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# ON THE ŁOJASIEWICZ EXPONENTS OF QUASI-HOMOGENEOUS FUNCTIONS

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**Résumé.** Les fonctions quasi-homogènes, en particulier les polynômes, possèdent des propriétés particulières au voisinage de l'origine qui permettent d'estimer les exposants dits de Łojasiewicz d'une manière tout à fait analogue à ceux des fonctions homogènes. En particulier on généralise un résultat antérieur pour les polynômes à deux variables concernant l'inégalité du gradient de Łojasiewicz optimale à l'origine.

**Abstract.** Quasi-homogeneous functions, and especially polynomials, enjoy some specific properties around the origin which allow to estimate the so-called Łojasiewicz exponents in a way quite similar to homogeneous functions. In particular we generalize a previous result for polynomials of two variables concerning the optimal Łojasiewicz gradient inequality at the origin.

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## 1. INTRODUCTION

In his pioneering papers [8], [9], Łojasiewicz established that any analytic function  $f$  of  $n$  real variables satisfies an inequality of the form

$$(1.1) \quad \|\nabla f(x)\| \geq c|f(x) - f(a)|^\beta$$

for  $\|x - a\|$  small enough with  $c > 0$ ,  $\beta \in (0, 1)$ . This inequality, known as the Łojasiewicz gradient inequality, is useful to establish trend to equilibrium of the general solutions of gradient systems

$$(1.2) \quad u' + \nabla f(u) = 0$$

and can also be used to evaluate the rate of convergence. It is therefore of interest to know as precisely as possible the connection between  $f$  and its gradient and in particular to determine the best(smallest) possible value of  $\beta$  in (1.1) when  $a$  is a critical point of  $f$ . This value is called the Łojasiewicz exponent at  $a$ . In [5] for instance, it was shown that if  $f$  is a homogeneous polynomial with degree  $d \geq 2$ , the Łojasiewicz exponent at the origin is exactly  $1 - \frac{1}{d}$  when  $n = 2$ . This property is no longer true if  $n > 2$ .

On the other hand, Gwoździwicz [4] considered the case of a real analytic function at an isolated zero and also found, in this case, an interesting relationship between various Łojasiewicz exponents, relative to different Łojasiewicz inequalities. In addition the case of general polynomials has been thoroughly investigated by D'Acunto and Kurdyka in [2].

Our paper is concerned to the extension of the result from [5] and several estimates of Łojasiewicz exponents at the origin when  $f$  is a quasi-homogeneous map (see, for example, [1]). It is divided in 5 sections. In Section 2, we state and prove some preliminary results, mainly concerning the local behavior of quasi-homogeneous maps near the origin. Section 3 contains more information in the specific case where the origin is an isolated zero of  $f$ . Section 4 deals with the Łojasiewicz gradient exponent of quasi-homogeneous polynomials, in particular we generalize the main result of [5]. Section 5 contains more precise estimates for quasi-homogeneous polynomials of two variables. These results are illustrated by typical examples and completed by a few remarks.

## 2. DEFINITIONS AND PRELIMINARY RESULTS

We first recall the concept of a quasi-homogeneous map. Let  $\mathbb{K} = \mathbb{C}$  or  $\mathbb{R}$ . We say that  $f: \mathbb{K}^n \rightarrow \mathbb{K}^k$  is a *(positively) quasi-homogeneous map with weight  $w :=$*

$(w_1, w_2, \dots, w_n) \in (\mathbb{R}_+ - \{0\})^n$  and quasi-degree  $d := (d_1, d_2, \dots, d_k) \in (\mathbb{R}_+ - \{0\})^k$  if

$$(2.1) \quad f_i(t^{w_1}x_1, t^{w_2}x_2, \dots, t^{w_n}x_n) = t^{d_i} f_i(x_1, x_2, \dots, x_n)$$

for each  $i = 1, 2, \dots, k$ , and all  $t > 0$ . Note that if  $w_j = 1$  for  $j = 1, 2, \dots, n$  then the above definition means that  $f_i$  are homogeneous functions of degree  $d_i$ . A function  $f_i$  satisfying (2.1) is called a (positively) quasi-homogeneous function with weight  $w = (w_1, w_2, \dots, w_n)$  and quasi-degree  $d_i$ . In the sequel we shall drop for simplicity the word ‘‘positively’’. Note that any monomial  $x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}$  is a quasi-homogeneous function with arbitrary weight  $w = (w_1, w_2, \dots, w_n) \in \mathbb{N} - \{0\}^n$  and quasi-degree  $\langle w, \alpha \rangle := w_1 \alpha_1 + w_2 \alpha_2 + \dots + w_n \alpha_n$ . Moreover, we have

**Proposition 2.1.** *Let  $f: \mathbb{K}^n \rightarrow \mathbb{K}$  be a polynomial function. Then  $f$  is quasi-homogeneous with weight  $w \in (\mathbb{N} - \{0\})^n$  and quasi-degree  $m \in \mathbb{N} - \{0\}$  if and only if all its constitutive monomials are quasi-homogeneous functions with weight  $w$  and quasi-degree  $m$ .*

*Proof.* Suppose that  $f$  is a quasi-homogeneous polynomial with weight  $w$  and quasi-degree  $m$ . We have the following finite expansion

$$f(x) := \sum_{\alpha} a_{\alpha} x^{\alpha}.$$

Then

$$\sum_{\alpha} a_{\alpha} t^{\langle w, \alpha \rangle} x^{\alpha} = t^m \sum_{\alpha} a_{\alpha} x^{\alpha}.$$

This gives

$$\sum_{\alpha} a_{\alpha} [t^{\langle w, \alpha \rangle - m} - 1] x^{\alpha} = 0$$

for all  $x \in \mathbb{K}^n$  and for all  $t > 0$ .

By a fundamental theorem of algebra, this identity implies that

$$\langle w, \alpha \rangle - m = 0$$

for all  $\alpha$  provided  $a_{\alpha} \neq 0$ . In other words, all constitutive monomials of  $f$  are quasi-homogeneous functions with weight  $w$  and quasi-degree  $m$ . The converse is clear.  $\square$

For a fixed weight  $w := (w_1, w_2, \dots, w_n) \in (\mathbb{R}_+ - \{0\})^n$  we set

$$\|x\|_w := \max_{j=1,2,\dots,n} |x_j|^{\frac{1}{w_j}}$$

for  $x := (x_1, x_2, \dots, x_n) \in \mathbb{K}^n$ . In the special case  $w_j = 1$  for all  $j = 1, 2, \dots, n$ , let  $\|x\| := \|x\|_w$ .

**Remark 2.2.** (i) It is worth noting that, in general,  $\|\cdot\|_w$  is not a norm.

(ii) It is easy to see that  $\|\cdot\|_w$  is a quasi-homogeneous function with weight  $w$  and quasi-degree 1.

The following basic properties will be used throughout the text.

