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On the projective geometry of the supercircle: a unified construction of the super cross-ratio and Schwarzian derivative

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Abstract

We consider the standard contact structure on the supercircle, $\mathrm{S}^{1|1}$, and the supergroups $\mathrm{E}(1|1)$, $\mathrm{Aff}(1|1)$ and $\mathrm{SpO}(2|1)$ of contactomorphisms, defining the Euclidean, affine and projective geometry respectively. Using the new notion of $p|q$-transitivity, we construct in synthetic fashion even and odd invariants characterizing each geometry, and obtain an even and an odd super cross-ratios.

Starting from the even invariants, we derive, using a superized Cartan formula, one-cocycles of the group of contactomorphisms, $K(1)$, with values in tensor densities $\mathcal{F}_\lambda(\mathrm{S}^{1|1})$. The even cross-ratio yields a $K(1)$ one-cocycle with values in quadratic differentials, $\mathcal{Q}(\mathrm{S}^{1|1})$, whose projection on $\mathcal{F}_{\frac{3}{2}}(\mathrm{S}^{1|1})$ corresponds to the super Schwarzian derivative arising in superconformal field theory. This leads to the classification of the cohomology spaces $H^1(K(1), \mathcal{F}_\lambda(\mathrm{S}^{1|1}))$.

The construction is extended to the case of $\mathrm{S}^{1|N}$. All previous invariants admit a prolongation for $N > 1$, as well as the associated Euclidean and affine cocycles. The super Schwarzian derivative is obtained from the even cross-ratio, for $N = 2$, as a projection to $\mathcal{F}_1(\mathrm{S}^{1|2})$ of a $K(2)$ one-cocycle with values in $\mathcal{Q}(\mathrm{S}^{1|2})$. The obstruction to obtain, for $N \geq 3$, a projective cocycle is pointed out.

Keywords: Projective structures, Contact geometry, Supermanifolds, Cross-ratio, Schwarzian derivative

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Contents

1 Introduction **3**

2 The supercircle $\mathrm{S}^{1|1}$ and its contactomorphisms: A compendium **6**
2.1 The supercircle $\mathrm{S}^{1|1}$ 6
2.2 The contact structure on $\mathrm{S}^{1|1}$ and its automorphisms 8
2.2.1 The super Lie algebra, $k(1)$, of contact vector fields 10
2.2.2 Tensor densities, 1-forms and quadratic differentials of $\mathrm{S}^{1|1}$ . . . 10
2.3 The orthosymplectic group $\mathrm{SpO}(2|1)$ 11

3 Main results **13**
3.1 Super Euclidean, affine and projective invariants 13
3.2 The associated 1-cocycles of $K(1)$ 15
3.3 The determination of $H^1(K(1), \mathcal{F}_\lambda)$ 17

4 Super Euclidean, affine and projective invariants of $\mathrm{S}^{1|1}$ **19**
4.1 Euclidean invariants 22
4.2 Affine invariants 23
4.3 Projective invariants 25

5 The Schwarzian derivative from the Cartan formula **30**
5.1 Preparation 30
5.2 Proof of Theorem 3.5 31
5.2.1 Euclidean and affine $K(1)$-cocycles, $\mathcal{E}, \mathcal{A}$ 31
5.2.2 The Schwarzian derivative, $\mathcal{S}$ 32
5.2.3 The kernels of the $K(1)$-cocycles $\mathcal{E}, \mathcal{A}, \mathcal{S}$ 34
5.3 Proof of Proposition 3.8 35

6 Super Euclidean, affine and projective invariants, and $K(N)$-cocycles for $\mathrm{S}^{1|N}$ **35**
6.1 Euclidean, affine and projective invariants 37
6.2 Associated cocycles from the Cartan formula 39
6.2.1 Euclidean and affine $K(N)$-cocycles 41
6.2.2 The Schwarzian $K(2)$-cocycle 42

7 Conclusion, discussion and outlook **46**

1 Introduction

The cross-ratio is the fundamental object of projective geometry; it is a projective invariant of the circle S^1 (or, rather, of $\mathbb{R}\mathrm{P}^1$). The main objective of this article is to propose and justify from a group theoretical analysis a super-analogue of the cross-ratio in the case of the supercircle $\mathrm{S}^{1|N}$, and to deduce then, from the Cartan formula (1.2), the associated Schwarzian derivative for $N = 1, 2$.

It is well-known that the circle, S^1, admits three different geometries, namely the Euclidean, affine and projective geometries, as highlighted by Ghys [14]. They are defined by the groups $(\mathbb{R}, +)$, $\mathrm{Aff}(1, \mathbb{R})$ and $\mathrm{PGL}(2, \mathbb{R})$, or equivalently by their characteristic invariants, the distance, the distance ratio, and the cross-ratio. From these invariants we can obtain, using Cartan-like formulæ, three 1-cocycles of $\mathrm{Diff}_+(\mathrm{S}^1)$ with coefficients in some tensorial density modules $\mathcal{F}_\lambda(\mathrm{S}^1)$ with $\lambda \in \mathbb{R}$; see [9]. They are the generators of the three nontrivial cohomology spaces $H^1(\mathrm{Diff}_+(\mathrm{S}^1), \mathcal{F}_\lambda)$, with $\lambda = 0, 1, 2$, as proved in [12].

The purpose of this article is to extend these results to the supercircle, $\mathrm{S}^{1|N}$, endowed with its standard contact structure. To that end, we use the embedding of the quotient, $\mathrm{PC}(2|N) = \mathrm{SpO}(2|N)/\{\pm\mathrm{Id}\}$, of the orthosymplectic supergroup $\mathrm{SpO}(2|N)$, into the group, $K(N)$, of contactomorphisms of $\mathrm{S}^{1|N}$. The supergroup $\mathrm{PC}(2|N)$ is the projective conformal supergroup introduced by Manin in [23], extending $\mathrm{PSL}(2, \mathbb{R})$. The two main objects of super projective geometry, namely the cross-ratio and the Schwarzian derivative, have, indeed, already been introduced in the general context of superstring theories, though in a somewhat independent fashion. This was mainly done in the framework of super Riemann surfaces, or in terms of the so-called SUSY structures. On the one hand, the even and odd cross-ratios, for $N = 1$, have been originally put forward by Aoki [2], and Nelson [24], respectively; these two references have opened the way to subsequent work of, e.g., Giddings [15], and Uehara and Yasui [33]. On the other hand, the super Schwarzian derivative has been introduced, in the framework of superconformal field theory, by Friedan [11], for $N = 1$, and by Cohn [6], for $N = 2$.

Quite independently, and from a more mathematical point of view, Manin [23] introduced the even and odd cross-ratios, for $N = 1, 2$, by resorting to linear super-

symplectic algebra. Also did Radul [28, 29] discover the formulæ for the super Schwarzian $K(N)$ 1-cocycles, for $N = 1, 2, 3$, using the transformation laws of the super Sturm-Liouville operators on $\mathrm{S}^{1|N}$.

Our first objective is to construct, in a systematic manner, invariants characterizing each supergroup $\mathrm{E}_+(1|N) \subset \mathrm{Aff}_+(1|N) \subset \mathrm{PC}(2|N)$ acting on the supercircle $\mathrm{S}^{1|N}$. To this end, we introduce the new notion of $p|q$-transitivity, well-adapted to supergroups, and state a general theorem, providing a way to build up characteristic invariant of a simply $p|q$-transitive group action. Applying this theorem to the three preceding supergroups, we obtain Euclidean, affine, and projective invariants, respectively I_{e}, I_{a} and I_{p}, with their even and odd part. In the case $N = 1$, the two components of I_{p} are, unsurprisingly, the even and odd above-mentioned super cross-ratios. Let us emphasize that, for arbitrary N, the even cross-ratio turns out to be given by the superfunction

$$[t_1, t_2, t_3, t_4] = \frac{[t_1, t_3][t_2, t_4]}{[t_2, t_3][t_1, t_4]} \tag{1.1}$$

of a quadruple of "points" (t_1, t_2, t_3, t_4) of $\mathrm{S}^{1|N}$, with even coordinates x_i, and odd ones $(\xi_i^1, \ldots, \xi_i^N)$, for $i = 1, \ldots, 4$; note that in (1.1) the two-point superfunction $[t_i, t_j] = x_j - x_i - \xi_j \cdot \xi_i$ is the Euclidean even invariant. The supergroups preserving I_{e}, I_{a} and I_{p} are respectively $\mathrm{E}_+(1|N)$, $\mathrm{Aff}_+(1|N)$ and $\mathrm{PC}(2|N)$, as expected.

Our second objective, is to link the three even parts of the previously found invariants to 1-cocycles of $K(N)$, by means of a natural superized version of the Cartan formula. It culminates in the projective case, where we get the super Schwarzian derivative (3.11) from the even cross-ratio. Let us go into some more details. Given a flow, $\phi_\varepsilon = \mathrm{Id} + \varepsilon X + O(\varepsilon^2)$, we posit $t_{i+1} = \phi_{i\varepsilon}(t_1)$, for $i = 0, \ldots, 3$. We contend that the Cartan formula [5, 27] can be consistently superized for $N = 1$, and $N = 2$, using the cross-ratio (1.1), namely by

$$\frac{\Phi^*[t_1, t_2, t_3, t_4]}{[t_1, t_2, t_3, t_4]} - 1 = \langle \varepsilon X \otimes \varepsilon X, \mathcal{S}(\Phi) \rangle + O(\varepsilon^3), \tag{1.2}$$

hence, providing us with a definition of the Schwarzian derivative, $\mathcal{S}(\Phi)$, of a contactomorphism Φ. In doing so, we naturally obtain a 1-cocycle of $K(N)$, for $N = 1, 2$ respectively, with values in the module, $\mathcal{Q}(\mathrm{S}^{1|N})$, of quadratic differentials. Their projections onto the modules $\mathcal{F}_{\frac{3}{2}}(\mathrm{S}^{1|1})$ and $\mathcal{F}_1(\mathrm{S}^{1|2})$, for $K(1)$ and $K(2)$ respectively, yield

the expressions of the super Schwarzian derivatives given in [11, 6, 29]. Remarkably enough, our formula allows us to recover the classical Schwarzian derivative on the circle, S^1, which would not be the case, had we started with Friedan's, Cohn's, and Radul's formulæ. Much in the same way, we define the Euclidean and affine 1-cocycles of $K(N)$ for any N, with the help of the Cartan-like formulæ (5.4), and (5.5). Using the results of Agrebaoui et al. [1] on the cohomology of the Lie superalgebra of contact vector field on $\mathrm{S}^{1|1}$, we can claim that our three 1-cocycles on $K(1)$ are, indeed, the generators of the three nontrivial cohomology spaces $H^1(K(1), \mathcal{F}_\lambda)$, where $\lambda = 0, \frac{1}{2}, \frac{3}{2}$.

The paper is organized as follows.

In Section 2, we recall the main definitions and facts related to the geometry of the supercircle $\mathrm{S}^{1|1}$, in particular its canonical contact structure and the action of the (special) orthosymplectic group $\mathrm{SpO}_+(2|1) \cong \mathrm{PC}(2|1)$, as a subgroup of the group, $K(1)$, of contactomorphisms of $\mathrm{S}^{1|1}$.

In Section 3, we review the main results of this article, namely the form of the invariants, and of the associated 1-cocycles of $K(1)$, obtained for each of the three above-mentioned geometries. This section also gives the classification of the cohomology spaces $H^1(K(1), \mathcal{F}_\lambda)$, for $\lambda \in \mathbb{C}$.

Sections 4 and 5 provide the proofs of the main results announced in Section 3. We first define the notion of $p|q$-transitivity and state the general Theorem 4.3, leading to the construction of the Euclidean, affine, and projective invariants, from the action of the corresponding subgroups of $K(1)$. Those invariants are then shown to yield, via a Taylor expansion, the sought 1-cocycles; in particular the Cartan formula readily leads to a new expression for the Schwarzian derivative, $\mathcal{S}(\Phi)$, of a contactomorphism, Φ, with values in the module of quadratic differentials, $\mathcal{Q}(\mathrm{S}^{1|1})$. The link with Friedan's and Radul's Schwarzian derivative is elucidated. The kernels of the three above 1-cocycles are shown to be, indeed, isomorphic to $\mathrm{E}(1|1)$, $\mathrm{Aff}(1|1)$, and $\mathrm{SpO}_+(2|1)$ respectively.

In Section 6, we present a detailed treatment of the general case, $N > 1$, along the same lines as before. As mentioned in Section 3, there is hardly no change in the construction and the resulting expressions of the invariants. The Euclidean and affine 1-cocycles of $K(N)$ are explicitly derived, as well as the Schwarzian derivative obtained

as a 1-cocycle of $K(2)$ with values in the module $\mathcal{Q}(\mathrm{S}^{1|2})$ of quadratic differentials. Upon projection of $\mathcal{Q}(\mathrm{S}^{1|2})$ onto the $K(2)$-module $\mathcal{F}_1(\mathrm{S}^{1|2})$ of 1-densities, we obtain Cohn's and Radul's formula for the Schwarzian derivative. Specific difficulties encountered in deriving the projective 1-cocycles for $N > 2$ are pointed out, together with those arising in the determination of the kernels of the Euclidean and affine 1-cocycles. At last, the kernel of the Schwarzian 1-cocycle of $K(2)$ is shown to be isomorphic to $\mathrm{PC}(2|2)$.

Section 7 gives us the opportunity to sum up the content of this article, and to draw several conclusions. It opens perspectives for future work related to the link between discrete projective invariants of the supercircle, and the cohomology of the group of its contactomorphisms.

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2 The supercircle $\mathrm{S}^{1|1}$ and its contactomorphisms: A compendium

We briefly define in this section the geometrical objects on $\mathrm{S}^{1|1}$ that will be needed for our purpose. This includes the basics of super differential geometry [21, 22, 7], the standard contact structure on the supercircle [29], and the orthosymplectic group $\mathrm{SpO}(2|1)$, see [23].

2.1 The supercircle $\mathrm{S}^{1|1}$

The supercircle $\mathrm{S}^{1|1}$ can be defined as the circle, S^1, endowed with the sheaf of the supercommutative associative algebra of superfunctions $\mathcal{C}^\infty(\mathrm{S}^{1|1}) = \mathcal{C}^\infty(\mathrm{S}^1)[\xi]$. Thus, $\mathrm{S}^{1|1}$ admits local coordinates $t = (x, \xi)$, where x is a local coordinate on S^1, and ξ is an odd (Grassmann) coordinate, i.e., such that $\xi^2 = 0$ and $x\xi = \xi x$. Then, a superfunction is of the form

$$f(x, \xi) = f_0(x) + \xi f_1(x) \tag{2.1}$$

with $f_0, f_1 \in \mathcal{C}^\infty(\mathrm{S}^1)$. There exists a $\mathbb{Z}_2$-grading on superfunctions, f_0 being the even part and ξf_1 the odd part of f. The parity is denoted by p, with the convention $p(f_0) = 0$ and $p(\xi f_1) = 1$. We define the projection

$$\pi : \mathcal{C}^\infty(\mathrm{S}^{1|1}) \to \mathcal{C}^\infty(\mathrm{S}^1) \tag{2.2}$$

by quotienting by the ideal of nilpotent elements; this gives an embedding of the circle into the supercircle.

Denote by $\mathrm{Diff}(\mathrm{S}^{1|1})$ the group of diffeomorphisms of $\mathrm{S}^{1|1}$, i.e., the group of automorphisms of $\mathcal{C}^\infty(\mathrm{S}^{1|1})$. Let $\Phi \in \mathrm{Diff}(\mathrm{S}^{1|1})$, then

$$\Phi(x, \xi) = (\varphi(x, \xi), \psi(x, \xi)) \tag{2.3}$$

where φ is an even superfunction and ψ an odd one, so Φ preserves parity and $(\varphi(x, \xi), \psi(x, \xi))$ become new coordinates on $\mathrm{S}^{1|1}$. For any morphism, i.e., algebra morphism preserving parity, the following diagram is commutative

$$\begin{tikzcd} \mathcal{C}^\infty(\mathrm{S}^{1|1}) \arrow[r, "\pi"] & \mathcal{C}^\infty(\mathrm{S}^1) \\ \mathcal{C}^\infty(\mathrm{S}^{1|1}) \arrow[r, "\pi"] \arrow[u, "\Phi"] & \mathcal{C}^\infty(\mathrm{S}^1) \arrow[u, "\Pi(\Phi)"'] \end{tikzcd} \tag{2.4}$$

So, every morphism of $\mathcal{C}^\infty(\mathrm{S}^{1|1})$ induces a morphism of $\mathcal{C}^\infty(\mathrm{S}^1)$, and we have a canonical morphism $\Pi : \mathrm{Diff}(\mathrm{S}^{1|1}) \to \mathrm{Diff}(\mathrm{S}^1)$.

A super vector field, X, on $\mathrm{S}^{1|1}$ is a superderivation of $\mathcal{C}^\infty(\mathrm{S}^{1|1})$, i.e., a linear operator satisfying super Leibniz rule, $X(fg) = X(f)g + (-1)^{p(f)p(X)} f X(g)$, for homogeneous elements. As in ordinary differential geometry, X can be locally written in terms of partial derivatives as

$$X = f(x, \xi)\partial_x + g(x, \xi)\partial_\xi \tag{2.5}$$

where $f, g \in \mathcal{C}^\infty(\mathrm{S}^{1|1})$, with $p(\partial_x) = 0$ and $p(\partial_\xi) = 1$. The space, $\mathrm{Vect}(\mathrm{S}^{1|1})$, of vector fields on $\mathrm{S}^{1|1}$ is thus a left-module over $\mathcal{C}^\infty(\mathrm{S}^{1|1})$. It has the structure of a super Lie algebra, $\mathrm{Vect}(\mathrm{S}^{1|1}) = \mathrm{Vect}(\mathrm{S}^{1|1})_0 \oplus \mathrm{Vect}(\mathrm{S}^{1|1})_1$, whose superbracket is denoted by $[\,\cdot\,,\,\cdot\,]$, and $[X, Y] = XY - (-1)^{p(X)p(Y)} YX$, for homogeneous elements.

Since the group $\mathrm{Diff}(\mathrm{S}^{1|1})$ of diffeomorphisms preserves parity, we can define the flow of $X \in \mathrm{Vect}(\mathrm{S}^{1|1})$, namely $\varphi_\varepsilon = \mathrm{Id} + \varepsilon X + \mathrm{O}(\varepsilon^2)$, only if $p(\varepsilon X) = 0$. For odd vector fields, X, the parameter ε must therefore be odd, see [7].

We can now define the $\mathcal{C}^\infty(\mathrm{S}^{1|1})$ right-module $\Omega^1(\mathrm{S}^{1|1})$ of 1-forms on $\mathrm{S}^{1|1}$, as the dual of the $\mathcal{C}^\infty(\mathrm{S}^{1|1})$ left-module $\mathrm{Vect}(\mathrm{S}^{1|1})$. The 1-forms dx and $d\xi$ will constitute the dual basis of ∂_x and ∂_ξ, that is $\langle \partial_x, dx\rangle = \langle \partial_\xi, d\xi\rangle = 1$ and $\langle \partial_\xi, dx\rangle = \langle \partial_x, d\xi\rangle = 0$. Then p is extended naturally to $\Omega^1(\mathrm{S}^{1|1})$ by $p(dx) = 0$ and $p(d\xi) = 1$. Using the exterior product we construct $\Omega^*(\mathrm{S}^{1|1})$, the space of all differential forms on $\mathrm{S}^{1|1}$, graded by $\mathbb{Z}$ with $|\cdot|$ the cohomological degree. Parity being also defined on this space, we have two choices for the Sign Rule, viz.,

$$\alpha \wedge \beta = (-1)^{(p(\alpha)+|\alpha|)\,(p(\beta)+|\beta|)} \beta \wedge \alpha \tag{2.6}$$

$$\alpha \wedge \beta = (-1)^{|\alpha||\beta|+p(\alpha)p(\beta)} \beta \wedge \alpha \tag{2.7}$$

where α, β are homogeneous elements of $\Omega^*(\mathrm{S}^{1|1})$. The second convention corresponds to a bigrading $\mathbb{Z} \times \mathbb{Z}_2$, and, following [7, 19], we will choose it from now on.

