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Christian Léonard

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MINIMIZATION OF ENTROPY FUNCTIONALS

CHRISTIAN LÉONARD

ABSTRACT. Entropy functionals (i.e. convex integral functionals) and extensions of these functionals are minimized on convex sets. This paper is aimed at reducing as much as possible the assumptions on the constraint set. Dual equalities and characterizations of the minimizers are obtained with weak constraint qualifications.

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1. INTRODUCTION

1.1. The entropy minimization problem. Let R be a positive measure on a space \mathcal{Z} . Take a $[0, \infty]$ -valued measurable function γ^* on $\mathcal{Z} \times \mathbb{R}$ such that $\gamma^*(z, \cdot) := \gamma_z^*$ is convex and lower semicontinuous for all $z \in \mathcal{Z}$. Denote $M_{\mathcal{Z}}$ the space of all signed measures Q on \mathcal{Z} . The entropy functional to be considered is defined by

$$I(Q) = \begin{cases} \int_{\mathcal{Z}} \gamma_z^* \left(\frac{dQ}{dR}(z) \right) R(dz) & \text{if } Q \prec R \\ +\infty & \text{otherwise} \end{cases}, \quad Q \in M_{\mathcal{Z}} \quad (1.1)$$

where $Q \prec R$ means that Q is absolutely continuous with respect to R . Assume that for each z there exists a unique $m(z)$ which minimizes γ_z^* with

$$\gamma_z^*(m(z)) = 0, \quad \forall z \in \mathcal{Z}. \quad (1.2)$$

Then, I is $[0, \infty]$ -valued, its unique minimizer is mR and $I(mR) = 0$.

This paper is concerned with the minimization problem

$$\text{minimize } I(Q) \text{ subject to } T_o Q \in C, \quad Q \in M_{\mathcal{Z}} \quad (1.3)$$

where $T_o : M_{\mathcal{Z}} \rightarrow \mathcal{X}_o$ is a linear operator which takes its values in a vector space \mathcal{X}_o and C is a convex subset of \mathcal{X}_o .

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1.2. Presentation of the results. *Our aim is to reduce as much as possible the restrictions on the convex set C .* Denoting the minimizer \hat{Q} of (1.3), the geometric picture is that some level set of I is tangent at \hat{Q} to the constraint set $T_o^{-1}C$. Since these sets are convex, they are separated by some affine hyperplane and the analytic description of this separation yields the characterization of \hat{Q} . Of course Hahn-Banach theorem is the key. Standard approaches require C to be open with respect to some given topology in order to be allowed to apply it. In the present paper, one chooses to use a topological structure which is designed for the level sets of I to “look like” open sets, so that Hahn-Banach theorem can be applied without assuming too much on C .

This strategy is implemented in [17] in an abstract setting suitable for several applications. It is a refinement of the standard saddle-point method [22] where convex conjugates play an important role. The proofs of the present article are applications of the general results of [17].

Clearly, for the problem (1.3) to be attained, $T_o^{-1}C$ must share a supporting hyperplane with some level set of I . This is the reason why it is assumed to be closed with respect to the above mentioned topological structure. This will be the only restriction to be kept together with the interior specification (1.4) below.

Dual equalities and primal attainment are obtained under the weakest possible assumption:

$$C \cap T_o \text{dom } I \neq \emptyset$$

where $\text{dom } I := \{Q \in M_{\mathcal{Z}}; I(Q) < \infty\}$ is the effective domain of I and $T_o \text{dom } I$ is its image by T_o . The main result of this article is the characterization of the minimizers of (1.3) in the interior case which is specified by

$$C \cap \text{icor}(T_o \text{dom } I) \neq \emptyset \quad (1.4)$$

where $\text{icor}(T_o \text{dom } I)$ is the intrinsic core of $T_o \text{dom } I$. The notion of intrinsic core does not rely on any topology; it gives the largest possible interior set. For comparison, a usual form of constraint qualification required for the representation of the minimizers of (1.3) is

$$\text{int}(C) \cap T_o \text{dom } I \neq \emptyset \quad (1.5)$$

where $\text{int}(C)$ is the interior of C with respect to a topology which is not directly connected to the “geometry” of I . In particular, $\text{int}(C)$ must be nonempty; this is an important restriction. The constraint qualification (1.4) is weaker.

An extension of Problem (1.3) is also investigated. One considers an extension \bar{I} of the entropy I to a vector space $L_{\mathcal{Z}}$ which contains $M_{\mathcal{Z}}$ and may also contain singular linear forms which are not σ -additive. The extended problem is

$$\text{minimize } \bar{I}(\ell) \text{ subject to } T_o \ell \in C, \quad \ell \in L_{\mathcal{Z}} \quad (1.6)$$

Even if I is strictly convex, \bar{I} isn't strictly convex in general so that (1.6) may admit several minimizers. There are situations where (1.3) is not attained in $M_{\mathcal{Z}}$ while (1.6) is attained in $L_{\mathcal{Z}}$. Other relations between these minimization problems are investigated by the author in [18] with probabilistic questions in mind.

1.3. Literature about entropy minimization. Entropy minimization problems appear in many areas of applied mathematics and sciences. The literature about the minimization of entropy functionals under convex constraints is considerable: many papers are concerned with an engineering approach, working on the implementation of numerical procedures in specific situations. In fact, entropy minimization is a popular method to solve ill-posed inverse problems.

