|\nabla u|^2) d\sigma.$$

So, using decomposition (10), the left term in this equality $I_\varepsilon(u)$ can be split in:

$$I_\varepsilon(u) = \sum_{x \in \Gamma} c_x^2 I_1^x(\varepsilon) + 2 \sum_{x \in \Gamma} c_x I_2^x(\varepsilon) + 2 \sum_{x \neq y \in \Gamma} c_x c_y I^{x,y}(\varepsilon)$$

with:

$$\begin{aligned}
I_1^x(\varepsilon) &= \int_{C_\varepsilon(x)} (2\partial_\nu u_S^x(m.\nabla u_S^x) - (m.\nu)|\nabla u_S^x|^2) d\sigma, \\
I_2^x(\varepsilon) &= \int_{C_\varepsilon(x)} (2\partial_\nu u_R(m.\nabla u_S^x) - (m.\nu)(\nabla u_S^x.\nabla u_R)) d\sigma, \\
I^{x,y}(\varepsilon) &= \int_{C_\varepsilon(x)} (2\partial_\nu u_S^y(m.\nabla u_S^x) - (m.\nu)(\nabla u_S^x.\nabla u_S^y)) d\sigma.
\end{aligned}$$

We will consider $I_1^x(\varepsilon)$ first.

If $\omega_x < \pi$, one has

$$2\partial_\nu u_S^x(m.\nabla u_S^x) - (m.\nu)|\nabla u_S^x|^2 = O(\varepsilon^{\frac{\pi}{2\omega_x}-2}) \text{ on } C_\varepsilon(x),$$

so, after integrating on $C_\varepsilon(x)$, we get $\lim_{\varepsilon \rightarrow 0} I_1^x(\varepsilon) = 0$.

If $\omega_x = \pi$, we will need the following identity:

$$2\partial_\nu u_S^x(m.\nabla u_S^x) - (m.\nu)|\nabla u_S^x|^2 = \frac{1}{4\varepsilon} m.\tau(x) \text{ on } C_\varepsilon(x).$$

If we note that $C_\varepsilon(x)$ behaves like an half-circle when $\varepsilon \rightarrow 0$, an integration brings

$$\lim_{\varepsilon \rightarrow 0} \int_{C_\varepsilon(x)} 2(\nu.\nabla u_S^x)(m.\nabla u_S^x) - (m.\nu)|\nabla u_S^x|^2 d\sigma = \frac{\pi}{4} m(x).\tau(x).$$

Considering $I_2^x(\varepsilon)$ and $I^{x,y}(\varepsilon)$ now, we have, thanks to Cauchy-Schwarz inequality, estimates of the form:

$$\begin{aligned}
& C \left(\int_{C_\varepsilon(x)} |\nabla u_S^x|^2 d\sigma \right)^{\frac{1}{2}} \left(\int_{C_\varepsilon(x)} |\nabla u_R|^2 d\sigma \right)^{\frac{1}{2}} \\
& \text{or } C \left(\int_{C_\varepsilon(x)} |\nabla u_S^x|^2 d\sigma \right)^{\frac{1}{2}} \left(\int_{C_\varepsilon(x)} |\nabla u_S^y|^2 d\sigma \right)^{\frac{1}{2}}.
\end{aligned}$$

If we note that $\int_{C_\varepsilon(x)} |\nabla u_S^x|^2 d\sigma = O(1)$ and $\nabla u_R, \nabla u_S^y u$ are bounded near x , we have then proved that these terms also tend to 0 when $\varepsilon \rightarrow 0$. The proof is now complete. ■

2.2 General case

Theorem 3 *Assume $n \geq 3$. Under the hypotheses (1) – (4), let $(u, v) \in D(A)$. Then, $2\partial_\nu u(m.\nabla u) - (m.\nu)|\nabla u|^2 \in L^1(\partial\Omega)$ and there exists $\zeta \in H^{\frac{1}{2}}(\Gamma)$ such that:*

$$\begin{aligned}
2 \int_{\Omega} \Delta u(m.\nabla u) dx &= d(n-2) \int_{\Omega} |\nabla u|^2 dx + \int_{\partial\Omega} (2\partial_\nu u(m.\nabla u) - (m.\nu)|\nabla u|^2) d\sigma \\
&\quad + \int_{\Gamma} (m.\tau)|\zeta|^2 d\gamma.
\end{aligned}$$

