Pseudoconvex regions of finite D'Angelo type in four dimensional almost complex manifolds

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PSEUDOCONVEX REGIONS OF FINITE D'ANGELO TYPE IN FOUR DIMENSIONAL ALMOST COMPLEX MANIFOLDS

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ABSTRACT. Let D be a J-pseudoconvex region in a smooth almost complex manifold (M, J) of real dimension four. We construct a local peak J-plurisubharmonic function at every point $p \in bD$ of finite D'Angelo type. As applications we give local estimates of the Kobayashi pseudometric, implying the local Kobayashi hyperbolicity of D at p. In case the point p is of D'Angelo type less than or equal to four, or the approach is nontangential, we provide sharp estimates of the Kobayashi pseudometric.

CONTENTS

INTRODUCTION

Analysis on almost complex manifolds recently became a fondamental tool in symplectic geometry with the work of M.Gromov in [15]. The local existence of pseudoholomorphic discs proved by A.Nijenhuis-W.Woolf in their paper [21], allows to define the Kobayashi pseudometric, which is crucial for local analysis.

In the present paper we study the behaviour of the Kobayashi pseudometric of a J-pseudoconvex region of finite D'Angelo type in an almost complex manifold (M, J) of dimension four. Finite D'Angelo

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type appeared naturally in complex manifolds when considering the boundary behaviour of the $\overline{\partial}$ operator (see [7],[8],[18],[4]). Moreover on complex manifolds of dimension two, the D'Angelo type unifies many type conditions as the finite regular type. Finite regular type was recently characterized intrinsically by J.-F.Barrault-E.Mazzilli [1] by means of Lie brackets, which generalizes in the non integrable case, a result of T.Bloom-I.Graham [4].

Our main result is the construction of a local peak J-plurisubharmonic function on pseudoconvex regions provided by Theorem A (see also Theorem 2.6):

Theorem A. *Let $D = \{\rho < 0\}$ be a domain of finite D'Angelo type in an almost complex manifold (M, J) of dimension four. We suppose that ρ is a $\mathcal{C}^2$ defining function of D, J-plurisubharmonic on a neighborhood of $\overline{D}$. Let $p \in \partial D$ be a boundary point. Then there exists a local peak J-plurisubharmonic function at p.*

Theorem A allows to localize pseudoholomorphic discs and to obtain lower estimates of the Kobayashi pseudometric which provide the local Kobayashi hyperbolicity of J-pseudoconvex regions of D'Angelo type $2m$ (Proposition 3.4 and Proposition 3.10). As an application we prove the $1/2m$-Hölder extension of biholomorphisms up to the boundary (Proposition 3.9). In order to obtain sharp lower estimates of the Kobayashi pseudometric similar to those given in complex manifolds by D.Catlin [5] (see also [3]), we consider a natural scaling method. However this reveals the fact that for a domain of finite D'Angelo type greater than four, the sequence of almost complex structures obtained by any polynomial scaling process does not converge generically to the standard structure; this is presented in the Appendix. This may be related to the fact that finite D'Angelo type is based on purely complex considerations, as the boundary behaviour of the Cauchy-Riemann equations. Hence we provide sharp lower estimates of the Kobayashi pseudometric for a region of finite D'Angelo type four (see also Theorem 4.1):

Theorem B. *Let $D = \{\rho < 0\}$ be a relatively compact domain of finite D'Angelo type less than or equal to four in an almost complex manifold (M, J) of dimension four, where ρ is a $\mathcal{C}^2$ defining function of D, J-plurisubharmonic on a neighborhood of $\overline{D}$. Then there is a positive constant C with the following property: for every $p \in D$ and every $v \in T_pM$ there exists a diffeomophism Φ_{p^*} in a neighborhood U of p, such that:*

$$K_{(D,J)}(p, v) \geq C \left(\frac{|\left(d_p \Phi_{p^*} v\right)_1|}{|\rho(p)|^{\frac{1}{4}}} + \frac{|\left(d_p \Phi_{p^*} v\right)_2|}{|\rho(p)|} \right).$$

We point out that the approach we use, based on some renormalization principle of pseudoholomorphic discs, gives also a different proof of precise lower estimates obtained by H.Gaussier-A.Sukhov in [12] for strictly J-pseudoconvex domains in arbitrary dimension. As an application of Theorem B, we obtain the (local) complete hyperbolicity of J-pseudoconvex regions of D'Angelo type less than or equal to four (Corollary 4.5) and we give a Wong-Rosay theorem for regions with noncompact automorphisms group (Corollary 4.6).

Finally, in order to obtain precise estimates near a point of arbitrary finite D'Angelo type, we are interested in the nontangential behaviour of the Kobayashi pseudometric (see also Theorem 4.7):

Theorem C. *Let $D = \{\rho < 0\}$ be a domain of finite D'Angelo type in an almost complex manifold (M, J) of dimension four, where ρ is a $\mathcal{C}^2$ defining function of D, J-plurisubharmonic on a neighborhood of $\overline{D}$. Let $q \in \partial D$ be a boundary point of D'Angelo type $2m$ and let $\Lambda \subset D$ be a cone with vertex at q and axis the inward normal axis. Then there exists a positive constant C such that for every $p \in D \cap \Lambda$ and every*

$v = v_n + v_t \in T_pM$*:*

$$K_{(D,J)}(p,v) \geq C\left(\frac{|v_n|}{|\rho(p)|^{\frac{1}{2m}}} + \frac{|v_t|}{|\rho(p)|}\right),$$

where v_n *and* v_t *are the normal and the tangential parts of* v *with respect to* q*.*

1. Preliminaries

We denote by Δ the unit disc of $\mathbb{C}$ and by Δ_r the disc of $\mathbb{C}$ centered at the origin of radius $r > 0$.

1.1. **Almost complex manifolds and pseudoholomorphic discs.** An *almost complex structure* J on a real smooth manifold M is a $(1,1)$ tensor field which satisfies $J^2 = -Id$. We suppose that J is smooth. The pair (M,J) is called an *almost complex manifold*. We denote by J_{st} the standard integrable structure on $\mathbb{C}^n$ for every n. A differentiable map $f : (M',J') \longrightarrow (M,J)$ beetwen two almost complex manifolds is said to be (J',J)*-holomorphic* if $J(f(p)) \circ d_pf = d_pf \circ J'(p)$, for every $p \in M'$. In case $M' = \Delta \subset \mathbb{C}$, such a map is called a *pseudoholomorphic disc*. If $f : (M,J) \longrightarrow M'$ is a diffeomorphism, we define an almost complex structure, f_*J, on M' as the direct image of J by f :

$$f_*J(q) := d_{f^{-1}(q)}f \circ J\left(f^{-1}(q)\right) \circ d_qf^{-1},$$

for every $q \in M'$.

The following lemma (see [12]) states that locally any almost complex manifold can be seen as the unit ball of $\mathbb{C}^n$ endowed with a small smooth pertubation of the standard integrable structure J_{st}.

Lemma 1.1. *Let* (M,J) *be an almost complex manifold, with* J *of class* $\mathcal{C}^k$*,* $k \geq 0$*. Then for every point* $p \in M$ *and every* $\lambda_0 > 0$ *there exist a neighborhood* U *of* p *and a coordinate diffeomorphism* $z : U \to \mathbb{B}$ *centered a* p *(ie* $z(p) = 0$*) such that the direct image of* J *satisfies* $z_*J(0) = J_{st}$ *and* $||z_*(J) - J_{st}||_{\mathcal{C}^k(\bar{\mathbb{B}})} \leq \lambda_0$*.*

This is simply done by considering a local chart $z : U \to \mathbb{B}$ centered a p (ie $z(p) = 0$), composing it with a linear diffeomorphism to insure $z_*J(0) = J_{st}$ and dilating coordinates.

So let J be an almost complex structure defined in a neighborhood U of the origin in $\mathbb{R}^{2n}$, and such that J is sufficiently closed to the standard structure in uniform norm on the closure $\overline{U}$ of U. The J-holomorphy equation for a pseudoholomorphic disc $u : \Delta \to U \subseteq \mathbb{R}^{2n}$ is given by

$$\frac{\partial u}{\partial y} - J(u)\frac{\partial u}{\partial x} = 0. \tag{1.1}$$

According to [21], for every $p \in M$, there is a neighborhood V of zero in T_pM, such that for every $v \in V$, there is a J-holomorphic disc u satisfying $u(0) = p$ and $d_0u(\partial/\partial x) = v$.

1.2. **Levi geometry.** Let ρ be a $\mathcal{C}^2$ real valued function on a smooth almost complex manifold (M,J). We denote by $d^c_J\rho$ the differential form defined by

$$d^c_J\rho(v) := -d\rho(Jv),$$

where v is a section of TM. The *Levi form* of ρ at a point $p \in M$ and a vector $v \in T_pM$ is defined by

$$\mathcal{L}_J\rho(p,v) := d(d^c_J\rho)(p)(v,J(p)v) = dd^c_J\rho(p)(v,J(p)v).$$

In case $(M,J) = (\mathbb{C}^n, J_{st})$, then $\mathcal{L}_{J_{st}}\rho$ is, up to a positive multiplicative constant, the usual standard Levi form :

$$\mathcal{L}_{J_{st}}\rho(p,v) = 4\sum \frac{\partial^2\rho}{\partial z_j\partial\overline{z_k}}v_j\overline{v_k}.$$

We investigate now how close is the Levi form with respect to J from the standard Levi form. For $p \in M$ and $v \in T_pM$, we easily get :

$$\mathcal{L}_J\rho\,(p,v) = \mathcal{L}_{J_{st}}\rho(p,v) + d(d^c_J - d^c_{J_{st}})\rho(p)(v,J(p)v) + dd^c_{J_{st}}\rho(p)(v,J(p)-J_{st})v). \tag{1.2}$$

In local coordinates $(t_1, t_2, \cdots, t_{2n})$ of $\mathbb{R}^{2n}$, (1.2) may be written as follows

$$\begin{aligned} \mathcal{L}_J\rho\,(p,v) &= \mathcal{L}_{J_{st}}\rho(p,v) + {}^tv(A - {}^tA)J(p)v + {}^t(J(p)-J_{st})vDJ_{st}v + \\ &\quad {}^t(J(p)-J_{st})vD(J(p)-J_{st})v \end{aligned} \tag{1.3}$$

where

$$A := \left(\sum_i \frac{\partial u}{\partial t_i}\frac{\partial J_{i,j}}{\partial t_k}\right)_{1\leq j,k\leq 2n} \quad \text{and} \quad D := \left(\frac{\partial^2 u}{\partial t_j \partial t_k}\right)_{1\leq j,k\leq 2n}.$$

Let f be a (J', J)-biholomorphism from (M', J') to (M, J). Then for every $p \in M$ and every $v \in T_pM$:

$$\mathcal{L}_{J'}\rho\,(p,v) = \mathcal{L}_J\rho \circ f^{-1}\,(f\,(p)\,, d_pf\,(v))\,.$$

This expresses the invariance of the Levi form under pseudobiholomorphisms.

The next proposition is useful in order to compute the Levi form (see [10], [16] and [17]).

Proposition 1.2. *Let $p \in M$ and $v \in T_pM$. Then*

$$\mathcal{L}_J\rho\,(p,v) = \Delta\,(\rho \circ u)\,(0)\,,$$

where $u : \Delta \to (M, J)$ is any J-holomorphic disc satisfying $u\,(0) = p$ and $d_0u\,(\partial/\partial_x) = v$.

Proposition 1.2 leads to the following proposition-definition :

Proposition 1.3. *The two statements are equivalent :*

(1) *$\rho \circ u$ is subharmonic for any J-holomorphic disc $u : \Delta \to M$.*
(2) *$\mathcal{L}_J\rho(p,v) \geq 0$ for every $p \in M$ and every $v \in T_pM$.*

If one of the previous statements is satisfied we say that ρ is *J-plurisubharmonic*. We say that ρ is *strictly J-plurisubharmonic* if $\mathcal{L}_J\rho(p,v)$ is positive for any $p \in M$ and any $v \in T_pM \setminus \{0\}$. J-plurisubharmonic functions play a very important role in almost complex geometry : they give attraction and localization properties for pseudoholomorphic discs. For this reason the construction of J-plurisubharmonic functions is crucial.

Similarly to the integrable case, one may define the notion of pseudoconvexity in almost complex manifolds. Let D be a domain in (M, J). We denote by $T^J\partial D := T\partial D \cap JT\partial D$ the J-invariant subbundle of $T\partial D$.

Definition 1.4.

(1) The domain D is J-pseudoconvex (resp. it strictly J-pseudoconvex) if $\mathcal{L}_J\rho(p,v) \geq 0$ (resp. > 0) for any $p \in \partial D$ and $v \in T^J_p\partial D$ (resp. $v \in T^J_p\partial D \setminus \{0\}$).
(2) A J-pseudoconvex region is a domain $D = \{\rho < 0\}$ where ρ is a $\mathcal{C}^2$ defining function, J-plurisubharmonic on a neighborhood of $\overline{D}$.

We recall that a defining function for D satisfies $d\rho \neq 0$ on ∂D.

The following Lemma is useful in order to compute the Levi form of some functions.

Lemma 1.5. *Assume that J is a diagonal almost complex structure on $\mathbb{R}^4$ that coincides with the standard structure J on $\mathbb{C}\times\{0\}$. To fix notations we suppose that its matricial representation is given by :*

$$J=\begin{pmatrix} a_1 & b_1 & 0 & 0\\ c_1 & -a_1 & 0 & 0\\ 0 & 0 & a_2 & b_2\\ 0 & 0 & c_2 & -a_2 \end{pmatrix}.$$

Then the Levi form of some smooth real valued function f at a point $z=(z_1,z_2)$ and $v=(1,0,0,0)$ is equal to

$$\mathcal{L}_J f(z,v)=-c_1\Delta_1 f+O(|z_2|)).$$

where $\Delta_1 f:=\dfrac{\partial^2 f}{\partial x_1\partial x_1}+\dfrac{\partial^2 f}{\partial y_1\partial y_1}$.

Proof. Let us compute the Levi form of some smooth real valued function f at a point $z=(z_1,z_2)$ and $v=(1,0,0,0)$:

$$\begin{aligned} c_1^{-1}\mathcal{L}_J f(z,v) &= -\Delta_1 f+\left[-2\frac{\partial^2 f}{\partial x_1\partial y_1}a_1+\frac{\partial^2 f}{\partial x_1\partial x_1}(1+b_1)+\frac{\partial^2 f}{\partial y_1\partial y_1}(c_1-1)\right]+\\ &\quad \frac{\partial f}{\partial x_1}\left[\frac{\partial b_1}{\partial x_1}-\frac{\partial a_1}{\partial y_1}\right]+\frac{\partial f}{\partial y_1}\left[\frac{\partial a_1}{\partial x_1}+\frac{\partial c_1}{\partial y_1}\right]\\ &= -\Delta_1 f+\left[-2\frac{\partial^2 f}{\partial x_1\partial y_1}O(|z_2|)+\frac{\partial^2 f}{\partial x_1\partial x_1}O(|z_2|)+\frac{\partial^2 f}{\partial y_1\partial y_1}O(|z_2|)\right]+\\ &\quad \frac{\partial f}{\partial x_1}O(|z_2|)+\frac{\partial f}{\partial y_1}O(|z_2|)\\ &= -\Delta_1 f+O(|z_2|). \end{aligned}$$

□

2. Construction of a local peak plurisubharmonic function

This section is devoted to the proof of Theorem A (see Theorem 2.6).

2.1. **Pseudoconvex regions of finite D'Angelo type.** In this subsection we describe a pseudonconvex region on a neighborhood of a boundary point of finite D'Angelo type. We point out that all our considerations are purely local. Assume that $D=\{\rho<0\}$ is a J-pseudoconvex region in $\mathbb{C}^2$ and that the structure J is defined on a fixed neighborhood U of $\overline{D}$. We suppose that the origin is a boundary point of D.

Definition 2.1. Let $u:(\Delta,0)\to\left(\mathbb{R}^4,0,J\right)$ be a J-holomorphic disc satisfying $u(0)=0$. The order of contact $\delta_0(\partial D,u)$ with ∂D at the origin is the degree of the first term in the Taylor expansion of $\rho\circ u$. We denote by $\delta(u)$ the multiplicity of u at the origin.

We now define the D'Angelo type and the regular type of the real hypersurface ∂D at the origin.

Definition 2.2.