Rigorous general results on this topic are quite recent. Let us cite, among others, the main contribution of Borwein and Lewis: [1], [2], [3], [4], [5], [6] together with the paper [23] by Teboulle and Vajda. In these papers, topological constraint qualifications of the type of (1.5) are required. Such restrictions are removed here.

With a geometric point of view, Csiszár [8, 9] provides a complete treatment of (1.3) with the relative entropy (see Section 6.1) under the weak assumption (1.4). The behavior of minimizing sequences of general entropy functionals is studied in [10].

By means of a method different from the saddle-point approach, the author has already studied in [15, 16] entropy minimization problems under affine constraints (corresponding to C reduced to a single point) and more restrictive assumptions on γ^* .

The present article extends these results.

Outline of the paper. The minimization problems (1.3) and (1.6) are described in details at Section 2. In Section 3, the main results of [17] about the extended saddle-point method are recalled. Section 4 is devoted to the extended problem (1.6) and Section 5 to (1.3). One presents important examples of entropies and constraints at Section 6.1.

Notation. Let X and Y be topological vector spaces. The algebraic dual space of X is X^* , the topological dual space of X is X' . The topology of X weakened by Y is $\sigma(X, Y)$ and one writes $\langle X, Y \rangle$ to specify that X and Y are in separating duality.

Let $f : X \rightarrow [-\infty, +\infty]$ be an extended numerical function. Its convex conjugate with respect to $\langle X, Y \rangle$ is $f^*(y) = \sup_{x \in X} \{\langle x, y \rangle - f(x)\} \in [-\infty, +\infty]$, $y \in Y$. Its subdifferential at x with respect to $\langle X, Y \rangle$ is $\partial_Y f(x) = \{y \in Y; f(x + \xi) \geq f(x) + \langle y, \xi \rangle, \forall \xi \in X\}$. If no confusion occurs, one writes $\partial f(x)$.

The intrinsic core of a subset A of a vector space is $\text{icor } A = \{x \in A; \forall x' \in \text{aff } A, \exists t > 0, [x, x + t(x' - x)] \subset A\}$ where $\text{aff } A$ is the affine space spanned by A . $\text{icordom } f$ is the intrinsic core of the effective domain of $f : \text{dom } f = \{x \in X; f(x) < \infty\}$.

The indicator of a subset A of X is defined by

$$\iota_A(x) = \begin{cases} 0, & \text{if } x \in A \\ +\infty, & \text{otherwise} \end{cases}, \quad x \in X.$$

The support function of $A \subset X$ is $\iota_A^*(y) = \sup_{x \in A} \langle x, y \rangle$, $y \in Y$.

One writes $I_\varphi(u) := \int_{\mathcal{Z}} \varphi(z, u(z)) R(dz) = \int_{\mathcal{Z}} \varphi(u) dR$ and $I = I_{\gamma^*}$ for short, instead of (1.1).

2. PRESENTATION OF THE MINIMIZATION PROBLEMS (P_C) AND (\bar{P}_C)

The problem (1.3) and its extension (1.6) are introduced. Their correct mathematical statements necessitate the notion of Orlicz spaces. The definitions of good and bad constraints are given and the main assumptions are collected at the end of this section.

2.1. Orlicz spaces. To state the minimization problem (1.3) and its extension correctly, one will need to talk in terms of Orlicz spaces related to the function γ^* .

Let us recall some basic definitions and results. A set \mathcal{Z} is furnished with a σ -finite nonnegative measure R on a σ -field which is assumed to be R -complete. A function $\rho : \mathcal{Z} \times \mathbb{R}$ is said to be a *Young function* if for R -almost every z , $\rho(z, \cdot)$ is a convex even $[0, \infty]$ -valued function on \mathbb{R} such that $\rho(z, 0) = 0$ and there exists a measurable function $z \mapsto s_z > 0$ such that $0 < \rho(z, s_z) < \infty$.

In the sequel, every numerical function on \mathcal{Z} is supposed to be measurable.

Definitions 2.1 (The Orlicz spaces $\mathcal{L}_\rho, \mathcal{E}_\rho, L_\rho$ and E_ρ). *The Orlicz space associated with ρ is defined by $\mathcal{L}_\rho(\mathcal{Z}, R) = \{u : \mathcal{Z} \rightarrow \mathbb{R}; \|u\|_\rho < +\infty\}$ where the Luxemburg norm $\|\cdot\|_\rho$ is defined by $\|u\|_\rho = \inf \left\{ \beta > 0 ; \int_{\mathcal{Z}} \rho(z, u(z)/\beta) R(dz) \leq 1 \right\}$. Hence,*

$$\mathcal{L}_\rho(\mathcal{Z}, R) = \left\{ u : \mathcal{Z} \rightarrow \mathbb{R} ; \exists \alpha_o > 0, \int_{\mathcal{Z}} \rho(z, \alpha_o u(z)) R(dz) < \infty \right\}.$$

A subspace of interest is

$$\mathcal{E}_\rho(\mathcal{Z}, R) = \left\{ u : \mathcal{Z} \rightarrow \mathbb{R} ; \forall \alpha > 0, \int_{\mathcal{Z}} \rho(z, \alpha u(z)) R(dz) < \infty \right\}.$$

Now, let us identify the R -a.e. equal functions. The corresponding spaces of equivalence classes are denoted $L_\rho(\mathcal{Z}, R)$ and $E_\rho(\mathcal{Z}, R)$.