2.2 The contact structure on $\mathrm{S}^{1|1}$ and its automorphisms

The standard contact structure on $\mathrm{S}^{1|1}$ is given by the conformal class of the 1-form

$$\alpha = dx + \xi d\xi \tag{2.8}$$

which satisfies $\alpha \wedge d\alpha \neq 0$. This contact structure is equivalently defined by the kernel of α, spanned by the odd vector field

$$D = \partial_\xi + \xi \partial_x, \tag{2.9}$$

whose square $D^2 = \frac{1}{2}[D, D] = \partial_x$ is the Reeb vector field of the structure.

Then D and ∂_x set up a basis of the $\mathcal{C}^\infty(\mathrm{S}^{1|1})$ left-module $\mathrm{Vect}(\mathrm{S}^{1|1})$, while α and $\beta = d\xi$ constitute the dual basis, with $d\alpha = \beta \wedge \beta$. Thus for any $f \in \mathcal{C}^\infty(\mathrm{S}^{1|1})$ we have

$$df = \alpha f' + \beta D f. \tag{2.10}$$

where $f' = \partial_x f$. The contact structure being given by the direction of α, it is therefore preserved by $\Phi \in \mathrm{Diff}(\mathrm{S}^{1|1})$ iff

$$\Phi^* \alpha = E_\Phi \alpha \tag{2.11}$$

for some superfunction E_Φ, which, following [29], we call the multiplier of Φ. We denote by $K(1)$ the subgroup of $\mathrm{Diff}(\mathrm{S}^{1|1})$ preserving the contact structure, its elements

are called contactomorphisms. From (2.3) and (2.8) we find $\Phi^*\alpha = d\varphi + \psi d\psi = \alpha(\varphi' + \psi\psi') + \beta(D\varphi - \psi D\psi)$.

Proposition 2.1. *Let $\Phi = (\varphi, \psi)$ be a diffeomorphism of $\mathrm{S}^{1|1}$; then $\Phi \in K(1)$ iff*

$$D\varphi - \psi D\psi = 0. \tag{2.12}$$

The multiplier of Φ is then given by $E_\Phi = \varphi' + \psi\psi'$, i.e., by

$$E_\Phi = \frac{\Phi^*\alpha}{\alpha} = (D\psi)^2. \tag{2.13}$$

Since α and β set up a basis of the $\mathcal{C}^\infty(\mathrm{S}^{1|1})$-module $\Omega^1(\mathcal{C}^\infty(\mathrm{S}^{1|1}))$, we will also need the expression of the action of $K(1)$ on the odd 1-form β; it reads

$$\Phi^*\beta = \alpha\psi' + \beta D\psi. \tag{2.14}$$

We might, as well, define $K(1)$ as the group of diffeomorphisms preserving the horizontal distribution spanned by D, denoted by $\langle D\rangle$. In the complex setting, D is interpreted as the covariant derivative of a super Riemann surface [11, 6], and $K(1)$ as the superconformal group; the distribution $\langle D\rangle$ is also often referred to as a SUSY structure [23, 7]. See also [18] for a review.

Using (2.11), we find that the transformation law (2.14) entails

$$\Phi^* D = \frac{1}{D\psi} D, \tag{2.15}$$

which makes sense as $D\psi \neq 0$ for any diffeomorphism Φ.

Remark 2.2. If $\Phi = (\varphi, \psi) \in K(1)$, see (2.3), we put $\varphi(x, \xi) = \varphi_0(x) + \xi\varphi_1(x)$, and $\psi(x, \xi) = \psi_1(x) + \xi\psi_0(x)$, with an index 0 for even functions and 1 for odd functions. The constraint (2.12) then reads $\varphi_0' = \psi_0^2 - \psi_1\psi_1'$ and $\varphi_1 = \psi_0\psi_1$. Using the natural projection $\Pi : K(1) \to \mathrm{Diff}(S^1)$, defined in (2.4), we note that Φ gives rise to a diffeomorphism of S^1, which is actually orientation-preserving since $\Pi(\Phi)' = \pi(\varphi_0') = \pi(\psi_0)^2 > 0$.

From the constraint (2.12), we can obtain an interesting property of contactomorphisms: they are essentially determined by their even part.

Lemma 2.3. *Let $\Phi = (\varphi, \psi) \in K(1)$ and $\widetilde{\Phi} = (\widetilde{\varphi}, \widetilde{\psi}) \in K(1)$, be two contactomorphisms such that their even part coincide, $\varphi = \widetilde{\varphi}$. We then have $\widetilde{\psi} = \pm\psi$.*

This can be checked by a direct calculation.

2.2.1 The super Lie algebra, $k(1)$, of contact vector fields

In view of the definition (2.11) of contactomorphisms, we will call $X \in \mathrm{Vect}(\mathrm{S}^{1|1})$ a contact vector field, $X \in k(1)$, if

$$L_X \alpha = e_X \, \alpha \tag{2.16}$$

for some superfunction e_X. The Lie derivative is still given by the derivative of the flow, so $k(1)$ is the Lie algebra of $K(1)$, and e is the derivative of E at the identity. Let us now recall the following classic result [17, 13]: if $X \in k(1)$, there exists a unique superfunction $f(x, \xi) = a(x) - 2\xi b(x)$, called the contact Hamiltonian, such that $X = X_f$, where

$$X_f = a(x)\partial_x + \frac{1}{2}a'(x)\xi\partial_\xi + b(x)(\partial_\xi - \xi\partial_x) \tag{2.17}$$

so that the associated (infinitesimal) multiplier is given by

$$e_{X_f} = f'. \tag{2.18}$$

2.2.2 Tensor densities, 1-forms and quadratic differentials of $\mathrm{S}^{1|1}$

Let us introduce now a 1-parameter family, $\mathcal{F}_\lambda(\mathrm{S}^{1|1})$ or $\mathcal{F}_\lambda$ for short, of $K(1)$-modules, which define the λ-densities associated with the contact structure, $\lambda \in \mathbb{C}$. As vector spaces, these modules are isomorphic to $\mathcal{C}^\infty(\mathrm{S}^{1|1})$, the $K(1)$ anti-action $(\Phi \mapsto \Phi_\lambda)$ on $\mathcal{F}_\lambda(\mathrm{S}^{1|1})$ being given by

$$\Phi_\lambda f = (E_\Phi)^\lambda \, \Phi^* f, \tag{2.19}$$

where $f \in \mathcal{C}^\infty(\mathrm{S}^{1|1})$. We may thus write a λ-density $F \in \mathcal{F}_\lambda$, symbolically, as $F = f\alpha^\lambda$. We will thus write $(\Phi \to \Phi^*)$ the $K(1)$ anti-action on $\mathcal{F}_\lambda$ with this identification.

Remark 2.4. In view of (2.13) and (2.15), we will regard, in conformity with the definition (2.19), the odd vector field D as a $(-\frac{1}{2})$-density.

There is an isomorphism of $K(1)$-modules: $\mathrm{Vect}(\mathrm{S}^{1|1}) \cong \mathcal{F}_{-1} \oplus \mathcal{F}_{-\frac{1}{2}}$, where $\mathcal{F}_{-1}$ corresponds to $k(1)$ and $\mathcal{F}_{-\frac{1}{2}}$ to the vector fields fD, with $f \in \mathcal{C}^\infty(\mathrm{S}^{1|1})$ and D as in (2.9). See [17, 13]. The space of 1-forms $\Omega^1(\mathrm{S}^{1|1})$ is generated, as $\mathcal{C}^\infty(\mathrm{S}^{1|1})$-module, by α and β.

Similarly the space $\mathcal{Q}(\mathrm{S}^{1|1})$ of quadratic differentials is generated, as a $\mathcal{C}^\infty(\mathrm{S}^{1|1})$-module by

$$\alpha^2 = \alpha \otimes \alpha \quad \text{and} \quad \alpha\beta = \frac{1}{2}(\alpha \otimes \beta + \beta \otimes \alpha), \tag{2.20}$$

where the tensor product is understood as the supersymmetric tensor product constructed via the commutativity isomorphism given by the Sign Rule [7]. This notation will be used throughout this paper.

Proposition 2.5. *The two $K(1)$-modules $\Omega^1(\mathrm{S}^{1|1})$ and $\mathcal{Q}(\mathrm{S}^{1|1})$, admit the following decomposition into $K(1)$-submodules, namely*

$$\Omega^1(\mathrm{S}^{1|1}) \cong \mathcal{F}_{\frac{1}{2}} \oplus \mathcal{F}_1, \tag{2.21}$$

$$\mathcal{Q}(\mathrm{S}^{1|1}) \cong \mathcal{F}_{\frac{3}{2}} \oplus \mathcal{F}_2. \tag{2.22}$$

The summands $\mathcal{F}_1$ (resp. $\mathcal{F}_2$) are naturally $K(1)$-submodules of $\Omega^1(\mathrm{S}^{1|1})$ (resp. $\mathcal{Q}(\mathrm{S}^{1|1})$). The projections $\Omega^1(\mathrm{S}^{1|1}) \to \mathcal{F}_{\frac{1}{2}}$ (resp. $\mathcal{Q}(\mathrm{S}^{1|1}) \to \mathcal{F}_{\frac{3}{2}}$) are given by $\alpha^{\frac{1}{2}}\langle D, \cdot\rangle$, and the corresponding sections by $\alpha^{\frac{1}{2}}L_D$ (resp. $\frac{2}{3}\alpha^{\frac{1}{2}}L_D$).

Proof. We have $\alpha^{\frac{1}{2}}\langle D, \alpha f + \beta g\rangle = \alpha^{\frac{1}{2}} g$ and $\alpha^{\frac{1}{2}}\langle D, \alpha^2 f + \alpha\beta g\rangle = \frac{1}{2}\alpha^{\frac{3}{2}} g$. The transformation rules (2.13) for α and (2.14) for β then entail that the projections $\alpha^{\frac{1}{2}}\langle D, \cdot\rangle$ actually define morphisms of $K(1)$-modules, $\Phi^*(\alpha^{\frac{1}{2}}\langle D, \omega\rangle) = \alpha^{\frac{1}{2}}\langle D, \Phi^*\omega\rangle$ for all $\omega \in \Omega^1(\mathrm{S}^{1|1})$, and for all $\omega \in \mathcal{Q}(\mathrm{S}^{1|1})$.

Moreover, since $L_D\alpha = 2\beta$ and $L_D\beta = 0$, we readily find $\alpha^{\frac{1}{2}}L_D(\alpha^{\frac{1}{2}}g) = \alpha Dg + \beta g$ and $\alpha^{\frac{1}{2}}L_D(\alpha^{\frac{3}{2}}g) = \alpha^2 Dg + 3\alpha\beta g$. Using, once more, (2.13) and (2.14), we then obtain that the inclusions $\alpha^{\frac{1}{2}}L_D$ define, again, morphisms of $K(1)$-modules. To have the identity $\mu\alpha^{\frac{1}{2}}\langle D, \alpha^{\frac{1}{2}}L_D F\rangle = F$, we choose $\mu = 1$ for F a $\frac{1}{2}$-density, and $\mu = \frac{2}{3}$ for F a $\frac{3}{2}$-density. The result follows. □

2.3 The orthosymplectic group $\mathrm{SpO}(2|1)$

To define the supergroup $\mathrm{SpO}(2|1)$ and its action on the supercircle we will introduce the notion of functor of points, following [7]. Let A be a supermanifold, an A-point of the supercircle is a morphism of supermanifolds $A \to \mathrm{S}^{1|1}$; we will denote by $\mathrm{S}^{1|1}(A)$ the set of A-points of $\mathrm{S}^{1|1}$. The assignation $A \to (A\text{-points})$ is the functor of points. An A-point of $\mathrm{S}^{1|1}$ is given by the image of the generators (x, ξ) of $\mathcal{C}^\infty(\mathrm{S}^{1|1})$ in $\mathcal{O}_A$, the

sheaf of functions defining A, see [7, 21]. By Yoneda's lemma, giving $f \in \mathrm{Diff}(\mathrm{S}^{1|1})$ is equivalent to giving, functorially in A, a map f_A on $\mathrm{S}^{1|1}(A)$.

For $\mathcal{A}$ any commutative superalgebra, $\mathrm{GL}_{p,q}(\mathcal{A})$ is the well-known group of even invertible linear transformation of the free $\mathcal{A}$-module of dimension $p|q$, see [21]. We define then the supergroup $\mathrm{GL}(p|q)$ by its functor of points, $\mathrm{GL}(p|q)(A) = \mathrm{GL}_{p,q}(\mathcal{O}_A)$, and this functor is representable by a supermanifold, $\mathrm{GL}(p|q)$. By Yoneda's lemma the action of $\mathrm{GL}(p|q)$ on $\mathbb{R}^{p|q}$ can be given by the action of $\mathrm{GL}(p|q)(A)$ on $\mathbb{R}^{p|q}(A)$.

If we restrict ourselves to the supermanifolds A whose underlying manifold is a point, then $\mathcal{O}_A$ is a Grassmann algebra, and we obtain the supermanifolds defined by Rogers [30] or the $\mathcal{A}$-manifolds of Tuynman [32].

From now on we will speak of points instead of A-points, and of the action of a supergroup on points, instead of the action of A-points of a supergroup on A-points.

The contact structure on $\mathrm{S}^{1|1}$ (or rather on $\mathbb{R}P^{1|1}$) defined by α, see (2.8), does stem from the 1-form on $\mathbb{R}^{2|1}$ given by $\varpi = \frac{1}{2}(pdq - qdp + \theta d\theta)$, via the formula $\varpi = \frac{1}{2}p^2\alpha$, with $p \neq 0$, expressed in affine coordinates $x = q/p$ and $\xi = \theta/p$. We define the orthosymplectic group [17, 23], denoted by $\mathrm{SpO}(2|1)$, via its functor of points; $\mathrm{SpO}(2|1)(A)$ is the group of all linear transformations of $\mathbb{R}^{2|1}(A)$, viz.,

$$h = \begin{pmatrix} a & b & \gamma \\ c & d & \delta \\ \alpha & \beta & e \end{pmatrix} \tag{2.23}$$

preserving the symplectic form $d\varpi$, i.e., such that [23]:

$$ad - bc - \alpha\beta = 1, \tag{2.24}$$

$$e^2 + 2\gamma\delta = 1, \tag{2.25}$$

$$\alpha e - a\delta + c\gamma = 0, \tag{2.26}$$

$$\beta e - b\delta + d\gamma = 0. \tag{2.27}$$

We easily find that $\mathrm{SpO}(2|1)$ also preserves ϖ. Since $\varpi = \frac{1}{2}p^2\alpha$, the orthosymplectic group acts by contactomorphisms, $\mathrm{SpO}(2|1) \to K(1)$, via the following projective action on $\mathrm{S}^{1|1}$, namely (in terms of A-points)

$$\widehat{h}(x,\xi) = \left(\frac{ax + b + \gamma\xi}{cx + d + \delta\xi}, \frac{\alpha x + \beta + e\xi}{cx + d + \delta\xi}\right) \tag{2.28}$$

where $h \in \mathrm{SpO}(2|1)$.

The Berezinian of h is $\mathrm{Ber}(h) = e + \alpha\beta e^{-1}$, see [23]. We introduce the special orthosymplectic group $\mathrm{SpO}_+(2|1)$ as the subgroup of $\mathrm{SpO}(2|1)$ of Berezinian 1, or as the quotient, $\mathrm{PC}(2|1)$, of $\mathrm{SpO}(2|1)$ by the kernel of the projective action (2.28), or as the connected component of the identity of $\mathrm{SpO}(2|1)$. So, $\mathrm{SpO}_+(2|1)$ is a super-extension of $\mathrm{Sp}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})$. We have the following (local) group-factorization

$$\mathrm{SpO}_+(2|1) \ni h = \begin{pmatrix} 1 & 0 & 0 \\ \tilde{c} & 1 & \tilde{\delta} \\ \tilde{\delta} & 0 & 1 \end{pmatrix} \begin{pmatrix} \tilde{a} & 0 & 0 \\ 0 & \tilde{a}^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \epsilon & \tilde{b} & -\tilde{\beta} \\ 0 & \epsilon & 0 \\ 0 & \epsilon\tilde{\beta} & 1 \end{pmatrix} \tag{2.29}$$

where $(\tilde{a}, \tilde{b}, \tilde{c}, \tilde{\beta}, \tilde{\delta}) \in \mathbb{R}^{3|2}$, with $\epsilon^2 = 1$, and $\tilde{a} > 0$. Thus, as read off in (2.29), every homography is the composition of an inversion, a dilatation and a translation. We will denote by $\mathrm{E}(1|1)$ the subgroup of translations and by $\mathrm{Aff}(1|1)$ the subgroup generated by translations and dilatations. The connected component of the identity of these subgroups of $\mathrm{SpO}_+(2|1)$, characterized by $\epsilon > 0$, will be denoted by $\mathrm{E}_+(1|1)$ and $\mathrm{Aff}_+(1|1)$, and referred to as special supergroups.

3 Main results

We expound in this section the two main results of this paper regarding the case of $\mathrm{S}^{1|1}$; the first one gives the invariants of the action on $\mathrm{S}^{1|1}$ of the special supergroups $\mathrm{E}_+(1|1)$, $\mathrm{Aff}_+(1|1)$ and $\mathrm{SpO}_+(2|1)$, and the second one provides, by means of a super version of the Cartan formula, the associated $K(1)$-cocycles. These results will be extended (whenever possible) to the case of $\mathrm{S}^{1|N}$ in Section 6.

3.1 Super Euclidean, affine and projective invariants

Let t_1, t_2, t_3, t_4 be four generic points of $\mathrm{S}^{1|1}$, $t_i = (x_i, \xi_i)$.

Theorem 3.1. *The following three couples, I_{e}, I_{a} and I_{p}, of superfunctions are the invariants of the action of Euclidean, affine and projective special supergroups on $\mathrm{S}^{1|1}$:*

- *Euclidean invariant: $I_{\mathsf{e}}(t_1, t_2) = ([t_1, t_2], \{t_1, t_2\})$ with*

$$[t_1, t_2] = x_2 - x_1 - \xi_2\xi_1, \tag{3.1}$$

$$\{t_1, t_2\} = \xi_2 - \xi_1. \tag{3.2}$$

- *Affine invariant,* $I_{\mathsf{a}}(t_1,t_2,t_3) = ([t_1,t_2,t_3],\{t_1,t_2,t_3\})$*, where, if* $x_1 < x_2$*,*

$$[t_1,t_2,t_3] = \frac{[t_1,t_3]}{[t_1,t_2]}, \tag{3.3}$$

$$\{t_1,t_2,t_3\} = [t_1,t_2,t_3]^{\frac{1}{2}}\,\frac{\{t_1,t_3\}}{[t_1,t_3]^{\frac{1}{2}}}. \tag{3.4}$$

- *Projective invariant,* $I_{\mathsf{p}}(t_1,t_2,t_3,t_4) = ([t_1,t_2,t_3,t_4],\pm\{t_1,t_2,t_3,t_4\})$*, where, when* $\mathrm{ord}(t_1,t_2,t_3) = 1$*, see (4.9),*

$$[t_1,t_2,t_3,t_4] = \frac{[t_1,t_3][t_2,t_4]}{[t_2,t_3][t_1,t_4]}, \tag{3.5}$$

$$\{t_1,t_2,t_3,t_4\} = [t_1,t_2,t_3,t_4]^{\frac{1}{2}}\,\frac{\{t_2,t_4\}[t_1,t_2]-\{t_1,t_2\}[t_2,t_4]}{([t_1,t_2][t_2,t_4][t_1,t_4])^{\frac{1}{2}}}. \tag{3.6}$$

If a bijective transformation of $\mathrm{S}^{1|1}$ *preserves one of these three couples of superfunctions, it can be identified with the action of an element of the corresponding supergroup,* $\mathrm{E}_+(1|1)$*,* $\mathrm{Aff}_+(1|1)$ *or* $\mathrm{SpO}_+(2|1)$*. Moreover, if a contactomorphism* $\Phi \in K(1)$ *preserves the even part of one of the invariants* I_{e}*,* I_{a}*, or* I_{p}*, respectively, then* $\Phi = \widehat{h}$ *for some* h *in* $\mathrm{E}(1|1)$*,* $\mathrm{Aff}(1|1)$*, or* $\mathrm{SpO}_+(2|1)$*, respectively.*

This theorem summarizes Theorems 4.8, 4.14, and 4.19 given below, as well as their corollaries. Their proofs rely on the $p|q$-transitivity of the action of these supergroups on $\mathrm{S}^{1|1}$; all details are given in Section 4.