(1) The D'Angelo type of ∂D at the origin is defined by:

$$\Delta^1(\partial D,0):=\sup\left\{\frac{\delta_0(\partial D,u)}{\delta(u)},\ u:\Delta\to\left(\mathbb{R}^4,J\right)\ J\text{-holomorphic}\ ,u(0)=0\right\}.$$

The point 0 is a point of finite D'Angelo type $2m$ if $\Delta^1(\partial D,0)=2m<+\infty$.

(2) The regular type of ∂D at origin is defined by:

$$\Delta^1_{\text{reg}}(\partial D,0) := \sup\{\delta_0(\partial D,u),\ u:\Delta\to\left(\mathbb{R}^4,J\right)\ J\text{-holomorphic},\ u(0)=0, d_0u\neq 0\}.$$

Since the regular type of ∂D at the origin consists in considering only regular discs we have:

$$\Delta^1_{\text{reg}}(\partial D,0)\leq\Delta^1(\partial D,0). \tag{2.1}$$

The type condition as defined in part 1 of Definition 2.2 was introduced by J.-P.D'Angelo [7], [8] who proved that this coincides with the regular type in complex manifolds of dimension two. After Proposition 2.3, we will also prove that the D'Angelo type and the regular type coincide in four dimensional almost complex manifolds (see Proposition 2.4).

We suppose that the origin is a point of finite regular type. Then let $u:\Delta\to\mathbb{R}^4$ be a regular J-holomorphic disc of maximal contact order $2m$. We choose coordinates such that u is given by $u(\zeta)=(\zeta,0)$, $J(z_1,0)=J_{st}$ and such that the complex tangent space $T_0\partial D\cap J(0)T_0\partial D$ is equal to $\{z_2=0\}$. Then by considering the family of vectors $(1,0)$ at base points $(0,t)$ for $t\neq 0$ small enough, we obtain a family of J holomorphic discs u_t such that $u_t(0)=(0,t)$ and $d_0u_t(\partial/\partial_x)=(0,1)$. Due to the parameters dependance of the solution to the J-holomorphy equation (1.1), we straighten these discs into the complex lines $\{z_2=t\}$. We then consider a transversal foliation by J-holomorphic discs and straighten these lines into $\{z_1=c\}$. In these new coordinates still denoted by z, the matricial representation of J is diagonal:

$$J=\begin{pmatrix} a_1 & b_1 & 0 & 0\\ c_1 & -a_1 & 0 & 0\\ 0 & 0 & a_2 & b_2\\ 0 & 0 & c_2 & -a_2\end{pmatrix}. \tag{2.2}$$

Since $J(z_1,0)=J_{st}$ we have

$$J=J_{st}+O(|z_2|). \tag{2.3}$$

In the next fundamental proposition we describe precisely the local expression of the defining function ρ.

Proposition 2.3. *The J-plurisubharmonic defining function for the domain D has the following local expression:*

$$\rho=\Re e z_2+H_{2m}(z_1,\overline{z_1})+\widetilde{H}(z_1,z_2)+O\left(|z_1|^{2m+1}+|z_2||z_1|^m+|z_2|^2\right)$$

where H_{2m} is a homogeneous polynomial of degree $2m$, subharmonic which is not harmonic and

$$\widetilde{H}(z_1,z_2)=\Re e\sum_{k=1}^{m-1}\rho_k z_1^k z_2.$$

Proof. Since $T_0\partial D\cap J(0)T_0\partial D=\{z_2=0\}$, we have

$$\rho=\Re e z_2+O(\|z\|^2).$$

Moreover the disc $\zeta\mapsto(\zeta,0)$ being a regular J-holomorphic disc of maximal contact order $2m$, the defining function ρ has the following local expression:

$$\rho=\Re e z_2+H_{2m}(z_1,\overline{z_1})+O\left(|z_1|^{2m+1}+|z_2|\|z\|\right),$$

where H_{2m} is a homogeneous polynomial of degree $2m$.

We prove that the polynomial H_{2m} is subharmonic using a standard dilation argument. Consider the non-isotropic dilation of $\mathbb{C}^2$

$$\Lambda_\delta(z_1, z_2) := \left(\delta^{-\frac{1}{2m}} z_1, \delta^{-1} z_2\right).$$

Due to Proposition 1.2, the domain

$$\Lambda_\delta(D) = \{\delta^{-1}\left(\rho \circ \Lambda_\delta^{-1}(z_1, z_2)\right) < 0\}$$

is $(\Lambda_\delta)_*(J)$-pseudoconvex. Moreover $\Lambda_\delta(D)$ converges in the sense of local Hausdorff set convergence to

$$\tilde{D} := \{Re(z_2) + H_{2m}(z_1, \overline{z_1}) < 0\},$$

as δ tends to zero and the sequence of structures $(\Lambda_\delta)_* J$ converges to the standard structure J_{st}. It follows that the limit domain $\tilde{D}$ is J_{st}-pseudoconvex implying that H_{2m} is subharmonic.

Now we prove H_{2m} that contains a nonharmonic part. By contradiction, we assume that H_{2m} is harmonic. Then H_{2m} can be written $\Re e z_1^{2m}$. According to Proposition 1.1 of [17], and since the structure J is smooth there exists, for a sufficiently small $\lambda > 0$, a pseudoholomorphic disc $u : \Delta \to (\mathbb{R}^4, J)$ such that:

$$\begin{cases} u(0) & = & 0 \\ \dfrac{\partial u}{\partial x}(0) & = & \left(\lambda^{\frac{1}{2m}}, 0, 0, 0\right) \\ \dfrac{\partial^k u}{\partial x^k}(0) & = & (0,0,0,0), \text{ for } 1 < k < 2m \\ \dfrac{\partial^{2m} u}{\partial x^{2m}}(0) & = & (0, 0, -\lambda(2m)!, 0). \end{cases}$$

We prove that the contact order of such a regular disc u is greater than $2m$ which contradicts the fact that D is of regular type $2m$. We denote by $[\rho \circ u]_{2m}$ the homogeneous part of degree $2m$ in the Taylor expansion of $\rho \circ u$ at the origin:

$$[\rho \circ u]_{2m}(x, y) = \sum_{k=0}^{2m} a_k x^k y^{2m-k}.$$

Let us prove that $a_k = \dfrac{\partial^k}{\partial x^k}\dfrac{\partial^{2m-k}}{\partial y^{2m-k}}\rho \circ u(0)$ is equal to zero for each $0 \leq k \leq 2m$.

For a_{2m}, we have:

$$\begin{aligned} \frac{\partial^{2m}}{\partial x^{2m}}\rho \circ u(0) &= \Re e \frac{\partial^{2m}}{\partial x^{2m}} u_2(0) + \Re e \frac{\partial^{2m}}{\partial x^{2m}} u_1^{2m}(0) \\ &= -\lambda(2m)! + \Re e \frac{\partial^{2m}}{\partial x^{2m}} u_1^{2m}(0). \end{aligned}$$

Since $u_1(0) = 0$, it follows that the only non vanishing term in $\Re e \dfrac{\partial^{2m}}{\partial x^{2m}} u_1^{2m}(0)$ is

$$(2m)! \Re e \left(\frac{\partial u_1}{\partial x}(0)\right)^{2m} = \lambda(2m)!.$$

This proves that $a_{2m} = 0$.

Then let $0 \leq k < 2m$:

$$\frac{\partial^k}{\partial x^k}\frac{\partial^{2m-k}}{\partial y^{2m-k}}\rho \circ u\,(0) \;=\; \Re e\frac{\partial^k}{\partial x^k}\frac{\partial^{2m-k}}{\partial y^{2m-k}}u_2\,(0) + \Re e\frac{\partial^k}{\partial x^k}\frac{\partial^{2m-k}}{\partial y^{2m-k}}u_1^{2m}\,(0)\,.$$

For the same reason as previously, the only term to consider in $\Re e\frac{\partial^k}{\partial x^k}\frac{\partial^{2m-k}}{\partial y^{2m-k}}u_1^{2m}\,(0)$ is

$$(2m)!\Re e\left(\frac{\partial}{\partial x}u_1\,(0)\right)^k\left(\frac{\partial}{\partial y}u_1\,(0)\right)^{2m-k} = \lambda^{\frac{k}{2m}}\,(2m)!\Re e\left(\frac{\partial u_1}{\partial y}\,(0)\right)^{2m-k}.$$

Then, since u is J-holomorphic, it satisfies the diagonal J-holomorphy equation:

$$\frac{\partial u_l}{\partial y} = J_l\,(u)\,\frac{\partial u_l}{\partial x},$$

for $l = 1, 2$, where

$$J_l = \begin{pmatrix} a_l & b_l \\ c_l & -a_l \end{pmatrix} \text{ (see (2.2) for notations).}$$

It follows that

$$\begin{aligned}\lambda^{\frac{k}{2m}}\,(2m)!\Re e\left(\frac{\partial u_1}{\partial y}\,(0)\right)^{2m-k} &= \lambda^{\frac{k}{2m}}\,(2m)!\Re e\left(J_1\,(u\,(0))\,\frac{\partial u_1}{\partial x}\,(0)\right)^{2m-k}\\ &= \lambda\,(2m)!\Re e\,(i)^{2m-k}\,.\end{aligned}$$

Moreover due to the condition $\frac{\partial^k u_2}{\partial x^k}\,(0) = (0,0)$, for $1 \leq k < 2m$, it follows that the only part we need to consider in $\frac{\partial^{2m-k}}{\partial y^{2m-k}}u_2\,(0)$ is $J_2\,(u)\,\frac{\partial}{\partial x}\frac{\partial^{2m-k-1}}{\partial y^{2m-k-1}}u_2\,(0)$ and by induction $(J_2\,(u))^{2m-k}\,\frac{\partial^{2m-k}}{\partial x^{2m-k}}u_2\,(0)$. Finally

$$\begin{aligned}\Re e\frac{\partial^k}{\partial x^k}\frac{\partial^{2m-k}}{\partial y^{2m-k}}u_2\,(0) &= \Re e\,(J_2\,(u\,(0)))^{2m-k}\,\frac{\partial^{2m}u_2}{\partial x^{2m}}\,(0)\\ &= -\lambda\,(2m)!\Re e\left(J_2\,(u\,(0))^{2m-k}\,(1,0)\right)\\ &= -\lambda\,(2m)!\Re e\,(i)^{2m-k}\,.\end{aligned}$$

This proves that the homogeneous part $[\rho \circ u]_{2m}$ is equal to zero.

For smaller order terms it is a direct consequence of $u\,(0) = 0$ and $\frac{\partial^k u}{\partial x^k}\,(0) = (0,0,0,0)$, for $1 < k < 2m$.

It remains to prove there are no term $\Re e\rho_k z_1^k\overline{z_2}$ with $k < m$ in the defining function ρ. This is done by contradiction and by computing the Levi form of ρ at a point $z_0 = (z_1, 0)$ and at a vector $v = (X_1, 0, X_2, 0)$. Assume that

$$\rho = \Re e z_2 + H_{2m}\,(z_1, \overline{z_1}) + \widetilde{H}(z_1, z_2) + \Re e\rho_{\overline{k}}z_1^k\overline{z_2} + O\left(|z_1|^{2m+1} + |z_2||z_1|^{k+1} + |z_2|^2\right),$$

with $k < m$. Replacing z_1 by $(\rho_{\overline{k}})^{\frac{1}{k}}z_1$ if necessary, we suppose $\rho_{\overline{k}} = 1$.

The Levi form of $\Re e z_2$ at a point $z_0 = (z_1, 0)$ and at a vector $v = (X_1, 0, X_2, 0)$ is equal to

$$\begin{aligned}\mathcal{L}_J \Re e z_2 (z_0, v) &= \left[(a_1 - a_2)(z_0) \frac{\partial a_2}{\partial x_1}(z_0) + c_1(z_0) \frac{\partial a_2}{\partial y_1}(z_0) - c_2(z_0) \frac{\partial b_2}{\partial x_1}(z_0) \right] X_1 X_2 + \\ & c_2(z_0) \left[\frac{\partial a_2}{\partial y_2}(z_0) - \frac{\partial b_2}{\partial x_2}(z_0) \right] X_2^2.\end{aligned}$$

Due to (2.3) we have

$$\begin{cases} a_1(z_0) = a_2(z_0) = 0, \\ c_2(z_0) = 1, \\ \dfrac{\partial a_2}{\partial y_1}(z_0) = \dfrac{\partial b_2}{\partial x_1}(z_0) = 0. \end{cases}$$

So the Levi form of $\Re e z_2$ at $z_0 = (x_1, 0, 0, 0)$ and at a vector $v = (X_1, 0, X_2, 0)$ is

$$\mathcal{L}_J \Re e z_2 (z_0, v) = \left[\frac{\partial a_2}{\partial y_2}(z_0) - \frac{\partial b_2}{\partial x_2}(z_0) \right] X_2^2.$$

According to Lemma 1.5, the Levi form of $H_{2m} + O(|z_1|^{2m+1})$ at z_0 and $v_1 = (X_1, 0, X_2, 0)$ is equal to

$$\mathcal{L}_J (H_{2m} + O(|z_1|^{2m+1}))(z_0, v) = \Delta \left(H_{2m} + O(|z_1|^{2m+1}) \right) X_1^2 + O(|z_1|^{2m-1}) X_1 X_2.$$

According to the fact that the Levi form for the standard structure of $\widetilde{H}(z_1, z_2)$ is identically equal to zero, and due to (1.3) and to (2.3), it follows that the Levi form of $\widetilde{H}(z_1, z_2)$ at z_0 is equal to

$$\mathcal{L}_J \widetilde{H}(z_0, v) = O(|z_1|) X_2^2.$$

Now the Levi form of $O(|z_2|^2)$ is equal to

$$\mathcal{L}_J O(|z_2|^2)(z_0, v) = O(1) X_2^2.$$

And the Levi form of $\Re e z_1^k \overline{z_2}$ is equal

$$\mathcal{L}_J \Re e z_1^k \overline{z_2}(z_0, v) = (k \Re e z_1^{k-1}) X_1 X_2 + O(|z_1|^k) X_2^2.$$

Finally the Levi form of the defining function ρ at a point $z_0 = (z1, 0)$ and at a vector $v = (X_1, 0, X_2, 0)$ is equal to:

$$\begin{aligned}\mathcal{L}_J \rho(z_0, v) &= O\left(|z_1|^{2m-2}\right) X_1^2 + \left[4k \Re e z_1^{k-1} + O(|z_1|^{2m-1}) \right] X_1 X_2 \\ &+ \left[\frac{\partial a_2}{\partial y_2}(z_0) - \frac{\partial b_2}{\partial x_2}(0) + O(1) + O(|z_1|) \right] X_2^2.\end{aligned}$$

It follows that since $k < m$ there are z_1, X_1 and X_2 such that $\mathcal{L}_J \rho(z_0, v)$ is negative, providing a contradiction.

□

Now we prove that the D'Angelo type coincides with the regular type in the non integrable case.

Proposition 2.4. *We have*

$$\Delta^1_{\text{reg}}(\partial D, 0) = \Delta^1(\partial D, 0).$$

Proof. We suppose that the origin is a point of finite D'Angelo type. According to (2.1) we may write:

$$\Delta^1_{\text{reg}}(\partial D, 0) = 2m < +\infty.$$

So we may assume that $u(\zeta) = (\zeta, 0)$ is a regular J-holomorphic disc of maximal contact order $2m$, and that the structure J satisfies (2.2) and (2.3). Moreover the defining function ρ has the following local expression:

$$\rho = \Re e z_2 + H_{2m}(z_1, \overline{z_1}) + O\left(|z_1|^{2m+1} + |z_2|\|z\|\right).$$

Now consider a J-holomorphic disc $v = (f_1, g_1, f_2, g_2) : (\Delta, 0) \to \left(\mathbb{R}^4, 0, J\right)$ of finite contact order satisfying $v(0) = 0$ and such that $\delta(v) \geq 2$ (see definition 2.1 for notations).