Of course $E_\rho \subset L_\rho$. Note that if ρ doesn't depend on z and $\rho(s_o) = \infty$ for some $s_o > 0$, E_ρ reduces to the null space and if in addition R is bounded, L_ρ is L_∞ . On the other hand, if ρ is a finite function which doesn't depend on z and R is bounded, E_ρ contains all the bounded functions.

Duality in Orlicz spaces is intimately linked with the convex conjugacy. The convex conjugate ρ^* of ρ is defined by $\rho^*(z, t) = \sup_{s \in \mathbb{R}} \{st - \rho(z, s)\}$. It is also a Young function so that one may consider the Orlicz space L_{ρ^*} .

Theorem 2.2 (Representation of E'_ρ). *Suppose that ρ is a finite Young function. Then, the dual space of E_ρ is isomorphic to L_{ρ^*} .*

Proof. For a proof of this result, see ([12], Thm 4.8). \square

A continuous linear form $\ell \in L'_\rho$ is said to be *singular* if for all $u \in L_\rho$, there exists a decreasing sequence of measurable sets (A_n) such that $R(\cap_n A_n) = 0$ and for all $n \geq 1$, $\langle \ell, u \mathbf{1}_{\mathcal{Z} \setminus A_n} \rangle = 0$. Let us denote L_ρ^s the subspace of L'_ρ of all singular forms.

Theorem 2.3 (Representation of L'_ρ). *Let ρ be any Young function. The dual space of L_ρ is isomorphic to the direct sum $L'_\rho = (L_{\rho^*} \cdot R) \oplus L_\rho^s$. This implies that any $\ell \in L'_\rho$ is uniquely decomposed as*

$$\ell = \ell^a + \ell^s \tag{2.4}$$

with $\ell^a \in L_{\rho^} \cdot R$ and $\ell^s \in L_\rho^s$.*

Proof. When $L_\rho = L_\infty$ this result is the usual representation of L'_∞ .

When ρ is a finite function, this result is ([13], Theorem 2.2).

The general result is proved in [19], with ρ not depending on z but the extension to a z -dependent ρ is obvious. \square

In the decomposition (2.4), ℓ^a is called the *absolutely continuous* part of ℓ while ℓ^s is its *singular* part.

Proposition 2.5. *Let us assume that ρ is finite. Then, $\ell \in L'_\rho$ is singular if and only if $\langle \ell, u \rangle = 0$, for all u in E_ρ .*

Proof. This result is ([13], Proposition 2.1). \square

The function ρ is said to satisfy the Δ_2 -condition if

$$\text{there exist } C > 0, s_o \geq 0 \text{ such that } \forall s \geq s_o, \rho(2s) \leq C\rho(s) \tag{2.6}$$

If $s_o = 0$, the Δ_2 -condition is said to be global. When R is bounded, in order that $E_\rho = L_\rho$, it is enough that ρ satisfies the Δ_2 -condition. When R is unbounded, this equality still holds if the Δ_2 -condition is global. Consequently, if ρ satisfies the Δ_2 -condition we have $L'_\rho = L_{\rho^*} \cdot R$ so that L_ρ^s reduces to the null vector space.

2.2. The minimization problem (P_C) . Before introducing an extended minimization problem, let us state properly the basic problem (1.3).

Relevant Orlicz spaces. Since γ_z^* is closed convex for each z , it is the convex conjugate of some closed convex function γ_z . Defining

$$\lambda(z, s) = \gamma(z, s) - m(z)s, \quad z \in \mathcal{Z}, s \in \mathbb{R}$$

where m satisfies (1.2), one sees that for R -a.e. z , λ_z is a nonnegative convex function and it vanishes at 0. Hence,

$$\lambda_\diamond(z, s) = \max[\lambda(z, s), \lambda(z, -s)] \in [0, \infty], \quad z \in \mathcal{Z}, s \in \mathbb{R}$$

is a Young function. We shall use Orlicz spaces associated with λ_\diamond and λ_\diamond^* .

We denote the space of R -absolutely continuous signed measures having a density in the Orlicz space $L_{\lambda_\diamond^*}$ by $L_{\lambda_\diamond^*}R$. The effective domain of I is included in $mR + L_{\lambda_\diamond^*}R$.

Constraint. In order to define the constraint, take \mathcal{X}_o a vector space and a function $\theta : \mathcal{Z} \rightarrow \mathcal{X}_o$. One wants to give a meaning to the formal constraint $\int_{\mathcal{Z}} \theta dQ = x$ with $Q \in L_{\lambda_\diamond^*}R$ and $x \in \mathcal{X}_o$. Suppose that \mathcal{X}_o is the algebraic dual space of some vector space \mathcal{Y}_o and define for all $y \in \mathcal{Y}_o$,

$$T_o^*y(z) := \langle y, \theta(z) \rangle_{\mathcal{Y}_o, \mathcal{X}_o}, \quad z \in \mathcal{Z}. \quad (2.7)$$