Proof. We will essentially follow Bey-Lohéac-Moussaoui([3]) for the proof of this relation. As in the plane case, we let $\Omega_\varepsilon = \{x \in \Omega; d(x, \Gamma) > \varepsilon\}$. For any given $\varepsilon > 0$, we have the identity of lemma 2:

$$2 \int_{\Omega_\varepsilon} \Delta u(m \cdot \nabla u) dx = d(n-2) \int_{\Omega_\varepsilon} |\nabla u|^2 dx + \int_{\partial\Omega_\varepsilon} (2\partial_\nu u(m \cdot \nabla u) - (m \cdot \nu) |\nabla u|^2) d\sigma,$$

and we will again analyse the behaviour of each term as $\varepsilon \rightarrow 0$.

Firstly, since $\Delta u \in L^2(\Omega)$ and $u \in H^1(\Omega)$, Lebesgue's dominated convergence immediatly gives

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega_\varepsilon} \Delta u(m \cdot \nabla u) dx = \int_{\Omega} \Delta u(m \cdot \nabla u) dx$$

and

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega_\varepsilon} |\nabla u|^2 dx = \int_{\Omega} |\nabla u|^2 dx.$$

Next, we will work on boundary terms. We define again $\widetilde{\partial\Omega_\varepsilon} = \partial\Omega_\varepsilon \cap \partial\Omega$ and $\partial\Omega_\varepsilon^* = \partial\Omega_\varepsilon \cap \Omega$. As in the plane case, there exists some constant $C > 0$ such that $|m \cdot \nu| \leq Cd(\cdot, \Gamma)$. Thus, using the fact that $d(\cdot, \Gamma) |\nabla u|^2 \in L^1(\partial\Omega)$ (see [3], Proposition 3), we can use again Lebesgue's theorem to conclude that, when $\varepsilon \rightarrow 0$,

$$\int_{\widetilde{\partial\Omega_\varepsilon}} (m \cdot \nu) |\nabla u|^2 d\sigma \rightarrow \int_{\partial\Omega} (m \cdot \nu) |\nabla u|^2 d\sigma.$$

For the second integral, if we note that

$$|\partial_\nu u(m \cdot \nabla u)| \leq (|(m \cdot \nu)v|^2 \|m\|_\infty^2 |\nabla u|^2)^{\frac{1}{2}} \leq C \|m\|_\infty^{\frac{3}{2}} (d(\cdot, \Gamma) |\nabla u|^2)^{\frac{1}{2}} |v| \text{ on } \partial\Omega,$$

Lebesgue's theorem gives again, when $\varepsilon \rightarrow 0$,

$$\int_{\widetilde{\partial\Omega_\varepsilon}} \partial_\nu u(m \cdot \nabla u) d\sigma \rightarrow \int_{\partial\Omega} \partial_\nu u(m \cdot \nabla u) d\sigma.$$

Now, we consider the terms on $\partial\Omega_\varepsilon^*$. We put:

$$I_\varepsilon(u) = 2 \int_{\partial\Omega_\varepsilon^*} \partial_\nu u(m \cdot \nabla u) d\sigma - \int_{\partial\Omega_\varepsilon^*} (m \cdot \nu) |\nabla u|^2 d\sigma.$$

To begin with, we shall use some definitions. For $\varepsilon > 0$ sufficiently small, every $x \in \partial\Omega_\varepsilon^*$ belongs to a unique plane $x^* + \langle \tau^*, \nu^* \rangle$ (putting: $\tau^* = \tau(x^*)$, $\nu^* = \nu(x^*)$) and more precisely to an arc-circle $C_\varepsilon(x^*)$ in this plane of center $x^* \in \Gamma$ and of radius ε . We put: $\omega_\varepsilon := \Omega \setminus \Omega_\varepsilon$ and $D_\varepsilon(x^*) := \omega_\varepsilon \cap (x^* + \langle \tau^*, \nu^* \rangle)$. We split $\nabla u(x) = \nabla_T u(x) + \nabla_2 u(x) \in T_{x^*} \Gamma \oplus \langle \tau^*, \nu^* \rangle$. This leads to a splitting of $I_\varepsilon(u) = I_\varepsilon^1(u) + 2I_\varepsilon^2(u) + I_\varepsilon^3(u)$ with:

$$\begin{aligned} I_\varepsilon^1(u) &= 2 \int_{\partial\Omega_\varepsilon^*} (\nu \cdot \nabla_T u)(m \cdot \nabla_T u) d\sigma - \int_{\partial\Omega_\varepsilon^*} (m \cdot \nu) |\nabla_T u|^2 d\sigma, \\ I_\varepsilon^2(u) &= \int_{\partial\Omega_\varepsilon^*} (\nu \cdot \nabla_2 u)(m \cdot \nabla_T u) + (\nu \cdot \nabla_T u)(m \cdot \nabla_2 u) - (m \cdot \nu) (\nabla_T u \cdot \nabla_2 u) d\sigma, \\ I_\varepsilon^3(u) &= \int_{\partial\Omega_\varepsilon^*} 2(\nu \cdot \nabla_2 u)(m \cdot \nabla_2 u) - (m \cdot \nu) |\nabla_2 u|^2 d\sigma. \end{aligned}$$

We will need the following result of behaviour of boundary integrals:

Lemma 4 *Let u be such that for all $x^* \in \Gamma$, $u(x^*, \cdot) \in H^1(D_{\varepsilon_0}(x^*))$ and $\|u(x^*, \cdot)\|_{H^1(D_{\varepsilon_0}(x^*))} \in L^2(\Gamma)$ for a fixed $\varepsilon_0 > 0$. Assume also that $u = 0$ on $\partial\omega_{\varepsilon_0} \cap \partial\Omega_D$, we have then for some $C > 0$ depending only on Ω and for any ε sufficiently small:*

$$\int_{\Gamma} \|u(x^*, \cdot)\|_{L^2(C_\varepsilon(x^*))}^2 d\gamma \leq C\varepsilon \int_{\Gamma} \|u(x^*, \cdot)\|_{H^1(D_\varepsilon(x^*))}^2 d\gamma.$$

Proof. We begin by using a change of coordinates as in [3]. For every $x_0^* \in \Gamma$, there exists $\rho_0 > 0$, Θ a C^2 diffeomorphism from an open neighbourhood W of x_0^* to $B(\rho_0) := B_{n-2}(\rho_0) \times B_2(\rho_0)$ such that:

$$\begin{aligned} \Theta(x_0^*) &= 0, \\ \Theta(W \cap \Omega) &= \{y \in B(\rho_0); y_n > 0\}, \\ \Theta(W \cap \partial\Omega_D) &= \{y \in B(\rho_0); y_{n-1} > 0, y_n = 0\}, \\ \Theta(W \cap \partial\Omega_N) &= \{y \in B(\rho_0); y_{n-1} < 0, y_n = 0\}, \\ \Theta(W \cap \Gamma) &= \{y \in B(\rho_0); y_{n-1} = 0, y_n = 0\} := \gamma(\rho_0). \end{aligned}$$

Reducing ε_0 if necessary, we can assume that $D_{\varepsilon_0}(x_0^*) \subset W$. We have then, writing for $x \in W$, $\Theta(x) = (y^*, Y)$ and $v := u \circ \Theta^{-1}$:

$$\int_{W \cap \Gamma} \left(\int_{C_\varepsilon(x^*)} u^2 dl \right) d\gamma(x^*) = \int_{\gamma(\rho_0)} \left(\int_{\Theta(C_\varepsilon(x^*))} v^2 dl(Y) \right) dy^*.$$

Setting

$$B_2^+(\rho) := \{Y = (y_1, y_2) \in B_2(\rho); y_2 > 0\}, \quad C_2^+(\rho) := \{Y = (y_1, y_2) \in \partial B_2(\rho); y_2 > 0\},$$

we first note that we can choose ρ_{x^*} such that $\{y^*\} \times B_2^+(\rho) \subset \Theta(D_\varepsilon(x^*))$. Consequently, one has, using the C^1 diffeomorphism

$$Y \in \pi_2(\Theta(C_\varepsilon(x^*))) \mapsto Y' = \rho \frac{Y}{\|Y\|} \in C_2^+(\rho),$$

the estimate:

$$\int_{\Theta(C_\varepsilon(x^*))} v^2(y^*, Y) dl(Y) \leq C \int_{C_2^+(\rho)} v^2(y^*, Y') dl(Y') \quad (10)$$

for a constant C depending only on x_0^* .