We set $v_1 := f_1 + ig_1$ and $v_2 := f_2 + ig_2$. The J-holomorphy equation for the disc v is given by:

$$\begin{cases} a_k(v)\dfrac{\partial f_k}{\partial x} + b_k(v)\dfrac{\partial g_k}{\partial x} &= \dfrac{\partial f_k}{\partial y}, \\ \\ c_k(v)\dfrac{\partial f_k}{\partial x} - a_k(v)\dfrac{\partial g_k}{\partial x} &= \dfrac{\partial g_k}{\partial y}, \end{cases}$$

for $k = 1, 2$. Since $J(v) = J_{st} + O(|v_2|)$ and $\delta(v) \geq 2$, it follows that:

$$\begin{cases} \delta(v_1) = \delta(f_1) = \delta(g_1), \\ \\ \delta(v_2) = \delta(f_2) = \delta(g_2). \end{cases} \tag{2.4}$$

Then consider

$$\rho \circ v(\zeta) = f_2(\zeta) + H_{2m}\left(v_1(\zeta), \overline{v_1(\zeta)}\right) + O\left(|v_1(\zeta)|^{2m+1} + |v_2(\zeta)|\|v(\zeta)\|\right). \tag{2.5}$$

Equation (2.4) implies that the term $O(|v_2|\|v\|)$ in (2.5) vanishes to order larger than f_2.

Case 1: $\delta(f_2) > \delta(H_{2m}(v_1, \overline{v_1}))$. In that case

$$\delta_0(\partial D, u) = \delta(H_{2m}(v_1, \overline{v_1})) = 2m\delta(v_1).$$

Thus we get:

$$\frac{\delta_0(\partial D, v)}{\delta(v)} = \frac{2m\delta(v_1)}{\delta(v_1)} = 2m.$$

Case 2: $\delta(f_2) \leq \delta(H_{2m}(v_1, \overline{v_1}))$. We have two subcases.

Subcase 2.1: $f_2 + H_{2m}(v_1, \overline{v_1}) \not\equiv 0$. Thus

$$\delta_0(\partial D, u) = \delta(\Re e v_2) = \delta(v_2),$$

and so

$$\frac{\delta_0(\partial D, v)}{\delta(v)} = \frac{\delta(v_2)}{\delta(v)} \leq \frac{\delta(H_{2m}(v_1, \overline{v_1}))}{\delta(v)} = \frac{2m\delta(v_1)}{\delta(v)}.$$

This means that:

$$\frac{\delta_0(\partial D, v)}{\delta(v)} = 1 \text{ if } \delta(v) = \delta(v_2)$$

or

$$\frac{\delta_0(\partial D, v)}{\delta(v)} \leq 2m \text{ if } \delta(v) = \delta(v_1).$$

Subcase 2.2: $f_2 + H_{2m}(v_1, \overline{v_1}) \equiv 0$. Let $w : \Delta \to \left(\mathbb{R}^4, J_{st}\right)$ be a standard holomorphic disc satisfying $w(0) = 0$ and:

$$\frac{\partial^k w}{\partial x^k}(0) = \frac{\partial^k v}{\partial x^k}(0),$$

for $k = 1, \cdots, 2m\delta(v)$. Since $\delta(v_2) = 2m\delta(v_1) = 2m\delta(v) < +\infty$ and since $J(v) = J_{st} + O(|v_2|)$, any differentiation of $J(v)$, of order smaller than $2m\delta(v)$, is equal to zero. Combining this with the J-holomorphy equation (1.1) of v we obtain:

$$\frac{\partial^{k+l} w}{\partial x^k \partial y^l}(0) = \frac{\partial^{k+l} v}{\partial x^k \partial y^l}(0),$$

for $k + l = 1, \cdots, 2m\delta(v)$. Since $\rho \circ v$ vanishes to an order greater than $2m\delta(v)$ at 0 and since it involves only the $2m\delta(v)$-jet of v, it follows that $\rho \circ w$ vanishes to an order greater than $2m\delta(v)$ at 0. Finally we have constructed a standard holomorphic disc w such that

$$\begin{cases} \delta(w) & = & \delta(v), \\ \\ \delta_0(\partial D, w) & > & 2m\delta(w), \end{cases}$$

which is not possible since, according Proposition 2.3, the type for the standard structure of ∂D at the origin is equal to $2m$. □

2.2. **Construction of a local peak plurisubharmonic function.** We first give the definition of a local peak J-plurisubharmonic function for a domain D.

Definition 2.5. Let D be a domain in an almost complex manifold (M, J). A function φ is called a local peak J-plurisubharmonic function at a boundary point $p \in \partial D$ if there exists a neighborhood U of p such that φ is continuous up to $\overline{D} \cap U$ and satisfies:

(1) φ is J-plurisubharmonic on $D \cap U$,
(2) $\varphi(p) = 0$,
(3) $\varphi < 0$ on $\overline{D} \cap U \backslash \{p\}$.

The existence of local peak J_{st}-plurisubharmonic functions was first proved by E.Fornaess and N.Sibony in [11]. For almost complex manifolds the existence was proved by S.Ivashkovich and J.-P.Rosay in [17] whenever the domain is strictly J-pseudoconvex. In the next Proposition we state the existence for J-pseudoconvex regions of finite D'Angelo type. As mentionned earlier our the considerations are purely local. In particular, the assumptions of J-plurisubharmonicity and of finite D'Angelo type may be restricted to a neighborhood of a boundary point. For convenience of writing, we state them globally.

Theorem 2.6. *Let $D = \{\rho < 0\}$ be a domain of finite D'Angelo type in a four dimensional almost complex manifold (M, J). We suppose that ρ is a $\mathcal{C}^2$ defining function of D, J-plurisubharmonic on a neighborhood of $\overline{D}$. Let $p \in \partial D$ be a boundary point. Then there exists a local peak J-plurisubharmonic function at p.*

Proof. Since the existence of a local peak function near a boundary point of type 2 was proved in [17], we assume that p is a boundary point of D'Angelo type $2m > 2$. The problem being purely local we assume that $D \subset \mathbb{C}^2$ and that $p = 0$. According to Proposition 2.3 the defining function ρ has the following local expression on a neighborhood U of the origin:

$$\rho = \Re e z_2 + H_{2m}(z_1, \overline{z_1}) + \widetilde{H}(z_1, z_2) + O\left(|z_1|^{2m+1} + |z_2||z_1|^m + |z_2|^2\right)$$

where H_{2m} is a subharmonic polynomial containing a nonharmonic part, denoted by H^*_{2m}, and

$$\widetilde{H}(z_1, z_2) = \Re e \sum_{k=1}^{m-1} \rho_k z_1^k z_2.$$

According to [11] (see Lemma 2.4), the polynomial H_{2m} satisfies the following Lemma:

Lemma 2.7. *There exist a positive $\delta > 0$ and a smooth function $g : \mathbb{R} \to \mathbb{R}$ with period 2π with the following properties:*

(1) $-2 < g(\theta) < -1$,
(2) $\|g\| < 1/\delta$,
(3) $\max\left(\Delta H_{2m}, \Delta\left(\|H^*_{2m}\| g(\theta)|z_1|^{2m}\right)\right) > \delta\|H^*_{2m}\||z_1|^{2(m-1)}$, *for* $z_1 = |z_1|e^{i\theta} \neq 0$ *and,*
(4) $\Delta\left(H_{2m} + \delta\|H^*_{2m}\| g(\theta)|z_1|^{2m}\right) > \delta^2\|H^*_{2m}\||z_1|^{2(m-1)}$.

We denote by P the function defined by

$$P(z_1, \overline{z_1}) := H_{2m}(z_1, \overline{z_1}) + \delta\|H^*_{2m}\| g(\theta)|z_1|^{2m}.$$

Theorem 2.6 will be proved by establishing the following claim.

Claim. There are positive constants L and C such that the function

$$\varphi := \Re e z_2 + 2L(\Re e z_2)^2 - L(\Im m z_2)^2 + P(z_1, \overline{z_1}) + \widetilde{H}(z_1, z_2) + C|z_1|^2|z_2|^2$$

is a local peak J-plurisubharmonic function at the origin.

Proof of the claim. We first prove that the function φ is J-plurisubharmonic. We set:

$$dd^c_J\varphi = \alpha_1 dx_1 \wedge dy_1 + \alpha_2 dx_2 \wedge dy_2 + \alpha_3 dx_1 \wedge dx_2 + \alpha_4 dx_1 \wedge dy_2 + \alpha_5 dy_1 \wedge dx_2 + \alpha_6 dy_1 \wedge dy_2,$$

where α_k, for $k = 1, \cdots, 6$, are real valued function. According to the matricial representation of J (see (2.2)), the Levi form of φ at a point $z \in D \cap U$ and at a vector $v = (X_1, Y_1, X_2, Y_2) \in T_z\mathbb{R}^4$ can be written

$$\begin{aligned}\mathcal{L}_J\varphi(z, v) &= c_1\alpha_1 X_1^2 - 2a_1\alpha_1 X_1Y_1 - b_1\alpha_1 Y_1^2 + \beta_3 X_1X_2 + \beta_4 X_1Y_2 + \\ &\beta_5 Y_1X_2 + \beta_6 Y_1Y_2 + c_2\alpha_2 X_2^2 - 2a_2\alpha_2 X_2Y_2 - b_2\alpha_2 Y_2^2,\end{aligned}$$

with

$$\begin{cases} \beta_3 := \alpha_3(a_2 - a_1) + \alpha_4 c_2 - \alpha_5 c_1 \\ \beta_4 := -\alpha_4(a_1 + a_2) + \alpha_3 b_2 - \alpha_6 c_1 \\ \beta_5 := \alpha_5(a_1 + a_2) - \alpha_3 b_1 + \alpha_6 c_2 \\ \beta_6 := \alpha_6(a_1 - a_2) - \alpha_4 b_1 + \alpha_5 b_2. \end{cases}$$

Moreover due to (2.3) we have for $k = 1, 2$

$$\begin{cases} a_k = O(|z_2|) \\ b_k = -1 + O(|z_2|) \\ c_k = 1 + O(|z_2|). \end{cases}$$

This implies that for $k = 1, 2$:

$$c_k\alpha_k X_k^2 - 2a_k\alpha_k X_kY_k - b_k\alpha_k Y_k^2 \geq \frac{\alpha_k}{2}\left(X_k^2 + Y_k^2\right).$$

Thus we obtain

$$\begin{aligned}\mathcal{L}_J\varphi(z,v) &\geq \frac{\alpha_1}{4}X_1^2+\beta_3X_1X_2+\frac{\alpha_2}{4}X_2^2+\frac{\alpha_1}{4}Y_1^2+\beta_5Y_1X_2+\frac{\alpha_2}{4}X_2^2+\\ &\quad \frac{\alpha_1}{4}X_1^2+\beta_4X_1Y_2+\frac{\alpha_2}{4}Y_2^2+\frac{\alpha_1}{4}Y_1^2+\beta_6Y_1Y_2+\frac{\alpha_2}{4}Y_2^2.\end{aligned}$$

In order to prove that φ is J-plurisubharmonic, we need to see that:

(1) $\alpha_k \geq 0$, for $k=1,2$,

(2) $4\beta_j^2 \leq \alpha_1\alpha_2$, for $j=3,\cdots,6$.

The coefficient α_2 is obtained by the differentiation of $\Re e z_2$, $2L\left(\Re e z_2\right)^2 - L\left(\Im m z_2\right)^2$, $\widetilde{H}(z_1,z_2)$ and $C|z_1|^2|z_2|^2$. Hence we have for z sufficiently close to the origin

$$\alpha_2 \geq L > 0.$$

The coefficient α_1 is obtained by differentiating P, $\widetilde{H}(z_1,z_2)$ and $C|z_1|^2|z_2|^2$. This is equal to

$$\begin{aligned}\alpha_1 &= \Delta P + O(|z_1|^{2m-2}|z_2|)+O(|z_2|^2)+C|z_2|^2+O(|z_2|^3)\\ &\geq \frac{\delta^2\|H^*_{2m}\|}{2}|z_1|^{2m-2}+\frac{C}{2}|z_2|^2,\end{aligned}$$

for z sufficiently small and $C>0$ large enough. Hence α_1 is nonnegative.

Finally it sufficient to prove that

$$4\beta_j^2 \leq L\left(\frac{\delta^2\|H^*_{2m}\|}{2}|z_1|^{2m-2}+\frac{C}{2}|z_2|^2\right),$$

to insure the J-plurisubharmonicity of φ. The coefficient $|\beta_j|$ is equal to

$$\begin{aligned}|\beta_j| &= O(|z_2|)+LO(|z_2|^2)+O(|z_1|^{2m-1})+CO(|z_1||z_2|)\\ &\leq C'(|z_2|+|z_1|^{2m-1}),\end{aligned}$$

for a positive constant C' (not depending on L and C). It follows that φ is J-plurisubharmonic on a neighborhood of the origin.

We prove now that φ is local peak at the origin, that is there exists $r>0$ such that $\overline{D}\cap\{0<\|z\|\leq r\}\subset\{\varphi<0\}$. Assuming that $z\in\{\rho=0\}\cap\{0<\|z\|\leq r\}$ we have:

$$\begin{aligned}\varphi(z) &= \delta\|H^*_{2m}\|g(\theta)|z_1|^{2m}+2L\left(\Re e z_2\right)^2-L\left(\Im m z_2\right)^2+C|z_1|^2|z_2|^2+\\ &\quad O\left(|z_1|^{2m+1}\right)+O\left(|z_2||z_1|^m\right)+O\left(|z_2|^2\right).\end{aligned}$$

Since $g<-1$ and increasing L if necessary we have

$$O\left(|\Im m z_2||z_1|^m\right)\leq -\frac{1}{2}\delta\|H^*_{2m}\|g(\theta)|z_1|^{2m}+\frac{1}{2}L\left(\Im m z_2\right)^2,$$

whenever z is sufficiently close to the origin. Thus

$$\begin{aligned}\varphi(z) &\leq -\frac{1}{2}\delta\|H^*_{2m}\||z_1|^{2m} + (2L + C|z_1|^2)(\Re e z_2)^2 - \frac{1}{2}L(\Im m z_2)^2 + C|z_1|^2(\Im m z_2)^2 + \\ &\quad O\left(|z_1|^{2m+1}\right) + O\left(|\Re e z_2||z_1|^m\right) + O\left(|z_2|^2\right) \\ &\leq -\frac{1}{4}\delta\|H^*_{2m}\||z_1|^{2m} + (2L + C|z_1|^2)(\Re e z_2)^2 - \frac{1}{4}L(\Im m z_2)^2 + O\left(|\Re e z_2||z_1|^m\right) + \\ &\quad O\left(|z_2|^2\right).\end{aligned}$$

There is a positive constant C'' such that

$$O\left(|z_2|^2\right) \leq C''|\Re e z_2|^2 + C''|\Im m z_2|^2.$$

Thus increasing L if necessary:

$$\begin{aligned}\varphi(z) &\leq -\frac{1}{4}\delta\|H^*_{2m}\||z_1|^{2m} + (2L + C|z_1|^2)(\Re e z_2)^2 + O(|\Re e z_2|^2) \\ &\quad -\left(\frac{1}{4}L - C''\right)(\Im m z_2)^2 + O\left(|\Re e z_2||z_1|^m\right) + O(|\Im m z_2|^2\|z\|). \\ &\leq -\frac{1}{4}\delta\|H^*_{2m}\||z_1|^{2m} + (2L + C|z_1|^2)(\Re e z_2)^2 + O(|\Re e z_2|^2) + O\left(|\Re e z_2||z_1|^m\right) \\ &\quad -\frac{1}{2}\left(\frac{1}{4}L - C''\right)(\Im m z_2)^2.\end{aligned}$$

Since

$$-\Re e z_2(1 + O(|z|)) = H_{2m}(z_1, \overline{z_1}) + O\left(|z_1|^{2m+1} + |\Im m z_2||z_1| + |\Im m z_2|^2\right),$$

we have

$$(\Re e z_2)^2(1 + O(|z|)) = O\left(|z_1|^{4m} + |\Im m z_2||z_1|^{2m+1} + |\Im m z_2|^2\|z\|\right).$$

We finally obtain for z small enough

$$\varphi(z) \leq -\frac{1}{8}\delta\|H^*_{2m}\||z_1|^{2m} - \frac{1}{4}\left(\frac{1}{4}L - C''\right)(\Im m z_2)^2.$$

Thus φ is negative for $z \in \{\rho = 0\} \cap \{0 < \|z\| \leq r\}$, with r small enough. It follows that, reducing r if necessary,

$$\overline{D} \cap \{0 < \|z\| \leq r\} \subset \{\varphi < 0\},$$

which achieves the proof of the claim and of Theorem 2.6. □

We notice that in case $\mathcal{L}_J \Re e z_2 \equiv 0$, we may give a simpler expression for a local peak J-plurisubharmonic function.