Assuming that

$$T_o^*\mathcal{Y}_o \subset \mathcal{L}_{\lambda_\diamond}, \quad (2.8)$$

Hölder's inequality in Orlicz spaces allows to define the constraint operator $T_o\ell := \int_{\mathcal{Z}} \theta d\ell$ for each $\ell \in L_{\lambda_\diamond^*}R$ by

$$\left\langle y, \int_{\mathcal{Z}} \theta d\ell \right\rangle_{\mathcal{Y}_o, \mathcal{X}_o} = \int_{\mathcal{Z}} \langle y, \theta(z) \rangle_{\mathcal{Y}_o, \mathcal{X}_o} \ell(dz), \quad \forall y \in \mathcal{Y}_o. \quad (2.9)$$

Minimization problem. Consider the minimization problem

$$\text{minimize } I(Q) \text{ subject to } \int_{\mathcal{Z}} \theta d(Q - mR) \in C_o, \quad Q \in mR + L_{\lambda_\diamond^*}R \quad (P_{C_o})$$

where C_o is a convex subset of \mathcal{X}_o . One sees with $\gamma_z^*(t) = \lambda_z^*(t - m(z))$ that $I_{\gamma^*}(Q) = I_{\lambda^*}(Q - mR)$. Therefore, the problem (P_{C_o}) is equivalent to

$$\text{minimize } I_{\lambda^*}(\ell) \text{ subject to } \int_{\mathcal{Z}} \theta d\ell \in C_o, \quad \ell \in L_{\lambda_\diamond^*}R \quad (2.10)$$

with $\ell = Q - mR$. If the function m satisfies $m \in L_{\lambda_\diamond^*}$, one sees with (2.8) and Hölder's inequality in Orlicz spaces that the vector $x_o = \int_{\mathcal{Z}} \theta m dR \in \mathcal{X}_o$ is well-defined in the weak sense. Therefore, (P_{C_o}) is

$$\text{minimize } I(Q) \text{ subject to } \int_{\mathcal{Z}} \theta dQ \in C, \quad Q \in L_{\lambda_\diamond^*}R \quad (P_C)$$

with $C = x_o + C_o$

2.3. The extended minimization problem (\bar{P}_C) . If the Young function λ_\diamond doesn't satisfy the Δ_2 -condition (2.6), for instance if it has an exponential growth at infinity as in (6.1) or even worse as in (6.3), the *small* Orlicz space $\mathcal{E}_{\lambda_\diamond}$ may be a proper subset of $\mathcal{L}_{\lambda_\diamond}$. Consequently, for some functions θ , the integrability property

$$T_o^* \mathcal{Y}_o \subset \mathcal{E}_{\lambda_\diamond} \quad (2.11)$$

or equivalently

$$\forall y \in \mathcal{Y}_o, \int_{\mathcal{Z}} \lambda(\langle y, \theta \rangle) dR < \infty \quad (A_\theta^\forall)$$

may not be satisfied while the weaker property (2.8): $T_o^* \mathcal{Y}_o \subset \mathcal{L}_{\lambda_\diamond}$, or equivalently

$$\forall y \in \mathcal{Y}_o, \exists \alpha > 0, \int_{\mathcal{Z}} \lambda(\alpha \langle y, \theta \rangle) dR < \infty \quad (A_\theta^\exists)$$

holds. In this situation, analytical complications occur (see Section 4). This is the reason why constraints satisfying (A_θ^\forall) are called *good constraints*, while constraints satisfying (A_θ^\exists) but not (A_θ^\forall) are called *bad constraints*.

If the constraint is bad, it may happen that (P_C) is not attained in $L_{\lambda_\diamond}^* R$. This is the reason why it is worth introducing its extension (\bar{P}_C) which may admit minimizers and is defined by

$$\text{minimize } \bar{I}(\ell) \text{ subject to } \langle \theta, \ell \rangle \in C, \quad \ell \in L'_{\lambda_\diamond} \quad (\bar{P}_C)$$

where L'_{λ_\diamond} is the topological dual space of L_{λ_\diamond} , \bar{I} and $\langle \theta, \ell \rangle$ are defined below.

The dual space L'_{λ_\diamond} admits the representation $L'_{\lambda_\diamond} \simeq L_{\lambda_\diamond}^* R \oplus L_{\lambda_\diamond}^s$. This means that any $\ell \in L'_{\lambda_\diamond}$ is uniquely decomposed as $\ell = \ell^a + \ell^s$ where $\ell^a \in L_{\lambda_\diamond}^* R$ and $\ell^s \in L_{\lambda_\diamond}^s$ are respectively the *absolutely continuous part* and the *singular part* of ℓ , see Theorem 2.3. The extension \bar{I} has the following form

$$\bar{I}(\ell) = I(\ell^a) + \iota_{\text{dom } I_\gamma}^*(\ell^s), \quad \ell \in L'_{\lambda_\diamond} \quad (2.12)$$

It will be shown that \bar{I} is the greatest convex $\sigma(L'_{\lambda_\diamond}, L_{\lambda_\diamond})$ -lower semicontinuous extension of I to $L'_{\lambda_\diamond} \supset L_{\lambda_\diamond}^*$. In a similar way to (2.9), the assumption (A_θ^\exists) allows to define $T_o \ell = \langle \theta, \ell \rangle$ for all $\ell \in L'_{\lambda_\diamond}$ by

$$\langle y, \langle \theta, \ell \rangle \rangle_{\mathcal{Y}_o, \mathcal{X}_o} = \langle \langle y, \theta \rangle, \ell \rangle_{L_{\lambda_\diamond}, L'_{\lambda_\diamond}}, \quad \forall y \in \mathcal{Y}_o.$$

Important examples of entropies with λ_\diamond not satisfying the Δ_2 -condition are the usual (Boltzmann) entropy and its variants, see Section 6.1 and (6.1) in particular.