We will now estimate this latter integral in terms of $\|\nabla_2 v\|_{L^2(\{y^*\} \times B_2^+(\rho))}$. If $v_\rho := v(y^*, \rho \cdot)$, one has $\nabla v_\rho \in L^2(B_2^+(1))$ and:

$$\|\nabla v_\rho\|_{L^2(B_2^+(1))} = \|\nabla_2 v\|_{L^2(\{y^*\} \times B_2^+(\rho))}, \quad \|v_\rho\|_{L^2(C_2^+(1))} = \rho^{-\frac{1}{2}} \|v\|_{L^2(\{y^*\} \times C_2^+(\rho))}.$$

Using now the fact that $v_\rho = 0$ on $B_2^{++}(1) := \{(y_1, y_2) \in B_2^+(1); y_1 > 0\}$, trace theorem and Poincaré's inequality give, for some universal constant $C > 0$, the estimate:

$$\int_{C_2^+(\rho)} v^2(y^*, Y') dl(Y') \leq C\rho \|\nabla_2 v\|_{L^2(\{y^*\} \times B_2^+(\rho))}^2.$$

So using (10), one gets

$$\int_{\Theta(C_\varepsilon(x^*))} v^2(y^*, Y) dl(Y) \leq C\rho_{x^*} \|\nabla_2 v\|_{L^2(\{y^*\} \times B_2^+(\rho_{x^*}))}^2.$$

Using the fact that ρ_{x^*} is uniformly $O(\varepsilon)$ on $W \cap \Gamma$ and the diffeomorphism $\Theta(x^*, \cdot)$, we can conclude that, for some constant $C_{x_0^*}$ depending only on x_0^* :

$$\begin{aligned} \int_{\Theta(C_\varepsilon(x^*))} v^2(y^*, Y) dl(Y) &\leq C_{x_0^*} \varepsilon \|u(x^*, \cdot)\|_{H^1(\Theta^{-1}(\{y^*\} \times B_2^+(\rho))}^2 \\ &\leq C_{x_0^*} \varepsilon \|u(x^*, \cdot)\|_{H^1(D_\varepsilon(x^*))}^2, \end{aligned}$$

so, after an integration on $W \cap \Gamma$:

$$\int_{W \cap \Gamma} \left(\int_{C_\varepsilon(x^*)} u^2 dl \right) d\gamma(x^*) \leq C_{x_0^*} \int_{W \cap \Gamma} \|u(x^*, \cdot)\|_{H^1(D_\varepsilon(x^*))}^2 d\gamma.$$

The proof finally ends using of a partition of unity on the open sets $(W_{x_0^*})_{x_0^* \in \Gamma}$.

■

Now, it suffices to show that $\nabla_T u$ satisfies the hypotheses of lemma 4 to get that $\lim_{\varepsilon \rightarrow 0} I_\varepsilon^1(u) = 0$. First, it is obvious that $\nabla_T u = 0$ on $\partial\omega_{\varepsilon_0} \cap \partial\Omega_D$. The other part is stated in the first part of theorem 4 in [3], thanks to Borel-Lebesgue theorem.

To treat $I_\varepsilon^2(u)$, we will have to use the initial work by Moussaoui([8]). Thanks to a suitable diffeomorphism and using again Borel-Lebesgue theorem, we can write $u = u_R + \eta \otimes u_S$ with u_S locally diffeomorphic to Shamir's function([13]), $\eta(x^*, \cdot) \in L^\infty(D_{\varepsilon_0}(x^*))$ for all $x^* \in \Gamma$, $\|\eta(x^*, \cdot)\|_\infty \in L^2(\Gamma)$ and $u_R \in H^2(\omega_{\varepsilon_0})$.