Proposition 2.8. *If $\mathcal{L}_J \Re e z_2 \equiv 0$, then there exists a real positive number L such that the function*

$$\varphi := \Re e z_2 + 2L(\Re e z_2)^2 - L(z_2)^2 + P(z_1, \overline{z_1})$$

is local peak J-plurisubharmonic at the origin.

3. Estimates of the Kobayashi pseudometric

In this section we prove standard estimates of the Kobayashi pseudometric on J-pseudoconvex regions of finite D'Angelo type in an almost complex manifold.

3.1. **The Kobayashi pseudometric.** The existence of local pseudoholomorphic discs proved in [21] allows to define the *Kobayashi pseudometric* $K_{(M,J)}$ for $p \in M$ and $v \in T_pM$:

$$K_{(M,J)}(p,v) := \inf\left\{\frac{1}{r} > 0, u : \Delta \to (M,J) \ J\text{-holomorphic}\ , u(0) = p, d_0u(\partial/\partial x) = rv\right\}.$$

Since the composition of pseudoholomorphic maps is still pseudoholomorphic, the Kobayashi (infinitesimal) pseudometric satisfies the following decreasing property :

Proposition 3.1. *Let $f : (M',J') \to (M,J)$ be a (J',J)-holomorphic map. Then for any $p \in M'$ and $v \in T_pM'$ we have*

$$K_{(M',J')}(p,v) \geq K_{(M,J)}(f(p), d_pf(v)).$$

Let $d_{(M,J)}$ be the integrated pseudodistance of $K_{(M,J)}$:

$$d_{(M,J)}(p,q) := \inf\left\{\int_0^1 K_{(M,J)}(\gamma(t), \dot\gamma(t))\,dt,\ \gamma : [0,1] \to M,\ \gamma(0) = p, \gamma(1) = q\right\}.$$

Similarly to the standard integrable case, B.Kruglikov (see [19]) proved that the integrated pseudodistance of the Kobayashi pseudometric coincides with the Kobayashi pseudodistance defined by chains of pseudholomorphic discs. Then we define :

Definition 3.2. (1) The manifold (M,J) is Kobayashi hyperbolic if the integrated pseudodistance $d_{(M,J)}$ is a distance.

(2) The manifold (M,J) is local Kobayashi hyperbolic at $p \in M$ if there exist a neighborhood U of p and a positive constant C such that

$$K_{(M,J)}(q,v) \geq C\|v\|$$

for every $q \in U$ and every $v \in T_qM$.

(3) A Kobayashi hyperbolic manifold (M,J) is complete hyperbolic if it is complete for the distance $d_{(M,J)}$.

3.2. **Hyperbolicity of pseudoconvex regions of finite D'Angelo type.** In order to localize pseudoholomorphic discs, we need the following technical Lemma (see [12] for a proof).

Lemma 3.3. *Let $0 < r < 1$ and let θ_r be a smooth nondecreasing function on $\mathbb{R}^+$ such that $\theta_r(s) = s$ for $s \leq r/3$ and $\theta_r(s) = 1$ for $s \geq 2r/3$. Let (M,J) be an almost complex manifold, and let p be a point of M. Then there exist a neighborhood U of p, positive constants $A = A(r) \geq 1$, $B = B(r)$, and a diffeomorphism $z : U \to \mathbb{B}$ such that $z(p) = 0$, $z_*J(p) = J_{st}$ and the function $\log\left(\theta_r\left(|z|^2\right)\right) + \theta_r(A|z|) + B|z|^2$ is J-plurisubharmonic on U.*

In the next Proposition we give a priori estimates and a localization principle of the Kobayashi pseudometric. This proves the local Kobayashi hyperbolicity of J-pseudoconvex $\mathcal{C}^2$ regions of finite D'Angelo type. If (M,J) admits a global J-plurisubharmonic function, then K.Diederich and A.Sukhov proved in [9] the (global) Kobayashi hyperbolicity of a relatively compact J-pseudoconvex domains (with $\mathcal{C}^3$ boundary) by constructing a bounded strictly J-plurisubharmonic exhaustion function. We notice that, in our case, if the manifold (M,J) admits a global J-plurisubharmonic function then J-pseudoconvex $\mathcal{C}^2$ relatively compact regions of finite D'Angelo type are also (globally) Kobayashi hyperbolic.

Proposition 3.4. *Let $D = \{\rho < 0\}$ be a domain of finite D'Angelo type in an almost complex manifold (M, J), where ρ is a $\mathcal{C}^2$ defining function of D, J-plurisubharmonic in a neighborhood of $\overline{D}$. Let $p \in \bar{D}$ and let U be a neighborhood of p in M. Then there exist positive constants C and s, and a neighborhood $V \subset U$ of p in M, such that for each $q \in D \cap V$ and each $v \in T_qM$:*

$$K_{(D,J)}(q,v) \geq C\|v\|, \tag{3.1}$$

$$K_{(D,J)}(q,v) \geq sK_{(D\cap U,J)}(q,v). \tag{3.2}$$

This Proposition is a classical application of Lemma 3.3. This is due to N.Sibony [22] (see also [2] and [12] for a proof). For convenience we give the proof.

Proof. According to Theorem 2.6, there exists a local peak J-plurisubharmonic function φ at p for D. We can choose constants $0 < \alpha < \alpha' < \beta' < \beta$ and $N > 0$ such that $\varphi \geq -\beta^2/N$ on $\{\|z\| < \alpha\}$ and $\varphi \leq -2\beta^2/N$ on $\overline{D} \cap \{\alpha' \leq \|z\| \leq \beta'\}$.

We define $\tilde{\varphi}$ by:

$$\tilde{\varphi} := \begin{cases} \max\left(N\varphi + \|z\|^2 - \beta^2, -2\beta^2\right) & \text{if } z \in D \cap \{\|z\| \leq \beta'\}, \\ -2\beta^2 & \text{on } D\backslash\{\|z\| \leq \beta'\}. \end{cases}$$

The function $\|z\|^2$ is J-plurisubharmonic on $\{q \in U : |z(q)| < 1\}$ if $\|z_*J - J_{st}\|_{\mathcal{C}^2(\mathbb{B})}$ is sufficiently small. Then it follows that $\tilde{\varphi}$ is J-plurisubharmonic on D. We may also suppose that $\tilde{\varphi}$ is negative on D. Moreover the function $\tilde{\varphi} - \|z\|^2$ is J-plurisubharmonic on $D \cap \{q \in U : |z(q)| \leq \alpha\}$.

Let θ_{α^2} be a smooth non decreasing function on $\mathbb{R}^+$ such that $\theta_{\alpha^2}(s) = s$ for $s \leq \alpha^2/3$ and $\theta_{\alpha^2}(s) = 1$ for $s \geq 2\alpha^2/3$. Set $V = \{q \in U : |z(q)| \leq \alpha^2\}$. According to Lemma 3.3, there are uniform positive constants $A \geq 1$ and B such that the function

$$\log\left(\theta_{\alpha^2}\left(|z - z(q)|^2\right)\right) + \theta_{\alpha^2}(A|z - z(q)|) + B\|z\|^2$$

is J-plurisubharmonic on U for every $q \in D \cap V$.

We define for each $q \in D \cap V$ the function:

$$\Psi_q := \begin{cases} \theta_{\alpha^2}\left(|z - z(q)|^2\right)\exp\left(\theta_{\alpha^2}(A|z - z(q)|)\right)\exp\left(B\tilde{\varphi}(z)\right) & \text{on } D \cap \{\|z\| < \alpha\}, \\ \exp\left(1 + B\tilde{\varphi}\right) & \text{on } D \setminus \{\|z\| < \alpha\}. \end{cases}$$

The function $\log\Psi_q$ is J-plurisubharmonic on $D \cap \{\|z\| < \alpha\}$ and, on $D \setminus \{\|z\| < \alpha\}$, it coincides with $1 + B\tilde{\varphi}$ which is J-plurisubharmonic. Finally $\log\Psi_q$ is J-plurisubharmonic on the whole domain D.

Let $q \in V$ and let $v \in T_qM$ and consider a J-holomorphic disc $u : \Delta \to D$ such that $u(0) = q$ and $d_0u(\partial/\partial x) = rv$ where $r > 0$. For ζ sufficiently close to 0 we have

$$u(\zeta) = q + d_0u(\zeta) + \mathcal{O}\left(|\zeta|^2\right).$$

We define the following function

$$\phi(\zeta) := \frac{\Psi_q(u(\zeta))}{|\zeta|^2}$$

which is subharmonic on $\Delta\backslash\{0\}$ since $\log\phi$ is subharmonic. If ζ close to 0, then

$$\phi(\zeta) = \frac{|u(\zeta) - q|^2}{|\zeta|^2}\exp\left(A|u(\zeta) - q|\right)\exp\left(B\tilde{\varphi}(u(\zeta))\right). \tag{3.3}$$

Setting $\zeta = \zeta_1 + i\zeta_2$ and using the J-holomorphy condition $d_0u \circ J_{st} = J \circ d_0u$, we may write :

$$d_0u(\zeta) = \zeta_1 d_0u(\partial/\partial x) + \zeta_2 J(d_0u(\partial/\partial x)).$$

$$|d_0u(\zeta)| \leq |\zeta| (\|I + J\| \, \|d_0u(\partial/\partial x)\|) \tag{3.4}$$

According to (3.3) and to (3.4), we obtain that $\limsup_{\zeta\to 0} \phi(\zeta)$ is finite. Moreover setting $\zeta_2 = 0$ we have

$$\limsup_{\zeta\to 0} \phi(\zeta) \geq \|d_0u(\partial/\partial x)\|^2 \exp(B\tilde{\varphi}(q)).$$

Applying the maximum principle to a subharmonic extension of ϕ on Δ we obtain the inequality

$$\|d_0u(\partial/\partial x)\|^2 \leq \exp(1 - B\tilde{\varphi}(q)).$$

Hence, by definition of the Kobayashi pseudometric, we obtain for every $q \in D \cap V$ and every $v \in T_qM$:

$$K_{(D,J)}(q, v) \geq (\exp(-1 + B\tilde{\varphi}(q)))^{\frac{1}{2}} \|v\|.$$

This gives estimate (3.1).

Now in order to obtain estimate (3.2), we prove that there is a neighborhood $V \subset U$ and a positive constant s such that for any J-holomorphic disc $u : \Delta \to D$ with $u(0) \in V$ then $u(\Delta_s) \subset D \cap U$. Suppose this is not the case. We obtain a sequence ζ_ν of Δ and a sequence of J-holomorphic discs u_ν such that ζ_ν converges to 0, $u_\nu(0)$ converges to p and $\|u_\nu(\zeta_\nu)\| \notin D \cap U$ for every ν. According to the estimate (3.1), we obtain for a positive constant $c > 0$:

$$c \leq d_{(D,J)}(u_\nu(0), u_\nu(\zeta_\nu)) \leq d_\Delta(\zeta_\nu, 0).$$

This contradicts the fact that ζ_ν converges to 0. □

The (global) Kobayahsi hyperbolicity is provided if we suppose that there is a global strictly J-plurisubharmonic function on (M, J).

Corollary 3.5. *Let $D = \{\rho < 0\}$ be a relatively compact domain of finite D'Angelo type in an almost complex manifold (M, J) of dimension four, ρ being a defining function of D, J-plurisubharmonic in a neighborhood of $\overline{D}$. Assume that (M, J) admits a global strictly J-plurisubharmonic function. Then (D, J) is Kobayahsi hyperbolic.*

As an application of the a priori estimate (3.1) of Proposition 3.4, we prove the tautness of D.

Corollary 3.6. *Let $D = \{\rho < 0\}$ be a relatively compact domain of finite D'Angelo type in an almost complex manifold (M, J) of dimension two. Assume that ρ is J-plurisubharmonic in a neighborhood of $\overline{D}$. Moreover suppose that (M, J) admits a global strictly J-plurisubharmonic function. Then D is taut.*

Proof. Let $(u_\nu)_\nu$ be a sequence of J-holomorphic discs in D. According to Corollary 3.5 the domain D is hyperbolic. Thus the sequence $(u_\nu)_\nu$ is equiconituous, and then by Ascoli Theorem, we can extract from this sequence a subsequence still denoted $(u_\nu)_\nu$ which converges to a map $u : \Delta \to \overline{D}$. Passing to the limit the equation of J-holomorphicity of each u_ν, it follows that u is a J-holomorphic disc. Since ρ is J-plurisubharmonic defining function for D, we have, by applying the maximun principle to $\rho \circ u$, the alternative: either $u(\Delta) \subset D$ or $u(\Delta) \subset \partial D$. □

We point out that the tautness of the domain D was proved, using a diferent method, by K.Diederich-A.Sukhov in [9].

3.3. **Uniform estimates of the Kobayashi pseudometric.** In order to obtain more precise estimates, we need to uniform estimates (3.1) of the Kobayashi pseudometric for a sequence of domains.

Proposition 3.7. *Assume that $D = \{\Re ez_2 + P(z_1, \overline{z_1}) < 0\}$ is a J_{st}-pseudoconvex region of $\mathbb{R}^4$, where P is a homogeneous polynomial of degree $2k \leq 2m$ admitting a nonharmonic part. Let D_ν be a sequence of J_ν-pseudoconvex region of $\mathbb{R}^4$ such that $0 \in \partial D_\nu$ is a boundary point of finite D'Angelo type $2l_\nu \leq 2m$. Suppose that D_ν converges in the sense of local Hausdorff set convergence to D when ν tends to $+\infty$ and that J_ν converges to J_{st} in the $\mathcal{C}^2$ topology when ν tends to $+\infty$. Then there exist a positive constant C and a neighborhood $V \subset U$ of the origin in $\mathbb{R}^4$, such that for large ν and for every $q \in D_\nu \cap V$ and every $v \in T_q\mathbb{R}^4$*

$$K_{(D_\nu, J)}(q, v) \geq C\|v\|.$$

Proof. Under the conditions of Proposition 3.7 we have the following Lemma:

Lemma 3.8. *For every large ν, there exists a diffeomorphism $\Phi_\nu : \mathbb{R}^4 \to \mathbb{R}^4$ with the following property:*

1. *The map $\zeta \mapsto (\zeta, 0)$ is a $(\Phi_\nu)_* J_\nu$-holomorphic disc of maximal contact order $2l_\nu$.*
2. *The almost complex structure $(\Phi_\nu)_* J_\nu$ satisfies conditions (2.2) and (2.3).*
3. *$\Phi_\nu(D_\nu) = \{\rho_\nu < 0\}$ with*

$$\rho_\nu = \Re ez_2 + \sum_{j=2l_\nu}^{2m} P_{j,\nu}(z_1, \overline{z_1}) + O\left(|z_1|^{2m+1} + |z_2|\|z\|\right) < 0,$$

   where $P_{j,\nu}$ are homogeneous polynomials of degree j and $P_{2l_\nu,\nu}$ contains a nonharmonic part denoted by $P^*_{2l_\nu,\nu} \neq 0$.
4. *we have $inf_\nu\{\|P_{2l_\nu,\nu}\|\} > 0$.*

Moreover the sequence of diffeomorphisms Φ_ν converges to the identity on any compact subsets of $\mathbb{R}^4$ in the $\mathcal{C}^2$ topology.

The crucial fact used to prove Proposition 3.7 is the point (4), which is a direct consequence of the convergence of $\Phi_\nu(D_\nu)$ to D. Hence the proof of Proposition 3.7 is similar to Theorem 2.6 and Theorem 3.4, where all the constants are uniform.

□

3.4. **Hölder extension of diffeomorphisms.** This subsection is devoted to the boundary continuity of diffeomorphisms. This is stated as follows:

Proposition 3.9. *Let $D = \{\rho < 0\}$ and $D' = \{\rho' < 0\}$ be two relatively compact domains of finite D'Angelo type $2m$ in four dimensional almost complex manifolds (M, J) and (M', J'). We suppose that ρ (resp. ρ') is a J(resp J')-plurisubharmonic defining function on a neighborhood of D (resp. $\overline{D'}$). Let $f : D \to D'$ be a (J, J')-biholomorphism. Then f extends as a Hölder homeomorphism with exponent $1/2m$ between $\overline{D}$ and $\overline{D'}$.*

Estimates of the Kobayashi pseudometric obtained by H.Gaussier and A.Sukhov in [12] provide the Hölder extension with exponent $1/2$ up to the boundary of a biholomorphism between two strictly pseudoconvex domains (see Proposition 3.3 of [6]). Similarly, in order to obtain Proposition 3.9, we begin by establishing a more precise estimate than (3.1) of Proposition 3.4.