**Proposition 2.3.** *Let*

$$\begin{aligned} w_* &:= \min_{j=1,2,\dots,n} w_j, \\ w^* &:= \max_{j=1,2,\dots,n} w_j. \end{aligned}$$

*Then the following hold*

(i) *For all  $\|x\| \geq 1$  we have*

$$\|x\|^{\frac{1}{w_*}} \geq \|x\|_w \geq \|x\|^{\frac{1}{w^*}}.$$

*In particular,  $\|x\| \rightarrow \infty$  if and only if  $\|x\|_w \rightarrow \infty$ .*

(ii) *For all  $\|x\| \leq 1$  we have*

$$\|x\|^{\frac{1}{w_*}} \leq \|x\|_w \leq \|x\|^{\frac{1}{w^*}},$$

*In particular,  $\|x\| \rightarrow 0$  if and only if  $\|x\|_w \rightarrow 0$ .*

*Proof.* The proof of the proposition is clear from the definitions. In fact, it follows from  $0 < w_* \leq w_j \leq w^*$  that

$$\frac{1}{w_*} \geq \frac{1}{w_j} \geq \frac{1}{w^*} > 0.$$

Hence, for  $|x_j| \geq 1$  we have

$$|x_j|^{\frac{1}{w_*}} \geq |x_j|^{\frac{1}{w_j}} \geq |x_j|^{\frac{1}{w^*}},$$

and for  $|x_j| \leq 1$  we have

$$|x_j|^{\frac{1}{w_*}} \leq |x_j|^{\frac{1}{w_j}} \leq |x_j|^{\frac{1}{w^*}}.$$

The results follow easily. □

In the sequel for  $t > 0$ , for any  $w := (w_1, w_2, \dots, w_n) \in (\mathbb{R}_+ - \{0\})^n$  and  $x := (x_1, x_2, \dots, x_n) \in \mathbb{K}^n$  we denote

$$t \bullet x := (t^{w_1} x_1, t^{w_2} x_2, \dots, t^{w_n} x_n),$$

and for  $d := (d_1, d_2, \dots, d_k) \in (\mathbb{R}_+ - \{0\})^k$  we set

$$\begin{aligned} d_* &:= \min_{i=1,2,\dots,k} d_i, \\ d^* &:= \max_{i=1,2,\dots,k} d_i. \end{aligned}$$

Let  $f: \mathbb{K}^n \rightarrow \mathbb{K}^k$  be quasi-homogeneous with weight  $w := (w_1, w_2, \dots, w_n) \in (\mathbb{R}_+ - \{0\})^n$  and quasi-degree  $d := (d_1, d_2, \dots, d_k) \in (\mathbb{R}_+ - \{0\})^k$ . If  $f_i \equiv 0$  for some  $i \in \{1, 2, \dots, k\}$ , then  $d_i$  can be replaced by any positive number. In the sequel we shall assume

$$(2.2) \quad \forall i \in \{1, 2, \dots, k\}, \quad f_i \not\equiv 0.$$

It is easy to check that in this case  $d_i$  is uniquely defined by (2.1) for all  $i \in \{1, 2, \dots, k\}$ . Then  $d_*$  and  $d^*$  are well defined.

The next two results summarize some important consequences of the quasi homogeneity property.

**Proposition 2.4.** *Let  $f := (f_1, f_2, \dots, f_k): \mathbb{K}^n \rightarrow \mathbb{K}^k$  be a quasi-homogeneous map with weight  $w := (w_1, w_2, \dots, w_n)$  and quasi-degree  $d := (d_1, d_2, \dots, d_k)$  satisfying (2.2). Then the following properties are equivalent.*

- (i) *The origin is an isolated zero of  $f$ .*
- (ii)  $f^{-1}(0) = \{0\}$ .

*Proof.* It is clear that (ii) implies (i). Conversely if (ii) is not satisfied, let  $a \neq 0$  be such that  $f(a) = 0$ . This implies that  $f_1(a) = f_2(a) = \dots = f_k(a) = 0$ . Hence

$$f_i(t \bullet a) = t^{d_i} f_i(a) = 0$$

for  $i = 1, 2, \dots, k$  and all  $t > 0$ . Note that  $\|t \bullet a\| \rightarrow 0$  as  $t \rightarrow 0$ . Thus, the origin is not an isolated zero of  $f$ , which is a contradiction.  $\square$

**Theorem 2.5.** *Let  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  be a quasi-homogeneous polynomial with weight  $w$  and quasi-degree  $m$ . Suppose that  $n \geq 2$ . Then the following conditions are equivalent*

- (i)  $f^{-1}(0) = \{0\}$ .
- (ii)  $f$  has a strict global extremum at the origin.
- (iii) For each  $\epsilon \geq 0$  we have  $\{x \in \mathbb{R}^n \mid |f(x)| = \epsilon\}$  is non-empty compact set.
- (iv)  $\min_{\|x\|=1} |f(x)| > 0$ .

*Proof.* (i)  $\Rightarrow$  (ii) If  $f^{-1}(0) = \{0\}$ , then by connectedness  $f$  has a constant sign (for instance  $f > 0$ ) on the unit euclidian sphere  $\mathbb{S} := \{x \in \mathbb{R}^n \mid \|x\| = 1\}$  of dimension  $(n - 1)$  if  $n > 1$ . But for any  $x \neq 0$ , there is clearly  $t > 0$  such that  $y := t \bullet x$  is in  $\mathbb{S}$ . Indeed the euclidian norm of  $t \bullet x$  is 0 for  $t = 0$ , tends to infinity with  $t$  and is a continuous function of  $t$ , hence it must take the value 1 for some finite positive  $t$ . Then  $f(x) = f(t^{-1} \bullet y) = t^{-m} f(y) > 0$ , which proves (ii).

(ii)  $\Rightarrow$  (i) is clear.

(i)  $\Rightarrow$  (iii) By contradiction, assume that the set  $\{x \in \mathbb{R}^n \mid |f(x)| = \epsilon\}$  is not compact for some  $\epsilon > 0$ . This means that there exists a sequence  $x^p \in \mathbb{R}^n$  such that

$\|x^p\| \rightarrow \infty$  and  $|f(x^p)| = \epsilon$ . Let  $t_p := \frac{1}{\|x^p\|_w} \rightarrow 0$ . Then the sequence  $|f(t_p \bullet x^p)| = t_p^m \epsilon$  tends to zero as  $k \rightarrow \infty$ . From the sequence of points  $t_p \bullet x^p$  lying on the compact set  $\{\|x\|_w = 1\}$  one can choose a subsequence convergent to some  $a$ ,  $\|a\|_w = 1$ . Clearly,  $f(a) = 0$  and  $a \neq 0$ , which contradicts the claim (i).

(iii)  $\Rightarrow$  (i) If  $f(a) = 0$  for some  $a \neq 0$ , then  $f(t \bullet a) = 0$  for all  $t > 0$ . This gives  $f^{-1}(0)$  is non-compact.

(i)  $\Rightarrow$  (iv) Suppose, by contradiction, that  $\min_{\|x\|=1} |f(x)| = 0$ . Then there exists a point  $a \in \mathbb{R}^n$  such that  $\|a\| = 1$  and  $f(a) = 0$ . This implies that  $f(t \bullet a) = 0$  for all  $t > 0$ , which contradicts (i).