Remark 3.2. The super cross-ratio, i.e., the even part (3.5) of the projective invariant, I_{p}, has already been introduced by Nelson [24], and used by Giddings [15] while studying the punctured super Riemann sphere, and also by Uehara and Yasui [33] to define coordinates on the super Teichmüller space. It has also been put forward by Manin in [23] from a somewhat different standpoint that we can summarize as follows in our formalism. Using the even symplectic form $d\varpi = dp \wedge dq + \frac{1}{2}d\theta \wedge d\theta$ on $\mathbb{R}^{2|1}$ one defines a $\mathrm{SpO}_+(2|1)$-invariant pairing $\langle Z_i, Z_j\rangle = d\varpi(Z_i,Z_j) = p_ip_j[t_j,t_i]$, for $Z_i = (p_i\ q_i\ \theta_i) \in \mathbb{R}^{2|1}$, where $t_i = (q_i/p_i, \theta_i/p_i)$. Positing $[Z_1,Z_2,Z_3,Z_4] = \frac{\langle Z_3,Z_1\rangle\langle Z_4,Z_2\rangle}{\langle Z_3,Z_2\rangle\langle Z_4,Z_1\rangle}$, one obtains a four-point function, not only $\mathrm{SpO}_+(2|1)$-invariant, but also invariant under rescalings of each variable. We then have $[Z_1,Z_2,Z_3,Z_4] = [t_1,t_2,t_3,t_4]$, see (3.5).

Remark 3.3. The odd part (3.6) of the projective invariant, I_{p}, can clearly be reduced to a three-point (almost) invariant function, corresponding to J_{p} given below in (4.15). The latter was already introduced by D'Hoker and Phong [8] and used in [15, 33] on the same footing as the super cross-ratio. We have written J_{p} as function of the Euclidean invariants, but it can be recast into the form

$$J_{\mathsf{p}}(t_1,t_2,t_3) = \pm\frac{\xi_1[t_2,t_3]+\xi_2[t_3,t_1]+\xi_3[t_1,t_2]-\xi_1\xi_2\xi_3}{([t_1,t_3][t_3,t_2][t_2,t_1])^{\frac{1}{2}}}, \tag{3.7}$$

which precisely corresponds to the expression originally given in [2, 8], where the cyclic symmetry is obvious. This invariant, J_{p}, has also been introduced by Manin in [23], using a construction akin to that developed by us in Section 4.

Remark 3.4. If we apply the projection $\pi : \mathcal{C}^\infty(\mathrm{S}^{1|1}) \to \mathcal{C}^\infty(S^1)$, see (2.2), to each invariant I_{e}, I_{a} and I_{p}, we obtain the usual Euclidean, affine and projective invariant, namely the distance, the distance-ratio and the cross-ratio.

3.2 The associated 1-cocycles of $K(1)$

Let $\Phi \in \mathrm{Diff}(\mathrm{S}^1)$ be a diffeomorphism of the circle, and $\phi_\varepsilon = \mathrm{Id} + \varepsilon X + O(\varepsilon^2)$ be the flow of a vector field X on the circle. We set $t_i = \phi_{(i-1)\varepsilon}(t_1)$ for $i = 1,2,3,4$. Then, the Schwarzian derivative can be defined in terms of the cross-ratio, as the quadratic differential $\mathcal{S}(\Phi) \in \mathcal{Q}(\mathrm{S}^1)$ appearing in the Cartan formula, see [5, 27]:

$$\frac{\Phi^*[t_1,t_2,t_3,t_4]}{[t_1,t_2,t_3,t_4]} - 1 = \langle \varepsilon X \otimes \varepsilon X, \mathcal{S}(\Phi)\rangle + O(\varepsilon^3). \tag{3.8}$$

For the group of contactomorphisms of $\mathrm{S}^{1|1}$, we will proceed by analogy with this method. Starting from the super cross-ratio (3.5), we will deduce the super Schwarzian derivative, $\mathcal{S}(\Phi) \in \mathcal{Q}(\mathrm{S}^{1|1})$, as a $K(1)$-cocycle with kernel $\mathrm{SpO}_+(2|1)$. Euclidean and affine $K(1)$-cocycles will, likewise, be obtained from the even Euclidean and affine invariants. We recall that $\Omega^1(\mathrm{S}^{1|1})$ is the space of 1-forms, $\mathcal{Q}(\mathrm{S}^{1|1})$ the space of quadratic differentials of the supercircle, and $E_\Phi = \frac{\Phi^*\alpha}{\alpha} = (D\psi)^2$, see Subsection 2.2.

Theorem 3.5. *From the Euclidean (3.1), affine (3.3), and projective (3.5) even invariants, we deduce via the Cartan formula (3.8) three* 1*-cocycles of* $K(1)$*, with kernel* $\mathrm{E}(1|1)$*,* $\mathrm{Aff}(1|1)$ *and* $\mathrm{SpO}_+(2|1)$ *respectively. They retain the following form:*

- *the Euclidean cocycle* $\mathcal{E} : K(1) \to \mathcal{F}_0(\mathrm{S}^{1|1})$*:*

$$\mathcal{E}(\Phi) = \log E_\Phi = \log(D\psi)^2, \tag{3.9}$$

- *the affine cocycle* $\mathcal{A} : K(1) \to \Omega^1(\mathrm{S}^{1|1})$*:*

$$\mathcal{A}(\Phi) = d\mathcal{E}(\Phi), \tag{3.10}$$

- *the projective Schwarzian cocycle* $\mathcal{S} : K(1) \to \mathcal{Q}(\mathrm{S}^{1|1})$*:*

$$\mathcal{S}(\Phi) = \frac{2}{3}\,\alpha^{\frac{1}{2}} L_D\, \mathrm{S}(\Phi), \tag{3.11}$$

where L_D *stands for the Lie derivative with respect to the vector field* D*, and* $\mathrm{S}(\Phi)$ *is given by Equation (3.13) below. Moreover, using the projections on tensor densities defined in Proposition 2.5, we obtain two new affine and projective* 1*-cocycles, namely*

- *the projection of the affine cocycle,* $\mathrm{A} : K(1) \to \mathcal{F}_{\frac{1}{2}}(\mathrm{S}^{1|1})$*:*

$$\mathrm{A}(\Phi) = \alpha^{\frac{1}{2}} \langle D, \mathcal{A}(\Phi)\rangle = \frac{DE_\Phi}{E_\Phi}\, \alpha^{\frac{1}{2}}, \tag{3.12}$$

- *the projection of the Schwarzian cocycle,* $\mathrm{S} : K(1) \to \mathcal{F}_{3/2}(\mathrm{S}^{1|1})$*:*

$$\mathrm{S}(\Phi) = \alpha^{\frac{1}{2}} \langle D, \mathcal{S}(\Phi)\rangle = \frac{1}{4}\left(\frac{D^3E_\Phi}{E_\Phi} - \frac{3}{2}\frac{DE_\Phi\, D^2E_\Phi}{E_\Phi^2}\right)\alpha^{3/2}. \tag{3.13}$$

We will give the proof of this theorem in Section 5.

Remark 3.6. As in the case of the Schwarzian cocycle (3.11), using Proposition 2.5, we can express the affine cocycle $\mathcal{A}$ in terms of its projection A, namely

$$\mathcal{A}(\Phi) = \alpha^{\frac{1}{2}} L_D\, \mathrm{A}(\Phi). \tag{3.14}$$

Remark 3.7. 1) The projection $\pi : \mathcal{C}^\infty(\mathrm{S}^{1|1}) \to \mathcal{C}^\infty(S^1)$, see (2.2), can be extended naturally to differential forms and quadratic differentials, sending α to dx and β to 0. So, we can project the $K(1)$-cocycle $\mathcal{S}(\Phi)$ given by (3.11) on $\mathcal{Q}(S^1)$, and as the result depends only on $f = \Pi(\Phi)$, see (2.4), we easily recover the classical Schwarzian derivative $\mathrm{S}_0 : \mathrm{Diff}_+(\mathrm{S}^1) \mapsto \mathcal{Q}(\mathrm{S}^1)$, namely

$$\mathrm{S}_0(f) = \left(\frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2\right) dx^2, \tag{3.15}$$

using the expression (5.9) where $\pi(E_\Phi) = f'$. See, e.g., [5, 9, 27]. The projections of the two other $K(1)$-cocycles, $\mathcal{E}$ and $\mathcal{A}$, lead to the Euclidean and affine cocycle of $\mathrm{Diff}_+(\mathrm{S}^1)$, namely $\mathcal{E}_0(f) = f'$ and $\mathcal{A}_0(f) = \frac{f''}{f'}dx$.

2) The $K(1)$-cocycle, S, given in (3.13), is the super Schwarzian derivative, independently introduced by Friedan [11] and Radul [29]. Recall that $E_\Phi = (D\psi)^2$, see (2.13), so we can also write

$$\mathrm{S}(\Phi) = \left(\frac{D^4\psi}{D\psi} - 2\frac{D^2\psi\, D^3\psi}{(D\psi)^2}\right)\alpha^{3/2}. \tag{3.16}$$

This is the form of the super Schwarzian derivative used in superconformal field theories [11], see also [23]. Gieres and Theisen use it in [16], as well as the affine cocycle A, to construct superconformal covariant operators.

It is well-known that the classical Schwarzian derivative (3.15) can be expressed in terms of the classical affine cocycle $\mathrm{A}_0(f) = (f''/f')dx$ on S^1, viz.,

$$\mathrm{S}_0(f) = dx\, L_{\partial_x}\mathrm{A}_0(f) - \frac{1}{2}\mathrm{A}_0(f)^2, \tag{3.17}$$

where $f \in \mathrm{Diff}_+(\mathrm{S}^1)$. A formula relating, in the super case, the expression of S and A can be found in [16]. The next proposition gives another formula for the 1-cocycle S in a form akin to (3.17).

Proposition 3.8. *Let $\mathcal{A}$ denote the affine $K(1)$-cocycle (3.10); the following holds true for the super Schwarzian derivative (3.13):*

$$\mathrm{S}(\Phi) = \frac{1}{4}\,\alpha^{\frac{1}{2}}\Big\langle D, (\alpha^{\frac{1}{2}}L_D)^2\mathcal{A}(\Phi) - \frac{1}{2}\mathcal{A}(\Phi)^2\Big\rangle. \tag{3.18}$$

3.3 The determination of $H^1(K(1), \mathcal{F}_\lambda)$

The following corollary of Theorem 3.5 is straightforward; its proof relies on the expression (2.18) of the Euclidean 1-cocycle of $k(1)$, the Lie superalgebra of infinitesimal contactomorphisms of $\mathrm{S}^{1|1}$.

Corollary 3.9. *The Lie algebra 1-cocycles associated with the $K(1)$-cocycles $\mathcal{E}$,* A, *and* S, *read $c_i : k(1) \to \mathcal{F}_{i/2}(\mathrm{S}^{1|1})$, with*

$$c_i(X_f) = (D^{i+2}f)\,\alpha^{i/2}, \tag{3.19}$$

where $i = 0, 1, 3$.

We recover, in this way, three of the four nontrivial 1-cocycles of $k(1)$ with coefficients in $\mathcal{F}_\lambda$ (see [1] for a classification). The fourth one, $\tilde{c}_0 : k(1) \to \mathcal{F}_0(\mathrm{S}^{1|1})$, defined by $\tilde{c}_0(X_f) = f - \frac{1}{2}\xi\partial_\xi f$, does not integrate as a group 1-cocycle, just like the $\mathrm{Vect}(\mathrm{S}^1)$-cocycle $X_f \mapsto f$. Indeed, suppose that $\tilde{c}_0$ does integrate as a $K(1)$-cocycle, $\tilde{C}_0$. Then $\partial_x \in k(1)$ induces, using an angular coordinate x, the flow $\Phi_t(x,\xi) = (x+t,\xi)$, and as $\tilde{c}_0(\partial_x) = 1$, we have $\tilde{C}_0(\Phi_t) = \frac{1}{2}t$, see e.g. [31]. But this is inconsistent with the periodicity condition $\Phi_t = \Phi_{t+2\pi}$. This is a straightforward generalization to the super-algebraic framework of the observation [26] that the only $\mathrm{Vect}(S^1)$ 1-cocycles that integrate as $\mathrm{Diff}_+(S^1)$-cocycles are those which are Euclidean-basic; see also [18]. Here, one checks that $\tilde{c}_0$ is not $\mathrm{E}(1|1)$-basic.

As the derivation of Lie group cocycle is an injection from the Lie group cocycle into the Lie algebra cocycle, we obtain the complete classification of the nontrivial 1-cocycles of $K(1)$ with values in $\mathcal{F}_\lambda$.

Theorem 3.10. *1. The cohomology spaces $H^1(K(1),\mathcal{F}_\lambda)$ are given by*

$$H^1(K(1),\mathcal{F}_\lambda) = \begin{cases} \mathbb{R} & \text{if } \lambda = 0,\ \frac{1}{2},\ \frac{3}{2} \\ \{0\} & \text{otherwise.} \end{cases} \tag{3.20}$$

These three cohomology spaces are respectively generated by $\mathcal{E}$, A *and* S.

2. Moreover, the two cohomology spaces

$$H^1(K(1),\Omega^1(\mathrm{S}^{1|1})) = \mathbb{R}, \tag{3.21}$$

$$H^1(K(1),\mathcal{Q}(\mathrm{S}^{1|1})) = \mathbb{R}, \tag{3.22}$$

are respectively generated by $\mathcal{A}$ and $\mathcal{S}$.

Proof. We have already proved (3.20) in the course of the above discussion. Let us now derive (3.21). Suffice it to notice that Proposition 2.5 yields the decomposition $\Omega^1(\mathrm{S}^{1|1}) = \mathcal{F}_{\frac{1}{2}} \oplus \mathcal{F}_1$ into $K(1)$-submodules. The classification (3.20) then shows that the image by the section $\alpha^{\frac{1}{2}}L_D$ of the generator A of $H^1(K(1),\mathcal{F}_{\frac{1}{2}})$ spans $H^1(K(1),\Omega^1(\mathrm{S}^{1|1}))$. The same argument holds for the proof of (3.22). □

Let us end up with the following synthesis of the results obtained in this section.

Remark 3.11. We have thus established a 1-1 correspondence between the set of nontrivial cohomology spaces $H^1(K(1), \mathcal{F}_\lambda)$ (or $H^1(K(1), \mathcal{M})$, with $\mathcal{M} = \Omega^0(\mathrm{S}^{1|1})$, $\Omega^1(\mathrm{S}^{1|1})$, $\mathcal{Q}(\mathrm{S}^{1|1})$) and the "natural" geometries of the supercircle, namely the Euclidean, affine, and projective geometries of $\mathrm{S}^{1|1}$. These geometries are defined by the kernels of the corresponding 1-cocycles $\mathcal{E}, \mathcal{A}, \mathcal{S}$. These groups, in turn, give rise to the invariants $I_{\mathsf{e}}, I_{\mathsf{a}}, I_{\mathsf{p}}$. At last, these invariants lead us back to the generators of the above cohomology spaces, with the help of the Cartan-like formulæ (5.4), (5.5), and (3.8).

4 Super Euclidean, affine and projective invariants of $\mathrm{S}^{1|1}$

In this section we construct the Euclidean, affine and projective invariants given by Theorem 3.1. We introduce an extension of the notion of transitivity, allowing us to formulate a theorem giving the sought invariants when applied to each supergroup: $\mathrm{E}_+(1|1)$, $\mathrm{Aff}_+(1|1)$, and $\mathrm{SpO}_+(2|1)$.

Let us first introduce an equivalence relation, on the n-tuples of a product set $E = E_0 \times E_1$. We denote by p_0 and p_1 the two canonical projections. Let $s = (s_1, \ldots, s_n)$ and $t = (t_1, \ldots, t_n)$ be two n-tuples of E, we will say that s and t are $p|q$ equivalent, $s \overset{p|q}{=} t$, where $n = \max(p, q)$, iff

$$\forall i \in [\![1, p]\!], \quad p_0(s_i) = p_0(t_i) \qquad \text{and} \qquad \forall i \in [\![1, q]\!], \quad p_1(s_i) = p_1(t_i). \tag{4.1}$$

We will use the notation $[t]$ for the class of t for this equivalence relation.

Definition 4.1. *Let G be a group acting on a set $E = E_0 \times E_1$ by $(g \mapsto \widehat{g})$. The action of G on E is $p|q$-transitive, $n = \max(p, q)$, if for any n-tuples s and t of distinct points, there exists an element $h \in G$ such that $\widehat{h}(t) \overset{p|q}{=} s$. If h is unique the action is said to be simply $p|q$-transitive.*

In particular a $p|q$-transitive action is $\min(p, q)$-transitive. To prove n-transitivity, we usually prove that any n-tuple t can be sent to a given n-tuple m. To prove $p|q$-transitivity we need an extra condition, this is specified by the next proposition.

Proposition 4.2. *Let G act on a set $E = E_0 \times E_1$ and choose m, a n-tuple of E. Suppose that for every n-tuple s, there exists $h \in G$ such that $\widehat{h}(s) \overset{p|q}{=} m$, where $n = \max(p, q)$, and $G.[s] \supseteq [m]$. Then the action of G on E is $p|q$-transitive.*

Proof. Let t and s be two n-tuples of E. We look for those $k \in G$ such that $\widehat{k}(t) \overset{p|q}{=} s$. By assumption, there exist $h, g \in G$ such that $\widehat{h}(t) \overset{p|q}{=} m$ and $\widehat{g}(s) \overset{p|q}{=} m$. Then, as $\widehat{h}(t) \in [m]$ and $G.[s] \supseteq [m]$, there exist $s' \overset{p|q}{=} s$ and $g' \in G$ such that $\widehat{g'}(s') = \widehat{h}(t)$. Finally $\widehat{g'}^{-1}(\widehat{h}(t)) \overset{p|q}{=} s$. □

Theorem 4.3. *Let $g \mapsto \widehat{g}$ denote the simply $p|q$-transitive action of a group G on a set $E = E_0 \times E_1$, and let m be a n-tuple, $n = \max(p, q)$, of distinct points of E. We can define the following $(n+1)$-point function of E with values in E, associated to the class of m, namely*

$$I_{[m]}(t_1, \ldots, t_{n+1}) = \widehat{h}(t_{n+1}) \tag{4.2}$$

where $\widehat{h}(t) \overset{p|q}{=} m$, and $t = (t_1, \ldots, t_n)$ is a n-tuple of distinct points of E. This function enjoys the following properties:

1. *$I_{[m]}$ is G-invariant.*
2. *If $\Phi \in E!$ preserves $I_{[m]}$, then $\Phi = \widehat{g}$ for some $g \in G$.*
3. *Let l be a n-tuple of E and $g \in G$, then $\widehat{g}[m] = [l]$ iff $I_{[l]} = \widehat{g} \circ I_{[m]}$.*

The first two properties assert that $I_{[m]}$ is a characteristic invariant of the action of G. Moreover, if $n = p > q$, we can define n-point invariant functions with values in E_1

$$J_{[m],j}(t) = p_1(\widehat{h}(t_j)) \tag{4.3}$$

for $j \in [\![q+1, p]\!]$. Any $(n+1)$-point G-invariant function I can be factorized through the invariants $I_{[m]}$ and $J_{[m],j}$, i.e., $I = f(J_{[m],q+1}, \ldots, J_{[m],p}, I_{[m]})$ for some function f, depending on the n-tuple m. Similarly, any n-point G-invariant function can be factorized through the invariants $J_{[m],j}$.