Proposition 3.10. *Let $D = \{\rho < 0\}$ be a domain of finite D'Angelo type in a four dimensional almost complex manifold (M, J), where ρ is a $\mathcal{C}^2$ defining function of D, J-plurisubharmonic in a neighborhood*

of $\overline{D}$. Let $p \in \partial D$ and let U be a neighborhood of p in M. Then there are positive constant C and a neighborhood $V \subset U$ of p in M, such that for every $q \in D \cap V$ and every $v \in T_q M$:

$$K_{(D,J)}(q,v) \geq C \frac{\|v\|}{\operatorname{dist}(q, \partial D)^{1/2m}}. \tag{3.5}$$

Proof of Proposition 3.10. Let $p \in \partial D$. We may suppose that $D \subset \mathbb{R}^4$, $p = 0$ and that J satisfies (2.2) and (2.3). Let q' be a boundary point in a neighborhood of the origin and let $\varphi_{q'}$ be the local peak J-plurisubharmonic function at q' given by Theorem 2.6. There are positive constants C_1 and C_2 such that

$$-C_1 \|z - q'\| \leq \varphi_{q'}(z) \leq -C_2 \Psi_{q'}(z), \tag{3.6}$$

where

$$\Psi_{q'}(z) := |z_1 - q'_1|^{2m} + |z_2 - q'_2|^2 + |z_1 - q'_1|^2 |z_2 - q'_2|^2$$

is a J-plurisubharmonic function on a neighborhood U of the origin.

Now consider a J-holomorphic disc $u : \Delta \to D$, such that $u(0)$ is sufficiently close to the origin and then, according to Proposition 3.4, we have $u(\Delta_s) \subset D \cap U$, for some $0 < s < 1$ depending only on $u(0)$. We assume that q' is such that $\operatorname{dist}(u(0), \partial D) = \|u(0) - q'\|$. According to the J-plurisubharmonicity of $\Psi_{q'}$, we have for $|\zeta| \leq s$:

$$\Psi_{q'}(u(\zeta)) \leq \frac{C_3}{2\pi} \int_0^{2\pi} \Psi_{q'}\left(u\left(re^{i\theta}\right)\right) d\theta,$$

for some positive constant C_3. Hence using (3.6) and the J-plurisubharmonicity of $\varphi_{q'}$ we obtain:

$$\Psi_{q'}(u(\zeta)) \leq -\frac{C_3}{2\pi C_2} \int_0^{2\pi} \varphi_{q'}\left(u\left(re^{i\theta}\right)\right) d\theta \leq -\frac{C_3}{C_2} \varphi_{q'}(u(0)).$$

Since there is a positive constant C_4 such that

$$\|u(\zeta) - q'\|^{2m} \leq C_4 \Psi_{q'}(u(\zeta))$$

and using (3.6), we finally obtain:

$$\|u(\zeta) - q'\|^{2m} \leq \frac{C_1 C_3 C_4}{C_2} \operatorname{dist}(u(0), \partial D).$$

Hence there exists a positive constant C_5 such that:

$$\operatorname{dist}(u(\zeta), \partial D) \leq C_5 \operatorname{dist}(u(0), \partial D)^{1/2m},$$

whenever $\zeta \leq s$.

According to Lemma 1.5 of [17] there is a positive constant C_6 such that:

$$\|\nabla u(0)\| \leq C_6 \sup_{|\zeta| < s} \|u(\zeta) - u(0)\| \leq C_5 C_6 \operatorname{dist}(u(0), \partial D)^{1/2m},$$

wich provides the desired estimate. □

We also need the two next lemmas provided by [6]:

Lemma 3.11. *Let D be a domain in an almost complex manifold (M, J). Then there is a positive constant C such that for any $p \in D$ and any $v \in T_p M$:*

$$K_{(D,J)}(p,v) \leq C \frac{\|v\|}{\operatorname{dist}(p, \partial D)}. \tag{3.7}$$

Lemma 3.12. *(Hopf lemma) Let D be a relatively compact domain with a $\mathcal{C}^2$ boundary on an almost complex manifold (M,J). Then for any negative J-plurisubharmonic function ρ on D there exists a constant $C>0$ such that for any $p\in D$:*

$$|\rho(p)|\geq C\mathrm{dist}(p,\partial D).$$

Now we can go on the proof of Proposition 3.9.

Proof of Proposition 3.9. Let $f:D\to D'$ be a (J,J')-biholomorphism. According to Proposition 3.10 and to the decreasing property of the Kobayashi pseudometric there is a positive constant C such that for every $p\in D$ sufficiently close to the boundary and every $v\in T_pM$

$$C\frac{\|d_pf\,(v)\,\|}{\mathrm{dist}\,(f\,(p)\,,\partial D')^{\frac{1}{2m}}}\leq K_{(D',J')}\,(f\,(p)\,,d_pf\,(v))=K_{(D,J)}\,(p,v)\,.$$

Due to Lemma 3.11 there exists a positive constant C_1 such that:

$$K_{(D,J)}\,(p,v)\leq C_1\frac{\|v\|}{\mathrm{dist}\,(p,\partial D)}.$$

This leads to:

$$\|d_pf\,(v)\,\|\leq\frac{C_1}{C}\frac{\mathrm{dist}\,(f\,(p)\,,\partial D')^{\frac{1}{2m}}}{\mathrm{dist}\,(p,\partial D)}\|v\|.$$

Moreover the Hopf lemma 3.12 for almost complex manifolds applied to $\rho'\circ f$ and $\rho\circ f^{-1}$ and the fact that ρ and ρ' are defining functions, provides the following boundary distance preserving property:

$$\frac{1}{C_2}\mathrm{dist}\,(p,\partial D)\leq\mathrm{dist}\left(f\,(p)\,,\partial D'\right)\leq C_2\mathrm{dist}\,(p,\partial D)\,,$$

for some positive constat C_2. Finally this implies:

$$\|d_pf\,(v)\,\|\leq\frac{C_1C_2}{C}\frac{\|v\|}{\mathrm{dist}\,(p,\partial D)^{\frac{2m-1}{2m}}}.$$

This gives the desired statement. □

4. Sharp estimates of the Kobayashi pseudometric

In this section we give sharp lower estimates of the Kobayashi pseudometric in a pseudoconvex region near a boundary point of finite D'Angelo type less than or equal to four. This condition will appear necessary, in our proof, as explained in the appendix. Moreover in order to give sharp estimates near a point of arbitrary finite D'Angelo type, we are also interested in the nontangential behaviour of the Kobayashi pseudometric.

The main result of this section is the following theorem (see also Theorem B):

Theorem 4.1. *Let $D=\{\rho<0\}$ be a relatively compact domain of finite D'Angelo type less than or equal to four in an almost complex manifold (M,J) of dimension four, where ρ is a $\mathcal{C}^2$ defining function of D, J-plurisubharmonic on a neighborhood of $\overline{D}$. Then there exists a positive constant C with the following property: for every $p\in D$ and every $v\in T_pM$ there is a diffeomophism, Φ_{p^*}, in a neighborhood U of p, such that:*

$$K_{(D,J)}\,(p,v)\geq C\left(\frac{|\,(d_p\Phi_{p^*}v)_1\,|}{\tau\,(p^*,|\rho\,(p)\,|)}+\frac{|\,(d_p\Phi_{p^*}v)_2\,|}{|\rho\,(p)\,|}\right), \tag{4.1}$$

where $\tau\,(p^*,|\rho\,(p)\,|)$ is defined by (4.3).

As a direct consequence we have:

$$K_{(D,J)}(p,v) \geq C' \left(\frac{|(d_p\Phi_{p^*}v)_1|}{|\rho(p)|^{\frac{1}{4}}} + \frac{|(d_p\Phi_{p^*}v)_2|}{|\rho(p)|} \right), \tag{4.2}$$

for a positive constant C'.

In complex manifolds, D.Catlin [5] first obtained such an estimate, based on lower estimates of the Carathéodory pseudometric. F.Berteloot [3] gave a different proof based on a Bloch principle. Our proof wich is inspired by the proof of F.Berteloot is based on some scaling method.

4.1. **The scaling method.** We consider here a pseudoconvex region $D = \{\rho < 0\}$ of finite D'Angelo type $2m$ in $\mathbb{R}^4$, where ρ has the following expression on a neighborhood U of the origin:

$$\rho(z_1,z_2) = \Re e z_2 + H_{2m}(z_1,\overline{z_1}) + O\left(|z_1|^{2m+1} + |z_2|\|z\|\right).$$

where H_{2m} is a homogeneous subharmonic polynomial of degree $2m$ admitting a nonharmonic part.

Assume that p_ν is a sequence of points in $D \cap U$ converging to the origin. For each p_ν sufficiently close to ∂D, there exists a unique point $p_\nu^* \in \partial D \cap U$ such that

$$p_\nu^* = p_\nu + (0,\delta_\nu),$$

with $\delta_\nu > 0$. Notice that for large ν, the quantity δ_ν is equivalent to $\operatorname{dist}(p_\nu, \partial D \cap U)$ and to $|\rho(p_\nu)|$.

We consider a diffeomorphism $\Phi^\nu : \mathbb{R}^4 \to \mathbb{R}^4$ satisfying:

(1) $\Phi^\nu(p_\nu^*) = 0$ and $\Phi^\nu(p_\nu) = (0,-\delta_\nu)$.
(2) Φ^ν converges to $Id : \mathbb{R}^4 \to \mathbb{R}^4$ on any compact subset of $\mathbb{R}^4$ in the $\mathcal{C}^2$ sense.
(3) When we denote by $D^\nu := \Phi^\nu(D \cap U)$ which admits the defining function is $\rho^\nu := \rho \circ (\Phi^\nu)^{-1}$ and by $J^\nu := (\Phi^\nu)_* J$, then ρ^ν is given by:

$$\rho^\nu(z_1,z_2) = \Re e z_2 + \sum_{k=2l_\nu}^{2m} P_k(z_1,\overline{z_1},p_\nu^*) + O\left(|z_1|^{2m+1} + |z_2|\|z\|\right),$$

where the polynomial P_{2l_ν} contains a nonharmonic part. Moreover J^ν satisfies (2.2) and (2.3).

This is done by considering first the translation T^ν of $\mathbb{R}^4$ given by $z \mapsto z - p_\nu^*$. According to J.-F.Barraud and E.Mazzilli [1] that the D'Angelo type is an upper semicontinuous function in a four dimensional almost complex manifold. Thus the D'Angelo type of points in a small enough neighborhood can only be smaller than at the point itself. Then we consider a $(T^\nu)_* J$-holomorphic disc u of maximal contact order $2l_\nu$, where $2l_\nu \leq 2m$ is the D'Angelo type of p_ν^*. We choose coordinates such that u is given by $u(\zeta) = (\zeta,0)$, and such that $(T^\nu)_* J(z_1,0) = J_{st}$ and $T_0(\partial T^\nu(D)) \cap J(0)T_0(\partial T^\nu(D)) = \{z_2 = 0\}$. Then by considering the family of vectors $(1,0)$ at base points $(0,t)$ for $t \neq 0$ small enough, we obtain a family of pseudoholomorphic discs u_t such that $u_t(0) = (0,t)$ and $d_0 u_t(\partial/\partial_x) = (0,1)$. Due to the parameters dependance of the solution to the J^ν-holomorphy equation, we straighten these discs into the lines $\{z_2 = t\}$. Next we consider a transversal foliation by pseudoholomorphic discs passing through $(t,0)$ and $(t,-\delta_\nu)$ for t small enough and we straighten these lines into $\{z_1 = c\}$. This leads to the desired diffeomorphism Φ^ν of $\mathbb{R}^4$.

Now, we need to remove harmonic terms from the polynomial

$$\sum_{k=2l_\nu}^{2m-1} P_k(z_1,\overline{z_1},p_\nu^*).$$

So we consider a biholomorphism (for the standard structure) of $\mathbb{C}^2$ with the following form:

$$\varphi_\nu(z_1,z_2) := \left(z_1, z_2 + \sum_{k=2l_\nu}^{2m-1} \Re e\left(c_{k,\nu} z_1^k\right)\right),$$

where $c_{k,\nu}$ are well chosen complex numbers. Then the diffeomorphism $\Phi_\nu := \varphi_\nu \circ \Phi^\nu$ satisfies:

(1) $\Phi_\nu(p_\nu^*) = 0$ and $\Phi_\nu(p_\nu) = (0, -\delta_\nu)$.
(2) Φ_ν converges to $Id : \mathbb{R}^4 \to \mathbb{R}^4$ on any compact subset of $\mathbb{R}^4$ in the $\mathcal{C}^2$ sense.
(3) If we denote by $D_\nu := \Phi_\nu(D \cap U)$ the domain with the defining function $\rho_\nu := \rho \circ (\Phi_\nu)^{-1}$, then ρ_ν is given by:

$$\rho_\nu(z_1,z_2) = \Re e z_2 + \sum_{k=2l_\nu}^{2m-1} P_k^*(z_1, \overline{z_1}, p_\nu^*) + P_{2m}(z_1, \overline{z_1}, p_\nu^*) + O\left(|z_1|^{2m+1} + |z_2|\|z\|\right),$$

where the polynomial

$$\sum_{k=2l_\nu}^{2m-1} P_k^*(z_1, \overline{z_1}, p_\nu^*)$$

does not contain any harmonic terms. Moreover the polynomial $P_{2l_\nu}^*$ is not idencally zero. Moreover, generically, $J_\nu := (\Phi_\nu)_* J$ is no more diagonal.

Since the origin is a boundary point of D'Angelo type $2m$ for D, it follows that, denoting by P_{2m}^* the nonharmonic part of P_{2m}, we have $P_{2m}^*(.,0) = H_{2m}^* \neq 0$, where H_{2m}^* is the nonharmonic part of H_{2m}. This allows to define for large ν:

$$\tau(p_\nu^*, \delta_\nu) := \min_{k=2l_\nu,\cdots,2m} \left(\frac{\delta_\nu}{\|P_k^*(.,p_\nu^*)\|}\right)^{\frac{1}{k}}. \tag{4.3}$$

Moreover the following inequalities hold:

$$\frac{1}{C}\delta_\nu^{\frac{1}{2}} \leq \tau(p_\nu^*, \delta_\nu) \leq C\delta_\nu^{\frac{1}{2m}}, \tag{4.4}$$

where C is a positive constant. The right inequality comes from the fact that $\|P_{2m}^*(.,p_\nu^*)\| \geq C_1 > 0$ for large ν. And the left one comes the fact that there exists a positive constant C_2 such that for every $2l_\nu \leq k \leq 2m$ we have $\|P_k^*(.,p_\nu^*)\| \leq C_2$.

Now we consider the nonisotropic dilation Λ_ν of $\mathbb{C}^2$:

$$\Lambda_\nu : (z_1,z_2) \mapsto \left(\tau(p_\nu^*,\delta_\nu)^{-1} z_1, \delta_\nu^{-1} z_2\right).$$

We set $\tilde{D_\nu} := \Lambda_\nu(D_\nu)$ the domain admitting the defining function $\tilde{\rho_\nu} := \delta_\nu^{-1}\rho_\nu \circ \Lambda_\nu^{-1}$ and $\tilde{J_\nu} := (\Lambda_\nu)_*(J_\nu)$ the direct image of J_ν under Λ_ν.

The next lemma is devoted to describe $(\tilde{D_\nu}, \tilde{J_\nu})$ when passing at the limit.

Lemma 4.2.

(1) *The domain $\tilde{D^\nu}$ converges in the sense of local Hausdorff set convergence to a (standard) pseudoconvex domain $\tilde{D} = \{\tilde{\rho} < 0\}$, with*

$$\tilde{\rho}(z) = \Re e z_2 + P(z_1, \overline{z_1}),$$

where P is a nonzero subharmonic polynomial of degree smaller than or equal to $2m$ which admits a nonharmonic part.

(2) ***In case the origin is of D'Angelo type four for*** D*, the sequence of almost complex structures* $\tilde{J}_\nu$ *converges on any compact subsets of* $\mathbb{C}^2$ *in the* $\mathcal{C}^2$ *sense to* J_{st}.