Working by approximation if necessary, we can suppose that $u_R \in C^1(\overline{\omega_{\varepsilon_0}})$. Now, using Fubini's theorem, one can write:

$$\begin{aligned} I_\varepsilon^2(u) = & \int_\Gamma \left(\int_{C_\varepsilon(x^*)} (\nu \cdot \nabla_T u)(m \cdot \nabla_2 u_R) dl \right) d\gamma + \int_\Gamma \left(\int_{C_\varepsilon(x^*)} (m \cdot \nabla_T u)(\nu \cdot \nabla_2 u_R) dl \right) d\gamma \\ & - \int_\Gamma \left(\int_{C_\varepsilon(x^*)} (m \cdot \nu)(\nabla_T u \cdot \nabla_2 u_R) dl \right) d\gamma + \int_\Gamma \left(\int_{C_\varepsilon(x^*)} \eta(\nu \cdot \nabla_T u)(m \cdot \nabla_2 u_S) dl \right) d\gamma \\ & + \int_\Gamma \left(\int_{C_\varepsilon(x^*)} \eta(m \cdot \nabla_T u)(\nu \cdot \nabla_2 u_S) dl \right) d\gamma - \int_\Gamma \left(\int_{C_\varepsilon(x^*)} \eta(m \cdot \nu)(\nabla_T u \cdot \nabla_2 u_S) dl \right) d\gamma. \end{aligned}$$

For the three first terms of this sum, Cauchy-Schwarz inequality brings an estimate of the form

$$C \left(\int_{\partial\Omega_\varepsilon^*} |\nabla_T u|^2 d\sigma \right)^{\frac{1}{2}} \left(\int_\Gamma \left(\int_{C_\varepsilon(x^*)} |\nabla_2 u_R|^2 dl \right) d\gamma \right)^{\frac{1}{2}}.$$

Using lemma 4 again to $\nabla_T u$, noting that the other integral term is dominated by $\|\nabla_2 u_R\|_\infty (2\pi\gamma(\Gamma)\varepsilon)^{\frac{1}{2}}$, we have now proved that these terms tend to 0 when $\varepsilon \rightarrow 0$. For the three last terms, a simple calculus shows that $\int_{C_\varepsilon(x^*)} |\nabla_2 u_S|^2 d\ell \leq 2\pi$. Consequently, Cauchy-Schwarz inequality brings an estimate of the form

$$C \left(\int_{\partial\Omega_\varepsilon^*} |\nabla_T u|^2 d\sigma \right)^{\frac{1}{2}} \left(\int_{\partial\Omega_\varepsilon^*} \eta^2 d\gamma \right)^{\frac{1}{2}}$$

and we have proved that these terms also tend to 0.

Using Fubini's theorem, the last term $I_\varepsilon^3(u)$ may be written

$$\begin{aligned} I_\varepsilon^3(u) = & \int_\Gamma \left(\int_{C_\varepsilon(x^*)} \eta^2 (2(\nu \cdot \nabla_2 u_S)(m \cdot \nabla_2 u_S) - (m \cdot \nu) |\nabla_2 u_S|^2) d\ell \right) d\gamma \\ & + 2 \int_\Gamma \left(\int_{C_\varepsilon(x^*)} \eta ((\nu \cdot \nabla_2 u_R)(m \cdot \nabla_2 u_S) - (m \cdot \nu)(\nabla_2 u \cdot \nabla_2 u_R) + (\nu \cdot \nabla_2 u_S)(m \cdot \nabla_2 u_R)) d\ell \right) d\gamma \\ & + \int_\Gamma \left(\int_{C_\varepsilon(x^*)} 2(\nu \cdot \nabla_2 u_R)(m \cdot \nabla_2 u_R) - (m \cdot \nu) |\nabla_2 u_R|^2 d\ell \right) d\gamma. \end{aligned}$$

The same technique as above shows that the two last integrals tend to 0 when $\varepsilon \rightarrow 0$. As in the plane case, and working in coordinates $x^* + \langle \tau^*, -\nu^* \rangle$, we have:

$$\lim_{\varepsilon \rightarrow 0} \int_{C_\varepsilon(x^*)} 2(\nu \cdot \nabla_2 u_S)(m \cdot \nabla_2 u_S) - (m \cdot \nu) |\nabla_2 u_S|^2 d\ell = \frac{\pi}{4} m \cdot \tau(x^*).$$

Moreover, this integral term is dominated by $\frac{\pi}{2} \|m\|_\infty$; so dominated convergence theorem applies and finally:

$$\lim_{\varepsilon \rightarrow 0} I_\varepsilon^3(u) = \frac{\pi}{4} \int_\Gamma \eta^2 m \cdot \tau d\gamma.$$

The proof is now complete with $\zeta = \frac{\sqrt{\pi}}{2} \eta$. ■

We will now see some applications of Rellich's relations to the stabilization of solutions of (S).