Proof. We first prove that $I_{[m]}(\widehat{g}(t_1), \ldots, \widehat{g}(t_{n+1})) = I_{[m]}(t_1, \ldots, t_{n+1})$ for all $g \in G$, i.e., $I_{[m]}$ is G-invariant. Since $\widehat{h}(t) \overset{p|q}{=} m$, we have $\widehat{h} \circ \widehat{g}^{-1}(\widehat{g}(t)) \overset{p|q}{=} m$. It follows that

$I_{[m]}(\widehat{g}(t_1),\dots,\widehat{g}(t_{n+1})) = \widehat{h}\circ\widehat{g}^{-1}(\widehat{g}(t_{n+1})) = \widehat{h}(t_{n+1})$, hence the result. The proof of the G-invariance of $J_{[m],j}$ is identical.

Secondly, we show that $I_{[m]}$ is a characteristic G-invariant. Let Φ be a bijection of E, such that $\Phi^* I_{[m]} = I_{[m]}$, we have to prove that Φ comes from an element of G. There exist $h, g \in G$, depending on t such that, $\widehat{h}(t) \overset{p|q}{=} m$ and $\widehat{g}(\Phi(t)) \overset{p|q}{=} m$. Since $\Phi^* I_{[m]} = I_{[m]}$, we have $\widehat{g}(\Phi(t_{n+1})) = \widehat{h}(t_{n+1})$ for all $t_{n+1} \in E$, and thus $\Phi = \widehat{k}$, with $k = g^{-1}h$.

Thirdly, suppose that there exists $g \in G$ such that $\widehat{g}[m] = [l]$. Let t be a n-tuple, we have $\widehat{h}(t) \overset{p|q}{=} m$ for a unique $h \in G$, then $\widehat{g}(\widehat{h}(t)) \overset{p|q}{=} l$, and it follows that $I_{[l]}(t_1,\dots,t_{n+1}) = \widehat{g}\circ\widehat{h}(t_{n+1})$. Conversely suppose that $I_{[l]} = \widehat{g}\circ I_{[m]}$ for some $g \in G$ and let $m' \in [m]$. For every n-tuple t, there exists $h \in G$ such that $\widehat{h}(t) \overset{p|q}{=} m$ and then $I_{[l]}(t, t_{n+1}) = \widehat{g}\circ\widehat{h}(t_{n+1})$, for all $t_{n+1} \in E$. As $I_{[l]}(t, t_{n+1}) = \widehat{k}(t_{n+1})$ for the unique $k \in G$ such that $\widehat{k}(t) \overset{p|q}{=} l$, we deduce that $\widehat{g}(\widehat{h}(t)) \overset{p|q}{=} l$. In particular for the n-tuple m', h is the identity, hence $\widehat{g}(m') \overset{p|q}{=} l$. It follows that $\widehat{g}[m] \subseteq [l]$, and as we also have $\widehat{g^{-1}}\circ I_{[l]} = I_{[m]}$, then $\widehat{g^{-1}}[l] \subseteq [m]$, leading to the result $\widehat{g}[m] = [l]$.

Fourthly, let I be an arbitrary $(n+1)$-point invariant function. For any n-tuple t there exists some $h \in G$ such that $I(t_1,\dots,t_{n+1}) = I(m'_1,\dots,m'_n,\widehat{h}(t_{n+1}))$ with $\widehat{h}(t) = m' \overset{p|q}{=} m$. Now $I_{[m]}(t_1,\dots,t_{n+1}) = \widehat{h}(t_{n+1})$ and since m' depends only on m and on $J_{[m],j}$, the result follows. □

Remark 4.4. This theorem generalizes the more common situation of a simply n-transitive action of a group, choosing $p = q$. In this case, $\overset{p|q}{=}$ reduces to the mere equality, $=$, and every $(n+1)$-point invariant can be factorized through the invariant I_m given by Theorem 4.3. In particular, the invariant I_l, for l another n-tuple, can be factorized $I_l = \widehat{g}\circ I_m$, with $g \in G$ such that $\widehat{g}(m) = l$.

Remark 4.5. In the definition of $p|q$-transitivity and in this theorem, we consider n-tuples of distinct points. The notion of distinct points of $E = E_0 \times E_1$ is well-known, but we will strengthen it by assuming distinct even coordinates when dealing with supergroups acting on the supercircle.

As direct and classical application of our result, the action of $\mathrm{PGL}(2,\mathbb{R})$ by homographies on the circle S^1, viewed as $\mathbb{R}P^1$, is simply 3-transitive, and choosing

$m = (\infty, 0, 1)$ as the distinguished triple of points, we obtain the usual cross-ratio: $I(x_1, x_2, x_3, x_4) = \frac{(x_1-x_3)(x_2-x_4)}{(x_2-x_3)(x_1-x_4)}$.

4.1 Euclidean invariants

We introduce the subgroups $\mathrm{E}(1|1)$ and $\mathrm{E}_+(1|1)$ of $\mathrm{SpO}_+(2|1)$ which act on $\mathrm{S}^{1|1}$ by translations in an affine coordinate system.

Definition 4.6. *Let us define* $\mathrm{E}(1|1)$ *as the subgroup of* $\mathrm{GL}(2|1)$ *whose elements are of the form*

$$g = \begin{pmatrix} \epsilon & \epsilon b & -\epsilon\beta \\ 0 & \epsilon & 0 \\ 0 & \beta & 1 \end{pmatrix} \tag{4.4}$$

where $(b, \beta) \in \mathbb{R}^{1|1}$*, and* $\epsilon^2 = 1$*. It acts on* $\mathbb{R}^{1|1} \subset \mathrm{S}^{1|1}$ *by translations, according to* $\widehat{g}(x, \xi) = (x + b - \beta\xi, \epsilon\beta + \epsilon\xi)$*. We will denote by* $\mathrm{E}_+(1|1)$ *the connected component of the identity characterized by* $\epsilon = 1$*.*

Remark 4.7. The Euclidean groups can be defined in an alternative manner, in terms of the transformation laws of the 1-forms α and β, and then directly as subgroups of $K(1)$. The group $\mathrm{E}(1|1)$ is the subgroup of those $\Phi \in \mathrm{Diff}(\mathrm{S}^{1|1})$ such that $\Phi^*\alpha = \alpha$ and $\Phi^*\beta = \epsilon\beta$, with $\epsilon = \pm 1$; restricting to $\epsilon = 1$ we obtain the subgroup $\mathrm{E}_+(1|1)$.

Proposition 4.8. *The action of* $\mathrm{E}_+(1|1)$ *on* $\mathbb{R}^{1|1} \subset \mathrm{S}^{1|1}$ *is simply* $1|1$*-transitive; choosing* $\mathsf{e} = (0, 0)$*, it defines a characteristic Euclidean invariant consisting of the following two-point couple of superfunctions*

$$I_{\mathsf{e}}(t_1, t_2) = ([t_1, t_2]\,,\, \{t_1, t_2\}) = (x_2 - x_1 - \xi_2\xi_1\,,\, \xi_2 - \xi_1) \tag{4.5}$$

where $t_1 = (x_1, \xi_1)$ *and* $t_2 = (x_2, \xi_2)$*.*

Proof. Following Theorem 4.3, we have to show that for any point t_1 of $\mathrm{S}^{1|1}$, there exists a unique $h \in \mathrm{E}_+(1|1)$ such that $\widehat{h}(t_1) = (0, 0)$, and then to compute $\widehat{h}(t_2) = ([t_1, t_2]\,,\, \{t_1, t_2\})$ for another point t_2.

The action of any $h \in \mathrm{E}_+(1|1)$ is given by $\widehat{h}(x, \xi) = (x + b - \beta\xi, \beta + \xi)$. Hence $\widehat{h}(t_1) = (0, 0)$ is equivalent to $x_1 + b - \beta\xi_1 = 0$ and $\beta + \xi_1 = 0$, i.e., $\beta = -\xi_1$ and $b = -x_1$. So, h is uniquely determined, and $\widehat{h}(t_2) = (x_2 - x_1 - \xi_2\xi_1, \xi_2 - \xi_1)$, as announced. $\square$

The choice of the point $e = (0,0)$ is immaterial, see Remark 4.4.

Remark 4.9. The even Euclidean invariant $[t_1, t_2]$ is the discretized version of the contact form $\alpha = dx + \xi d\xi$, while the odd Euclidean invariant $\{t_1, t_2\}$ is that of $\beta = d\xi$. This will be specified in Lemma 5.1.

Corollary 4.10. *The even part of I_{e} is invariant under* $\mathrm{E}(1|1)$*, and characterizes this subgroup of $K(1)$, namely if $\Phi \in K(1)$ satisfies $\Phi^*[t_1, t_2] = [t_1, t_2]$, then $\Phi = \widehat{h}$ for some $h \in \mathrm{E}(1|1)$.*

Proof. Let $\iota \in K(1)$ be defined by $\iota : (x, \xi) \mapsto (x, -\xi)$. Identifying $\mathrm{E}(1|1)$ with its image in $K(1)$ we have $\mathrm{E}(1|1) = \mathrm{E}_+(1|1) \sqcup \iota(\mathrm{E}_+(1|1))$. Since $[t_1, t_2]$ is invariant under $\mathrm{E}_+(1|1)$ as well as under the action of ι, this is a $\mathrm{E}(1|1)$-invariant.

Let $\Phi = (\varphi, \psi) \in K(1)$ be such that $\Phi^*[t_1, t_2] = [t_1, t_2]$. There exists t_1 such that $\Phi(t_1) = (0,0)$, and $h \in \mathrm{E}_+(1|1)$ such that $\widehat{h}(t_1) = \Phi(t_1)$. Since Φ leaves $[t_1, t_2]$ invariant, we have $\varphi(t_2) = [\Phi(t_1), \Phi(t_2)] = [t_1, t_2] = \widehat{h}_0(t_2)$ in view of (4.2); hence $\varphi = \widehat{h}_0$, with $\widehat{h} = (\widehat{h}_0, \widehat{h}_1)$. Using Lemma 2.3, we obtain $\Phi = \widehat{h}$ or $\Phi = \iota(\widehat{h})$, and Φ is then a (super) translation. $\square$

4.2 Affine invariants

Let us start with the definitions of $\mathrm{Aff}(1|1)$ and $\mathrm{Aff}_+(1|1)$ and with their action on $\mathrm{S}^{1|1}$.

Definition 4.11. *The affine supergroup,* $\mathrm{Aff}(1|1)$*, is the subgroup of* $\mathrm{GL}(2|1)$ *whose elements are of the form*

$$g = \begin{pmatrix} a & ab & -a\beta \\ 0 & a^{-1} & 0 \\ 0 & \beta & 1 \end{pmatrix} \tag{4.6}$$

where $(a, b, \beta) \in \mathbb{R}^{2|1}$, and $a \neq 0$. This supergroup acts on $\mathbb{R}^{1|1} \subset \mathrm{S}^{1|1}$ by translations and dilatations, $\widehat{g}(x, \xi) = (a^2 x + a^2 b - a^2 \beta \xi, a\beta + a\xi)$. We will denote by $\mathrm{Aff}_+(1|1)$ *the connected component of the identity, characterized by $a > 0$.*

Remark 4.12. The affine groups can be defined in an alternative manner, in terms of the transformation laws of the 1-forms α and β, and then directly as subgroups of $K(1)$. The group $\mathrm{Aff}(1|1)$ is the subgroup of those $\Phi \in K(1)$ which satisfy $\Phi^*\beta = F_\Phi \beta$, with F_Φ a superfunction; restricting to $\pi(F_\Phi) > 0$ we obtain the subgroup $\mathrm{Aff}_+(1|1)$.

Acting by contactomorphisms on $\mathrm{S}^{1|1}$, $\mathrm{Aff}_+(1|1)$ preserves the orientation of the underlying circle, see Remark 2.2. Moreover, two points on the supercircle t_1 and t_2 define an orientation given by the sign of $x_2 - x_1$ (in the chosen affine coordinate system). Hence, the action of $\mathrm{Aff}_+(1|1)$ cannot be $2|1$-transitive, but for all couples s and t defining the same orientation there exists a unique $h \in \mathrm{Aff}_+(1|1)$ such that $\widehat{h}(t) \overset{2|1}{=} s$. So, let us introduce $\widetilde{\mathrm{Aff}_+(1|1)}$ as the group generated by $\mathrm{Aff}_+(1|1)$ and the orientation-reversing transformation $r : (x, \xi) \mapsto (-x, \xi)$.

Lemma 4.13. *The action of* $\widetilde{\mathrm{Aff}_+(1|1)}$ *on* $\mathbb{R}^{1|1} \subset \mathrm{S}^{1|1}$ *is simply* $2|1$*-transitive.*

Proof. Let $a_1 = (0, 0)$, $a_2 = (1, \zeta)$ and t_1, t_2 be two distinct points of $\mathrm{S}^{1|1}$, with $x_1 < x_2$, a condition which can always been satisfied, using the transformation r, if necessary. We look for $h \in \mathrm{Aff}_+(1|1)$ such that $\widehat{h}((t_1, t_2)) \overset{2|1}{=} (a_1, a_2)$. We thus have to solve the system: $a^2x_1 + a^2b - a^2\beta\xi_1 = 0$, $a\beta + a\xi_1 = 0$ and $a^2x_2 + a^2b - a^2\beta\xi_2 = 1$. In doing so, we obtain $\beta = -\xi_1$, $b = -x_1$ and $a^2 = [t_1, t_2]^{-1}$, see (4.5). This entails that h is uniquely determined and $\widehat{h}(t_3) = \left(\frac{[t_1,t_3]}{[t_1,t_2]}, \frac{\{t_1,t_3\}}{[t_1,t_2]^{\frac{1}{2}}}\right)$ for any point t_3 of $\mathrm{S}^{1|1}$. Here, p_1 is the projection to the odd component, hence $p_1(\widehat{h}(t_2))$ is given by $\frac{\{t_1,t_2\}}{[t_1,t_2]^{\frac{1}{2}}}$. The function $p_1 \circ \widehat{h}$ is thus surjective from $[t]$ onto $\mathbb{R}^{0|1}$ and Proposition 4.2 applies, proving the simply $2|1$-transitivity of the action of $\widetilde{\mathrm{Aff}_+(1|1)}$. □

Now, the action of $\widetilde{\mathrm{Aff}_+(1|1)}$ on $\mathrm{S}^{1|1}$ satisfies all assumptions of Theorem 4.3, and then, restricting ourselves to $x_1 < x_2$, we obtain affine invariants with all properties stated in Theorem 4.3.

Proposition 4.14. *Choosing* a*, the class of a couple* $a = ((0, 0), (1, \zeta))$ *for the relation* $\overset{2|1}{=}$*, Theorem 4.3 gives rise to a characteristic affine invariant consisting of the following three-point couple of superfunctions, defined, for* $x_1 < x_2$*, by*

$$I_{\mathsf{a}}(t_1, t_2, t_3) = ([t_1, t_2, t_3]\,,\, \{t_1, t_2, t_3\}) = \left(\frac{[t_1, t_3]}{[t_1, t_2]}, \frac{\{t_1, t_3\}}{[t_1, t_2]^{\frac{1}{2}}}\right). \tag{4.7}$$

We, likewise, have a two-point odd invariant, defined, for $x_1 < x_2$*, by*

$$J_{\mathsf{a}}(t_1, t_2) = \frac{\{t_1, t_2\}}{[t_1, t_2]^{\frac{1}{2}}}, \tag{4.8}$$

which is fundamental in that it generates all other two-point invariants.

Proof. The action of $\widetilde{\mathrm{Aff}_+(1|1)}$ being simply 2|1-transitive, we can apply Theorem 4.3. Let $t = (t_1, t_2)$ be a couple; if $x_1 < x_2$, we obtain, resorting to the proof of the last lemma, $I_{\mathsf{a}}(t_1, t_2, t_3) = \left(\frac{[t_1,t_3]}{[t_1,t_2]}, \frac{\{t_1,t_3\}}{[t_1,t_2]^{\frac{1}{2}}}\right)$ and $J_{\mathsf{a}}(t_1, t_2) = \frac{\{t_1,t_2\}}{[t_1,t_2]^{\frac{1}{2}}}$. For $x_1 < x_2$, I_{a} and J_{a} are invariants (with all properties given in Theorem 4.3) of the subgroup of $\widetilde{\mathrm{Aff}_+(1|1)}$ preserving the condition $x_1 < x_2$, i.e., $\mathrm{Aff}_+(1|1)$. □

For $x_2 < x_1$, we can easily show that I_{a} and J_{a} are simply obtained by exchanging t_1 and t_2.

Remark 4.15. The invariants I_{a} and J_{a} depend on a; for another class, b, of a couple of points, we have, following the third assertion of Theorem 4.3, $I_{\mathsf{b}} = \widehat{g} \circ I_{\mathsf{a}}$ and $J_{\mathsf{b}} = \widehat{g} \circ J_{\mathsf{a}}$ iff $p_1(b_1) = 0$. For $p_1(b_1) \neq 0$, I_{b} and J_{b} depend on I_{a} and J_{a} in a more involved way.

Remark 4.16. We can rewrite the odd three-point invariant, $p_1(I_{\mathsf{a}})$, as $\{t_1, t_2, t_3\} = [t_1, t_2, t_3]^{\frac{1}{2}} \frac{\{t_1,t_3\}}{[t_1,t_3]^{\frac{1}{2}}}$, showing that it is a function of the odd two-point invariant function, J_{a}, and of the even three-point invariant, $p_0(I_{\mathsf{a}})$. Hence, every affine three-point invariant function is a function of J_{a} and $p_0(I_{\mathsf{a}})$.

Corollary 4.17. *The even part, $p_0(I_{\mathsf{a}})$, of I_{a} is invariant under* $\mathrm{Aff}(1|1)$*, and characterizes this subgroup of $K(1)$, namely if $\Phi \in K(1)$ satisfies $\Phi^*[t_1, t_2, t_3] = [t_1, t_2, t_3]$, then $\Phi = \widehat{h}$ for some $h \in \mathrm{Aff}(1|1)$.*

The proof is identical to that of Corollary 4.10, in the Euclidean case.

4.3 Projective invariants

Once more, we will follow the previous method, and derive the super cross-ratio as the even part of the $\mathrm{SpO}_+(2|1)$-invariant given by Theorem 4.3.

We begin by the introduction of an orientation index, ord, on the oriented circle, defined on triples of distinct points by

$$\begin{aligned} \mathrm{ord}(x_1, x_2, x_3) &= +1 \text{ if } x_2 \in [x_1, x_3] \\ &= -1 \text{ if } x_2 \in [x_3, x_1]. \end{aligned} \tag{4.9}$$

It is uniquely preserved by orientation-preserving diffeomorphisms of the circle and enjoys the property: $\mathrm{ord}(\sigma(x_1), \sigma(x_2), \sigma(x_3)) = \varepsilon(\sigma)\mathrm{ord}(x_1, x_2, x_3)$ for any permutation σ

whose parity is denoted by $\varepsilon(\sigma)$, see [3]. This index, ord, can be extended to triples of points of the supercircle by $\mathrm{ord}(t_1,t_2,t_3)=\mathrm{ord}(x_1,x_2,x_3)$.

As $\mathrm{SpO}_+(2|1)$ acts by contactomorphisms on $\mathrm{S}^{1|1}$, it preserves the orientation of the underlying circle, see Remark 2.2. Hence, the action of $\mathrm{SpO}_+(2|1)$ cannot be $3|2$-transitive, a triple of distinct points defining an orientation. However, if s and t are two triples defining the same orientation, there exist exactly two elements $h_\pm\in\mathrm{SpO}_+(2|1)$ such that $\widehat{h_\pm}(t)\overset{3|2}{=}s$. So, let us introduce $\widetilde{\mathrm{SpO}_+(2|1)}$, the group generated by $\mathrm{SpO}_+(2|1)$ already considered, and the orientation-reversing transformation $r:(x,\xi)\mapsto(-x,\xi)$.