Proof. We first prove part 1. Due to inequalities (4.4), the defining function of $\tilde{D_\nu}$ satisfies:

$$\tilde{\rho_\nu} = \Re e z_2 + \sum_{k=2l_\nu}^{2m} \delta_\nu^{-1}\tau\left(p_\nu^*, \delta_\nu\right)^k P_k^*\left(z_1, \overline{z_1}, p_\nu^*\right) + \delta_\nu^{-1}\tau\left(p_\nu^*, \delta_\nu\right)^{2m} P_{2m}\left(z_1, \overline{z_1}, p_\nu^*\right) + O\left(\tau\left(\delta_\nu\right)\right).$$

Passing to a subsequence, we may assume that the polynomial

$$\sum_{k=2l_\nu}^{2m} \delta_\nu^{-1}\tau\left(p_\nu^*, \delta_\nu\right)^k P_k^*\left(z_1, \overline{z_1}, p_\nu^*\right) + \delta_\nu^{-1}\tau\left(p_\nu^*, \delta_\nu\right)^{2m} P_{2m}\left(z_1, \overline{z_1}, p_\nu^*\right)$$

converges uniformly on compact subsets of $\mathbb{C}^2$ to a nonzero polynomial P of degree $\leq 2m$ admitting a nonharmonic part. Since the pseudoconvexity is invariant under diffeomorphisms, it follows that the domains $\tilde{D^\nu}$ are $\tilde{J_\nu}$-pseudoconvex, and then passing to the limit, the domain $\tilde{D}$ is J_{st}-pseudoconvex. Thus the polynomial P is subharmonic.

We next prove part 2. The complexification of the almost complex structure J_ν is given by

$$\begin{aligned}
(J_\nu)_{\mathbb{C}} &= \sum_{l=1}^{2}\Big(A_{l,l}(z)\,dz_l \otimes \frac{\partial}{\partial z_l} + B_{l,l}(z)\,dz_l \otimes \frac{\partial}{\partial \overline{z_l}} + \overline{B_{l,l}}(z)\,d\overline{z_l} \otimes \frac{\partial}{\partial z_l} + \\
&\quad \overline{A_{l,l}}(z)\,d\overline{z_l} \otimes \frac{\partial}{\partial \overline{z_l}}\Big) + A_{1,2}(z)\,dz_1 \otimes \frac{\partial}{\partial z_2} + B_{1,2}(z)\,dz_1 \otimes \frac{\partial}{\partial \overline{z_2}} + \\
&\quad \overline{B_{1,2}}(z)\,d\overline{z_1} \otimes \frac{\partial}{\partial z_2} + \overline{A_{1,2}}(z)\,d\overline{z_1} \otimes \frac{\partial}{\partial \overline{z_2}},
\end{aligned}$$

where

$$\begin{cases}
A_{l,l}(z) &= i + O\left(\left|z_2 + \sum_{k=2}^{3} c_{k,\nu} z_1^k\right|^2\right) \text{ for } l = 1,2, \\
B_{l,l}(z) &= O\left(\left|z_2 + \sum_{k=2}^{3} c_{k,\nu} z_1^k\right|\right) \text{ for } l = 1,2, \\
A_{1,2}(z) &= \sum_{k=2}^{3} k c_{k,\nu} z_1^{k-1} O\left(\left|z_2 + \sum_{k=2}^{3} c_{k,\nu} z_1^k\right|^2\right), \\
B_{1,2}(z) &= \sum_{k=2}^{3} k\left(c_{k,\nu} z_1^{k-1} - \overline{c_{k,\nu} z_1^{k-1}}\right) O\left(\left|z_2 + \sum_{k=2}^{3} c_{k,\nu} z_1^k\right|\right).
\end{cases}$$

By a direct computation, the complexification of $\tilde{J_\nu}$ is equal to:

$$\begin{aligned}\left(\tilde{J_\nu}\right)_{\mathbb{C}} &= \sum_{l=1}^{2}(A_{l,l}(\Lambda_\nu^{-1}(z))dz_l\otimes\frac{\partial}{\partial z_l}+B_{l,l}(\Lambda_\nu^{-1}(z))dz_l\otimes\frac{\partial}{\partial\overline{z_l}}+\\ &\overline{B_{l,l}}(\Lambda_\nu^{-1}(z))d\overline{z_l}\otimes\frac{\partial}{\partial z_l}+\overline{A_{l,l}}(\Lambda_\nu^{-1}(z))d\overline{z_l}\otimes\frac{\partial}{\partial\overline{z_l}})+\\ &\tau(p_\nu^*,\delta_\nu)\delta_\nu^{-1}A_{1,2}(\Lambda_\nu^{-1}(z))dz_1\otimes\frac{\partial}{\partial z_2}+\tau(p_\nu^*,\delta_\nu)\delta_\nu^{-1}B_{1,2}(\Lambda_\nu^{-1}(z))dz_1\otimes\frac{\partial}{\partial\overline{z_2}}+\\ &\tau(p_\nu^*,\delta_\nu)\delta_\nu^{-1}\overline{B_{1,2}}(\Lambda_\nu^{-1}(z))d\overline{z_1}\otimes\frac{\partial}{\partial z_2}+\tau(p_\nu^*,\delta_\nu)\delta_\nu^{-1}\overline{A_{1,2}}(\Lambda_\nu^{-1}(z))d\overline{z_1}\otimes\frac{\partial}{\partial\overline{z_2}}.\end{aligned}$$

According to (4.4) and since $c_{k,\nu}$ converges to zero when ν tends to $+\infty$ for $k=2,3$, it follows that $\tilde{J_\nu}$ converges to J_{st}. This proves part (2). □

4.2. **Complete hyperbolicity in D'Angelo type four condition.** In this subsection we prove Theorem 4.1. Keeping notations of the previous subsection; we start by establishing the following lemma which gives a precise localization of pseudoholomorphic discs in boxes.

Lemma 4.3. *Assume the origin $\in\partial D$ is a point of D'Angelo type four. There are positive constants C_0, δ_0 and r_0 such that for any $0<\delta<\delta_0$, for any large ν and for any J_ν-holomorphic disc $g_\nu:\Delta\to D_\nu$ we have :*

$$g_\nu(0)=(0,-\delta_\nu)\Rightarrow g_\nu(r_0\Delta)\subset Q(0,C_0\delta_\nu),$$

where $Q(0,\delta_\nu):=\{z\in\mathbb{C}^2:|z_1|\leq\tau(p_\nu^*,\delta_\nu),|z_2|\leq\delta_\nu\}$.

Proof. Proof of Lemma 4.3. Assume by contradiction that there are a sequence $(C_\nu)_\nu$ that tends to $+\infty$ as ζ_ν converges to 0 in Δ, and J_ν-holomorphic discs $g_\nu:\Delta\to D_\nu$ such that $g_\nu(0)=(0,-\delta_\nu)$ and $g_\nu(\zeta_\nu)\notin Q(0,C_\nu\delta_\nu)$. We consider the nonisotropic dilations of $\mathbb{C}^2$:

$$\Lambda_\nu^r:(z_1,z_2)\mapsto\left(r^{\frac{1}{4}}\tau(p_\nu^*,\delta_\nu)^{-1}z_1,r\delta_\nu^{-1}z_2\right),$$

where r is a positive constant to be fixed. We set $h_\nu:=\Lambda_\nu^r\circ g_\nu$, $\tilde{\rho_\nu^r}:=r\delta_\nu^{-1}\rho_\nu\circ(\Lambda_\nu^r)^{-1}$ and $\tilde{J_\nu^r}:=(\Lambda_\nu^r)_*J_\nu$. It follows from Lemma 4.2 that $\tilde{\rho_\nu^r}$ converges to

$$\tilde{\rho}=Re(z_2)+P(z_1,\overline{z_1})$$

uniformly on any compact subset of $\mathbb{C}^2$ and $\tilde{J_\nu^r}$ converges to J_{st}, uniformly on any compact subset of $\mathbb{C}^2$. According to the stability of the Kobayashi pseudometric stated in Proposition 3.7, there exist a positive constant C and a neighborhood V of the origin in $\mathbb{R}^4$, such that for every large ν, for every $q\in\tilde{D_\nu}\cap V$ and every $v\in T_q\mathbb{R}^4$:

$$K_{(\tilde{D_\nu},\tilde{J_\nu})}(q,v)\geq C\|v\|.$$

Therefore, there exists a constant $C'>0$ such that

$$\|\,dh_\nu(\zeta)\,\|\leq C'$$

for any $\zeta\in(1/2)\,\Delta$ satisfying $h_\nu(\zeta)\in\tilde{D_\nu}\cap V'$, with $V'\subset V$. Now we choose the constant r such that $h_\nu(0)=(0,-r)\in Int(V')$. On the other hand, the sequence $|h_\nu(\zeta_\nu)|$ tends to $+\infty$. Denote by $[0,\zeta_\nu]$ the segment (in $\mathbb{C}$) joining the origin and ζ_ν and let $\zeta_\nu'=r_\nu e^{i\theta_\nu}\in[0,\zeta_\nu]$ be the point closest to the origin such that $h_\nu([0,\zeta_\nu'])\subset\tilde{D_\nu}\cap V$ and $h_\nu(\zeta_\nu')\in\partial V$. Since $h_\nu(0)\in Int(V')$, we have

$$\|h_\nu(0)-h_\nu(\zeta_\nu')\|\geq C''$$

for some constant $C'' > 0$. It follows that:

$$\|h_\nu(0) - h_\nu(\zeta'_\nu)\| \leq \int_0^{r_\nu} \left\| dh_\nu\left(te^{i\theta_\nu}\right)\right\| dt \leq C' r_\nu \longrightarrow 0.$$

This contradiction proves Lemma 4.3. □

Now we go on the proof of Theorem 4.1.

Proof of Theorem 4.1. Due to the localization of the Kobayashi pseudometric established in Proposition 3.4, it suffices to prove Theorem 4.1 in a neighborhood U of $q \in \partial D$. Choosing local coordinates $z : U \to \mathbb{B} \subset \mathbb{R}^4$ centered at q, we may assume that $D \cap U = \{\rho < 0\}$ is a J-pseudonconvex region of $(\mathbb{R}^4, J)$, that $q = 0 \in \partial D$ and that J satisfies (2.2) and (2.3). We also suppose that the complex tangent space $T_0\partial D \cap J(0)T_0\partial D$ at 0 of ∂D is given by $\{z_2 = 0\}$. Moreover the defining function ρ is expressed by:

$$\rho(z) = \Re e z_2 + H_{2m}(z_1, \overline{z_1}) + O\left(|z_1|^{2m+1} + |z_2|\|z\|\right)$$

For $p \in D \cap U$ be sufficiently close to the boundary ∂D, there exists a unique point $p^* \in \partial D \cap U$ such that

$$p^* = p + (0, \delta),$$

with $\delta > 0$. We define an infinitesimal pseudometric N on $D \cap U \subseteq \mathbb{R}^4$ by:

$$N(p, v) := \frac{|(d_p\Phi_{p^*}v)_1|}{\tau(p^*, |\rho(p)|)} + \frac{|(d_p\Phi_{p^*}v)_2|}{|\rho(p)|}, \tag{4.5}$$

for every $p \in D \cap U$ and every $v \in T_p\mathbb{R}^4$, where Φ_{p^*} is defined as diffeomorphisms Φ^ν (of previous subsection) for p^* instead of p^*_ν.

To prove estimate (4.1) of Theorem 4.1, it suffices to find a positive constant C such that for any J-holomorphic disc $u : \Delta \to D \cap U$, we have:

$$N(u(0), d_0u(\partial/\partial_x)) \leq C. \tag{4.6}$$

Indeed, for a J-holomorphic disc u such that $u(0) = p$ and $d_0u(\partial/\partial_x) = rv$, (4.6) leads to

$$\frac{1}{r} = \frac{N(p, v)}{N(u(0), d_0u(\partial/\partial_x))} \geq \frac{N(p, v)}{C}.$$

Suppose by contradiction that (4.6) is not true, that is, there is a sequence of J-holomorphic discs $u_\nu : \Delta \to D \cap U$ such that $N(u_\nu(0), d_0u_\nu(\partial/\partial_x)) \geq \nu^2$. Then we consider a sequence $(y_\nu)_\nu$ of points in $\overline{\Delta_{1/2}}$ such that:

(1) $|y_\nu| \leq \dfrac{2\nu}{N(u_\nu(y_\nu), d_{y_\nu}u_\nu(\partial/\partial x))}$,

(2) $N(u_\nu(y_\nu), d_{y_\nu}u_\nu(\partial/\partial x)) \geq \nu^2$, and

(3) $y_\nu + \Delta_{\nu/N(u_\nu(y_\nu), d_{y_\nu}u_\nu(\partial/\partial x))} \subseteq \overline{\Delta_{1/2}}$ for sufficiently large ν.

This allows to define a sequence of J-holomorphic discs $g_\nu : \Delta_\nu \to D \cap U$ by

$$g_\nu(\zeta) := u_\nu\left(y_\nu + \frac{\zeta}{2N(u_\nu(y_\nu), d_{y_\nu}u_\nu(\partial/\partial x))}\right).$$

Consider the sequence $g_\nu = u_\nu(y_\nu)$ in $D\cap U$. Since $|y_\nu| \leq 2/\nu$ and since the $\mathcal{C}^1$ norm of any J-holomorphic disc u_ν is uniformally bounded it follows that $g_\nu(0)$ converges to the origin.

We apply the scaling method to the sequence $g_\nu\,(0)$. We denote by $g_\nu\,(0)^*$ the boundary point given by $g_\nu\,(0)^* := g_\nu\,(0) + (0, \delta_\nu)$. We set the scaled disc $\tilde{g_\nu} := \Lambda_\nu \circ \Phi_\nu \circ g_\nu$, where diffeomorphisms Λ_ν and Φ_ν are define in the subsection about the scaling method. In order to extract from $\tilde{g_\nu}$ a subsequence which converges to a Brody curve, we need the following Lemma.

Lemma 4.4. *There is a positive constant r_0 such that:*

(1) *There exists a positive constant C_1 such that*

$$\tilde{g_\nu}\,(r_0\Delta_\nu) \subset \Delta_{C_1} \times \Delta_{C_1}. \tag{4.7}$$

(2) *There exists a positive constant C_2 such that for every large ν we have :*

$$\|d\tilde{g_\nu}\|_{\mathcal{C}^0(r_0\Delta_\nu)} \leq C_2. \tag{4.8}$$

Proof. We prove the first part. We define a J_ν-holomorphic disc $h_\nu\,(\zeta) := \Phi_\nu \circ g_\nu\,(\nu\zeta)$ from the unit disc Δ to D_ν. According to Lemma 4.3, since $h_\nu\,(0) = \Phi_\nu \circ g_\nu\,(0) = (0, -\delta_\nu)$, we have

$$h_\nu\,(r_0\Delta) \subseteq Q\,(0, C_0\delta_\nu)$$

for some positive constants r_0 and C_0. Hence

$$\Phi_\nu \circ g_\nu\,(r_0\Delta_\nu) \subseteq Q\,(0, C_0\delta_\nu)\,.$$

After dilations, this leads to (4.7).

Then we prove the second part. According to Lemma 4.2, the sequence of almost complex structures $\tilde{J_\nu}$ converges on any compact subsets of $\mathbb{C}^2$ in the $\mathcal{C}^2$ sense to J_{st}. Then for sufficiently large ν, the norm $\|\tilde{J_\nu} - J_{st}\|_{\mathcal{C}^1(\overline{\Delta_{C_1}\times\Delta_{C_1}})}$ is as small as necessary. So for large ν, and due to Proposition $2.3.6$ of J.-C.Sikorav in [23] there exists $C_2 > 0$ such that (4.8) holds. □

Hence according to Lemmas 4.2 and 4.4 we may extract from $\tilde{g_\nu}$ a subsequence, still denoted by $\tilde{g_\nu}$ wich converges in $\mathcal{C}^1$ topology to a standard complex line

$$\tilde{g}\ : \mathbb{C} \to (\{Re z_2 + P\,(z_1, \overline{z_1}) < 0\}, J_{st})\,.$$

The polynomial P is subharmonic and contains a nonharmonic part; this implies that the domain $(\{Re z_2 + P\,(z_1, \overline{z_1}) < 0\}, J_{st})$ is Brody hyperbolic and so the complex line $\tilde{g}$ is constant. To obtain a contradiction, we prove that the derivative of $\tilde{g}$ at the origin is nonzero:

$$\frac{1}{2} = N\,(g_\nu\,(0)\,, d_0 g_\nu\,(\partial/\partial_x)) = \frac{|\,(d_0\,(\Phi_\nu \circ g_\nu)\,(\partial/\partial_x))_1\,|}{\tau\,(g_\nu\,(0)^*\,, |\rho\,(g_\nu\,(0))\,|)} + \frac{|\,(d_0\,(\Phi_\nu \circ g_\nu)\,(\partial/\partial_x))_2\,|}{|\rho\,(g_\nu\,(0))\,|}.$$

Since $|\rho\,(g_\nu\,(0))\,|$ is equivalent to δ_ν, it follows that for some positive constant C_3 and for large ν, we have:

$$\frac{1}{2} \leq C_3\left(\frac{|\,(d_0\,(\Phi_\nu \circ g_\nu)\,(\partial/\partial_x))_1\,|}{\tau\,(g_\nu\,(0)^*\,, \delta_\nu)} + \frac{|\,(d_0\,(\Phi_\nu \circ g_\nu)\,(\partial/\partial_x))_2\,|}{\delta_\nu}\right) = C_3\|d_0\tilde{g_\nu}\,(\partial/\partial_x)\,\|_1.$$

Since $\tilde{g_\nu}$ converges to $\tilde{g}$ in the $\mathcal{C}^1$ sense, it follows that $d_0\tilde{g}\,(\partial/\partial_x) \neq 0$, providing a contradiction. This achieves the proof of Theorem 4.1. □

Estimate (4.2) of the Kobayashi pseudometric allows to study the completness of the Kobayashi pseudodistance D.