(iv)  $\Rightarrow$  (i) If  $f(a) = 0$  for some  $a \neq 0$ , then  $f(t \bullet a) = 0$  for all  $t > 0$ . Consequently,  $\min_{\|x\|=1} |f(x)| = 0$ , which contradicts (iv).  $\square$

**Remark 2.6.** (i) Checking that a given polynomial function is lower (or upper) bounded function is far from trivial (see [11]).

(ii) Checking that a given hypersurface  $f^{-1}(\epsilon)$  is compact set is far from trivial (see [12]).

### 3. THE ŁOJASIEWICZ INEQUALITY FOR A QUASI-HOMOGENEOUS MAP WHICH VANISHES ONLY AT THE ORIGIN

In this section we are interested in the first Łojasiewicz inequality which relates in general the size of  $f(u)$  and the distance of  $u$  to the set  $f^{-1}(0)$ . However we essentially restrict our study to the case where this set is reduced to 0.

**Proposition 3.1.** *Let  $f := (f_1, f_2, \dots, f_k): \mathbb{K}^n \rightarrow \mathbb{K}^k$  be a continuous quasi homogeneous map with weight  $w := (w_1, w_2, \dots, w_n)$  and quasi-degree  $d := (d_1, d_2, \dots, d_k)$ . Then the following statements hold.*

(i) *There exists a positive constant  $c_1$  such that<sup>1</sup>*

$$\|f(x)\| \leq c_1 \|x\|_w^{d_*}, \quad \text{as } \|x\| \ll 1.$$

(ii) *If  $f^{-1}(0) = \{0\}$ , then there exists a positive constant  $c_2$  such that*

$$c_2 \|x\|_w^{d_*} \leq \|f(x)\|, \quad \text{as } \|x\| \ll 1.$$

*Proof.* Consider the family of topological closed spheres

$$S_t := \{(x_1, x_2, \dots, x_n) \in \mathbb{K}^n \mid \|x\|_w = t\}.$$

Then, by Proposition 2.3(i), for each  $t > 0$  we have that  $S_t$  is a compact set. Let  $x$  be an element of  $\mathbb{K}^n$ ,  $x \neq 0$ . Let  $t := \frac{1}{\|x\|_w}$ . Then it is easy to check that  $t \bullet x \in S_1$ .

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<sup>1</sup> $\varphi(x) \ll \psi(x)$  (when  $x$  tends to 0) means that  $\lim_{x \rightarrow 0} \frac{\varphi(x)}{\psi(x)} = 0$ .

(i) Let

$$c_1 := \max_{i=1,2,\dots,k} \max_{x \in S_1} |f_i(x)|.$$

We have the estimate

$$c_1 \geq |f_i(t \bullet x)| = |t^{d_i} f_i(x)| \quad \text{for } i = 1, 2, \dots, k.$$

It follows that

$$c_1 \|x\|_w^{d_i} \geq |f_i(x)| \quad \text{for } i = 1, 2, \dots, k.$$

Consequently,

$$c_1 \max_{i=1,2,\dots,k} \|x\|_w^{d_i} \geq \|f(x)\|.$$

But

$$\|x\|_w^{d^*} = \max_{i=1,2,\dots,k} \|x\|_w^{d_i}, \quad \text{as } \|x\| \ll 1.$$

Thus

$$c_1 \|x\|_w^{d^*} \geq \|f(x)\|, \quad \text{as } \|x\| \ll 1,$$

which proves (i).

(ii) Let  $c_2 := \min_{x \in S_1} \|f(x)\| > 0$ . By definition, we have

$$\begin{aligned} c_2 := \min_{x \in S_1} \|f(x)\| &\leq \|f(t \bullet x)\| = \max_{i=1,2,\dots,k} |t^{d_i} f_i(x)| \\ &\leq \max_{i=1,2,\dots,k} |t^{d_i}| \max_{i=1,2,\dots,k} |f_i(x)|. \end{aligned}$$

Hence

$$c_2 \leq |t^{d^*}| \|f(x)\|$$

for  $t := \frac{1}{\|x\|_w}$  large enough. Consequently,

$$c_2 \|x\|_w^{d^*} \leq \|f(x)\| \quad \text{for } \|x\| \ll 1,$$

which proves (ii). The proposition is proved.  $\square$

**Theorem 3.2.** *Let  $f := f^0 + f^1 + \dots + f^l: \mathbb{K}^n \rightarrow \mathbb{K}^k$ , where  $f^0, f^1, \dots, f^l$  are continuous quasi-homogeneous maps with weight  $w = (w_1, w_2, \dots, w_n)$  and quasi-degrees  $d^0, d^1, \dots, d^l$  respectively such that*

$$(d^0)^* < (d^1)_* \leq (d^2)_* \leq \dots \leq (d^l)_*.$$

*If the origin is an isolated zero of  $f^0$  then there exists a positive constant  $c$  such that*

$$c \|x\|_w^{(d^0)^*} \leq \|f(x)\|, \quad \text{as } \|x\| \ll 1.$$

*Proof.* By Proposition 3.1(ii), there exists a positive constant  $c_0$  such that

$$c_0 \|x\|_w^{(d^0)*} \leq \|f^0(x)\|, \quad \text{as } \|x\| \ll 1.$$

On the other hand, from Proposition 3.1(i), there exist positive constants  $c_1, c_2, \dots, c_l$  such that for  $i = 1, 2, \dots, l$ ,

$$\|f^i(x)\| \leq c_i \|x\|_w^{(d^i)*}, \quad \text{as } \|x\| \ll 1.$$

We have for  $\|x\| \ll 1$  the next estimate

$$\begin{aligned} \|f^1(x) + f^2(x) + \dots + f^l(x)\| &\leq \|f^1(x)\| + \|f^2(x)\| + \dots + \|f^l(x)\| \\ &\leq c_1 \|x\|_w^{(d^1)*} + c_2 \|x\|_w^{(d^2)*} + \dots + c_l \|x\|_w^{(d^l)*}. \end{aligned}$$

Thus it follows from  $(d^0)* < (d^1)* \leq (d^2)* \leq \dots \leq (d^l)*$  that

$$\|f^1(x) + f^2(x) + \dots + f^l(x)\| \ll \|x\|_w^{(d^0)*}, \quad \text{as } \|x\| \ll 1.$$

Therefore, we have for  $\|x\| \ll 1$  the next inequality

$$\begin{aligned} \|f(x)\| &\geq \|f^0(x)\| - \|f^1(x) + f^2(x) + \dots + f^l(x)\| \\ &\geq c_0 \|x\|_w^{(d^0)*} - c' \|x\|_w^{(d^0)*} \quad (0 < c' \ll c_0). \end{aligned}$$

This gives

$$\|f(x)\| \geq (c_0 - c') \|x\|_w^{(d^0)*}, \quad \text{as } \|x\| \ll 1,$$

which proves the theorem.  $\square$

The following is a direct consequence from Proposition 3.1 and Theorem 3.2:

**Corollary 3.3.** *Under the hypothesis of Theorem 3.2, there exists a positive constant  $c$  such that*

$$c \|x\|_w^{\frac{(d^0)*}{w_*}} \leq \|f(x)\|, \quad \text{as } \|x\| \ll 1.$$

We define the *Lojasiewicz exponent*  $\alpha_0(f)$  of the map  $f$  = the infimum of the set of all real numbers  $l > 0$  which satisfy the condition: there exists a positive constant  $c$  such that

$$c \|x\|^l \leq \|f(x)\|, \quad \text{as } \|x\| \ll 1.$$

If the set of all the exponents is empty we put  $\alpha_0(f) := +\infty$ .