Lemma 4.18. *1. The action of* $\widetilde{\mathrm{SpO}_+(2|1)}$ *on* $\mathrm{S}^{1|1}$ *is* $3|2$*-transitive.*

2. Moreover, let p *be the class of* $p=((\infty,0),(0,0),(1,\zeta))$ *for the relation* $\overset{3|2}{=}$*, then for any triple* t*, there exist exactly two elements of* $\widetilde{\mathrm{SpO}_+(2|1)}$*,* k_+,k_-*, such that* $\widehat{k_\pm}(t)\overset{3|2}{=}p$*, and* $\widehat{k_-}=\iota\circ\widehat{k_+}$*, with* $\iota:(x,\xi)\mapsto(x,-\xi)$*.*

Proof. Assume first that the triple $t=(t_1,t_2,t_3)$ be such that $x_1<x_2<x_3$, even if it means to apply r and an element of $\mathrm{SpO}_+(2|1)$ inducing a cyclic permutation on t. Then, Proposition 4.14 insures that there exists a unique $g\in\mathrm{Aff}_+(1|1)$ such that: $\widehat{g}(t_2)=(0,0)$, and $\widehat{g}(t_3)=(1,\zeta')$, with $\zeta'=\frac{\{t_2,t_3\}}{[t_2,t_3]^{\frac12}}$. Since $g\in\mathrm{SpO}_+(2|1)$, we just have to determine all $h\in\mathrm{SpO}_+(2|1)$ such that $\widehat{h}(0,0)=(0,0)$, $p_0(\widehat{h}(1,\zeta'))=1$, and $\widehat{h}(\widehat{g}(t_1))=(\infty,0)$, implying that $hg=k$ are the sought transformations such that $\widehat{hg}(t)\overset{3|2}{=}p$.

As h is an element of $\mathrm{SpO}_+(2|1)$, $\widehat{h}$ is of the form $\widehat{h}(x,\xi)=\left(\frac{ax+b+\gamma\xi}{cx+d+\delta\xi},\frac{\alpha x+\beta+e\xi}{cx+d+\delta\xi}\right)$, with the relations (2.24) to (2.27). Since $\widehat{h}(0,0)=(0,0)$, we have $b=\beta=0$, and the relations become $ad=1$, $e^2=1$, $\alpha e=a\delta$ and $\gamma=0$; now $e=1$ since we restrict us to special transformations, i.e., of Berezinian 1. The equation $\widehat{h}(\widehat{g}(t_1))=(\infty,0)$ gives $ac\frac{[t_2,t_1]}{[t_2,t_3]}+1+\alpha\frac{\{t_2,t_1\}}{[t_2,t_3]^{\frac12}}=0$ and $\alpha\frac{[t_2,t_1]}{[t_2,t_3]}+\frac{\{t_2,t_1\}}{[t_2,t_3]^{\frac12}}=0$, where we have used the fact that $\widehat{g}(t_1)=I_{\mathsf{a}}(t_2,t_3,t_1)$ as given by (4.7). Hence, we have

$$\alpha=-\frac{\{t_1,t_2\}}{[t_2,t_3]^{\frac12}}\frac{[t_2,t_3]}{[t_1,t_2]}\qquad\text{and}\qquad ac=\frac{[t_2,t_3]}{[t_1,t_2]}.\tag{4.10}$$

There is one extra equation to satisfy, namely $p_0(\widehat{h}(1,\zeta'))=1$; it yields explicitly $a^2=ac+1+\alpha\zeta'$, giving $a^2=\frac{[t_2,t_3]}{[t_1,t_2]}+1-\frac{\{t_1,t_2\}\{t_2,t_3\}}{[t_1,t_2]}$, since $\zeta'=J_{\mathsf{a}}(t_2,t_3)$, see (4.3), as given

by (4.8). We then get, with the help of the identity $[t_2,t_3]+[t_1,t_2]-\{t_1,t_2\}\{t_2,t_3\}=[t_1,t_3]$,

$$a^2=\frac{[t_1,t_3]}{[t_1,t_2]}, \tag{4.11}$$

so a is determined up to an overall sign. We have proved that h is therefore given by

$$\widehat{h}(x,\xi)=\left(\frac{a^2x}{acx+1+\alpha\xi},\frac{a(\alpha x+\xi)}{acx+1+\alpha\xi}\right), \tag{4.12}$$

the sign of $a\neq 0$ remaining unspecified. This proves the existence and uniqueness of $h_\pm$, as stated above. Moreover, $J_{\mathsf{p}}(t_1,t_2,t_3)=p_1(\widehat{h}(t_3))$ is a surjective function from $[t]$ to $\mathbb{R}^{0|1}$, see (4.15). Using Proposition 4.2, we conclude that the action of $\widetilde{\mathrm{SpO}_+(2|1)}$ is 3|2-transitive. □

Now, even if the action of $\widetilde{\mathrm{SpO}_+(2|1)}$ is not simply 3|2-transitive, we can construct, following Theorem 4.3, invariants in the same way as before, and restricting consideration to $\mathrm{ord}(t_1,t_2,t_3)=1$, we will end up with projective invariants.

Proposition 4.19. *Let $p=((\infty,0),(0,0),(1,\zeta))$ be a triple of points of $\mathrm{S}^{1|1}$, and denote by p the class of p for the relation $\overset{3|2}{=}$. Theorem 4.3 then yields the projective invariant $I_{\mathsf{p}}(t_1,t_2,t_3,t_4)=([t_1,t_2,t_3,t_4],\pm\{t_1,t_2,t_3,t_4\})$, given, if $\mathrm{ord}(t_1,t_2,t_3)=1$, by*

$$[t_1,t_2,t_3,t_4] = \frac{[t_1,t_3][t_2,t_4]}{[t_2,t_3][t_1,t_4]}, \tag{4.13}$$

$$\{t_1,t_2,t_3,t_4\} = [t_1,t_2,t_3,t_4]^{\frac12}\,\frac{\{t_2,t_4\}[t_1,t_2]-\{t_1,t_2\}[t_2,t_4]}{([t_1,t_2][t_2,t_4][t_1,t_4])^{\frac12}}, \tag{4.14}$$

which characterizes the group $\mathrm{SpO}_+(2|1)$ within the diffeomorphisms of $\mathrm{S}^{1|1}$.

We also have an odd projective invariant, namely, if $\mathrm{ord}(t_1,t_2,t_3)=1$,

$$J_{\mathsf{p}}(t_1,t_2,t_3)=\pm\frac{\{t_2,t_3\}[t_1,t_2]-\{t_1,t_2\}[t_2,t_3]}{([t_1,t_2][t_2,t_3][t_1,t_3])^{\frac12}}. \tag{4.15}$$

which is fundamental in that it generates all other three-point invariants.

Proof. Assume that $\mathrm{ord}(t_1,t_2,t_3)=1$. Using Lemma 4.18, we know that there exist exactly two elements $k_+,k_-\in\mathrm{SpO}_+(2|1)$ such that $\widehat{k_\pm}(t)\overset{3|2}{=}p$. We set $I_{\mathsf{p}}(t_1,\dots,t_4)=\widehat{k_\pm}(t_4)$ and $J_{\mathsf{p}}(t_1,t_2,t_3)=p_1(\widehat{k_\pm}(t_3))$, as suggested by Theorem 4.3. Despite the non

uniqueness of k, all conclusions of Theorem 4.3 apply just as well, and the proofs are identical, except for I_{p} being a characteristic invariant. The proof of Theorem 4.3 shows that any bijection, Φ, of the supercircle such that $\Phi^* I_{\mathsf{p}} = I_{\mathsf{p}}$, satisfies $\Phi(t_4) = \widehat{k_\pm}(t_4)$ for all t_4. We have to impose that Φ be a diffeomorphism to obtain $\Phi = \widehat{k_+}$ or $\Phi = \widehat{k_-}$.

It then remains to compute $\widehat{k_\pm}(t_4)$; using the proof of Lemma 4.18 we will easily calculate $\widehat{k_\pm} = \widehat{h_\pm} \circ \widehat{g}$, for the specific case $x_1 < x_2 < x_3$. Starting with the even part of $\widehat{k_\pm}(t_4)$, we obtain, see (4.12),

$$\begin{aligned}
[t_1, t_2, t_3, t_4] &= \frac{a^2[t_2, t_3, t_4]}{ac[t_2, t_3, t_4] + 1 + \alpha\{t_2, t_3, t_4\}} \\
&= \frac{[t_1, t_3][t_2, t_4]}{[t_1, t_2][t_2, t_3]\left(\dfrac{[t_2, t_4]}{[t_1, t_2]} + 1 - \dfrac{\{t_1, t_2\}\{t_2, t_4\}}{[t_1, t_2]}\right)},
\end{aligned}$$

where we have used (4.10) and (4.11). With the help of the identity $[t_2, t_4] + [t_1, t_2] - \{t_1, t_2\}\{t_2, t_4\} = [t_1, t_4]$, we find the announced result, viz., Equation (4.13).

We then compute the odd part of $\widehat{h_\pm}(\widehat{g}(t_4))$, which is determined up to global sign governed by the sign of a (see proof of Lemma 4.18). For $a > 0$, we find, using (4.12),

$$\begin{aligned}
\{t_1, t_2, t_3, t_4\} &= \frac{a\left(\alpha[t_2, t_3, t_4] + \{t_2, t_3, t_4\}\right)}{ac[t_2, t_3, t_4] + 1 + \alpha\{t_2, t_3, t_4\}} \\
&= \frac{([t_1, t_2][t_1, t_3])^{\frac{1}{2}}\left(-\dfrac{\{t_1, t_2\}}{[t_2, t_3]^{\frac{1}{2}}}\dfrac{[t_2, t_4]}{[t_1, t_2]} + \dfrac{\{t_2, t_4\}}{[t_2, t_3]^{\frac{1}{2}}}\right)}{[t_1, t_4]} \\
&= [t_1, t_2, t_3, t_4]^{\frac{1}{2}}\left([t_1, t_2, t_4]^{-\frac{1}{2}}\frac{\{t_2, t_4\}}{[t_2, t_4]^{\frac{1}{2}}} - [t_4, t_2, t_1]^{-\frac{1}{2}}\frac{\{t_1, t_2\}}{[t_1, t_2]^{\frac{1}{2}}}\right),
\end{aligned}$$

with the help of the equalities (4.10) and (4.11). For $x_1 < x_2 < x_3$, we can write

$$\{t_1, t_2, t_3, t_4\} = [t_1, t_2, t_3, t_4]^{\frac{1}{2}}\frac{\{t_2, t_4\}[t_1, t_2] - \{t_1, t_2\}[t_2, t_4]}{([t_1, t_2][t_2, t_4][t_1, t_4])^{\frac{1}{2}}}, \tag{4.16}$$

which is the announced result, viz., Equation (4.14).

For the more general case $\mathrm{ord}(t_1, t_2, t_3) = 1$, we still have to compute $\widehat{k_\pm}(t_4)$ for $x_3 < x_1 < x_2$ and $x_2 < x_3 < x_1$. Let us introduce the homography $\widehat{c}(x, \xi) = (\frac{x-1+\zeta\xi}{x}, \frac{\zeta x-\xi}{x})$, which cyclically permutes $(0, 0)$, $(\infty, 0)$ and $(1, \zeta)$. Start with the case

$x_3 < x_1 < x_2$; we can assume that $x_3 < 0 < x_1 < x_2$, even if it means to apply a translation, and then $\widehat{c}(x_1) < \widehat{c}(x_2) < \widehat{c}(x_3)$. As I_{p} is invariant under the action of $\mathrm{SpO}_+(2|1)$, we have $I_{\mathsf{p}} = \widehat{c}^* I_{\mathsf{p}}$, and using the above results, we deduce that $\widehat{k_\pm}(t_4) = \widehat{c}^* \left(\frac{[t_1,t_3][t_2,t_4]}{[t_2,t_3][t_1,t_4]}, [t_1,t_2,t_3,t_4]^{\frac{1}{2}} \frac{\{t_2,t_4\}[t_1,t_2]-\{t_1,t_2\}[t_2,t_4]}{([t_1,t_2][t_2,t_4][t_1,t_4])^{\frac{1}{2}}} \right)$. The Euclidean invariants are transformed by $\widehat{c}$ as follows $\widehat{c}^*[t_i,t_j] = \frac{[t_i,t_j]}{x_i x_j}$ and $\widehat{c}^*\{t_i,t_j\} = \frac{t_i}{x_i} - \frac{t_j}{x_j}$, we then have $\widehat{k_\pm}(t_4) = \left(\frac{[t_1,t_3][t_2,t_4]}{[t_2,t_3][t_1,t_4]}, [t_1,t_2,t_3,t_4]^{\frac{1}{2}} \frac{\{t_2,t_4\}[t_1,t_2]-\{t_1,t_2\}[t_2,t_4]}{([t_1,t_2][t_2,t_4][t_1,t_4])^{\frac{1}{2}}} \right)$. The case $x_2 < x_3 < x_1$ is similar, except for the fact that we have to apply $\widehat{c^2}$ instead of $\widehat{c}$. □

For $\mathrm{ord}(t_1,t_2,t_3) = -1$, the projective invariants I_{p} and J_{p} are simply given by the exchange of t_1 and t_2 in Formulæ (4.13), (4.14), and (4.15).

Corollary 4.20. *The cross-ratio (4.13) is invariant under* $\mathrm{SpO}_+(2|1)$*, and characterizes this subgroup of* $K(1)$*, namely if* $\Phi \in K(1)$ *satisfies* $\Phi^*[t_1,t_2,t_3,t_4] = [t_1,t_2,t_3,t_4]$*, then* $\Phi = \widehat{h}$ *for some* $h \in \mathrm{SpO}_+(2|1)$*.*

The proof is the same as in the Euclidean case, except for the fact that $\mathrm{SpO}_+(2|1)$ contains now the transformation $\iota : (x,\xi) \mapsto (x,-\xi)$.

Remark 4.21. Projective groups and projective invariants of the circle and of the supercircle share various properties. Like $\mathrm{SpO}_+(2|1)$ in the super case, the action of $\mathrm{PSL}(2,\mathbb{R})$ preserves the orientation of the circle. The action of $\mathrm{PSL}(2,\mathbb{R})$ on the circle is thus not simply 3-transitive, in contradistinction to that of $\mathrm{PGL}(2,\mathbb{R})$. The cross-ratio can also be defined following Theorem 4.3, leading to the classical expression $[x_1,x_2,x_3,x_4] = \frac{(x_1-x_3)(x_2-x_4)}{(x_2-x_3)(x_1-x_4)}$, which is invariant under $\mathrm{PSL}(2,\mathbb{R})$ only. The $\mathrm{PGL}(2,\mathbb{R})$-invariant is given either by this last expression or by the same expression where x_1 and x_2 have been exchanged, depending on $\mathrm{ord}(x_1,x_2,x_3)$.

Remark 4.22. Again, the odd four-point invariant $p_1(I_{\mathsf{p}})$ is a function of the odd three-point invariant J_{p} and of the even four-point invariant $p_0(I_{\mathsf{p}})$. So every four-point invariant is a function of these two invariants.

5 The Schwarzian derivative from the Cartan formula

This section provides the proof of Theorem 3.5. We will begin by two preliminary lemmas and then give the proof for the Euclidean and affine cases, and, finally, for the projective one.

5.1 Preparation

Let us first recall the formula for the Taylor expansion of a smooth superfunction $f \in \mathcal{C}^\infty(\mathrm{S}^{1|1})$ as given in [21, 7], namely

$$\begin{aligned} f(t_2) - f(t_1) &= \sum_{i=1}^{n} \frac{1}{i!} \left((x_2 - x_1)^i \partial_x^i f(t_1) + i(\xi_2 - \xi_1)(x_2 - x_1)^{i-1} \partial_x^{i-1} \partial_\xi f(t_1) \right) \\ &\quad + O((x_2 - x_1)^{n+1}, (\xi_2 - \xi_1)(x_2 - x_1)^n) \\ &= \sum_{i=1}^{n} \frac{1}{i!} \left([t_1, t_2]^i \partial_x^i f(t_1) + i\{t_1, t_2\}[t_1, t_2]^{i-1} \partial_x^{i-1} D f(t_1) \right) \\ &\quad + O((x_2 - x_1)^{n+1}, (\xi_2 - \xi_1)(x_2 - x_1)^n). \end{aligned} \tag{5.1}$$

The following lemma linking discrete variations and forms, will enable us to write Taylor expansions in terms of the differential forms α and β. We will skip its straightforward proof.

Lemma 5.1. *Let $X \in \mathrm{Vect}(\mathrm{S}^{1|1})$, and ϕ_ε the associated flow. Putting $t_2 = \phi_\varepsilon(t_1)$, we have*

$$[t_1, t_2] = \langle \varepsilon X, \alpha \rangle (t_1) + O(\varepsilon^2), \quad \textit{and} \quad \{t_1, t_2\} = \langle \varepsilon X, \beta \rangle (t_1) + O(\varepsilon^2). \tag{5.2}$$

The next result is of central importance in the subsequent proof of Theorem 3.5.

Lemma 5.2. *Let $\Phi = (\varphi, \psi) \in K(1)$ be a contactomorphism of $\mathrm{S}^{1|1}$, and let $t_2 = \phi_\varepsilon(t_1)$, where ϕ_ε is the flow of a vector field X, then*

$$\begin{aligned} \frac{\Phi^*[t_1, t_2]}{E_\Phi(t_1)[t_1, t_2]} &= 1 + \frac{1}{2} \left([t_1, t_2] \frac{E'_\Phi}{E_\Phi}(t_1) + \{t_1, t_2\} \frac{D E_\Phi}{E_\Phi}(t_1) \right) \\ &\quad + \left\langle \varepsilon X \otimes \varepsilon X, \alpha^2 \frac{A}{6E_\Phi} + \alpha\beta \frac{B}{2E_\Phi} \right\rangle (t_1) + O(\varepsilon^3) \end{aligned} \tag{5.3}$$

where $A = \varphi''' + \psi\psi'''$ and $B = D\varphi'' - \psi D\psi''$, α^2 and $\alpha\beta$ being as in (2.20).

Proof. We have $\Phi^*[t_1,t_2] = [\Phi(t_1),\Phi(t_2)] = \varphi(t_2)-\varphi(t_1)-(\psi(t_2)-\psi(t_1))\psi(t_1)$, by virtue of (4.5). Using Taylor's formula (5.1), we obtain

$$\begin{aligned}\Phi^*[t_1,t_2] &= \{t_1,t_2\}(D\varphi-\psi D\psi)(t_1)+[t_1,t_2](\varphi'+\psi\psi')(t_1)\\ &\quad+[t_1,t_2]\left(\frac{1}{2}[t_1,t_2](\varphi''+\psi\psi'')(t_1)+\{t_1,t_2\}(D\varphi'-\psi D\psi')(t_1)\right)\\ &\quad+[t_1,t_2]\left(\frac{1}{6}[t_1,t_2]^2(\varphi'''+\psi\psi''')(t_1)+\frac{1}{2}\{t_1,t_2\}[t_1,t_2](D\varphi''-\psi D\psi'')(t_1)\right)\\ &\quad+O(\varepsilon^4).\end{aligned}$$

Then, as $\Phi\in K(1)$, Proposition 2.1 yields $D\varphi-\psi D\psi=0$, and $\varphi'+\psi\psi'=E_\Phi$. This entails that $\varphi''+\psi\psi''=E'_\Phi$, and $D\varphi'-\psi D\psi'=\frac{1}{2}DE_\Phi$. Lemma 5.1 then leads to the result. □

At first order in ε we obtain simply: $\frac{\Phi^*[t_1,t_2]}{[t_1,t_2]}=\left[E_\Phi+\frac{1}{2}\langle\varepsilon X,dE_\Phi\rangle\right](t_1)+O(\varepsilon^2)$.

5.2 Proof of Theorem 3.5

5.2.1 Euclidean and affine $K(1)$-cocycles, $\mathcal{E},\mathcal{A}$

The Cartan formula (3.8) yields a privileged means to define the Schwarzian derivative via a Taylor expansion of the cross-ratio. Much in the same way, we will construct 1-cocycles via the Euclidean and affine even invariants. Thanks to the last lemma, we have

$$\frac{\Phi^*[t_1,t_2]}{[t_1,t_2]}=E_\Phi(t_1)+O(\varepsilon). \tag{5.4}$$

Hence, $\mathcal{E}:\Phi\mapsto\log(E_\Phi)$ is a 1-cocycle of $K(1)$, with values in $\mathcal{F}_0(\mathrm{S}^{1|1})$; this justifies (3.9). Note that $\log(E_\Phi)$ is well-defined since the reduced function $\pi(E_\Phi)=\pi(D\psi)^2$, see (2.2), is positive.