3 Linear and non-linear stabilization

We begin by a classical elementary result due to Komornik([6]):

Lemma 5 *Let $E : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ a non-decreasing function such that there exists $\alpha \geq 0$ and $C > 0$ which fulfill:*

$$\forall t \geq 0, \int_t^\infty E^{\alpha+1}(s) ds \leq CE(t).$$

Then, putting $T = CE^\alpha(0)$, one has:

$$\begin{aligned} \text{if } \alpha &= 0: & E(t) &\leq E(0) \exp\left(1 - \frac{t}{T}\right) \quad \forall t \geq T, \\ \text{if } \alpha &> 0: & E(t) &\leq E(0) \left(\frac{T + \alpha T}{T + \alpha t}\right)^{\frac{1}{\alpha}} \quad \forall t \geq T. \end{aligned}$$

We will now show the following stabilization result:

Theorem 6 Assume (1) – (3), (6) and $n = 2$, (4), (7) or $n \geq 3$, (5), (8), and suppose that there exist $p \in \mathbb{N}^*$ and $k > 0$ such that

$$\forall s \in \mathbb{R}, |g(s)| \geq k \min(|s|, |s|^p). \quad (11)$$

Then for every $(u_0, u_1) \in H$, there exists $T > 0$ such that the energy of the solution u of (S) satisfies:

$$\begin{aligned} \text{if } p = 1: \quad E(u, t) &\leq E(u, 0) \exp\left(1 - \frac{dt}{C}\right) \quad \forall t > \frac{T}{d}, \\ \text{if } p > 1: \quad E(u, t) &\leq Ct^{2/(1-p)} \quad \forall t > \frac{T}{d}, \end{aligned}$$

where C depends on the initial energy $E(u, 0)$ in the second case, not in the first.

Proof. Following [6], we will prove the estimates for $(u_0, u_1) \in D(\mathcal{A})$ which, using density argument, will be sufficient. Putting $Mu = 2m \cdot \nabla u + d(n-1)u$, we prove the following result:

Lemma 7 For any $0 \leq S < T < \infty$, one has:

$$\begin{aligned} 2d \int_S^T E^{\frac{p+1}{2}} dt &\leq - \left[E^{\frac{p-1}{2}} \int_{\Omega} u' M u dx \right]_S^T + \frac{p-1}{2} \int_S^T E^{\frac{p-3}{2}} E' \left(\int_{\Omega} u' M u dx \right) dt \\ &\quad + \int_S^T E^{\frac{p-1}{2}} \left(\int_{\partial\Omega_N} (m \cdot \nu) ((u')^2 - |\nabla u|^2 - g(u') M u) d\sigma \right) dt. \end{aligned}$$

Proof. Using the fact that u is solution of (S) and noting that $u'' M u = (u' M u)' - u' M u'$, an integration by parts gives:

$$\begin{aligned} 0 &= \int_S^T E^{\frac{p-1}{2}} \int_{\Omega} (u'' - \Delta u) M u dx dt \\ &= \left[E^{\frac{p-1}{2}} \int_{\Omega} u' M u dx \right]_S^T - \frac{p-1}{2} \int_S^T E^{\frac{p-3}{2}} E' \left(\int_{\Omega} u' M u dx \right) dt \\ &\quad - \int_S^T E^{\frac{p-1}{2}} \int_{\Omega} (u' M u' + \Delta u M u) dx dt. \end{aligned}$$