**Corollary 3.4.** *Let  $f := (f_1, f_2, \dots, f_k): \mathbb{K}^n \rightarrow \mathbb{K}^k$  be a continuous quasi-homogeneous map with weight  $w := (w_1, w_2, \dots, w_n)$  and quasi-degree  $d := (d_1, d_2, \dots, d_k)$ . Suppose that  $f^{-1}(0) = \{0\}$ . Then*

$$\frac{d_*}{w_*} \leq \alpha_0(f) \leq \frac{d^*}{w_*}.$$

*Proof.* It follows from Corollary 3.3 that

$$\alpha_0(f) \leq \frac{d^*}{w_*}.$$

In order to prove the left inequality, let  $i, j$  be such that  $d_i = d_*$  and  $w_j = w_*$ . Take  $a \in \mathbb{K}^n$  with the property that  $a_j f_i(a) \neq 0$ . Then, asymptotically as  $t \rightarrow 0$ , we have<sup>2</sup>

$$\begin{aligned} \|f(t \bullet a)\| &\simeq t^{d_*}, \\ \|t \bullet a\| &\simeq t^{w_*}. \end{aligned}$$

Consequently,

$$\|f(t \bullet a)\| \simeq \|t \bullet a\|^{\frac{d_*}{w_*}}.$$

By the definition of the Łojasiewicz exponent  $\alpha_0(f)$ , we find that

$$\frac{d_*}{w_*} \leq \alpha_0(f).$$

□

**Example 3.5.** (i) Let  $f := (f_1 := x^2 + y^4, f_2 := (x^2 - y^4)^2): \mathbb{R}^2 \rightarrow \mathbb{R}^2$ . It is easy to check that  $f$  is a positive quasi-homogeneous map with weight  $w := (2, 1)$  and quasi-degree  $d := (4, 8)$ . Moreover,  $\alpha_0(f) = 4 (= \frac{d_*}{w_*})$ .

(ii) Let  $f := (f_1 := x^2 - y^4, f_2 := (x^2 + y^4)^2): \mathbb{R}^2 \rightarrow \mathbb{R}^2$ . Then  $f$  is a positive quasi-homogeneous map with weight  $w := (2, 1)$  and quasi-degree  $d := (4, 8)$ . Moreover,  $\alpha_0(f) = 8 (= \frac{d^*}{w_*})$ .

**Corollary 3.6.** *Let  $f: \mathbb{K}^n \rightarrow \mathbb{K}$  be a continuous quasi-homogeneous function with weight  $w$  and quasi-degree  $m$ . If  $f^{-1}(0) = \{0\}$  then*

$$\alpha_0(f) = \frac{m}{w_*}.$$

*Proof.* The claim comes from  $d^* = d_* = m$ . □

#### 4. THE ŁOJASIEWICZ GRADIENT INEQUALITY FOR QUASI-HOMOGENEOUS POLYNOMIALS

We now consider the case  $k = 1$  and let  $f: \mathbb{K}^n \rightarrow \mathbb{K}$  be a  $C^1$  quasi-homogeneous function with weight  $w := (w_1, w_2, \dots, w_n)$  and quasi-degree  $m$ :

$$f(t^{w_1} x_1, t^{w_2} x_2, \dots, t^{w_n} x_n) = t^m f(x_1, x_2, \dots, x_n).$$

---

<sup>2</sup>Where  $A \simeq B$  means that  $A/B$  lies between two positive constants.

We define the *Lojasiewicz gradient exponent*  $\beta_0(f)$  of the map  $f$  as the infimum of the set of all real numbers  $l > 0$  which satisfy the condition: there exists a positive constant  $c$  such that

$$c|f(x)|^l \leq \|\nabla f(x)\| \quad \text{for} \quad \|x\| \ll 1.$$

If the set of all the exponents is empty we put  $\beta_0(f) := +\infty$ .

We start with a general result valid for  $C^1$  functions.

**Theorem 4.1.** *Let  $f: \mathbb{K}^n \rightarrow \mathbb{K}$  be a  $C^1$  quasi-homogeneous function with weight  $w$  and quasi-degree  $m \geq w^*$ . Then*

$$1 - \frac{w^*}{m} \leq \beta_0(f) \leq 1$$

*the inequality on the right being strict if  $f$  is analytic.*

*Proof.* Since  $f$  is a  $C^1$ -positive quasi-homogeneous function with weight  $w$  and quasi-degree  $m \geq w^*$ ,

$$mt^{m-1}f(x) = \sum_{j=1}^n w_j t^{w_j-1} x_j \frac{\partial f}{\partial x_j}(t \bullet x).$$

In particular, we have the generalized Euler identity

$$(4.1) \quad mf(x) = \sum_{j=1}^n w_j x_j \frac{\partial f}{\partial x_j}(x).$$

As a consequence, there exists a positive constant  $c_1$  such that

$$c_1|f(x)| \leq \|x\| \|\nabla f(x)\| \quad \text{for all} \quad \|x\| \ll 1.$$

This implies that  $\beta_0(f) < 1$ , so  $\beta_0(f)$  is finite. And hence we can write

$$(4.2) \quad c_2|f(x)|^{\beta_0(f)} \leq \|\nabla f(x)\| \quad \text{for all} \quad \|x\| \ll 1$$

for some  $c_2 > 0$ .

We have for all  $t > 0$  the following relation

$$t^{w_j} \frac{\partial f}{\partial x_j}(t \bullet x) = t^m \frac{\partial f}{\partial x_j}(x) \quad \text{for} \quad j = 1, 2, \dots, n.$$

This shows that

$$(4.3) \quad \frac{\partial f}{\partial x_j}(t \bullet x) = t^{m-w_j} \frac{\partial f}{\partial x_j}(x) \quad \text{for} \quad j = 1, 2, \dots, n.$$

Then we have for all  $t > 0$  the following equations

$$\begin{aligned} f(t \bullet x) &= t^m f(x), \\ \frac{\partial f}{\partial x_j}(t \bullet x) &= t^{m-w_j} \frac{\partial f}{\partial x_j}(x) \quad \text{for} \quad j = 1, 2, \dots, n. \end{aligned}$$

Since  $f \not\equiv 0$  there is  $a \in \mathbb{K}^n$  such that  $0 \neq \nabla f^2(a) = 2f(a)\nabla f(a)$ . This implies  $f(a) \neq 0$  and  $\nabla f(a) \neq 0$ . Then, asymptotically as  $t \rightarrow +0$ , we have

$$\begin{aligned} |f(t \bullet a)| &\simeq t^m, \\ \|\nabla f(t \bullet a)\| &\simeq t^{m-w_j} \quad \text{for some } j \in \{1, 2, \dots, n\}, \end{aligned}$$

Therefore (4.2) implies the existence of  $c_3 > 0$  such that

$$c_3 t^{m\beta_0(f)} \leq t^{m-w_j} \leq t^{m-w^*}$$

as  $t \rightarrow +0$ . This in turn implies that  $m\beta_0(f) \geq m - w^*$ , which is equivalent to

$$\beta_0(f) \geq 1 - \frac{w^*}{m}.$$

□

In the special case where 0 is the only critical point of  $f$  we have a more precise estimation as follows.