For the affine even invariant (4.7), we have, putting $t_2=\phi_\varepsilon(t_1)$ and $t_3=\phi_{2\varepsilon}(t_1)$,

$$\begin{aligned}\frac{\Phi^*[t_1,t_2,t_3]}{[t_1,t_2,t_3]}-1 &= \frac{1+\frac{1}{2}\langle 2\varepsilon X,d(\log E_\Phi)\rangle(t_1)+O(\varepsilon^2)}{1+\frac{1}{2}\langle\varepsilon X,d(\log E_\Phi)\rangle(t_1)+O(\varepsilon^2)}-1\\ &= \frac{1}{2}\langle\varepsilon X,d(\log E_\Phi)\rangle(t_1)+O(\varepsilon^2).\end{aligned} \tag{5.5}$$

This implies that $\mathcal{A}:\Phi\mapsto d(\log E_\Phi)$ is a 1-cocycle of the group $K(1)$ of contactomorphisms, with values in the space, $\Omega^1(\mathrm{S}^{1|1})$, of 1-forms on $\mathrm{S}^{1|1}$. Using the projection on

half-densities $\mathcal{F}_{\frac{1}{2}}(\mathrm{S}^{1|1})$ given by $\alpha^{\frac{1}{2}}\langle D,\,\cdot\,\rangle$, see Proposition 2.5, we still obtain an affine 1-cocycle: $\mathrm{A}:\Phi\mapsto\alpha^{\frac{1}{2}}\,\langle D, d(\log E_\Phi)\rangle=\frac{DE_\Phi}{E_\Phi}\,\alpha^{\frac{1}{2}}$. The justification of (3.10) and (3.12) is complete.

5.2.2 The Schwarzian derivative, $\mathcal{S}$

We will now resort, *verbatim*, to the Cartan formula (3.8) in order to derive the expression of the Schwarzian derivative (3.11) of a diffeomorphism $\Phi\in K(1)$. This formula involves the cross-ratio $[t_1,t_2,t_3,t_4]$ of four close by points; we will, hence, posit $t_2=\phi_\varepsilon(t_1), t_3=\phi_{2\varepsilon}(t_1)$, and $t_4=\phi_{3\varepsilon}(t_1)$, where $\phi_\varepsilon=\mathrm{Id}+\varepsilon X+O(\varepsilon^2)$ is the flow of a vector field X of $\mathrm{S}^{1|1}$.

Let us then expand in powers of ε the following expression:

$$\frac{\Phi^*[t_1,t_2,t_3,t_4]}{[t_1,t_2,t_3,t_4]}-1=\frac{\dfrac{\Phi^*[t_1,t_3]}{E_\Phi(t_1)[t_1,t_3]}\dfrac{\Phi^*[t_2,t_4]}{E_\Phi(t_2)[t_2,t_4]}-\dfrac{\Phi^*[t_2,t_3]}{E_\Phi(t_2)[t_2,t_3]}\dfrac{\Phi^*[t_1,t_4]}{E_\Phi(t_1)[t_1,t_4]}}{1+O(\varepsilon)}. \tag{5.6}$$

We note that the terms $\frac{\Phi^*[t_1,t_3]}{E_\Phi(t_1)[t_1,t_3]}$ and $\frac{\Phi^*[t_1,t_4]}{E_\Phi(t_1)[t_1,t_4]}$, with base point t_1, are explicitly given by Lemma 5.2. The remaining terms, with base point t_2, will be computed separately, using again Equation (5.3) and the Taylor formula (5.1), viz.,

$$f(t_2)=f(t_1)+[t_1,t_2]f'(t_1)+\{t_1,t_2\}Df(t_1)+O(\varepsilon^2), \tag{5.7}$$

for a superfunction $f\in\mathcal{C}^\infty(\mathrm{S}^{1|1})$.

We have

$$\begin{aligned}\frac{\Phi^*[t_2,t_3]}{E_\Phi(t_2)[t_2,t_3]} &= 1+\frac{1}{2}\left([t_2,t_3]\frac{E'_\Phi}{E_\Phi}(t_1)+\{t_2,t_3\}\frac{DE_\Phi}{E_\Phi}(t_1)\right)\\ &\quad+\frac{1}{2}[t_1,t_2]\left([t_2,t_3]\left(\frac{E'_\Phi}{E_\Phi}\right)'(t_1)+\{t_2,t_3\}\left(\frac{DE_\Phi}{E_\Phi}\right)'(t_1)\right)\\ &\quad+\frac{1}{2}\{t_1,t_2\}\left([t_2,t_3]D\left(\frac{E'_\Phi}{E_\Phi}\right)(t_1)+\{t_2,t_3\}D\left(\frac{DE_\Phi}{E_\Phi}\right)(t_1)\right)\\ &\quad+\left\langle\varepsilon X\otimes\varepsilon X,\alpha^2\,\frac{A}{6E_\Phi}+\alpha\beta\,\frac{B}{2E_\Phi}\right\rangle(t_1)+O(\varepsilon^3),\end{aligned}$$

where the terms A and B are defined in Lemma 5.2. The other term $\frac{\Phi^*[t_2,t_4]}{E_\Phi(t_2)[t_2,t_4]}$ is, likewise, obtained by replacing in the latter expression t_3 by t_4, and εX by $2\varepsilon X$.

From Lemma 5.1 and Taylor's formula (5.7), we get $[t_2,t_3]=\langle\varepsilon X,\alpha\rangle(t_1)+O(\varepsilon^2)$ and $\{t_2,t_3\}=\langle\varepsilon X,\beta\rangle(t_1)+O(\varepsilon^2)$. In particular $\{t_2,t_3\}\{t_1,t_2\}$ is thus of third order in ε, since $\langle\varepsilon X,\beta\rangle$ is an odd superfunction. We finally have

$$\begin{aligned}\frac{\Phi^*[t_2,t_3]}{E_\Phi(t_2)[t_2,t_3]} &= 1+\frac{1}{2}\left([t_2,t_3]\frac{E'_\Phi}{E_\Phi}(t_1)+\{t_2,t_3\}\frac{DE_\Phi}{E_\Phi}(t_1)\right)\\ &\quad+\frac{1}{2}\left\langle\varepsilon X\otimes\varepsilon X,\alpha^2\left(\frac{E'_\Phi}{E_\Phi}\right)'+2\alpha\beta\left(\frac{DE_\Phi}{E_\Phi}\right)'\right\rangle(t_1)\\ &\quad+\left\langle\varepsilon X\otimes\varepsilon X,\alpha^2\frac{A}{6E_\Phi}+\alpha\beta\frac{B}{2E_\Phi}\right\rangle(t_1)+O(\varepsilon^3).\end{aligned}$$

This formula and Lemma 5.2 help us find the contribution of the first order terms of each product in the numerator of Equation (5.6); this contribution is found as $([t_1,t_3]+[t_2,t_4]-[t_2,t_3]-[t_1,t_4])\frac{E'_\Phi}{2E_\Phi}(t_1)+(\{t_1,t_3\}+\{t_2,t_4\}-\{t_2,t_3\}-\{t_1,t_4\})\frac{DE_\Phi}{2E_\Phi}(t_1)=(\xi_1-\xi_2)(\xi_3-\xi_4)\frac{E'_\Phi}{2E_\Phi}(t_1)$, which is of third order in ε, since $\xi_3-\xi_4=\xi_1-\xi_2+O(\varepsilon^2)$. The right-hand side of (5.6) is of second order in ε and we now compute it. We find

$$\begin{aligned}\frac{\Phi^*[t_1,t_2,t_3,t_4]}{[t_1,t_2,t_3,t_4]}-1 &= \frac{1}{4}\left\langle\varepsilon X,\alpha\frac{E'_\Phi}{E_\Phi}+\beta\frac{DE_\Phi}{E_\Phi}\right\rangle^2(t_1)\\ &\quad+\frac{1}{2}\left\langle\varepsilon X\otimes\varepsilon X,\alpha^2\left(\frac{E'_\Phi}{E_\Phi}\right)'+2\alpha\beta\left(\frac{DE_\Phi}{E_\Phi}\right)'\right\rangle(t_1)\\ &\quad-2\left\langle\varepsilon X\otimes\varepsilon X,\alpha^2\frac{A}{6E_\Phi}+\alpha\beta\frac{B}{2E_\Phi}\right\rangle(t_1)+O(\varepsilon^3).\end{aligned}$$

Collecting the terms involving α^2 and $\alpha\beta$, we put the latter expression in a nicer form, namely

$$\begin{aligned}\frac{\Phi^*[t_1,t_2,t_3,t_4]}{[t_1,t_2,t_3,t_4]}-1 &= \left\langle\varepsilon X\otimes\varepsilon X,\alpha^2\left(\frac{E''_\Phi}{2E_\Phi}-\frac{A}{3E_\Phi}-\frac{1}{4}\left(\frac{E'_\Phi}{E_\Phi}\right)^2\right)\right\rangle(t_1)\\ &\quad+\left\langle\varepsilon X\otimes\varepsilon X,\alpha\beta\left(\frac{DE'_\Phi}{E_\Phi}-\frac{B}{E_\Phi}-\frac{E'_\Phi\,DE_\Phi}{2E_\Phi^2}\right)\right\rangle(t_1).\end{aligned}\tag{5.8}$$

Since $D\varphi=\psi D\psi$ and $E_\Phi=(D\psi)^2$, see Proposition 2.1, we calculate the terms A and B whose expression is given in Lemma 5.2; we find $A=\varphi'''+\psi\psi'''=(E_\Phi-\psi\psi')''+\psi\psi'''=E''_\Phi-\psi'\psi''$, together with $B=D\varphi''-\psi D\psi''=\psi''D\psi+2\psi'D\psi'=\frac{1}{2}D^3E_\Phi+\frac{1}{4}\frac{E'_\Phi\,DE_\Phi}{E_\Phi}$.

Plugging these quantities into (5.8), we obtain

$$\begin{aligned}\frac{\Phi^*[t_1,t_2,t_3,t_4]}{[t_1,t_2,t_3,t_4]}-1 &= \left\langle \varepsilon X\otimes\varepsilon X, \alpha^2\left(\frac{1}{6}\frac{E''_\Phi}{E_\Phi}-\frac{1}{4}\left(\frac{E'_\Phi}{E_\Phi}\right)^2+\frac{1}{3}\frac{\psi'\psi''}{E_\Phi}\right)\right\rangle(t_1)\\ &\quad+\left\langle \varepsilon X\otimes\varepsilon X, \alpha\beta\left(\frac{1}{2}\frac{DE'_\Phi}{E_\Phi}-\frac{3}{4}\frac{E'_\Phi\,DE_\Phi}{E_\Phi^2}\right)\right\rangle(t_1). \qquad (5.9)\end{aligned}$$

Upon defining

$$\widetilde{S}(\Phi)=\frac{DE'_\Phi}{E_\Phi}-\frac{3}{2}\frac{E'_\Phi\,DE_\Phi}{E_\Phi^2}, \qquad (5.10)$$

we find $D\widetilde{S}(\Phi)=\frac{E''_\Phi}{E_\Phi}-\frac{3}{2}\left(\frac{E'_\Phi}{E_\Phi}\right)^2-\frac{1}{2}\frac{DE'_\Phi\,DE_\Phi}{E_\Phi^2}$. We also have $DE'_\Phi\,DE_\Phi=-4\psi'\psi''E_\Phi$. Inserting the latter result into (5.9) and using the Cartan formula (3.8) to define the Schwarzian derivative, $\mathcal{S}(\Phi)$, of the contactomorphism Φ, we obtain

$$\mathcal{S}(\Phi)=\frac{1}{6}\alpha^2 D\widetilde{S}(\Phi)+\frac{1}{2}\alpha\beta\widetilde{S}(\Phi). \qquad (5.11)$$

Thus, $\mathcal{S}$ defines a 1-cocycle of $K(1)$ with values in the space, $\mathcal{Q}(\mathrm{S}^{1|1})$, of quadratic differentials, cf. Subsection 2.2.2. Using the projection onto the $\frac{3}{2}$-densities, $\mathcal{F}_{\frac{3}{2}}(\mathrm{S}^{1|1})$, given by $\alpha^{\frac{1}{2}}\langle D,\,\cdot\,\rangle$, see Proposition 2.5, we still obtain a projective 1-cocycle of $K(1)$, viz.,

$$\mathrm{S}(\Phi)=\alpha^{\frac{1}{2}}\,\langle D,\mathcal{S}(\Phi)\rangle=\frac{1}{4}\left(\frac{DE'_\Phi}{E_\Phi}-\frac{3}{2}\frac{E'_\Phi\,DE_\Phi}{E_\Phi^2}\right)\alpha^{3/2}. \qquad (5.12)$$

This ends the proof of (3.13).

Equation (3.11) can now be deduced from (5.11) and (5.12). Indeed, using

$$L_D\alpha=2\beta, \qquad (5.13)$$

we find $\alpha^{\frac{1}{2}}L_D\mathrm{S}(\Phi)=\frac{1}{4}\alpha^{\frac{1}{2}}L_D(\widetilde{S}(\Phi)\alpha^{\frac{3}{2}})=\frac{3}{2}\mathcal{S}(\Phi)$.

5.2.3 The kernels of the $K(1)$-cocycles $\mathcal{E},\mathcal{A},\mathcal{S}$

- The subgroup of those $\Phi\in K(1)$ such that $\mathcal{E}(\Phi)=0$ is characterized by the equation $E_\Phi=1$, see (3.9). Writing $\Phi=(\varphi,\psi)$, and using (2.13), we find $D\psi=\epsilon$, with $\epsilon^2=1$. This entails that $\psi(x,\xi)=\epsilon(\beta+\xi)$, with $\beta\in\mathbb{R}^{0|1}$. The constraint (2.12) then leads to $\varphi(x,\xi)=x+b-\beta\xi$, with $b\in\mathbb{R}$. This proves that $\ker(\mathcal{E})=\mathrm{E}(1|1)$.

- The kernel of the 1-cocycle $\mathcal{A}$, given by (3.10), is determined by the equation $E_\Phi = a^2$, with $a \in \mathbb{R}^*$. The kernel of A is given by the same equation, hence is equal to the kernel of $\mathcal{A}$. The same computation as before clearly leads to $\Phi(x,\xi) = (a^2x + a^2b - a^2\beta\xi, a\beta + a\xi)$. Hence, $\ker(\mathrm{A}) = \ker(\mathcal{A}) = \mathrm{Aff}(1|1)$.

- The kernels of the 1-cocycles $\mathcal{S}$ and S, given respectively by (3.11), and (3.13), clearly coincide. Suffice it to determine ker(S). Let us consider $\Phi \in K(1)$, then its Schwarzian derivative (3.13) reads alternatively

$$\mathrm{S}(\Phi) = -\frac{1}{2} E_\Phi^{\frac{1}{2}}\, D^3(E_\Phi^{-\frac{1}{2}})\, \alpha^{3/2}. \tag{5.14}$$

Hence, $\mathrm{S}(\Phi) = 0$ iff $\partial_x D(E_\Phi^{-\frac{1}{2}}) = 0$. As $\partial_x D\chi_0 = 0$ implies, for χ_0 an even super-function, $\chi_0 = c'x + d' + \delta'\xi$, where $(c', d', \delta') \in \mathbb{R}^{2|1}$; we obtain $E_\Phi = (c'x + d' + \delta'\xi)^{-2}$. Consider now $h \in \mathrm{SpO}_+(2|1)$, whose action is given by (2.28), then $E_{\widehat{h}} = (cx+d+\delta\xi)^{-2}$. We thus have $E_\Phi = E_{\widehat{h}}$ for some $h \in \mathrm{SpO}_+(2|1)$, so that $\Phi = \widehat{h} \circ \widehat{g}$ with $g \in \mathrm{E}(1|1)$ in view of the above result; this implies that $\Phi \in \mathrm{SpO}_+(2|1)$. The conclusion, $\ker(\mathrm{S}) = \mathrm{SpO}_+(2|1)$, easily follows.

The proof of Theorem 3.5 is complete.

5.3 Proof of Proposition 3.8

With the help of (2.10), the affine cocycle $\mathcal{A}$, given by (3.10), can be recast into the form $\mathcal{A}(\Phi) = E_\Phi^{-1} dE_\Phi = \alpha E_\Phi^{-1} E'_\Phi + \beta E_\Phi^{-1} DE_\Phi$. Using Equation (5.13), we obtain $(\alpha^{\frac{1}{2}} L_D)^2 = \alpha L_{D^2} + \beta L_D$.

Straightforward calculation yields the expressions of $L_D\mathcal{A}$, $L_{D^2}\mathcal{A}$, and $\mathcal{A}^2$, so that

$$(\alpha^{\frac{1}{2}} L_D)^2 \mathcal{A}(\Phi) - \frac{1}{2}\mathcal{A}(\Phi)^2 = \alpha^2 \left(D\widetilde{S}(\Phi) - \frac{1}{2}\frac{DE_\Phi DE'_\Phi}{E_\Phi^2} \right) + 2\alpha\beta\widetilde{S}(\Phi). \tag{5.15}$$

This formula leads directly to (3.18), using $\langle D, \alpha^2 \rangle = 0$ and $\langle D, \alpha\beta \rangle = \frac{1}{2}\alpha$, together with the expressions (5.10) and (5.12) for $\widetilde{S}$ and S.

6 Super Euclidean, affine and projective invariants, and $K(N)$-cocycles for $\mathrm{S}^{1|N}$

The aim of this section is to extend to $\mathrm{S}^{1|N}$ the previous constructions, namely those of the Euclidean, affine and projective invariants, of the Euclidean and affine cocycles,

and of the Schwarzian derivative for $N = 2$. For $N \geq 3$, the cross-ratio is badly transformed by contactomorphisms, which prevents the construction of a Schwarzian derivative along the same lines as before (see Remark 6.6 below).

Let us define the notation used throughout this section. Except if otherwise stated, all indices i, j of odd objects will run from 1 to N, and Einstein's summation convention will be freely used. The space of superfunctions $\mathcal{C}^\infty(\mathrm{S}^{1|N})$, defining $\mathrm{S}^{1|N}$, is the superalgebra $\mathcal{C}^\infty(\mathrm{S}^1)[\xi^1, \ldots, \xi^N]$ where the ξ^i are odd indeterminates. It is topologically generated, as an algebra, by the coordinates (x, ξ) with $\xi = (\xi^1, \ldots, \xi^N)$. The diffeomorphisms retain the form $\Phi = (\varphi, \psi)$, with $\psi = (\psi^1, \ldots, \psi^N)$ and $\varphi, \psi^j \in \mathcal{C}^\infty(\mathrm{S}^{1|N})$, such that (φ, ψ) is a new coordinate system. Let $F, G \in \mathcal{C}^\infty(\mathrm{S}^{1|N})^N$, we denote their pairing with values in $\mathcal{C}^\infty(\mathrm{S}^{1|N})$ by

$$F \cdot G = F_i G^i \tag{6.1}$$

where F^i and G^i are the i-th components of F and G, and $F_i = \delta_{ij} F^j$ (with the choice of an Euclidean signature). The $\mathcal{C}^\infty(\mathrm{S}^{1|N})$-module $\Omega^1(\mathrm{S}^{1|N})$ is generated by the 1-forms

$$\alpha = dx + \xi_i d\xi^i = dx + \xi \cdot d\xi \qquad \text{and} \qquad \beta^i = d\xi^i, \tag{6.2}$$

with dual vectors ∂_x and $D_i = \partial_{\xi_i} + \xi_i \partial_x$. For $f \in \mathcal{C}^\infty(\mathrm{S}^{1|N})$ we therefore have

$$df = \alpha f' + \beta^i D_i f, \tag{6.3}$$

see (2.10). We furthermore denote by $K(N)$ the group of contactomorphisms, Φ, characterized by $\Phi^*\alpha = E_\Phi\, \alpha$ for some superfunction E_Φ. Let $\Phi = (\varphi, \psi) \in K(N)$, then $\Phi^*\alpha = d\varphi + \psi \cdot d\psi = \alpha(\varphi' + \psi \cdot \psi') + \beta^i(D_i\varphi - \psi \cdot D_i\psi)$. It follows that $\Phi \in K(N)$ iff

$$D_i\varphi - \psi \cdot D_i\psi = 0, \tag{6.4}$$

for all $i = 1, \ldots, N$. The multiplier of Φ is then given by $E_\Phi = \varphi' + \psi \cdot \psi'$, i.e., by

$$E_\Phi = \frac{\Phi^*\alpha}{\alpha} = (D_i\psi)^2 \tag{6.5}$$

for any $i = 1, \ldots, N$. The expression $(D_i\psi)^2$ stands for $D_i\psi \cdot D_i\psi$. This has been first developed in the framework of super Riemann surfaces by Cohn [6]; we will nevertheless refer to work of Radul [29], whose geometric approach, in terms of contact structure, is closer to our viewpoint. See also [18].