Now, thanks to theorems 1, 3 and (7) or (8), we have :

$$\begin{aligned} \int_{\Omega} \Delta u M u dx &\leq d(n-1) \int_{\Omega} \Delta u u dx + d(n-2) \int_{\Omega} |\nabla u|^2 dx \\ &\quad + \int_{\partial\Omega} (2\partial_{\nu} u (m \cdot \nabla u) - (m \cdot \nu) |\nabla u|^2) d\sigma. \end{aligned}$$

Consequently, Green-Riemann formula leads to:

$$\int_{\Omega} \Delta u M u dx \leq -d \int_{\Omega} |\nabla u|^2 dx + \int_{\partial\Omega} (\partial_{\nu} u M u - (m \cdot \nu) |\nabla u|^2) d\sigma.$$

Using boundary conditions and the fact that $\nabla u = \partial_\nu u \nu$ on $\partial\Omega_D$, we have then:

$$\int_{\Omega} \Delta u M u dx \leq -d \int_{\Omega} |\nabla u|^2 dx - \int_{\partial\Omega_N} (m \cdot \nu)(g(u') M u + |\nabla u|^2) d\sigma.$$

On the other hand, using $\operatorname{div}(m) = nd$, an other use of Green formula gives us:

$$\int_{\Omega} u' M u' dx = -d \int_{\Omega} (u')^2 dx + \int_{\partial\Omega_N} (m \cdot \nu) |u'|^2 d\sigma.$$

The proof is now complete. \blacksquare

Thanks to Young's inequality, one gets: $|\int_{\Omega} u' M u dx| \leq CE(t)$. So, lemma 7 shows that:

$$\begin{aligned} 2d \int_S^T E^{\frac{p+1}{2}} dt &\leq C(E^{\frac{p+1}{2}}(T) + E^{\frac{p+1}{2}}(S)) + C \int_S^T E^{\frac{p-1}{2}} E' dt \\ &\quad + \int_S^T E^{\frac{p-1}{2}} \left(\int_{\partial\Omega_N} (m \cdot \nu) ((u')^2 - |\nabla u|^2 - g(u') M u) d\sigma \right) dt \end{aligned}$$

Set $d\sigma_m = (m \cdot \nu) d\sigma$. If we note that $E'(t) = -\int_{\partial\Omega_N} g(u') u' d\sigma_m \leq 0$, we have then for a constant $C > 0$ independent of $E(0)$ if $p = 1$:

$$2d \int_S^T E^{\frac{p+1}{2}} dt \leq CE(S) + \int_S^T E^{\frac{p-1}{2}} \left(\int_{\partial\Omega_N} (u')^2 - |\nabla u|^2 - g(u') M u d\sigma_m \right) dt.$$

Using the definition of Mu and Young's inequality, we get for any $\varepsilon > 0$:

$$2d \int_S^T E^{\frac{p+1}{2}} dt \leq CE(S) + \int_S^T E^{\frac{p-1}{2}} \left(\int_{\partial\Omega_N} (u')^2 + (\|m\|_{\infty}^2 + \frac{d^2(n-1)^2}{4\varepsilon}) g(u')^2 + \varepsilon u^2 d\sigma_m \right) dt.$$

Now, using Poincaré's inequality, we can choose $\varepsilon > 0$ such that:

$$\varepsilon \int_{\partial\Omega_N} (m \cdot \nu) u^2 d\sigma \leq \frac{d}{2} \int_{\Omega} |\nabla u|^2 dx \leq dE,$$

so we conclude that:

$$d \int_S^T E^{\frac{p+1}{2}} dt \leq CE(S) + C \int_S^T E^{\frac{p-1}{2}} \left(\int_{\partial\Omega_N} (u')^2 + g(u')^2 d\sigma_m \right) dt.$$

We are going to make a partition of $\partial\Omega_N$ to majorize the two last terms of this estimation. Set $\partial\Omega_N^1 = \{x \in \partial\Omega_N; |u'(x)| > 1\}$ and $\partial\Omega_N^2 = \{x \in \partial\Omega_N; |u'(x)| \leq 1\}$.

Using (6) and (11), we get:

$$\begin{aligned} \int_S^T E^{\frac{p-1}{2}} \left(\int_{\partial\Omega_N^1} (u')^2 + g(u')^2 d\sigma_m \right) dt &\leq C \int_S^T E^{\frac{p-1}{2}} \left(\int_{\partial\Omega_N} u' g(u') d\sigma_m \right) dt, \\ &\leq CE(S), \end{aligned}$$

where C neither depend on $E(0)$ if $p = 1$.