When λ_\diamond satisfies the Δ_2 -condition (2.6), (\bar{P}_C) is (P_C) .

2.4. Assumptions. Let us collect the assumptions on R, γ^* and θ .

Assumptions (A).

(A_R) It is assumed that the reference measure R is a σ -finite nonnegative measure on a space \mathcal{Z} endowed with some R -complete σ -field.

(A_{γ^*}) Assumptions on γ^* .

- (1) $\gamma^*(\cdot, t)$ is z -measurable for all t and for R -almost every $z \in \mathcal{Z}$, $\gamma^*(z, \cdot)$ is a lower semicontinuous strictly convex $[0, +\infty]$ -valued function on \mathbb{R} which attains its (unique) minimum at $m(z)$ with $\gamma^*(z, m(z)) = 0$.
- (2) $\int_{\mathcal{Z}} \lambda^*(\alpha m) dR + \int_{\mathcal{Z}} \lambda^*(-\alpha m) dR < \infty$, for some $\alpha > 0$.

(A_θ) Assumptions on θ .

- (1) for any $y \in \mathcal{Y}_o$, the function $z \in \mathcal{Z} \mapsto \langle y, \theta(z) \rangle \in \mathbb{R}$ is measurable;
- (2) for any $y \in \mathcal{Y}_o$, $\langle y, \theta(\cdot) \rangle = 0$, R -a.e. implies that $y = 0$;

$$(\exists) \forall y \in \mathcal{Y}_o, \exists \alpha > 0, \int_{\mathcal{Z}} \lambda(\alpha \langle y, \theta \rangle) dR < \infty.$$

Remarks 2.13. Some technical remarks about the assumptions.

- (a) Since γ_z^* is a convex function on \mathbb{R} , it is continuous on the interior of its domain. Under our assumptions, γ^* is (jointly) measurable, and so are γ and m . Hence, λ is also measurable.
- (b) As γ_z^* is strictly convex, γ_z is differentiable.
- (c) Assumption $(A_{\gamma^*}^2)$ is $m \in L_{\lambda^*}^*$. It allows to consider Problem (P_C) rather than (P_{C_o}) . If this assumption is not satisfied, our results still hold for (P_{C_o}) , but their statement is a little heavier, see Remark 4.10-d below.
- (d) Since \mathcal{X}_o and \mathcal{Y}_o are in separating duality, (A_θ^2) states that the vector space spanned by the range of θ “is essentially” \mathcal{X}_o . This is not an effective restriction.

3. PRELIMINARY RESULTS

The aim of this section is to recall for the convenience of the reader some results of [14, 16, 17].

3.1. Convex minimization problems under weak constraint qualifications. The main results of [17] are presented.

Basic diagram. Let \mathcal{U}_o be a vector space, $\mathcal{L}_o = \mathcal{U}_o^*$ its algebraic dual space, Φ a $(-\infty, +\infty]$ -valued convex function on \mathcal{U}_o and Φ^* its convex conjugate for the duality $\langle \mathcal{U}_o, \mathcal{L}_o \rangle$:

$$\Phi^*(\ell) := \sup_{u \in \mathcal{U}_o} \{ \langle u, \ell \rangle - \Phi(u) \}, \quad \ell \in \mathcal{L}_o$$

Let \mathcal{Y}_o be another vector space, $\mathcal{X}_o = \mathcal{Y}_o^*$ its algebraic dual space and $T_o : \mathcal{L}_o \rightarrow \mathcal{X}_o$ a linear operator. We consider the convex minimization problem

$$\text{minimize } \Phi^*(\ell) \text{ subject to } T_o \ell \in C, \ell \in \mathcal{L}_o \quad (\mathcal{P}_o)$$

where C is a convex subset of \mathcal{X}_o .

This will be used later with $\Phi = I_\lambda$ on the Orlicz space $\mathcal{U}_o = \mathcal{E}_{\lambda_o}(\mathcal{Z}, R)$ or $\mathcal{U}_o = \mathcal{L}_{\lambda_o}(\mathcal{Z}, R)$. It is useful to define the constraint operator T_o by means of its adjoint $T_o^* : \mathcal{Y}_o \rightarrow \mathcal{L}_o^*$ for each $\ell \in \mathcal{L}_o$, by $\langle T_o^* y, \ell \rangle_{\mathcal{L}_o^*, \mathcal{L}_o} = \langle y, T_o \ell \rangle_{\mathcal{Y}_o, \mathcal{X}_o}$, $\forall y \in \mathcal{Y}_o$.

Hypotheses. Let us give the list of the main hypotheses.

- (H_Φ) 1- $\Phi : \mathcal{U}_o \rightarrow [0, +\infty]$ is $\sigma(\mathcal{U}_o, \mathcal{L}_o)$ -lower semicontinuous, convex and $\Phi(0) = 0$
- 2- $\forall u \in \mathcal{U}_o, \exists \alpha > 0, \Phi(\alpha u) < \infty$
- 3- $\forall u \in \mathcal{U}_o, u \neq 0, \exists t \in \mathbb{R}, \Phi(tu) > 0$
- (H_T) 1- $T_o^*(\mathcal{Y}_o) \subset \mathcal{U}_o$
- 2- $\ker T_o^* = \{0\}$
- (H_C) $C \cap \mathcal{X}$ is a convex $\sigma(\mathcal{X}, \mathcal{Y})$ -closed subset of \mathcal{X}

The definitions of the vector spaces \mathcal{X} and \mathcal{Y} which appear in the last assumption are stated below. For the moment, let us only say that if C is convex and $\sigma(\mathcal{X}_o, \mathcal{Y}_o)$ -closed, then (H_C) holds.