Proposition 6.1. *Let $\Phi \in K(N)$, then*

$$D_j D_i \varphi + \psi \cdot D_j D_i \psi = D_i \psi \cdot D_j \psi = E_\Phi \delta_{ij}. \tag{6.6}$$

Hence $(E_\Phi^{-\frac{1}{2}} D_i \psi)_{i=1,\dots,N}$ is an "orthonormal basis" for the pairing (6.1) on $\mathcal{C}^\infty(\mathrm{S}^{1|N})^N$.

Proof. As $\Phi \in K(N)$, we have $D_j D_i \varphi = D_j(\psi \cdot D_i \psi) = D_j \psi \cdot D_i \psi - \psi \cdot D_j D_i \psi$, in view of (6.4); by exchanging i and j, we deduce $D_i \psi \cdot D_j \psi = 0$ if $i \neq j$. For $i = j$, the result is given by (6.5) and the equality $E_\Phi = \varphi' + \psi \cdot \psi'$. □

6.1 Euclidean, affine and projective invariants

We now extend to $\mathrm{S}^{1|N}$, where $N \geq 2$, the content of Subsection 2.3. Now, α (6.2) stems from the 1-form on $\mathbb{R}^{2|N}$ given by $\varpi = \frac{1}{2}(pdq - qdp + \theta_i d\theta^i)$, via the formula $\varpi = \frac{1}{2}p^2\alpha$ $(p \neq 0)$, expressed in affine coordinates $x = q/p$ and $\xi^i = \theta^i/p$. We define the orthosymplectic group [17, 23], $\mathrm{SpO}(2|N)$, as the supergroup whose A-points are all linear transformations of $\mathcal{O}_A^{2|N}$, see Subsection 2.3,

$$h = \begin{pmatrix} a & b & \gamma \\ c & d & \delta \\ \alpha & \beta & e \end{pmatrix} \tag{6.7}$$

preserving the symplectic form $d\varpi$. If we demand that these linear transformations preserve the direction of $d\varpi$, only, we end up with the conformal supergroup $\mathrm{C}(2|N)$, see [23]. In the expression (6.7), the entries a, b, c, d are even elements, α, β are odd column vectors of size N, while δ, γ are odd row vectors of size N, and e is an even matrix of size $N \times N$. Moreover, as $d\varpi$ is preserved, we have

$$ad - bc - \alpha^t \beta = 1, \tag{6.8}$$

$$e^t e + 2\gamma^t \delta = 1, \tag{6.9}$$

$$\alpha^t e - a\delta + c\gamma = 0, \tag{6.10}$$

$$\beta^t e - b\delta + d\gamma = 0, \tag{6.11}$$

where the superscript t denotes transposition. We easily find that $\mathrm{SpO}(2|N)$ also preserves ϖ. Again, since $\varpi = \frac{1}{2}p^2\alpha$, the orthosymplectic group acts by contactomorphisms, $\mathrm{SpO}(2|N) \to K(N)$, via the following projective action on $\mathrm{S}^{1|N}$, namely

$$\widehat{h}(x, \xi) = \left(\frac{ax + b + \gamma\xi}{cx + d + \delta\xi}, \frac{\alpha x + \beta + e\xi}{cx + d + \delta\xi} \right), \tag{6.12}$$

where $h \in \mathrm{SpO}(2|N)$, and ξ is understood as a column vector.

The kernel of this action is $\{\mathrm{Id}, -\mathrm{Id}\}$, hence the action is effective for the supergroup $\mathrm{SpO}(2|N)/\{\pm\mathrm{Id}\} = \mathrm{PC}(2|N)$ of conformal projective transformations. If N is odd, this supergroup coincides with the special orthosymplectic group $\mathrm{SpO}_+(2|N)$, which is the subgroup of $\mathrm{SpO}(2|N)$ of Berezinian 1. We still can define Euclidean and affine subgroups of $\mathrm{SpO}(2|N)$, whose elements are

$$g = \begin{pmatrix} a & ab & -a\beta^t \\ 0 & a^{-1} & 0 \\ 0 & \beta & 1 \end{pmatrix}, \tag{6.13}$$

where $(a, b, \beta) \in \mathbb{R}^{2|N}$, $a > 0$ defining $\mathrm{Aff}_+(1|N)$ and $a = 1$ defining $\mathrm{E}_+(1|N)$.

Remark 6.2. The group $\mathrm{Aff}_+(1|N)$ may be defined as the subgroup of those $\widehat{h} \in K(N)$ that preserve the direction of each β_i, namely $\widehat{h}^*\beta_i = \beta_i f_i$, for some superfunction f_i, with $i = 1, \ldots, N$. Its subgroup $\mathrm{E}_+(1|N)$ is characterized by exactly preserving α.

Let t_1, t_2, t_3, t_4 be four generic points of $\mathrm{S}^{1|N}$.

Theorem 6.3. *We have three invariants, I_{e}, I_{a} and I_{p}, of the action of the Euclidean, affine and projective supergroups on the supercircle $\mathrm{S}^{1|N}$.*

- *Euclidean invariant: $I_{\mathsf{e}}(t_1, t_2) = ([t_1, t_2], \{t_1, t_2\})$ with*

$$[t_1, t_2] = x_2 - x_1 - \xi_2 \cdot \xi_1, \qquad \{t_1, t_2\} = \xi_2 - \xi_1. \tag{6.14}$$

- *Affine invariant: $I_{\mathsf{a}}(t_1, t_2, t_3) = ([t_1, t_2, t_3], \{t_1, t_2, t_3\})$, where, if $x_1 < x_2$,*

$$[t_1, t_2, t_3] = \frac{[t_1, t_3]}{[t_1, t_2]}, \qquad \{t_1, t_2, t_3\} = \frac{\{t_1, t_3\}}{[t_1, t_2]^{\frac{1}{2}}}. \tag{6.15}$$

- *Projective invariant: $I_{\mathsf{p}}(t_1, t_2, t_3, t_4) = ([t_1, t_2, t_3, t_4], \mathrm{O}(N).\{t_1, t_2, t_3, t_4\})$, where, if $\mathrm{ord}(t_1, t_2, t_3) = 1$,*

$$[t_1, t_2, t_3, t_4] = \frac{[t_1, t_3][t_2, t_4]}{[t_2, t_3][t_1, t_4]}, \tag{6.16}$$

$$\{t_1, t_2, t_3, t_4\} = [t_1, t_2, t_3, t_4]^{\frac{1}{2}} \frac{\{t_2, t_4\}[t_1, t_2] - \{t_1, t_2\}[t_2, t_4]}{([t_1, t_2][t_2, t_4][t_1, t_4])^{\frac{1}{2}}}. \tag{6.17}$$

The odd invariant, denoted by $\mathrm{O}(N).\{t_1,t_2,t_3,t_4\}$*, is the orbit of* $\{t_1,t_2,t_3,t_4\}$ *under the natural group action of* $\mathrm{O}(N)$*. If a bijective transformation,* Φ*, of* $\mathrm{S}^{1|N}$ *leaves* I_{e} *(resp.* I_{a}*) invariant, it can be identified with the action of an element* $h \in \mathrm{E}_+(1|N)$ *(resp.* $h \in \mathrm{Aff}_+(1|N)$*), i.e.,* $\Phi = \widehat{h}$*. If* $\Phi \in \mathrm{Diff}(\mathrm{S}^{1|N})$ *preserves* I_{p}*, then* $\Phi = \rho \circ \widehat{h}$*, with* $h \in \mathrm{SpO}(2|N)$ *and* $\rho(x,\xi) = (x, R\xi)$*,* $R \in \mathcal{C}^\infty(\mathrm{S}^{1|N}, \mathrm{O}(N))$*.*

The proof of this Theorem can be carried out along the same lines as in the proof of Theorem 3.1, we will skip it and just provide some hints for it. As in the case $N = 1$, we can show that the action of $\mathrm{E}_+(1|N)$ is simply $1|1$-transitive, while that of $\widetilde{\mathrm{Aff}_+(1|N)}$ is simply $2|1$-transitive, on $\mathbb{R}^{1|N} \subset \mathrm{S}^{1|N}$. Moreover, the action of $\widetilde{\mathrm{PC}(2|N)}$ is $3|2$-transitive on $\mathrm{S}^{1|N}$ and satisfies the following property: for any triple t, and $g, h \in \widetilde{\mathrm{PC}(2|N)}$, $\widehat{g}(t) \overset{3|2}{=} p \overset{3|2}{=} \widehat{h}(t)$ is equivalent to $\widehat{g} = \widehat{k} \circ \widehat{h}$, with $\widehat{k}(x,\xi) = (x, e\xi)$, $e \in \mathrm{O}(N)$. As in Section 4, the tilde denotes the extension of the group by the involution $\iota : (x,\xi) \mapsto (-x,\xi)$. We can now apply Theorem 4.3 and the claims of Theorem 6.3 follow.

Remark 6.4. For $N = 1$, the Corollaries 4.10, 4.17 and 4.20 have been obtained thanks to Lemma 2.3. They cannot be prolonged for $N > 1$ as there exists no such lemma in this case. However, the supergroup preserving each even invariant is included in the kernel of the associated $K(N)$-cocycles, and for $N = 2$, see Remark 6.9 and Theorem 6.10, one can easily check the converse inclusion. So, for $N = 2$, the preserving supergroups of the even part of I_{e}, I_{a} and I_{p} are respectively $\mathrm{EO}(1|2)/\{\pm\mathrm{Id}\}$, $\mathrm{AO}(1|2)/\{\pm\mathrm{Id}\}$ and $\mathrm{PC}(2|2)$.

6.2 Associated cocycles from the Cartan formula

The following calculation will rely on Proposition 6.1, and on the relation $[D_i, D_j] = D_iD_j + D_jD_i = 2\delta_{ij}\partial_x$, for $i,j = 1,\dots,N$, which results from a direct calculation. As in the case $N = 1$, we need a lemma giving the third-order Taylor expansion of $\Phi^*[t_1,t_2]$. To that end, we will be using the notation:

$$\beta^i\beta^j = \frac{1}{2}(\beta^i \otimes \beta^j - \beta^j \otimes \beta^i), \tag{6.18}$$

and $\beta^i\beta^j\beta^k = \frac{1}{6}(\sum_{\sigma\in\mathfrak{S}_3} \varepsilon(\sigma)\beta^{\sigma(i)} \otimes \beta^{\sigma(j)} \otimes \beta^{\sigma(k)})$, i.e., the symmetrized tensor product of odd elements; see [7].

Lemma 6.5. *Let $\Phi = (\varphi, \psi) \in K(N)$, and $t_2 = \phi_\varepsilon(t_1)$, with ϕ_ε the flow of a vector field X, and t_1 a point of $\mathrm{S}^{1|N}$; we then have*

$$\begin{aligned}\Phi^*[t_1,t_2] &= [t_1,t_2]E_\Phi(t_1)\left(1+\frac{1}{2}[t_1,t_2]\frac{E'_\Phi}{E_\Phi}(t_1)+\frac{1}{2}\{t_1,t_2\}^i\frac{D_iE_\Phi}{E_\Phi}(t_1)\right) \\ &\quad +[t_1,t_2]\left\langle \varepsilon X\otimes\varepsilon X, \alpha^2\frac{A}{6}+\alpha\beta^i\frac{B_i}{2}+\beta^i\beta^j\frac{C_{ij}}{2}\right\rangle(t_1) \\ &\quad +\frac{1}{6}\left\langle \varepsilon X\otimes\varepsilon X\otimes\varepsilon X, \beta^i\beta^j\beta^k\left[D_kD_jD_i\varphi-\psi\cdot D_kD_jD_i\psi\right]\right\rangle(t_1)+O(\varepsilon^4),\end{aligned} \tag{6.19}$$

where $A = \varphi''' + \psi\cdot\psi'''$, $B_i = D_i\varphi'' - \psi\cdot D_i\psi''$ and $C_{ij} = D_jD_i\varphi' + \psi\cdot D_jD_i\psi'$.

Proof. By definition we have: $\Phi^*[t_1,t_2] = \varphi(t_2)-\varphi(t_1)-(\psi(t_2)-\psi(t_1))\cdot\psi(t_1)$. Using the formula (5.1), trivially extended to the case $N \geq 2$, we obtain

$$\begin{aligned}\Phi^*[t_1,t_2] &= [t_1,t_2]\left[E_\Phi+\frac{1}{2}[t_1,t_2](\varphi''+\psi\cdot\psi'')+\frac{1}{2}\{t_1,t_2\}^iD_i(\varphi'+\psi\cdot\psi')\right](t_1) \\ &\quad +\frac{1}{2}\{t_1,t_2\}^i\{t_1,t_2\}^j\left[D_jD_i\varphi+\psi\cdot D_jD_i\psi\right](t_1) \\ &\quad +[t_1,t_2]\left[\frac{1}{6}[t_1,t_2]^2(\varphi'''+\psi\cdot\psi''')+\frac{1}{2}\{t_1,t_2\}^i[t_1,t_2](D_i\varphi''-\psi\cdot D_i\psi'')\right](t_1) \\ &\quad +\frac{1}{2}[t_1,t_2]\{t_1,t_2\}^i\{t_1,t_2\}^j\left[D_jD_i\varphi'+\psi\cdot D_jD_i\psi')\right](t_1) \\ &\quad +\frac{1}{6}\{t_1,t_2\}^i\{t_1,t_2\}^j\{t_1,t_2\}^k\left[D_kD_jD_i\varphi-\psi\cdot D_kD_jD_i\psi\right](t_1) \\ &\quad +O(\varepsilon^4).\end{aligned} \tag{6.20}$$

The coefficient of $\{t_1,t_2\}^i\{t_1,t_2\}^j$ on the second line of (6.20) vanishes if $i \neq j$, using (6.6). The analog of Lemma 5.1 holds true, namely $[t_1,t_2] = \langle \varepsilon X, \alpha\rangle + O(\varepsilon^2)$, and $\{t_1,t_2\}^i = \langle \varepsilon X, \beta^i\rangle + O(\varepsilon^2)$. Hence, we are done. □

Remark 6.6. The last term in (6.20) definitely does not vanish in the case $N \geq 3$, implying that $\Phi^*[t_1,t_2]$ is not proportional to $[t_1,t_2]$ at third order in ε. This entails that the Cartan formula fails to provide an expression of the Schwarzian derivative for $N \geq 3$.

6.2.1 Euclidean and affine $K(N)$-cocycles

Up to the second order in ε, $\Phi^*[t_1, t_2]$ is proportional to $[t_1, t_2]$; this enables us to obtain 1-cocycles from Euclidean and affine invariants, as was done in Subsection 5.2.1.

Theorem 6.7. *From the Euclidean and affine even invariants, we construct the two following $K(N)$ nontrivial 1-cocycles:*

- *The Euclidean cocycle $\mathcal{E} : K(N) \to \mathcal{F}_0(\mathrm{S}^{1|N})$:*

$$\mathcal{E}(\Phi) = \log(E_\Phi) = \log(D_i\psi)^2, \tag{6.21}$$

  where the equality holds for any $i = 1, \ldots, N$.

- *The affine cocycle $\mathcal{A} : K(N) \to \Omega^1(\mathrm{S}^{1|N})$:*

$$\mathcal{A}(\Phi) = d\mathcal{E}(\Phi) = \frac{dE_\Phi}{E_\Phi}. \tag{6.22}$$

The proof is the same as in the case $N = 1$, it relies on Lemma 6.5.

Remark 6.8. The directions of the individual vector fields D_i are no longer preserved by the contactomorphisms; only that of $D_1 \otimes \cdots \otimes D_N$ is preserved. Hence, the projection of $\mathcal{A}$ on D_i is no longer a $K(N)$-cocycle.

Let us introduce $\mathrm{AO}(1|N)$, the ortho-affine subgroup of $\mathrm{SpO}(2|N)$ whose elements are

$$g = \begin{pmatrix} a & ab & -a\beta^t \\ 0 & a^{-1} & 0 \\ 0 & \beta & e \end{pmatrix} \tag{6.23}$$

where $(a, b, \beta) \in \mathbb{R}^{2|N}$, $e \in \mathrm{O}(N)$, and restricting us to $a = \pm 1$, we obtain the ortho-Euclidean subgroup $\mathrm{EO}(1|N)$. Since the action of $\mathrm{SpO}(2|N)$ on the supercircle has a kernel equal to $\{\pm\mathrm{Id}\}$, the same holds for its above introduced subgroups.

Remark 6.9. For $N = 2$, a direct computation shows that the kernel of the two cocycles $\mathcal{E}$ and $\mathcal{A}$, are, respectively, $\mathrm{EO}(1|2)/\{\pm\mathrm{Id}\}$ and $\mathrm{AO}(1|2)/\{\pm\mathrm{Id}\}$. This groups are also the groups preserving the even part of I_{e} and I_{a}, see Remark 6.4. But for $N \geq 3$, this is no longer the case, i.e., the subgroup of $K(N)$ preserving the even invariant

and the kernel of the associated cocycle are no longer the same defining groups. For example, if $N = 3$, the contactomorphism $\Phi = (\varphi, \psi)$, with $\varphi(x, \xi) = x + \xi_1\xi_2\xi_3\lambda$ and $\psi(x, \xi) = \xi - (\xi_2\xi_3, \xi_3\xi_1, \xi_1\xi_2)\lambda$, where $\lambda \in \mathbb{R}^{0|1}$, does not preserve $p_0(I_{\mathsf{e}})$ although $E(\Phi) = 1$. Moreover, Φ is not even an homography.

6.2.2 The Schwarzian $K(2)$-cocycle

For $N = 2$, the expression (6.5) of $\Phi^*[t_1, t_2]$ is proportional to $[t_1, t_2]$. This enables us to use the Cartan formula to define the projective 1-cocycle, $\mathcal{S}$, from the cross-ratio (6.16). By construction, our projective 1-cocycle will take its values in the $K(2)$-module of quadratic differentials, $\mathcal{Q}(\mathrm{S}^{1|2})$, generated by α^2, $\alpha\beta^1$, $\alpha\beta^2$ and $\beta^1\beta^2$, where α^2 and $\alpha\beta^i$ are as in (2.20), and $\beta^1\beta^2$ as in (6.18). One can check that the linear mapping

$$\alpha\langle D_2 \otimes D_1, .\rangle : \mathcal{Q}(\mathrm{S}^{1|2}) \to \mathcal{F}_1(\mathrm{S}^{1|2}) \tag{6.24}$$

intertwines the natural action of $K(2)$, see Remark 6.8.

Now, the Schwarzian derivative given by Radul [29], or Cohn [6], for $N = 2$, has again coefficients in tensor densities. Projecting the 1-cocycle, $\mathcal{S}$, via (6.24), we will readily recover Radul's and Cohn's Schwarzian derivative.