**Corollary 4.2.** *Let  $f: \mathbb{K}^n \rightarrow \mathbb{K}$  be a  $C^1$  quasi-homogeneous function with weight  $w$  and quasi-degree  $m$ . Suppose that  $\nabla f^{-1}(0) = \{0\}$ . Then*

$$1 - \frac{w^*}{m} \leq \beta_0(f) \leq 1 - \frac{w_*}{m}.$$

*Proof.* One has only to show that

$$\beta_0(f) \leq 1 - \frac{w_*}{m}.$$

Indeed, by the generalized Euler identity (4.1), there exists a positive constant  $c_1$  such that

$$|f(x)| \leq c_1 \|\nabla f(x)\| \|x\|.$$

On the other hand, it follows from (4.3) that the following

$$\nabla f(x): \mathbb{K}^n \rightarrow \mathbb{K}^n, \quad x \mapsto \left( \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n} \right),$$

is a continuous quasi-homogeneous map with weight  $w$  and quasi-degree  $(m - w_1, m - w_2, \dots, m - w_n)$ .

Therefore, by Proposition 2.3 and then Proposition 3.1(ii), for all  $\|x\| \ll 1$  we have

$$\begin{aligned} |f(x)| &\leq c_1 \|\nabla f(x)\| \|x\|_w^{w_*} \\ &\leq c_2 \|\nabla f(x)\| \|\nabla f(x)\|^{\frac{w_*}{m-w_*}} = c_2 \|\nabla f(x)\|^{1+\frac{w_*}{m-w_*}}, \end{aligned}$$

for some  $c_2 > 0$ . Hence there exists a positive constant  $c$  such that

$$c|f(x)|^{1-\frac{w_*}{m}} \leq \|\nabla f(x)\| \quad \text{for } \|x\| \ll 1.$$

Consequently, by the definition of the Lojasiewicz gradient exponent  $\beta_0(f)$ , we obtain

$$\beta_0(f) \leq 1 - \frac{w_*}{m},$$

which completes the proof.  $\square$

**Example 4.3.** (i) (see [7]). Let  $f: \mathbb{C}^2 \rightarrow \mathbb{C}, (x, y) \mapsto x^3 + 3xy^k, k \geq 3$ , be a complex polynomial. It is clear that  $f$  is a quasi-homogeneous polynomial with weight  $w = (k, 2)$  and quasi-degree  $m = 3k$ . A direct computation shows that the origin in  $\mathbb{C}^2$  is an isolated critical point of  $f$ . Moreover, it follows from the results in [7] and [13] that

$$\begin{aligned} \alpha_0(\nabla f) &= \frac{3k}{2} - 1, \\ \beta_0(f) &= \frac{\alpha_0(\nabla f)}{1 + \alpha_0(\nabla f)} = 1 - \frac{2}{3k}. \end{aligned}$$

(ii) Let  $f: \mathbb{C}^2 \rightarrow \mathbb{C}, (x, y) \mapsto x^4 - 4xy$ , be a complex polynomial. It is clear that  $f$  is a quasi-homogeneous polynomial with weight  $w = (1, 3)$  and quasi-degree  $m = 4$ . A direct computation shows that the origin in  $\mathbb{C}^2$  is an isolated critical point of  $f$ . Moreover, it follows from the results in [7] and [13] that

$$\begin{aligned} \alpha_0(\nabla f) &= 1, \\ \beta_0(f) &= \frac{\alpha_0(\nabla f)}{1 + \alpha_0(\nabla f)} = \frac{1}{2}. \end{aligned}$$

**Remark 4.4.** Let  $f: \mathbb{C}^n \rightarrow \mathbb{C}$  be a complex polynomial function with an isolated singularity at 0. Then from the works of Teissier [13, Corollary 2] we have the following equation

$$\beta_0(f) = \frac{\alpha_0(\nabla f)}{1 + \alpha_0(\nabla f)}.$$

Moreover, Gwoździewicz has remarked, [4], that the above relation fails to hold for some real polynomial functions with an isolated singularity at 0. However, we have the following.

**Corollary 4.5.** (see also [4, Theorem 1.3]) *Let  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  be a quasi-homogeneous polynomial function with weight  $w$  and quasi-degree  $m$ . If  $f^{-1}(0) = \{0\}$ , then*

$$\begin{aligned} \alpha_0(f) &= \frac{m}{w_*}, \\ \alpha_0(\nabla f) &= \alpha_0(f) - 1 = \frac{m}{w_*} - 1, \\ \beta_0(f) &= \frac{\alpha_0(\nabla f)}{1 + \alpha_0(\nabla f)} = 1 - \frac{w_*}{m}. \end{aligned}$$

*Proof.* In fact, by Corollary 3.6, we have

$$\alpha_0(f) = \frac{m}{w_*}.$$

Then the remained relations follow from [4, Theorem 1.3]. We will give below a direct proof in order to keep our paper self-contained.

We first note that the origin is an isolated critical point of  $f$ . Indeed, if  $\nabla f(a) = 0$  for some  $a \neq 0$ , then it follows easily from the generalized Euler identity (4.1) that  $f(a) = 0$ , which is a contradiction.

Without loss of generality, we may suppose that  $f(x) \geq 0$  for all  $x \in \mathbb{R}^n$  with equality if and only if  $x = 0$ . For each  $\delta > 0$ , the restriction of  $f$  to the sphere  $\{x \in \mathbb{R}^n \mid \|x\| = \delta\}$  attains its minimum at at least one point. Let

$$\Gamma := \{u \in \mathbb{R}^n \mid f(u) = \min_{\|x\|=\|u\|} f(x)\}.$$

It follows from the Tarski-Seidenberg theorem (see, for example, [3, Theorem 2.3.4]), that  $\Gamma$  is semi-algebraic. Hence the Curve Selection Lemma [10] is applicable. Together with Lagrange's Multipliers Theorem, this implies that there exists a continuous semi-algebraic map  $(\lambda, \varphi): [0, \epsilon) \rightarrow \mathbb{R} \times \mathbb{R}^n, \tau \mapsto (\lambda(\tau), \varphi(\tau))$ , such that

- (i)  $\varphi(\tau) = 0$  if and only if  $\tau = 0$ ;
- (ii) the restriction of the map  $(\lambda, \varphi)$  to  $(0, \epsilon)$  is analytic;
- (iii)  $\varphi(\tau) \in \Gamma$  for all  $\tau \in (0, \epsilon)$ ; and
- (iv)  $\nabla f(\varphi(\tau)) = \lambda(\tau)\varphi(\tau)$ .