Several primal and dual problems. These variants are expressed below in terms of new spaces and functions. Let us first introduce them.

- **The norms $|\cdot|_\Phi$ and $|\cdot|_\Lambda$.** Let $\Phi_\pm(u) = \max(\Phi(u), \Phi(-u))$. By (H_{Φ_1}) and (H_{Φ_2}) , $\{u \in \mathcal{U}_o; \Phi_\pm(u) \leq 1\}$ is a convex absorbing balanced set. Hence its gauge functional which is defined for all $u \in \mathcal{U}_o$ by $|u|_\Phi := \inf\{\alpha > 0; \Phi_\pm(u/\alpha) \leq 1\}$ is a seminorm. Thanks to hypothesis (H_{Φ_3}) , it is a norm. Taking (H_{T1}) into account, one can define

$$\Lambda_o(y) := \Phi(T_o^*y), y \in \mathcal{Y}_o. \quad (3.1)$$

Let $\Lambda_\pm(y) = \max(\Lambda_o(y), \Lambda_o(-y))$. The gauge functional on \mathcal{Y}_o of the set $\{y \in \mathcal{Y}_o; \Lambda_\pm(y) \leq 1\}$ is $|y|_\Lambda := \inf\{\alpha > 0; \Lambda_\pm(y/\alpha) \leq 1\}, y \in \mathcal{Y}_o$. Thanks to (H_Φ) and (H_T) , it is a norm and

$$|y|_\Lambda = |T_o^*y|_\Phi, \quad y \in \mathcal{Y}_o.$$

- **The spaces.** Let

\mathcal{U} be the $|\cdot|_\Phi$ -completion of \mathcal{U}_o and let

$\mathcal{L} := (\mathcal{U}_o, |\cdot|_\Phi)'$ be the topological dual space of $(\mathcal{U}_o, |\cdot|_\Phi)$.

Of course, we have $(\mathcal{U}, |\cdot|_\Phi)' \cong \mathcal{L} \subset \mathcal{L}_o$ where any ℓ in \mathcal{U}' is identified with its restriction to \mathcal{U}_o . Similarly, we introduce

\mathcal{Y} the $|\cdot|_\Lambda$ -completion of \mathcal{Y}_o and

$\mathcal{X} := (\mathcal{Y}_o, |\cdot|_\Lambda)'$ the topological dual space of $(\mathcal{Y}_o, |\cdot|_\Lambda)$.

We have $(\mathcal{Y}, |\cdot|_\Lambda)' \cong \mathcal{X} \subset \mathcal{X}_o$ where any x in \mathcal{Y}' is identified with its restriction to \mathcal{Y}_o .

We also have to consider the *algebraic* dual spaces \mathcal{L}^* and \mathcal{X}^* of \mathcal{L} and \mathcal{X} .

- **The operators T and T^* .** Let us denote T the restriction of T_o to $\mathcal{L} \subset \mathcal{L}_o$. One can show that under $(H_{\Phi \& T})$, $T_o\mathcal{L} \subset \mathcal{X}$. Hence $T : \mathcal{L} \rightarrow \mathcal{X}$. Let us define its adjoint $T^* : \mathcal{X}^* \rightarrow \mathcal{L}^*$ for all $\omega \in \mathcal{X}^*$ by: $\langle \ell, T^*\omega \rangle_{\mathcal{L}, \mathcal{L}^*} = \langle T\ell, \omega \rangle_{\mathcal{X}, \mathcal{X}^*}, \forall \ell \in \mathcal{L}$. We have the inclusions $\mathcal{Y}_o \subset \mathcal{Y} \subset \mathcal{X}^*$. The adjoint operator T_o^* is the restriction of T^* to \mathcal{Y}_o .
- **The functionals.** They are:

$$\begin{aligned} \bar{\Phi}(\zeta) &:= \sup_{\ell \in \mathcal{L}} \{\langle \zeta, \ell \rangle - \Phi^*(\ell)\}, & \zeta \in \mathcal{L}^* \\ \Lambda(y) &:= \bar{\Phi}(T_o^*y), & y \in \mathcal{Y} \\ \bar{\Lambda}(\omega) &:= \bar{\Phi}(T^*\omega), & \omega \in \mathcal{X}^* \\ \Lambda_o^*(x) &:= \sup_{y \in \mathcal{Y}_o} \{\langle y, x \rangle - \Lambda_o(y)\}, & x \in \mathcal{X}_o \\ \Lambda^*(x) &:= \sup_{y \in \mathcal{Y}} \{\langle y, x \rangle - \Lambda(y)\}, & x \in \mathcal{X} \end{aligned}$$

- **The optimization problems.** They are:

$$\text{minimize } \Phi^*(\ell) \quad \text{subject to } T_o\ell \in C, \quad \ell \in \mathcal{L}_o \quad (\mathcal{P}_o)$$

$$\text{minimize } \Phi^*(\ell) \quad \text{subject to } T\ell \in C, \quad \ell \in \mathcal{L} \quad (\mathcal{P})$$

$$\text{maximize } \inf_{x \in C \cap \mathcal{X}} \langle y, x \rangle - \Lambda(y), \quad y \in \mathcal{Y} \quad (\mathcal{D})$$

$$\text{maximize } \inf_{x \in C \cap \mathcal{X}} \langle x, \omega \rangle - \bar{\Lambda}(\omega), \quad \omega \in \mathcal{X}^* \quad (\bar{\mathcal{D}})$$

Statement of the results. It is assumed that (H_Φ) , (H_T) and (H_C) hold.