On the other hand, using (6), (11) and Jensen's inequality, one successively obtains:

$$\begin{aligned} \int_{\partial\Omega_N^2} (u')^2 + g(u')^2 d\sigma_m &\leq C \int_{\partial\Omega_N^2} (u'g(u'))^{2/(p+1)} d\sigma_m, \\ &\leq C \left(\int_{\partial\Omega_N^2} u'g(u') d\sigma_m \right)^{\frac{2}{p+1}}, \\ &\leq C(-E')^{\frac{2}{p+1}}. \end{aligned}$$

Hence, using Young's inequality again, we get for every $\varepsilon > 0$:

$$\begin{aligned} \int_S^T E^{\frac{p-1}{2}} \left(\int_{\partial\Omega_N^2} (u')^2 + g(u')^2 d\sigma_m \right) dt &\leq \int_S^T \varepsilon E^{\frac{p+1}{2}} - C(\varepsilon)E' dt, \\ &\leq \varepsilon \int_S^T E^{\frac{p+1}{2}} dt + C(\varepsilon)E(S), \end{aligned}$$

so finally, we have for some $C(\varepsilon)$ and C independent of $E(0)$ if $p = 1$:

$$d \int_S^T E^{\frac{p+1}{2}} dt \leq C(\varepsilon)E(S) + \varepsilon C \int_S^T E^{\frac{p+1}{2}} dt.$$

Choosing now $\varepsilon C < d$, the theorem results from lemma 5. ■

References

- [1] BANASIAK, J., ROACH, G.-F., 1989, On mixed boundary value problems of Dirichlet oblique-derivative type in plane domains with piecewise differentiable boundary. *J. Diff. Equations*, **79**, no 1, 111-131.
- [2] BARDOS, C., LEBEAU, G., RAUCH, J., 1992, Sharp sufficient conditions for the observation, control and stabilization of waves from the boundary. *SIAM J. Control Optim.*, **30**, no 5, 1024-1065.
- [3] BEY, R., LOHÉAC, J.-P., MOUSSAOUI, M., 1999, Singularities of the solution of a mixed problem for a general second order elliptic equation and boundary stabilization of the wave equation. *J. Math. pures et appl.*, **78**, 1043-1067.
- [4] GRISVARD, P., 1985, Elliptic problems in nonsmooth domains. *Pitman, London*.
- [5] GRISVARD, P., 1989, Contrôlabilité exacte des solutions de l'équation des ondes en présence de singularités. *J. Math. pures et appl.*, **68**, 215-259.
- [6] V. KOMORNIK, 1994, Exact controllability and stabilization ; the multiplier method. *Masson-John Wiley, Paris*.

- [7] KOMORNIK, V., ZUAZUA, E., 1990, A direct method for the boundary stabilization of the wave equation. *J. Math. pures et appl.*, **69**, 33-54.
- [8] MOUSSAOUI M., 1996, Singularités des solutions du problème mêlé, contrôlabilité exacte et stabilisation frontière. *ESAIM Proceedings, Élasticité, Viscoélasticité et Contrôle optimal, Huitièmes Entretiens du Centre Jacques Cartier*, pp 157-168.
- [9] NEČAS, J., 1967, Les méthodes directes en théorie des équations elliptiques. *Masson, Paris*.
- [10] OSSES, A., 1998, Une nouvelle famille de multiplicateurs et applications à la contrôlabilité exacte de l'équation des ondes. *C. R. Acad. Sci. Paris*, **326**, Série I, 1099-1104.
- [11] OSSES, A., 2001, A rotated multiplier applied to the controllability of waves, elasticity and tangential Stokes control. *SIAM J. Control Optim.*, **40**, no 3, 777-800.
- [12] RELICH F., 1940, Darstellung der Eigenwerte von $\Delta u + \lambda u$ durch ein Randintegral. *Math. Zeitschrift*, **46**, 635-636.
- [13] SHAMIR, E., 1968, Regularity of mixed second order elliptic problems. *Israel Math. Journal*, **6**, 150-168.