Let  $at^p, a > 0$ , be the leading term of the Taylor expansion of  $\|\varphi(\tau)\|^2$ , and  $bt^q, b \neq 0$ , be that of  $|f(\varphi(\tau))|^2$ . Then, asymptotically as  $t \rightarrow 0$ , we have

$$|f(\varphi(\tau))| \simeq \|\varphi(\tau)\|^{\frac{q}{p}}.$$

Consequently, by the definition of  $\alpha_0(f)$ , we get

$$\alpha_0(f) \geq \frac{q}{p}.$$

On the other hand, we may assume (taking  $\epsilon > 0$  small enough if necessary) that the function  $\tau \mapsto \|\varphi(\tau)\|$  is strictly increasing. Together with the condition (i), we find that for each  $x \in \mathbb{R}^n, \|x\| \ll 1$ , there exists a positive number  $\tau \in [0, \epsilon)$  satisfying the relation  $\|\varphi(\tau)\| = \|x\|$ . Hence,

$$|f(x)| = f(x) \geq \min_{\|u\|=\|x\|} f(u) = f(\varphi(\tau)) \simeq \|\varphi(\tau)\|^{\frac{q}{p}} = \|x\|^{\frac{q}{p}}.$$

By the definition, thus

$$\alpha_0(f) \leq \frac{q}{p}.$$

Therefore,

$$\alpha_0(f) = \frac{q}{p}.$$

Moreover, it follows from the generalized Euler identity (4.1) that

$$mf(\varphi(\tau)) = \sum_{j=1}^n w_j \varphi_j(\tau) \frac{\partial f}{\partial x_j}(\varphi(\tau)).$$

By the condition (iv), hence

$$\begin{aligned} |mf(\varphi(\tau))| &= |\lambda(\tau)| \sum_{j=1}^n w_j [\varphi_j(\tau)]^2 \\ &= \frac{\|\nabla f(\varphi(\tau))\|}{\|\varphi(\tau)\|} \sum_{j=1}^n w_j [\varphi_j(\tau)]^2 \\ &\simeq \|\varphi(\tau)\| \|\nabla f(\varphi(\tau))\|. \end{aligned}$$

In particular, we get

$$\|\nabla f(\varphi(\tau))\| \simeq \|\varphi(\tau)\|^{\frac{q}{p}-1} \simeq |f(\varphi(\tau))|^{1-\frac{p}{q}}.$$

By definitions, hence

$$\begin{aligned} \alpha_0(\nabla f) &\geq \frac{q}{p} - 1 = \alpha_0(f) - 1 = \frac{m}{w_*} - 1, \\ \beta_0(f) &\geq 1 - \frac{p}{q} = 1 - \frac{1}{\alpha_0(f)} = 1 - \frac{w_*}{m}. \end{aligned}$$

Then the corollary follows immediately from Corollaries 3.4 and 4.2.  $\square$

The following result is of general interest but we shall only use it to prove Theorem 4.7 below.

**Lemma 4.6.** *Let  $f: \mathbb{K}^n \rightarrow \mathbb{K}$  be a  $C^1$ -function. For each  $k$  positive integer, consider the function  $\tilde{f}: \mathbb{K}^n \rightarrow \mathbb{K}, x \mapsto [f(x)]^k$ . Suppose that there exist  $c > 0$  and  $\theta \in (0, 1]$  such that*

$$c|\tilde{f}(x)|^{1-\theta} \leq \|\nabla \tilde{f}(x)\| \quad \text{for } \|x\| \ll 1.$$

Then

$$\frac{c}{k}|f(x)|^{1-k\theta} \leq \|\nabla f(x)\| \quad \text{for } \|x\| \ll 1.$$

*Proof.* We have

$$\nabla \tilde{f}(x) = k[f(x)]^{k-1} \nabla f(x).$$

Hence

$$c|[f(x)]^k|^{1-\theta} \leq k|[f(x)]^{k-1}| \|\nabla f(x)\|.$$

This implies

$$\frac{c}{k}|f(x)|^{1-k\theta} \leq \|\nabla f(x)\|,$$

concluding the proof of the lemma. □

The following is a generalization of [5, Theorem 2.1].

**Theorem 4.7.** *Let  $f: \mathbb{R}^2 \rightarrow \mathbb{R}$  be a quasi-homogeneous polynomial function with weight  $w := (w_1, w_2)$  and quasi-degree  $m$ . Then there exists a positive constant  $c$  such that*

$$(4.4) \quad c|f(x, y)|^{1-\frac{w_*}{m}} \leq \|\nabla f(x, y)\|, \quad \text{as } \|(x, y)\| \ll 1.$$

*Proof.* Without loss of generality, we may suppose that

$$1 \leq w_* = w_1 \leq w_2 \leq m.$$

There are two cases to be considered.

**Case 1.**  $m$  is divisible by  $w_1$ ; i.e.,  $q := \frac{m}{w_1}$  is a positive integer number.

Consider the following function

$$g(x, y) := f(x, y^{\frac{w_2}{w_1}}).$$

Then, by Proposition 2.1, we can see that  $g$  is a homogeneous polynomial on  $\mathbb{R} \times \mathbb{R}_+$  of degree  $q = \frac{m}{w_1}$ . Indeed we can write for some finite set  $S \subset \mathbb{N} \times \mathbb{N}$ :

$$f(x, y) = \sum_{\alpha:=(\alpha_1, \alpha_2) \in S} a_\alpha x^{\alpha_1} y^{\alpha_2}$$

with

$$w_1 \alpha_1 + w_2 \alpha_2 = m = q w_1.$$

Hence

$$\alpha_2 = (q - \alpha_1) \frac{w_1}{w_2};$$

and therefore

$$f(x, y) = \sum_{\alpha \in S} a_\alpha x^{\alpha_1} y^{(q - \alpha_1) \frac{w_1}{w_2}}$$

which provides

$$g(x, y) = \sum_{\alpha \in S} a_\alpha x^{\alpha_1} y^{q - \alpha_1}.$$

It now follows from [5, Theorem 2.1] that there exists a positive constant  $c$  such that

$$c|g(x, y)|^{1-\frac{w_1}{m}} \leq \|\nabla g(x, y)\|, \quad \text{as } \|(x, y)\| \ll 1 \text{ and } y \geq 0.$$

On the other hand, by the definition

$$\begin{aligned} \frac{\partial g}{\partial x}(x, y) &= \frac{\partial f}{\partial x}(x, y^{\frac{w_2}{w_1}}), \\ \frac{\partial g}{\partial y}(x, y) &= \frac{w_2}{w_1} y^{\frac{w_2}{w_1} - 1} \frac{\partial f}{\partial y}(x, y^{\frac{w_2}{w_1}}). \end{aligned}$$

Therefore, asymptotically as  $(x, y) \rightarrow (0, 0)$  and  $y \geq 0$ ,

$$\begin{aligned} c|f(x, y^{\frac{w_2}{w_1}})|^{1-\frac{w_1}{m}} &\leq \left\| \left( \frac{\partial f}{\partial x}(x, y^{\frac{w_2}{w_1}}), \frac{w_2}{w_1} y^{\frac{w_2}{w_1}-1} \frac{\partial f}{\partial y}(x, y^{\frac{w_2}{w_1}}) \right) \right\|. \\ &\leq \left\| \left( \frac{\partial f}{\partial x}(x, y^{\frac{w_2}{w_1}}), \frac{\partial f}{\partial y}(x, y^{\frac{w_2}{w_1}}) \right) \right\|, \end{aligned}$$

because  $w_2 \geq w_1$ .