Theorem 3.2 (Primal attainment and dual equality).

- (a) *The problems (\mathcal{P}_o) and (\mathcal{P}) are equivalent: they have the same solutions and $\inf(\mathcal{P}_o) = \inf(\mathcal{P}) \in [0, \infty]$.*

(b) We have the dual equalities

$$\inf(\mathcal{P}_o) = \inf(\mathcal{P}) = \sup(\mathcal{D}) = \sup(\overline{\mathcal{D}}) = \inf_{x \in C} \Lambda_o^*(x) = \inf_{x \in C \cap \mathcal{X}} \Lambda^*(x) \in [0, \infty]$$

(c) If in addition $\{\ell \in \mathcal{L}_o; T_o \ell \in C\} \cap \text{dom } \Phi^* \neq \emptyset$, then (\mathcal{P}_o) is attained in \mathcal{L} . Moreover, any minimizing sequence for (\mathcal{P}_o) has $\sigma(\mathcal{L}, \mathcal{U})$ -cluster points and every such cluster point solves (\mathcal{P}_o) .

Theorem 3.3 (Dual attainment and representation. Interior convex constraint).

Assume that $C \cap \text{icor}(T_o \text{dom } \Phi^*) \neq \emptyset$.

Then, the primal problem (\mathcal{P}_o) is attained in \mathcal{L} and the extended dual problem $(\overline{\mathcal{D}})$ is attained in \mathcal{X}^* . Any solution $\hat{\ell} \in \mathcal{L}$ of (\mathcal{P}_o) is characterized by the existence of some $\bar{\omega} \in \mathcal{X}^*$ such that

$$\begin{cases} (a) & T\hat{\ell} \in C \\ (b) & \langle T^*\bar{\omega}, \hat{\ell} \rangle \leq \langle T^*\bar{\omega}, \ell \rangle \text{ for all } \ell \in \{\ell \in \mathcal{L}; T\ell \in C\} \cap \text{dom } \Phi^* \\ (c) & \hat{\ell} \in \partial_{\mathcal{L}} \bar{\Phi}(T^*\bar{\omega}) \end{cases} \quad (3.4)$$

Moreover, $\hat{\ell} \in \mathcal{L}$ and $\bar{\omega} \in \mathcal{X}^*$ satisfy (3.4) if and only if $\hat{\ell}$ solves (\mathcal{P}_o) and $\bar{\omega}$ solves $(\overline{\mathcal{D}})$.

The assumption $C \cap \text{icor}(T_o \text{dom } \Phi^*) \neq \emptyset$ is equivalent to $C \cap \text{icordom } \Lambda_o^* \neq \emptyset$ and the representation formula (3.4-c) is equivalent to Young's identity

$$\Phi^*(\hat{\ell}) + \bar{\Phi}(T^*\bar{\omega}) = \langle \bar{\omega}, T\hat{\ell} \rangle = \Lambda^*(\hat{x}) + \bar{\Lambda}(\bar{\omega}). \quad (3.5)$$

Formula (3.4-c) can be made a little more precise by means of the following regularity result.

Theorem 3.6. Any solution $\bar{\omega}$ of $(\overline{\mathcal{D}})$ shares the following properties

- (a) $\bar{\omega}$ is in the $\sigma(\mathcal{X}^*, \mathcal{X})$ -closure of $\text{dom } \Lambda$;
- (b) $T^*\bar{\omega}$ is in the $\sigma(\mathcal{L}^*, \mathcal{L})$ -closure of $T^*(\text{dom } \Lambda)$.

If in addition the level sets of Φ are $|\cdot|_{\Phi}$ -bounded, then

- (a') $\bar{\omega}$ is in \mathcal{Y}'' . More precisely, it is in the $\sigma(\mathcal{Y}'', \mathcal{X})$ -closure of $\text{dom } \Lambda$;
- (b') $T^*\bar{\omega}$ is in \mathcal{U}'' . More precisely, it is in the $\sigma(\mathcal{U}'', \mathcal{L})$ -closure of $T^*(\text{dom } \Lambda)$

where \mathcal{Y}'' and \mathcal{U}'' are the topological bidual spaces of \mathcal{Y} and \mathcal{U} . This occurs if Φ , or equivalently Φ^* , is an even function.

3.2. Convex conjugates in a Riesz space. The following results are taken from [14, 16]. For the basic definitions and properties of Riesz spaces, see [7, Chapter 2].

Let U be a Riesz vector space for the order relation \leq . Since U is a Riesz space, any $u \in U$ admits a nonnegative part: $u_+ := u \vee 0$, and a nonpositive part: $u_- := (-u) \vee 0$. Of course, $u = u_+ - u_-$ and as usual, we state: $|u| = u_+ + u_-$.