Theorem 6.10. *From the cross-ratio (6.16), we deduce, via the Cartan formula (3.8), the following projective* 1*-cocycle* $\mathcal{S} : K(2) \to \mathcal{Q}(\mathrm{S}^{1|2})$*, which reads*

$$\mathcal{S} = \frac{1}{6}\alpha^2 \left(D_1 D_2 \mathrm{S}_{12} + \frac{1}{2}\mathrm{S}_{12}^2 \right) + \frac{1}{2}\alpha(\beta^1 D_2 + \beta^2 D_1)\mathrm{S}_{12} + \beta^1\beta^2\, \mathrm{S}_{12}, \tag{6.25}$$

where we have put $\mathrm{S}_{12} = 2\,\mathrm{S}\,\alpha^{-1}$*, see (6.26).*

Moreover, using the projection (6.24) of the quadratic differentials on 1*-densities, we obtain the Schwarzian derivative* $\mathrm{S} : K(2) \to \mathcal{F}_1(\mathrm{S}^{1|2})$ *given by*

$$\mathrm{S}(\Phi) = \left(\frac{D_2 D_1 E_\Phi}{E_\Phi} - \frac{3}{2}\frac{D_2 E_\Phi D_1 E_\Phi}{E_\Phi^2} \right) \alpha. \tag{6.26}$$

These two 1*-cocycles are nontrivial; their kernels coincide and are isomorphic to* $\mathrm{PC}(2|2)$*.*

Proof. The formula of the cross-ratio being similar to that of the case $N = 1$, we have to compute, again, the expression (5.6), the term $\frac{\Phi^*[t_1,t_2]}{E_\Phi(t_1)[t_1,t_2]}$ being now given by

Lemma 6.5. Straightforward calculation, essentially the same as in Subsection 5.2.2, leads to

$$\begin{aligned}
\frac{\Phi^*[t_2,t_3]}{E_\Phi(t_2)[t_2,t_3]} &= 1+\frac{1}{2}\left([t_2,t_3]\frac{E'_\Phi}{E_\Phi}(t_1)+\{t_2,t_3\}^i\frac{D_iE_\Phi}{E_\Phi}(t_1)\right)\\
&+\frac{1}{2}\left\langle \varepsilon X\otimes\varepsilon X,\alpha^2\left(\frac{E'_\Phi}{E_\Phi}\right)'+2\alpha\beta^i\left(\frac{D_iE_\Phi}{E_\Phi}\right)'+\beta^i\beta^jD_j\left(\frac{D_iE_\Phi}{E_\Phi}\right)\right\rangle(t_1)\\
&+\left\langle \varepsilon X\otimes\varepsilon X,\alpha^2\frac{A}{6E_\Phi}+\alpha\beta^i\frac{B_i}{2E_\Phi}+\beta^i\beta^j\frac{C_{ij}}{2E_\Phi}\right\rangle(t_1)+O(\varepsilon^3).
\end{aligned}$$

The combinatorics is the same as before; we thus obtain

$$\begin{aligned}
\frac{\Phi^*[t_1,t_2,t_3,t_4]}{[t_1,t_2,t_3,t_4]}-1 &= \frac{1}{4}\left\langle \varepsilon X,\alpha\frac{E'_\Phi}{E_\Phi}+\beta^i\frac{D_iE_\Phi}{E_\Phi}\right\rangle^2(t_1)\\
&+\frac{1}{2}\left\langle \varepsilon X\otimes\varepsilon X,\alpha^2\left(\frac{E'_\Phi}{E_\Phi}\right)'+2\alpha\beta^i\left(\frac{D_iE_\Phi}{E_\Phi}\right)'+\beta^i\beta^jD_j\left(\frac{D_iE_\Phi}{E_\Phi}\right)\right\rangle(t_1)\\
&-2\left\langle \varepsilon X\otimes\varepsilon X,\alpha^2\frac{A}{6E_\Phi}+\alpha\beta^i\frac{B_i}{2E_\Phi}+\beta^i\beta^j\frac{C_{ij}}{2E_\Phi}\right\rangle(t_1)+O(\varepsilon^3).
\end{aligned}$$

As in the case $N=1$, see (5.8) and (5.9), we still have $A=E''_\Phi-\psi'\cdot\psi''$ and also $B_i=\frac{1}{2}D_iE'_\Phi+\psi'\cdot D_i\psi'$. We now collect the terms according to

$$\begin{aligned}
\frac{\Phi^*[t_1,t_2,t_3,t_4]}{[t_1,t_2,t_3,t_4]}-1 &= \left\langle \varepsilon X\otimes\varepsilon X,\alpha^2\left(\frac{1}{6}\frac{E''_\Phi}{E_\Phi}+\frac{\psi'\cdot\psi''}{3E_\Phi}-\frac{1}{4}\left(\frac{E'_\Phi}{E_\Phi}\right)^2\right)\right\rangle(t_1)\\
&+\left\langle \varepsilon X\otimes\varepsilon X,\alpha\beta^i\left(\frac{1}{2}\frac{D_iE'_\Phi}{E_\Phi}-\frac{\psi'\cdot D_i\psi'}{E_\Phi}-\frac{E'_\Phi D_iE_\Phi}{2E_\Phi^2}\right)\right\rangle(t_1)\\
&+\left\langle \varepsilon X\otimes\varepsilon X,\beta^1\beta^2\left(\frac{D_2D_1E_\Phi}{E_\Phi}-\frac{2C_{12}}{E_\Phi}-\frac{D_2E_\Phi D_1E_\Phi}{2E_\Phi^2}\right)\right\rangle(t_1)\\
&+O(\varepsilon^3). \qquad (6.27)
\end{aligned}$$

We denote by $S(\Phi)$ the coefficient of α^2, $S_i(\Phi)$ that of $\alpha\beta^i$ and $S_{12}(\Phi)$ that of $\beta^1\beta^2$. Let us start with the computation of $S_{12}(\Phi)$, our goal being to write C_{12} as a function of E_Φ and its derivatives.

We first give a useful lemma.

Lemma 6.11. *For any $\phi, \widetilde{\phi} \in \mathcal{C}^\infty(\mathrm{S}^{1|2})^2$, the following relations hold:*

$$(\phi \cdot D_1\psi)(\widetilde{\phi} \cdot D_1\psi) + (\phi \cdot D_2\psi)(\widetilde{\phi} \cdot D_2\psi) = \phi \cdot \widetilde{\phi}\, E_\Phi, \tag{6.28}$$

and also,

$$\phi \times D_2\psi = \lambda\phi \cdot D_1\psi \qquad \text{and} \qquad \phi \times D_1\psi = -\lambda\phi \cdot D_2\psi, \tag{6.29}$$

with $\lambda^2 = 1$, and where the cross-product is defined by $\phi \times \widetilde{\phi} = \phi_1\widetilde{\phi}_2 - \phi_2\widetilde{\phi}_1$. Moreover, ψ' being odd, for even ϕ and $\widetilde{\phi}$, we have

$$(\psi' \cdot \phi)(\psi' \cdot \widetilde{\phi}) = \psi'_1\psi'_2(\phi \times \widetilde{\phi}). \tag{6.30}$$

Proof. Proposition 6.1 proves the first equality. As $D_2\psi \cdot D_2\psi = E_\Phi$, either $D_2\psi_1$ or $D_2\psi_2$ is invertible, where $\psi = (\psi_1, \psi_2)$. Suppose that $D_2\psi_2$ is invertible, we then have $D_1\psi_1 = \lambda D_2\psi_2$ for some λ. Using $D_i\psi \cdot D_j\psi = \delta_{ij}E_\Phi$, we obtain $D_1\psi_2 = -\lambda D_2\psi_1$ and $\lambda^2 = 1$. If $D_2\psi_1$ is invertible the same equalities hold. Then, easy calculation ends the proof. □

We have $C_{12} = D_2D_1\varphi' + \psi \cdot D_2D_1\psi'$, as given by Lemma 6.5. Differentiating the constraint $D_i\varphi = \psi \cdot D_i\psi$, see (6.4), we find $D_1\varphi' = \psi' \cdot D_1\psi + \psi \cdot D_1\psi'$, and then $D_2D_1\varphi' = D_2\psi' \cdot D_1\psi - \psi' \cdot D_2D_1\psi + D_2\psi \cdot D_1\psi' - \psi \cdot D_2D_1\psi'$. Plugging the latter expression into C_{12}, and using $D_i\psi \cdot D_j\psi = 0$, for $i \neq j$, we obtain $C_{12} = -\psi' \cdot D_2D_1\psi$. Using the proof of Lemma 6.11, we have $D_1\psi_1 = \lambda D_2\psi_2$ and $D_1\psi_2 = -\lambda D_2\psi_1$, and then $\psi' \cdot D_2D_1\psi = 2\lambda\psi'_1\psi'_2$. Moreover, as $\frac{1}{4}D_1E_\Phi D_2E_\Phi = (\psi' \cdot D_1\psi)(\psi' \cdot D_2\psi)$, we find, using (6.30) and (6.29), $\frac{1}{4}D_1E_\Phi D_2E_\Phi = \lambda\psi'_1\psi'_2E_\Phi$. We thus have $C_{12} = -\frac{1}{2E_\Phi}D_1E_\Phi D_2E_\Phi$, and replacing this in the last expression of S_{12}, as given by (6.27), we finally get

$$S_{12}(\Phi) = \frac{D_2D_1E_\Phi}{E_\Phi} - \frac{3}{2}\frac{D_2E_\Phi D_1E_\Phi}{E_\Phi^2}. \tag{6.31}$$

We will show that $S_1 = \frac{1}{2}D_2S_{12}$, and then, exchanging D_1 and D_2, readily obtain $S_2 = -\frac{1}{2}D_1S_{12}$. Let us first recall the expression of S_1, given in (6.27),

$$S_1(\Phi) = \frac{1}{2}\left(\frac{D_1E'_\Phi}{E_\Phi} - \frac{E'_\Phi D_1E_\Phi}{E_\Phi^2} - \frac{2\psi' \cdot D_1\psi'}{E_\Phi}\right).$$

Secondly, we find

$$D_2S_{12}(\Phi) = \frac{D_1E'_\Phi}{E_\Phi} - \frac{D_2E_\Phi D_2D_1E_\Phi}{E_\Phi^2} + \frac{3}{2}\frac{D_2E_\Phi D_2D_1E_\Phi}{E_\Phi^2} - \frac{3}{2}\frac{E'_\Phi D_1E_\Phi}{E_\Phi^2}.$$

We then have to show that the following expression vanishes, namely

$$2S_1(\Phi) - D_2S_{12}(\Phi) = -\frac{2\psi' \cdot D_1\psi'}{E_\Phi} + \frac{1}{2}\frac{D_2E_\Phi D_1D_2E_\Phi}{E_\Phi^2} + \frac{1}{2}\frac{E'_\Phi D_1E_\Phi}{E_\Phi^2}. \tag{6.32}$$

To that end, let us use Formula (6.5) to rewrite the last two terms as $D_2E_\Phi D_1D_2E_\Phi = 4(\psi' \cdot D_2\psi)(D_1\psi' \cdot D_2\psi + \psi' \cdot D_2D_1\psi)$, and $E'_\Phi D_1E_\Phi = 4(\psi' \cdot D_1\psi)(D_1\psi' \cdot D_1\psi)$, respectively. We have already proved that $\psi' \cdot D_2D_1\psi = 2\lambda\psi'_1\psi'_2$, and using (6.28), we thus obtain

$$2S_1 - D_2S_{12} = 0. \tag{6.33}$$

At last, we want to show that $6S = D_1D_2S_{12} + \frac{1}{2}S_{12}^2$. Begin by writing explicitly

$$6S(\Phi) = \frac{E''_\Phi}{E_\Phi} - \frac{3}{2}\left(\frac{E'_\Phi}{E_\Phi}\right)^2 + \frac{2\psi' \cdot \psi''}{E_\Phi},$$

with the help of 6.27, and also

$$\begin{aligned} D_1D_2S_{12}(\Phi) &= \frac{E''_\Phi}{E_\Phi} - \frac{D_1E_\Phi D_1E'_\Phi}{E_\Phi^2} - \frac{3}{2}\left(\frac{E'_\Phi}{E_\Phi}\right)^2 - \frac{3}{2}\frac{D_1E'_\Phi D_1E_\Phi}{E_\Phi^2} \\ &\quad + \frac{1}{2}\frac{D_2E_\Phi D_2E'_\Phi}{E_\Phi^2} - \frac{1}{2}\left(\frac{D_2D_1E_\Phi}{E_\Phi}\right)^2 - \frac{D_1E_\Phi D_2E_\Phi D_2D_1E_\Phi}{E_\Phi^3}. \end{aligned}$$

We now compute the difference:

$$\begin{aligned} D_1D_2S_{12}(\Phi) - 6S(\Phi) &= \frac{1}{2}\frac{D_1E_\Phi D_1E'_\Phi + D_2E_\Phi D_2E'_\Phi}{E_\Phi^2} - \frac{D_1E_\Phi D_2E_\Phi D_2D_1E_\Phi}{E_\Phi^3} \\ &\quad - \frac{1}{2}\left(\frac{D_2D_1E_\Phi}{E_\Phi}\right)^2 - 2\frac{\psi' \cdot \psi''}{E_\Phi}. \end{aligned}$$

Using Equation (6.5), we get $\frac{1}{2}(D_1E_\Phi D_1E'_\Phi + D_2E_\Phi D_2E'_\Phi) = 2[(\psi' \cdot D_1\psi)(\psi'' \cdot D_1\psi) + (1 \leftrightarrow 2)] + 2[(\psi' \cdot D_1\psi)(\psi' \cdot D_1\psi') + (1 \leftrightarrow 2)]$. Thanks to Formula (6.28), the first term reduces to $2\psi' \cdot \psi''E_\Phi$, and using (6.30) the second one turns out to be $2\psi'_1\psi'_2(D_1\psi \times D_1\psi' + D_2\psi \times D_2\psi')$, which is equal to $4\lambda\psi'_1\psi'_2(D_2\psi \cdot D_1\psi')$, in view of (6.29). On the other hand, using the previous equalities, we find $D_1E_\Phi D_2E_\Phi D_2D_1E_\Phi = -8\lambda E_\Phi\psi'_1\psi'_2(D_1\psi' \cdot D_2\psi)$; hence, we obtain

$$D_1D_2S_{12}(\Phi) - 6S(\Phi) = -\frac{3}{2}\frac{D_1E_\Phi D_2E_\Phi D_2D_1E_\Phi}{E_\Phi^3} - \frac{1}{2}\left(\frac{D_2D_1E_\Phi}{E_\Phi}\right)^2,$$

which is the desired result: $6S = D_1 D_2 S_{12} + \frac{1}{2} S_{12}^2$. Together with (6.33) and (6.31), the latter equation finishes the derivation of (6.25). Using the projection (6.24) on 1-densities, we obtain the Schwarzian derivative S, given by (6.26).

An argument analogous to that of Remark 3.7 helps us to prove that the cocycles S and $\mathcal{S}$ are nontrivial.

To find the kernel of the cocycle S, whence that of $\mathcal{S}$, we have to use another form of S, namely

$$\mathrm{S}(\Phi) = -2E_\Phi^{\frac{1}{2}}(D_2 D_1(E_\Phi^{-\frac{1}{2}}))\alpha.$$

Hence, if $\Phi \in \ker(\mathrm{S})$, we have $D_2 D_1(E_\Phi^{-\frac{1}{2}}) = 0$. As was the case for $N = 1$, the solution is $E_\Phi = (cx+d+\delta\cdot\xi)^{-2}$. A direct computation shows that there exists $h \in \mathrm{PC}(2|2)$ such that $E_\Phi = E_{\widehat{h}}$. Since the kernel of the Euclidean cocycle is $\widehat{\mathrm{E}}_+1|N/\{\pm\mathrm{Id}\}$ for $N = 2$, we obtain as announced: $\ker(\mathrm{S}) = \mathrm{PC}(2|2)$. The proof of Theorem 6.10 is complete. □

7 Conclusion, discussion and outlook

Starting off with the orthosymplectic group, $\mathrm{SpO}(2|N)$, and two nested subgroups $\mathrm{E}_+(1|N) \subset \mathrm{Aff}_+(1|N) \subset \mathrm{SpO}(2|N)$, we have been able to uniquely characterize Euclidean, I_{e}, affine, I_{a}, and projective, I_{p}, invariants for their actions as contactomorphisms of the supercircle $\mathrm{S}^{1|N}$, using the central notion of $p|q$-transitivity. Moreover, these invariants are characteristic of their defining groups. For $N = 0, 1, 2$, their even part does characterize the image of the supergroups $\mathrm{EO}(1|N)$, $\mathrm{AO}(1|N)$ and $\mathrm{SpO}(2|N)$, by the projective action on $\mathrm{S}^{1|N}$, within the group, $K(N)$, of all contactomorphisms. In doing so, we have recovered in a systematic fashion the previously introduced [2, 24] even and odd cross-ratios.

Then, using a natural super extension of the Cartan formula (1.2), we have provided a novel construction of the nontrivial 1-cocycles $\mathcal{E}$, $\mathcal{A}$, and $\mathcal{S}$ of $K(N)$, associated with the even invariants, for $N = 0, 1, 2$. We have also succeeded to recover the known expressions [11, 6, 28, 29] of the Schwarzian derivatives for $N = 0, 1, 2$. The kernels of the above-mentioned 1-cocycles have been shown to coincide with the groups defining the invariants leading to them. So, for each geometry, the group action, the even invariant and the 1-cocycle are three equivalent geometric objects on the supercircle $\mathrm{S}^{1|N}$

(where $N = 0, 1, 2$), endowed with its standard contact structure.

In the cases $N = 0, 1$, a complete classification of the subgeometries of the contact geometry of $\mathrm{S}^{1|N}$ is given by that of the nontrivial cohomology spaces $H^1(K(N), \mathcal{F}_\lambda)$, see Theorem 3.10 and Remark 3.11. A similar classification for $N = 2$ is still lacking. Work in progress related to the determination of $H^1(K(2), \mathcal{F}_\lambda)$ should provide a first insight into this classification, as well as that of the 1-cohomology spaces of $K(2)$ with coefficients in other natural modules such as $\Omega^1(\mathrm{S}^{1|2})$ and $\mathcal{Q}(\mathrm{S}^{1|2})$. In doing so, we will resort to the computation of $H^1(k(2), \mathcal{F}_\lambda)$ carried out by Ben Fraj [4].

For $N > 2$, our method yields, indeed, the Euclidean and affine 1-cocycles of $K(N)$. There is, however, no way to obtain, in our approach, Radul's Schwarzian integro-differential operator for $N = 3$, since there exists no projection from $\mathcal{Q}(\mathrm{S}^{1|3})$ to $\mathcal{F}_{\frac{1}{2}} = k(3)^*_{\mathrm{reg}}$ intertwining the $K(3)$ action. Moreover, our study provides a clear cut explanation of the fact that $\mathcal{S}(\Phi)$ cannot be derived as a quadratic differential by the Cartan formula (see Remark 6.6) for $N \geq 3$, and therefore help us understand why the Radul expression for $N = 3$ involves pseudo-differential operators.

We have, so far, studied the supercircle $\mathrm{S}^{1|N}$; but there are in fact two super-extensions of the circle, namely $\mathrm{S}^{1|N}$ and $\mathrm{S}^{1|N}_+$, see [10, 29, 17]. Let us discuss the case $N = 1$. The only difference between these two supermanifolds is that the functions on $\mathrm{S}^{1|1}$ are, indeed, functions on $\mathbb{R}^{1|1}$ invariant with respect to the transformation $(x, \xi) \mapsto (x + 2\pi, \xi)$, whereas functions on the Möbius supercircle, $\mathrm{S}^{1|1}_+$, can be viewed as functions on $\mathbb{R}^{1|1}$ invariant under the transformation $(x, \xi) \mapsto (x + 2\pi, -\xi)$. Here the coordinate x is regarded as an angular coordinate on S^1. The canonical contact structure on $\mathbb{R}^{1|1}$ define a contact structure on both $\mathrm{S}^{1|1}$ and $\mathrm{S}^{1|1}_+$ [25]. All our cocycles, prior to projections, are left invariant by the map $(x, \xi) \mapsto (x, -\xi)$, as well as the projections themselves; then $\mathcal{E}$, $\mathcal{A}$, $\mathcal{S}$, and A, S still define cocycles on $\mathrm{S}^{1|1}_+$. This can be generalized for $N > 1$ along the same line as before.

We expect that our approach will help us express the Bott-Thurston cocycles of $K(1)$ and $K(2)$ given by Radul in terms of the Berezin integral of the cup product of the 1-cocycles $\mathcal{E}$ and $\mathcal{A}$ introduced above (in a manner similar to the case of $\mathrm{Diff}_+(\mathrm{S}^1)$ spelled out in [18]). This should, hence, extend the classical formula worked out in [9] using a contact 1-form on $\mathrm{Diff}_+(\mathrm{S}^1) \times \mathbb{R}$.

Another plausible development would be the superization of the hyperboloid of one sheet in $\mathrm{sl}(2,\mathbb{R})^*$ whose conformal geometry is related to the projective geometry of null infinity [20, 9].

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