Corollary 4.5. *Let $D = \{\rho < 0\}$ be a relatively compact domain of finite D'Angelo type less than or equal to four in an almost complex manifold (M, J) of dimension four, where ρ is a defining function of D, J-plurisubharmonic in a neighborhood of $\overline{D}$. Assume that (M, J) admits a global strictly J-plurisubharmonic function. Then (D, J) is complete hyperbolic.*

Proof. The fact that (M, J) admits a global strictly J-plurisubharmonic function and estimate (3.1) of Proposition 3.4 leads to the Kobayashi hyperbolicity of D. Then estimate (4.2) of the Kobayashi pseudometric stated in Theorem 4.1 gives the completness of the metric space $\left(D, d_{(D,J)}\right)$ by a classical integration argument. □

4.3. **Regions with noncompact automorphisms group.** The next corollary is devoted to regions with noncompact automorphisms group.

Corollary 4.6. *Let $D = \{\rho < 0\}$ be a relatively compact domain in a four dimensional almost complex manifold (M, J) of finite D'Angelo type less than or equal to four. Assume that ρ is a $\mathcal{C}^2$ defining function of D, J-plurisubharmonic on a neighborhood of $\overline{D}$. If there is an automorphism of D with orbit accumulating at a boundary point then there exists a polynomial P of degree at most four, without harmonic part such that (D, J) is biholomorphic to $(\{\Re e z_2 + P(z_1, \overline{z_1}) < 0\}, J_{st})$.*

If the domain D is a relatively compact strictly J-pseudoconvex domain with noncompact automorphisms group then (D, J) is biholomorphic to a model domain. This was proved by H.Gaussier and A.Sukhov in [12] in dimension four and by K.H.Lee in [20] in arbitrary (even) dimension.

Sketch of the proof. We suppose that for some point $p_0 \in D$, there is a sequence f_ν of automorphisms of (D, J) such that $p_\nu := f_\nu(p_0)$ converges to $0 \in \partial D$. We apply the scaling method to the sequence p_ν. Still keeping notations of subsection 4.1, we set

$$F_\nu := \Lambda_\nu \circ \Phi_\nu \circ f_\nu : f_\nu^{-1}(D \cap U) \to \tilde{D}_\nu.$$

This sequence of biholomorphisms is such that :

(1) $\left(f_\nu^{-1}(D \cap U)\right)_\nu$ converges to D.
(2) $\tilde{D}_\nu$ converges to a pseudoconvex domain $\tilde{D} = \{Re z_2 + P(z_1, \overline{z_1}) < 0\}$, where P is a nonzero subharmonic polynomial of degree ≤ 4 which contains a nonharmonic part. Changing $\tilde{D}$ by applying a standard biholomorphism if necessary, we may suppose that $P(z_1, \overline{z_1})$ is without harmonic terms.
(3) For any compact subset $K \subset D$, the sequence $\left(\|F_\nu\|_{\mathcal{C}^1(K)}\right)_\nu$ is bounded.

Hence, we may extract from $(F_\nu)_\nu$ a subsequence converging, on any compact subset of D in the $\mathcal{C}^\infty$ sense, to a (J, J_{st})-holomorphic map $F : D \longrightarrow \bar{\tilde{D}}$. Finally F is a (J, J_{st})-biholomorphism from D to $\tilde{D}$. □

4.4. **Nontangential approach in the general setting.** In this subsection, refering to I.Graham [14], we give a sharp estimate of the Kobayashi pseudometric of a pseudoconvex region in a cone with vertex at a boundary point of arbitrary finite D'Angelo type. We denote by $\Lambda := \{-\Re e z_2 > k\|z\|\}$, where $0 < k < 1$, the cone with vertex at the origin and axis the negative real z_2 axis.

Theorem 4.7. *Let $D = \{\rho < 0\}$ be a domain of finite D'Angelo type in $\left(\mathbb{R}^4, J\right)$, where*

$$\rho(z_1, z_2) = \Re e z_2 + H_{2m}(z_1, \overline{z_1}) + O\left(|z_1|^{2m+1} + |z_2|\|z\|\right),$$

is a $\mathcal{C}^2$ defining function of D, J-plurisubharmonic on a neighborhood of $\overline{D}$. We suppose that H_{2m} is a homogeneous subharmonic polynomial of degree $2m$ admitting a nonharmonic part. Then there exists a positive constant C such that for every $p \in D \cap \Lambda$ and every $v \in T_pM$:

$$K_{(D,J)}(p, v) \geq C\left(\frac{|v_1|}{|\rho(p)|^{\frac{1}{2m}}} + \frac{|v_2|}{|\rho(p)|}\right).$$

Before proving Theorem 4.7 we need the following crucial lemma.

Lemma 4.8. *There exist a neighborhood U of the origin and a positive constant C such that if $p \in D \cap U \cap \Lambda$ then*

$$p \in \left\{ z \in \mathbb{C}^2 \ : |z_1| < C_1 \text{dist}\,(p, \partial D)^{\frac{1}{2m}}, |z_2| < C_1 \text{dist}\,(p, \partial D) \right\}.$$

Proof. According to the fact that $\text{dist}\,(z, \partial D)$ is equivalent to $|\rho(z)| = -\Re e z_2 + O\left(\|z\|^2\right)$ and to the definition of the cone Λ, we have:

$$\lim_{z \to 0, z \in D \cap \Lambda} \frac{-\Re e z_2}{\text{dist}\,(z, \partial D)} = 1.$$

This implies the existence of a positive constant C_1 such that

$$\|p\| < -\frac{1}{k} \Re e p_2 \leq C_1 \text{dist}\,(p, \partial D),$$

whenever $p \in D \cap \Lambda$ is sufficiently close to the origin. Thus

$$p \in \left\{ z \in \mathbb{C}^2 \ : |z_1| < C_1 \text{dist}\,(p, \partial D)^{\frac{1}{2m}}, |z_2| < C_1 \text{dist}\,(p, \partial D) \right\},$$

for $p \in D \cap \Lambda$ sufficiently close to the origin.

□

The proof of Theorem 4.7 is similar and easier than proof of Theorem 4.1. For convenience, we write it.

Proof of Theorem 4.7. Let U be a neighborhood of the origin. We define an infinitesimal pseudometric N on $D \cap U \subseteq \mathbb{R}^4$ by:

$$N(p, v) := \frac{|v_1|}{|\rho(p)|^{\frac{1}{2m}}} + \frac{|v_2|}{|\rho(p)|},$$

for every $p \in D \cap U$ and every $v \in T_p \mathbb{C}^2$.

We have to find a positive constant C such that for every J-holomorphic disc $u : \Delta \to D \cap U$, such that if $u(0) \in \Lambda$ then:

$$N\left(u(0), d_0 u\left(\partial/\partial_x\right)\right) \leq C.$$

Suppose by contradiction that this inequality is not true, that is, there exists a sequence of J-holomorphic discs $u_\nu : \Delta \to D \cap U$ such that

$$u_\nu(0) \in \Lambda \quad \text{and} \quad N\left(u_\nu(0), d_0 u_\nu\left(\partial/\partial_x\right)\right) \geq \nu^2.$$

Then consider a sequence $(y_\nu)_\nu$ of points in $\overline{\Delta_{1/2}}$ such that

(1) $|y_\nu| \leq \dfrac{2\nu}{N\left(u_\nu(y_\nu), d_{y_\nu} u_\nu\left(\partial/\partial x\right)\right)}$,

(2) $N\left(u_\nu(y_\nu), d_{y_\nu} u_\nu\left(\partial/\partial x\right)\right) \geq \nu^2$, and

(3) $y_\nu + \Delta_{\nu/N(u_\nu(y_\nu), d_{y_\nu} u_\nu(\partial/\partial x))} \subseteq \overline{\Delta_{1/2}}$ for sufficiently large ν.

Then we define a sequence of J-holomorphic discs $g_\nu : \Delta_\nu \to D \cap U$ by

$$g_\nu(\zeta) := u_\nu\left(y_\nu + \frac{\zeta}{2N\left(u_\nu(y_\nu), d_{y_\nu} u_\nu\left(\partial/\partial x\right)\right)}\right).$$

For large ν, we have $g_\nu(0) = u_\nu(y_\nu)$ in $D \cap U \cap \Lambda$ and $g_\nu(0)$ converges to the origin. Set

$$\delta_\nu := \operatorname{dist}(g_\nu(0), \partial D),$$

and consider the following dilations of $\mathbb{C}^2$:

$$\Lambda_\nu : (z_1, z_2) \mapsto \left(\delta_\nu^{\frac{-1}{2m}} z_1, \delta_\nu^{-1} z_2\right).$$

In order to extract from $\Lambda_\nu \circ g_\nu$ a subsequence which converges to a Brody curve, we need the following Lemma.

Lemma 4.9. *There exists a positive constant r_0 such that:*

(1) *there exists a positive constant C_1 such that:*

$$\Lambda_\nu \circ g_\nu(r_0 \Delta_\nu) \subset \Delta_{C_1} \times \Delta_{C_1}, \tag{4.9}$$

(2) *there is a positive constant C_2 such that for every large ν we have :*

$$\|d(\Lambda_\nu \circ g_\nu)\|_{\mathcal{C}^0(r_0\Delta_\nu)} \leq C_2. \tag{4.10}$$

Proof. We first prove (4.9). We define a new J-holomorphic disc $h_\nu(\zeta) := g_\nu(\nu\zeta)$ from the unit disc Δ to D_ν. According to Lemma 4.8, we have

$$h_\nu(0) = g_\nu(0) \in \{z \in \mathbb{C}^2 : |z_1| \leq C_1 \delta_\nu^{\frac{1}{2m}}, |z_2| \leq C_1 \delta_\nu\}.$$

This implies:

$$h_\nu(r_0\Delta) \subseteq \{z \in \mathbb{C}^2 : |z_1| \leq C_0 \delta_\nu^{\frac{1}{2m}}, |z_2| < C_0 \delta_\nu\},$$

for positive constants r_0 and C_0, since Lemma 4.3 is true if we replace $\tau(p_\nu^*, \delta_\nu)$ by $\delta_\nu^{\frac{1}{2m}}$. Hence

$$g_\nu(r_0\Delta_\nu) \subseteq \{z \in \mathbb{C}^2 : |z_1| < C_0 \delta_\nu^{\frac{1}{2m}}, |z_2| \leq C_0 \delta_\nu\}.$$

After dilations, this leads to (4.9).

The proof of (4.10) is similar to (4.8) of Lemma 4.4, since the sequence of structures $(\Lambda_\nu)_* J$ converges on any compact subset of $\mathbb{C}^2$ in the $\mathcal{C}^1$ sense to J_{st} because J is diagonal. □

Hence according to Lemma 4.9 we may extract from $\Lambda_\nu \circ g_\nu$ a subsequence, still denoted by $\Lambda_\nu \circ g_\nu$ wich converges in the $\mathcal{C}^1$ sense to a standard complex line $\tilde{g} : \mathbb{C} \to (\{Re z_2 + H_{2m}(z_1, \overline{z_1}) < 0\}, J_{st})$, where the domain $(\{Re z_2 + P(z_1, \overline{z_1}) < 0\}, J_{st})$ is Brody hyperbolic since $H_{2m}(z_1, \overline{z_1})$ contains a nonharmonic part. Then the standard complex line $\tilde{g}$ is constant. To obtain a contradiction, we prove that the derivative of $\tilde{g}$ is nonzero:

$$\frac{1}{2} = N(g_\nu(0), d_0 g_\nu(\partial/\partial_x)) = \frac{|(d_0 g_\nu(\partial/\partial_x))_1|}{|\rho(g_\nu(0))|^{\frac{1}{2m}}} + \frac{|(d_0 g_\nu(\partial/\partial_x))_2|}{|\rho(g_\nu(0))|}.$$

Since $|\rho(g_\nu(0))|$ is equivalent to δ_ν, it follows that for some positive constant C_3 we have for large ν:

$$\frac{1}{2} \leq C_3 \left(\frac{|(d_0(g_\nu)(\partial/\partial_x))_1|}{\delta_\nu^{\frac{1}{2m}}} + \frac{|(d_0(g_\nu)(\partial/\partial_x))_2|}{\delta_\nu}\right) = C_3 \|d_0(\Lambda_\nu \circ g_\nu)(\partial/\partial_x)\|_1.$$

This provide a contradiction. □

5. Appendix : Convergence of the structures involved by the scaling method.

In this appendix, we prove that, generically, the convergence of the sequence of structures involved by the scaling method to the standard structure J_{st} occurs only on a neighborhood of boundary points of D'Angelo type less than or equal to four.

Let $D = \{\rho < 0\}$ be a pseudoconvex region of finite D'Angelo type $2m$ in $\mathbb{R}^4$, where ρ has the following expression on a neighborhood U of the origin:

$$\rho(z_1, z_2) = \Re e z_2 + H_{2m}(z_1, \overline{z_1}) + O\left(|z_1|^{2m+1} + |z_2|\|z\|\right),$$

where H_{2m} is a homogeneous subharmonic polynomial of degree $2m$ admitting a nonharmonic part. Assume that p_ν is a sequence of points in $D \cap U$ converging to the origin, and, for large ν, consider the sequence of diffeomorphisms $\Phi^\nu : \mathbb{R}^4 \to \mathbb{R}^4$ given in the scaling method. We suppose that the function $\rho^\nu = \rho \circ (\Phi^\nu)^{-1}$ is given by:

$$\rho^\nu(z_1, z_2) = \Re e z_2 + \Re e\left(\alpha_\nu z_1^2\right) + \beta\nu|z_1|^2 + \sum_{k=3}^{2m} P_k(z_1, \overline{z_1}, p_\nu^*) + O\left(|z_1|^{2m+1} + |z_2|\|z\|\right).$$

Moreover the structure $J^\nu := (\Phi^\nu)_* J$ satisfies (2.2) and (2.3). To fix notations, we set:

$$J^\nu = \begin{pmatrix} a_1^\nu & b_1^\nu & 0 & 0 \\ c_1^\nu & -a_1^\nu & 0 & 0 \\ 0 & 0 & a_2^\nu & b_2^\nu \\ 0 & 0 & c_2^\nu & -a_2^\nu \end{pmatrix}.$$

Now, consider the following diffeomorphism of $\mathbb{R}^4$ defined by:

$$\Psi_\nu^{-1}(x_1, y_1, x_2, y_2) = (x_1 + R_{1,\nu}, y_1 + S_{1,\nu}, x_2 + R_{2,\nu}, y_2 + S_{2,\nu}) \tag{5.1}$$

converging to the identity and such that $d_0\Psi_\nu^{-1} = Id$. We suppose that $R_{k,\nu}$ and $S_{k,\nu}$, for $k = 1, 2$ are real functions depending smoothly on x_1, y_1 and y_2 and that $R_{2,\nu}$ and $S_{2,\nu}$ are given by:

$$\begin{cases} R_{2,\nu} &= -\alpha_\nu x_1^2 + \alpha_\nu y_1^2 + O\left(|z_1|^3 + y_2^2 + |y_2|\|z\|\right), \\ S_{2,\nu} &= -2\alpha_\nu x_1 y_1 + O\left(|z_1|^3 + y_2^2 + |y_2|\|z\|\right). \end{cases} \tag{5.2}$$