Let  $u := y^{\frac{w_2}{w_1}} \geq 0$ . Then

$$c|f(x, u)|^{1-\frac{w_1}{m}} \leq \left\| \left( \frac{\partial f}{\partial x}(x, u), \frac{\partial f}{\partial y}(x, u) \right) \right\|.$$

By an entirely analogous argument but replacing  $g(x, y) = f(x, y^{\frac{w_2}{w_1}})$  by  $f(x, -y^{\frac{w_2}{w_1}})$  we can show that the above inequality also holds for all  $u \leq 0$ . These prove the theorem in Case 1.

**Case 2.**  $m$  is not divisible by  $w_1$ .

Let  $\tilde{f}(x, y) := [f(x, y)]^{w_1}$ . Then it is clear that  $\tilde{f}(x, y)$  is a positive quasi-homogeneous polynomial with weight  $(w_1, w_2)$  and quasi-degree  $\tilde{m} := mw_1$ . Since  $\frac{\tilde{m}}{w_1} = m$  is an integer number, by applying Case 1 for the polynomial  $\tilde{f}$  we get

$$\tilde{c}|\tilde{f}(x, y)|^{1-\frac{w_1}{m}} \leq \|\nabla \tilde{f}(x, y)\|, \quad \text{as } \|(x, y)\| \ll 1,$$

for some  $\tilde{c} > 0$ .

By Lemma 4.6, we get

$$\frac{\tilde{c}}{w_1} |f(x, y)|^{1-\frac{w_1}{m}} \leq \|\nabla f(x, y)\|,$$

which completes the proof of the theorem.  $\square$

**Remark 4.8.** As we see in the next proposition, the result of Theorem 4.7 is no longer valid in dimensions  $n > 2$ .

**Proposition 4.9.** (Compare with [5, Remark 2.4]) *Let  $f: \mathbb{R}^3 \rightarrow \mathbb{R}$  be given by*

$$f(x, y, z) := x^4 + x^2 z^2 - 2xy^2 z + y^4 = x^4 + (xz - y^2)^2.$$

*Then there exists a curve  $\varphi: [0, \epsilon) \rightarrow \mathbb{R}^n, t \mapsto \varphi(t)$ , such that*

$$\|\nabla f[\varphi(t)]\| \ll |f[\varphi(t)]|^{1-\frac{1}{4}} \quad \text{for } 0 < t \ll 1.$$

*In particular,  $\beta_0(f) > 1 - \frac{w_*}{m} = 1 - \frac{1}{4}$ .*

*Proof.* It is clear that  $f$  is a weighted quasi-homogeneous polynomial with weight  $w = (1, 1, 1)$  and quasi-degree  $m = 4$ . Moreover,  $f$  has non-isolated zero at the origin; namely,  $f^{-1}(0) = \{(0, 0, t) \mid t \in \mathbb{R}\}$ . Define the polynomial curve  $\varphi: [0, \epsilon) \rightarrow \mathbb{R}^3, t \mapsto (x(t), y(t), z(t))$ , by

$$\begin{aligned} x(t) &:= t^2, \\ y(t) &:= t + t^5, \\ z(t) &:= 1. \end{aligned}$$

One easily verifies that

$$\begin{aligned} f[\varphi(t)] &= t^8 + 4t^{12} + 4t^{16} + t^{20}, \\ \frac{\partial f}{\partial x}[\varphi(t)] &= -2t^{10}, \\ \frac{\partial f}{\partial y}[\varphi(t)] &= 8t^7 + 12t^{11} + 4t^{15}, \\ \frac{\partial f}{\partial z}[\varphi(t)] &= -4t^8 - 2t^{12}. \end{aligned}$$

Hence, asymptotically as  $t \rightarrow 0$ ,

$$\|\nabla f[\varphi(t)]\| \simeq t^7 \ll t^6 \simeq |f[\varphi(t)]|^{1-\frac{1}{4}}.$$

This completes the proof. □

**Remark 4.10.** The polynomial  $x^4 + x^2z^2 - 2xy^2z + y^4$  in the above proposition is the homogenized of  $x^4 + (x - y^2)^2$  by the new variable  $z$ . The last one is a polynomial in the class of polynomials which was considered by János Kollár ([6]).

The following is a direct consequence of Theorem 4.1 and Theorem 4.7.

**Corollary 4.11.** *Let  $f: \mathbb{R}^2 \rightarrow \mathbb{R}$  be a quasi-homogeneous polynomial function with weight  $w := (w_1, w_2)$  and quasi-degree  $m$ . Then*

$$1 - \frac{w^*}{m} \leq \beta_0(f) \leq 1 - \frac{w_*}{m}.$$

## 5. ADDITIONAL RESULTS, REMARKS AND EXAMPLES IN DIMENSION 2

In this section we will denote by  $f: \mathbb{R}^2 \rightarrow \mathbb{R}$  a quasi-homogeneous polynomial with weight  $w = (w_1, w_2)$  and quasi-degree  $m$  such that

$$w_* = w_1 \leq w_2 = w^* \leq m.$$

We now apply Corollary 4.11 in special cases.

**Corollary 5.1.** *If the origin is an isolated zero of  $f_y$  then*

$$\beta_0(f) = 1 - \frac{w_2}{m}.$$

*Proof.* It is well known that  $f_y$  is quasi-homogeneous polynomial with weight  $w = (w_1, w_2)$  and quasi-degree  $m - w_2$ . Since  $f_y^{-1}(0) = \{0\}$ , it follows from Theorem 2.5 that the polynomial  $f_y$  has a strict global extremum at the origin. Thus we can assume that  $f_y > 0$  on  $\mathbb{R}^n - \{0\}$ . By again Theorem 2.5, the set  $\{f_y = 1\}$  is nonempty compact. Hence

$$\infty > c := \max_{f_y(u,v)=1} |f(u, v)| > 0.$$

Take any  $(x, y) \in \mathbb{R}^2$ . Let  $\epsilon := [f_y(x, y)]^{\frac{1}{m-w_2}}$ . Then  $\{(u, v) \in \mathbb{R}^2 \mid f_y(u, v) = \epsilon^{m-w_2}\}$  is a non empty compact set. Moreover,

$$\begin{aligned} |f(x, y)| &\leq \max_{f_y(u,v)=\epsilon^{m-w_2}} |f(u, v)| = \max_{f_y(\epsilon^{-w_1}u, \epsilon^{-w_2}v)=1} |f(u, v)| \\ &= \max_{f_y(\tilde{u}, \tilde{v})=1} |f(\epsilon^{w_1}\tilde{u}, \epsilon^{w_2}\tilde{v})| \\ &= \max_{f_y(\tilde{u}, \tilde{v})=1} |f(\tilde{u}, \tilde{v})|\epsilon^m \\ &= c[f_y(x, y)]^{\frac{m}{m-w_2}}. \end{aligned}$$

This gives

$$f_y(x, y) \geq c'|f(x, y)|^{1-\frac{w_2}{m}},$$

here  $c' := c^{1-\frac{w_2}{m}} > 0$ . Therefore

$$\|\nabla f(x, y)\| \geq |f_y(x, y)| \geq c'|f(x, y)|^{1-\frac{w_2}{m}}.$$

By the definition of  $\beta_0(f)$ , we get

$$\beta_0(f) \leq 1 - \frac{w_2}{m}.$$

Then, by Corollary 4.11,  $\beta_0(f) = 1 - \frac{w_2}{m}$ . □

**Example 5.2.** Let  $f: \mathbb{R}^2 \rightarrow \mathbb{R}, (x, y) \mapsto y^3 + 3x^4y + 2x^6$  be a real polynomial. It is clear that  $f$  is a quasi-homogeneous polynomial with weight  $w = (1, 2)$  and quasi-degree  $m = 6$ . A direct computation shows that  $f_y^{-1}(0) = \{(0, 0)\}$ . Hence  $\beta_0(f) = 1 - \frac{2}{6}$ .

**Corollary 5.3.** *Suppose that the origin is not an isolated zero of  $f_y$ . If there exists  $(a, b) \in \mathbb{R}^2$  such that  $f(a, b) \neq 0, f_x(a, b) \neq 0$  and  $f_y(a, b) = 0$ , then*

$$\beta_0(f) = 1 - \frac{w_1}{m}.$$

*Proof.* By the hypothesis, we have for all  $t > 0$

$$\begin{aligned} f(t^{w_1}a, t^{w_2}b) &\simeq t^m, \\ f_x(t^{w_1}a, t^{w_2}b) &\simeq t^{m-w_1}, \\ f_y(t^{w_1}a, t^{w_2}b) &\equiv 0. \end{aligned}$$

Asymptotically as  $t \rightarrow 0$ , hence

$$\begin{aligned} |f(t^{w_1}a, t^{w_2}b)| &\simeq t^m, \\ \|\nabla f(t^{w_1}a, t^{w_2}b)\| &\simeq t^{m-w_1}. \end{aligned}$$

This implies that

$$\|\nabla f(t^{w_1}a, t^{w_2}b)\| \simeq |f(t^{w_1}a, t^{w_2}b)|^{1-\frac{w_1}{m}}.$$

Then, by the definition of  $\beta_0(f)$ ,

$$\beta_0(f) \geq 1 - \frac{w_1}{m}.$$

On the other hand, by Corollary 4.11,  $\beta_0(f) \leq 1 - \frac{w_1}{m}$ . Therefore  $\beta_0(f) = 1 - \frac{w_1}{m}$ .  $\square$

**Example 5.4.** Let  $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ ,  $(x, y) \mapsto x^2y - y^2$  be a real polynomial. It is clear that  $f$  is a quasi-homogeneous polynomial with weight  $w = (1, 2)$  and quasi-degree  $m = 4$ . A direct computation shows that  $f_y^{-1}(0) = \{x^2 - 2y = 0\}$  and the origin in  $\mathbb{R}^2$  is an isolated critical point of  $f$ . Moreover, it is easy to see that the conditions of Corollary 5.3 are satisfied. Hence,  $\beta_0(f) = 1 - \frac{1}{4}$ .

**Remark 5.5.** (i) All results in this paper allow to compute the Lojasiewicz exponents for some functions which are not quasi-homogeneous, for instance, the function  $f(x) := P(Ax)$ , where  $P$  is a quasi-homogeneous polynomial of two variables and  $A$  a nonsingular  $2 \times 2$  square matrix. As an example the polynomial  $P(x, y) := ax^4 + by^2 + cx^2y$  is quasi-homogeneous with weight  $(1, 2)$  and quasi-degree 4. The polynomial  $Q(x, y) = P(x, x + y)$  is not quasi-homogeneous if  $bc \neq 0$ .

(ii) On the other hand, there are of course polynomials of two variables which cannot be put in the form  $= P(Ax)$  with,  $P, A$  as above. For instance the polynomial  $Q(x, y) := x^2(1 + y)$  is such that no polynomial  $P = Q \circ A$  with  $A$  a nonsingular  $2 \times 2$  square matrix is quasi-homogeneous.

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## REFERENCES

- [1] V. I. Arnold, A. N. Varchenko, and S. M. Gusein-Zade, *Singularities of differentiable maps*, Birkhauser, 1985.
- [2] D. D'Acunto, K. Kurdyka, *Explicit bounds for the Lojasiewicz exponent in the gradient inequality for polynomials*, Ann. Pol. Math., Vol. **87** (2005), 51-61.
- [3] Benedetti R. and J. J. Risler, *Real algebraic and semi-algebraic sets*, Hermann, 1990.
- [4] J. Gwoździwicz, *The Lojasiewicz exponent of an analytic function at an isolated zero*, Comment. Math. Helv. Vol. **74** (1999), No. 3, 364-375.
- [5] A. Haraux, *Positively homogeneous functions and the Lojasiewicz gradient inequality*, Ann. Polon. Math. Vol. **87** (2005), 165-174.
- [6] J. Kollar, *An effective Lojasiewicz inequality for real polynomials*, Period. Math. Hungar., Vol. **38** (1999), N. 3, 213-221.
- [7] T. C. Kuo and Y. C. Lu, *On analytic function germs of two complex variables*, Topology, Vol. **16** (1977), 299-310.
- [8] S. Lojasiewicz, *Une propriété topologique des sous ensembles analytiques réels*, Colloques internationaux du C.N.R.S, **117**. Les Équations aux Dérivées Partielles, (1963) 87-89.
- [9] S. Lojasiewicz, *Ensembles semi-analytiques*, preprint IHES, 1965.
- [10] J. Milnor, *Singular points of complex hypersurfaces*, Annals of Mathematics Studies **61**, Princeton University Press, 1968.
- [11] N. Z. Shor, *Class of global minimum bounds of polynomial functions*, Cybernetics, **23** (1987), No. 6, 731-734.
- [12] J. G. Stalker, *A compactness criterion for real plane algebraic curves*, arXiv:math.AG/0404279, 2006.
- [13] B. Teissier, *Variétés polaires I - Invariants polaires des singularités d'hypersurfaces*, Inventiones Mathematicae, Vol. **40** (1977), 267-293.