We write:

$$\begin{cases} R_{1,\nu} &= r_{5,\nu}x_1^2 + r_{6,\nu}x_1y_1 + r_{7,\nu}y_1^2 + r_{1,\nu}x_1^3 + r_{2,\nu}x_1^2y_1 + r_{3,\nu}x_1y_1^2 + \\ & \quad r_{4,\nu}y_1^3 + O\left(|z_1|^4 + y_2^2 + |y_2|\|z\|\right) \\ S_{1,\nu} &= s_{5,\nu}x_1^2 + s_{6,\nu}x_1y_1 + s_{7,\nu}y_1^2 + s_{1,\nu}x_1^3 + s_{2,\nu}x_1^2y_1 + s_{3,\nu}x_1y_1^2 + \\ & \quad s_{4,\nu}y_1^3 + O\left(|z_1|^4 + y_2^2 + |y_2|\|z\|\right). \end{cases} \tag{5.3}$$

It follows that:

$$\rho_\nu \circ \Psi_\nu^{-1}(z_1, z_2) = \Re e z_2 + \beta_\nu|z_1^2| + \sum_{k=3}^{2m} P_k'(z_1, \overline{z_1}, \nu) + O\left(|z_1|^{2m+1} + |z_2|\|z\|\right).$$

Then we define

$$\tau_\nu := \min\left(\left(\frac{\delta_\nu}{|\beta_\nu|}\right)^{\frac{1}{2}}, \min_{k=3,\cdots,2m-1}\left(\frac{\delta_\nu}{\|P'_k(.,\nu)\|}\right)^{\frac{1}{k}}, \delta_\nu^{\frac{1}{2m}}\right).$$

And we consider the following anisotropic dilations of $\mathbb{C}^2$:

$$\Lambda_\nu(z_1,z_2) := \left(\tau_\nu^{-1}z_1, \delta_\nu^{-1}z_2\right).$$

If we write $J_\nu := (\Psi_\nu)_* J^\nu$ as:

$$J_\nu = \begin{pmatrix} J_{1,\nu} & B_{1,\nu} \\ C_{1,\nu} & J_{2,\nu} \end{pmatrix} \text{ with } C_{1,\nu} := \begin{pmatrix} (J_\nu)_1^3 & (J_\nu)_2^3 \\ (J_\nu)_1^4 & (J_\nu)_2^4 \end{pmatrix},$$

then we have:

$$(\Lambda_\nu)_* J_\nu(z) = \begin{pmatrix} J_{1,\nu}(\tau_\nu z_1, \delta_\nu z_2) & \tau_\nu^{-1}\delta_\nu B_{1,\nu}(\tau_\nu z_1, \delta_\nu z_2) \\ \tau_\nu\delta_\nu^{-1}C_{1,\nu}(\tau_\nu z_1, \delta_\nu z_2) & J_{2,\nu}(\tau_\nu z_1, \delta_\nu z_2) \end{pmatrix}.$$

We have generically the following situation:

Proposition 5.1. *The sequence of structures $(\Lambda_\nu)_* J_\nu$ converges to the standard structure J_{st} if and only if the D'Angelo type of the origin is less than or equal to four.*

Proof. We notice that $(\Lambda_\nu)_* J_\nu$ converges to J_{st} if and only if $C_{1,\nu} = O\left(|z_1|^{2m-1}\right) + O(|z_2|)$. Indeed if $C_{1,\nu} = O\left(|z_1|^{2m-1}\right) + O(|z_2|)$ then

$$\tau_\nu\delta_\nu^{-1}C_{1,\nu}(\tau_\nu z_1, \delta_\nu z_2) = \tau_\nu^{2m}\delta_\nu^{-1}O|z_1|^{2m} + \tau_\nu^{2m}O|z_1|^{2m},$$

which converges to the zero 2 by 2 matrix since $\tau_\nu \leq \delta_\nu^{\frac{1}{2m}}$ and since $C_{1,\nu}$ tends to the zero 2 by 2 matrix. Conversely if $C_{1,\nu} = O\left(|z_1|^k\right) + O(|z_2|)$, with $k < 2m-1$, then $(\Lambda_\nu)_* J_\nu$ converges to a polynomial integrable structure $\tilde{J} = J_{st} + O|z_1|^2$ wich is generically different from J_{st}.

We have proved in Lemma 4.2 that when the origin is a point of D'Angelo type four, then $C_{1,\nu} = O\left(|z_1|^3\right) + O(|z_2|)$ and so $(\Lambda_\nu)_* J_\nu = (\Lambda_\nu \circ \Psi_\nu)_* J^\nu$ converges to J_{st} when ν tends to $+\infty$, with:

$$\begin{cases} R_{1,\nu} &= S_{1,\nu} = 0, \\ R_{2,\nu} &= -\alpha_\nu x_1^2 + \alpha_\nu y_1^2, \\ S_{2,\nu} &= -2\alpha_\nu x_1 y_1. \end{cases}$$

In case the D'Angelo type of the origin is greater than four, we cannot guarantee the convergence of $\tau_\nu\delta_\nu^{-1}C_1^\nu(\tau_\nu z_1, \delta_\nu z_2)$ when we only remove harmonic terms. So we need to find a more general sequence of diffeomorphisms Ψ_ν defined by (5.1), (5.2) and (5.3) and such that $C_{1,\nu} = O\left(|z_1|^{2m-1}\right) + O(|z_2|)$.

Claim. There are no polynomial $R_{1,\nu}, S_{1,\nu}, R_{2,\nu}$ and $S_{2,\nu}$ such that $C_{1,\nu}$ does not contain any order three terms in x_1 and y_1.

A direct computation leads to:

$$\begin{aligned}\alpha_\nu^{-1}\left(J_\nu\right)_1^3(z) &= \left(a_2^\nu-a_1^\nu\right)\left(\Psi_\nu^{-1}(z)\right)x_1-\left(c_1^\nu+b_2^\nu\right)\left(\Psi_\nu^{-1}(z)\right)y_1-y_1\frac{\partial R_{1,\nu}}{\partial x_1}\\ &\quad -x_1\frac{\partial R_{1,\nu}}{\partial y_1}-x_1\frac{\partial S_{1,\nu}}{\partial x_1}+y_1\frac{\partial S_{1,\nu}}{\partial y_1}+x_1\frac{\partial R_{1,\nu}}{\partial x_1}\frac{\partial S_{1,\nu}}{\partial x_1}+y_1\frac{\partial R_{1,\nu}}{\partial x_1}\frac{\partial S_{1,\nu}}{\partial y_1}\\ &\quad -y_1\frac{\partial R_{1,\nu}}{\partial y_1}\frac{\partial S_{1,\nu}}{\partial x_1}+x_1\frac{\partial R_{1,\nu}}{\partial y_1}\frac{\partial S_{1,\nu}}{\partial y_1}-y_1\left(\frac{\partial S_{1,\nu}}{\partial x_1}\right)^2-y_1\left(\frac{\partial S_{1,\nu}}{\partial y_1}\right)^2\\ &\quad -x_1\frac{\partial R_{1,\nu}}{\partial x_1}\frac{\partial R_{2,\nu}}{\partial y_2}+x_1\frac{\partial S_{1,\nu}}{\partial y_1}\frac{\partial R_{2,\nu}}{\partial y_2}+y_1\frac{\partial R_{1,\nu}}{\partial y_1}\frac{\partial R_{2,\nu}}{\partial y_2}+\\ &\quad y_1\frac{\partial S_{1,\nu}}{\partial x_1}\frac{\partial R_{2,\nu}}{\partial y_2}+O\left(|z_1|^4+|z_2|\|z\|\right)\end{aligned}$$

and to

$$\begin{aligned}\alpha_\nu^{-1}\left(J_\nu\right)_2^3(z) &= \left(b_1^\nu-b_2^\nu\right)\left(\Psi_\nu^{-1}(z)\right)x_1+\left(a_1^\nu+a_2^\nu\right)\left(\Psi_\nu^{-1}(z)\right)y_1+x_1\frac{\partial R_{1,\nu}}{\partial x_1}\\ &\quad -y_1\frac{\partial R_{1,\nu}}{\partial y_1}-y_1\frac{\partial S_{1,\nu}}{\partial x_1}-x_1\frac{\partial S_{1,\nu}}{\partial y_1}-x_1\left(\frac{\partial R_{1,\nu}}{\partial x_1}\right)^2-x_1\left(\frac{\partial R_{1,\nu}}{\partial y_1}\right)^2+\\ &\quad y_1\frac{\partial R_{1,\nu}}{\partial x_1}\frac{\partial S_{1,\nu}}{\partial x_1}+x_1\frac{\partial R_{1,\nu}}{\partial x_1}\frac{\partial S_{1,\nu}}{\partial y_1}-x_1\frac{\partial R_{1,\nu}}{\partial y_1}\frac{\partial S_{1,\nu}}{\partial x_1}+y_1\frac{\partial R_{1,\nu}}{\partial y_1}\frac{\partial S_{1,\nu}}{\partial y_1}\\ &\quad -x_1\frac{\partial R_{1,\nu}}{\partial y_1}\frac{\partial R_{2,\nu}}{\partial y_2}-x_1\frac{\partial S_{1,\nu}}{\partial x_1}\frac{\partial R_{2,\nu}}{\partial y_2}-y_1\frac{\partial R_{1,\nu}}{\partial x_1}\frac{\partial R_{2,\nu}}{\partial y_2}+y_1\frac{\partial S_{1,\nu}}{\partial y_1}\frac{\partial R_{2,\nu}}{\partial y_2}+\\ &\quad O\left(|z_1|^4+|z_2|\|z\|\right).\end{aligned}$$

The only order two terms in x_1 and y_1 of $\alpha_\nu^{-1}\left(J^\nu\right)_1^3(z)$ and of $\alpha_\nu^{-1}\left(J^\nu\right)_2^3(z)$ are those contained, respectively, in

$$-y_1\frac{\partial R_{1,\nu}}{\partial x_1}-x_1\frac{\partial R_{1,\nu}}{\partial y_1}-x_1\frac{\partial S_{1,\nu}}{\partial x_1}+y_1\frac{\partial S_{1,\nu}}{\partial y_1}$$

and

$$x_1\frac{\partial R_{1,\nu}}{\partial x_1}-y_1\frac{\partial R_{1,\nu}}{\partial y_1}-y_1\frac{\partial S_{1,\nu}}{\partial x_1}-x_1\frac{\partial S_{1,\nu}}{\partial y_1}.$$

Vanishing these order two terms leads to:

$$\begin{cases} R_{1,\nu} &= r_{5,\nu}x_1^2-2s_{5,\nu}x_1y_1-r_{5,\nu}y_1^2+r_{1,\nu}x_1^3+r_{2,\nu}x_1^2y_1+r_{3,\nu}x_1y_1^2+r_{4,\nu}y_1^3+\\ &\quad O\left(|z_1|^4+y_2^2+|y_2|\|z\|\right)\\ S_{1,\nu} &= s_{5,\nu}x_1^2+2s_{5,\nu}x_1y_1-s_{5,\nu}y_1^2+s_{1,\nu}x_1^3+s_{2,\nu}x_1^2y_1+s_{3,\nu}x_1y_1^2+s_{4,\nu}y_1^3+\\ &\quad O\left(|z_1|^3+y_2^2+|y_2|\|z\|\right).\end{cases}$$

Then it follows that:

$$\begin{aligned}\alpha_\nu^{-1}\left(J_\nu\right)_1^3(z) &= \left(a_2^\nu-a_1^\nu\right)\left(\Psi_\nu^{-1}(z)\right)x_1-\left(c_1^\nu+b_2^\nu\right)\left(\Psi_\nu^{-1}(z)\right)y_1-y_1\frac{\partial R_{1,\nu}}{\partial x_1}\\ &\quad -x_1\frac{\partial R_{1,\nu}}{\partial y_1}-x_1\frac{\partial S_{1,\nu}}{\partial x_1}+y_1\frac{\partial S_{1,\nu}}{\partial y_1}+O\left(|z_1|^4+|z_2|\|z\|\right),\end{aligned}$$

and that

$$\begin{aligned}\alpha_\nu^{-1}\left(J_\nu\right)_2^3(z) &= \left(b_1^\nu-b_2^\nu\right)\left(\Psi_\nu^{-1}(z)\right)x_1+\left(a_1^\nu+a_2^\nu\right)\left(\Psi_\nu^{-1}(z)\right)y_1+x_1\frac{\partial R_{1,\nu}}{\partial x_1}\\ &\quad -y_1\frac{\partial R_{1,\nu}}{\partial y_1}-y_1\frac{\partial S_{1,\nu}}{\partial x_1}-x_1\frac{\partial S_{1,\nu}}{\partial y_1}+O\left(|z_1|^4+|z_2|\|z\|\right).\end{aligned}$$

Since J^ν satisfies (2.3), we have:

$$\begin{cases}\left(a_2^\nu-a_1^\nu\right)\left(\Psi_\nu^{-1}(z)\right)x_1-\left(c_1^\nu+b_2^\nu\right)\left(\Psi_\nu^{-1}(z)\right)y_1 = H_{3,\nu}(x_1,y_1)+\\ \qquad\qquad O\left(|z_1|^4+|z_2|\|z\|\right)\\ \left(b_1^\nu-b_2^\nu\right)\left(\Psi_\nu^{-1}(z)\right)x_1+\left(a_1^\nu+a_2^\nu\right)\left(\Psi_\nu^{-1}(z)\right)y_1 = H'_{3,\nu}(x_1,y_1)+\\ \qquad\qquad O\left(|z_1|^4+|z_2|\|z\|\right),\end{cases}$$

where $H_{3,\nu}(x_1,y_1)$ and $H'_{3,\nu}(x_1,y_1)$ are real homogeneous polynomials of degree three in x_1 and y_1 which are generically non identically zero. Since we cannot insure the convergence of

$$\alpha_\nu\tau_\nu\delta_\nu^{-1}H_{3,\nu}(\tau_\nu x_1,\tau_\nu y_1)=\alpha_\nu\tau_\nu^4\delta_\nu^{-1}H_{3,\nu}(x_1,y_1)$$

and

$$\alpha_\nu\tau_\nu\delta_\nu^{-1}H'_{3,\nu}(\tau_\nu x_1,\tau_\nu y_1)=\alpha_\nu\tau_\nu^4\delta_\nu^{-1}H'_{3,\nu}(x_1,y_1),$$

we want to cancel polynomials $H_{3,\nu}(x_1,y_1)$ and $H'_{3,\nu}(x_1,y_1)$ by order three terms in x_1 and y_1 contained in

$$-y_1\frac{\partial R_{1,\nu}}{\partial x_1}-x_1\frac{\partial R_{1,\nu}}{\partial y_1}-x_1\frac{\partial S_{1,\nu}}{\partial x_1}+y_1\frac{\partial S_{1,\nu}}{\partial y_1}$$

and

$$x_1\frac{\partial R_{1,\nu}}{\partial x_1}-y_1\frac{\partial R_{1,\nu}}{\partial y_1}-y_1\frac{\partial S_{1,\nu}}{\partial x_1}-x_1\frac{\partial S_{1,\nu}}{\partial y_1}.$$

Finally, vanishing order three terms in x_1 and y_1 of $\alpha_\nu^{-1}\left(J^\nu\right)_1^3(z)$ and of $\alpha_\nu^{-1}\left(J^\nu\right)_2^3(z)$ involve the following system of linear equations:

$$\begin{pmatrix}3&0&2&0&0&1&0&0\\3&0&0&0&0&-1&0&0\\0&1&0&0&3&0&0&0\\0&2&0&3&0&0&-1&0\\0&1&0&0&-3&0&-2&0\\0&0&1&0&0&0&0&-3\\0&0&1&0&0&2&0&3\\0&0&0&3&0&0&1&0\end{pmatrix}\begin{pmatrix}r_{1,\nu}\\r_{2,\nu}\\r_{3,\nu}\\r_{4,\nu}\\s_{1,\nu}\\s_{2,\nu}\\s_{3,\nu}\\s_{4,\nu}\end{pmatrix}=Y$$

Since this 8×8 system of linear equations is not a Cramer system, it follows that there does not exist, generically, polynomials $R_{1,\nu}$ and $S_{1,\nu}$ such that there are no order three term in x_1 and y_1 in $(J^\nu)^3_1(z)$ and $(J^\nu)^3_2(z)$. □

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