Remark 3.7. Recall that there is a natural order on the algebraic dual space E^* of a Riesz vector space E which is defined by: $e^* \leq f^* \Leftrightarrow \langle e^*, e \rangle \leq \langle f^*, e \rangle$ for any $e \in E$ with $e \geq 0$. A linear form $e^* \in E^*$ is said to be *relatively bounded* if for any $f \in E$, $f \geq 0$, we have $\sup_{e: |e| \leq f} |\langle e^*, e \rangle| < +\infty$. Although E^* may not be a Riesz space in general, the vector space E^b of all the relatively bounded linear forms on E is always a Riesz space. In particular, the elements of E^b admit a decomposition in positive and negative parts $e^* = e_+^* - e_-^*$.

Let Φ be a $[0, \infty]$ -valued function on U which satisfies the following conditions:

$$\forall u \in U, \Phi(u) = \Phi(u_+ - u_-) = \Phi(u_+) + \Phi(-u_-) \quad (3.8)$$

$$\forall u, v \in U, \begin{cases} 0 \leq u \leq v & \implies \Phi(u) \leq \Phi(v) \\ u \leq v \leq 0 & \implies \Phi(u) \geq \Phi(v) \end{cases} \quad (3.9)$$

Clearly (3.8) implies $\Phi(0) = 0$, (3.8) and (3.9) imply that for any $u \in U$, $\Phi(u) = \Phi(u_+) + \Phi(-u_-) \geq \Phi(0) + \Phi(0) = 0$. Therefore, Φ^* is $[0, \infty]$ -valued and $\Phi^*(0) = 0$.

For all $u \in U$, $\Phi_+(u) = \Phi(|u|)$, $\Phi_-(u) = \Phi(-|u|)$. The convex conjugates of Φ , Φ_+ and Φ_- with respect to $\langle U, U^* \rangle$ are denoted Φ^* , Φ_+^* and Φ_-^* . Let L be the vector space spanned by $\text{dom } \Phi^*$. The convex conjugates of Φ^* , Φ_+^* and Φ_-^* with respect to $\langle L, L^* \rangle$ are denoted $\bar{\Phi}$, $\bar{\Phi}_+$ and $\bar{\Phi}_-$. The space of relatively bounded linear forms on U and L are denoted by U^b and L^b , whenever L is a Riesz space.

One writes $a_{\pm} \in A_{\pm}$ for $[a_+ \in A_+ \text{ and } a_- \in A_-]$.

Proposition 3.10. *Assume (3.8) and (3.9) and suppose that L is a Riesz space.*

(a) *For all $\ell \in U^*$,*

$$\Phi^*(\ell) = \begin{cases} \Phi_+^*(\ell_+) + \Phi_-^*(\ell_-) & \text{if } \ell \in U^b \\ +\infty & \text{otherwise} \end{cases}$$

(b) *Denoting L_+ and L_- the vector subspaces of L spanned by $\text{dom } \Phi_+^*$ and $\text{dom } \Phi_-^*$, we have*

$$\bar{\Phi}(\zeta) = \begin{cases} \bar{\Phi}_+(\zeta_{+|L_+}) + \bar{\Phi}_-(\zeta_{-|L_-}) & \text{if } \zeta \in L^b \\ +\infty & \text{otherwise} \end{cases}$$

which means that $\bar{\Phi}_{\pm}(\zeta_{\pm}) = \bar{\Phi}_{\pm}(\zeta'_{\pm})$ if ζ_{\pm} and ζ'_{\pm} match on L_{\pm} .

(c) *Let $\ell \in L$, $\zeta \in L^*$ be such that $\ell \in \partial_L \bar{\Phi}(\zeta)$. Then, $\ell_{\pm} \in \partial_{L_{\pm}} \bar{\Phi}_{\pm}(\zeta_{\pm}|_{L_{\pm}}) \subset L_{\pm}$.*

Proof. (a) and (b) are proved at [14, Proposition 4.4] under the additional assumption that for all $u \in U$ there exists $\lambda > 0$ such that $\Phi(\lambda u) < +\infty$. But it can be removed. Indeed, if for instance Φ_- is null, Φ_-^* is the convex indicator of $\{0\}$ whose domain is in U^b . The statement about $\bar{\Phi}$ is an iteration of this argument.

The last statement of (b) about $\zeta_{\pm}|_{L_{\pm}}$ directly follows from $\text{dom } \Phi_{\pm}^* \subset L_{\pm}$.

For (c), see the proof of [16, Proposition 4.5]. \square

4. SOLVING (\bar{P}_C)

The general assumptions (A) are imposed and we study (\bar{P}_C) .

4.1. Several function spaces and cones. To state the extended dual problem (\bar{D}_C) below, notation is needed. If λ is not an even function, one has to consider

$$\begin{cases} \lambda_+(z, s) = \lambda(z, |s|) \\ \lambda_-(z, s) = \lambda(z, -|s|) \end{cases} \quad (4.1)$$

which are Young functions and the corresponding Orlicz spaces.

Definitions 4.2. *For any relatively bounded linear form ζ on $L'_{\lambda_{\diamond}}$ i.e. $\zeta \in L'^b_{\lambda_{\diamond}}$, one writes:*

- $\zeta \in K''_{\lambda}$ to specify that $\zeta_{\pm}|_{L'_{\lambda_{\pm}} \cap L'_{\lambda_{\diamond}}} \in L''_{\lambda_{\pm}}$
- $\zeta \in K'_{\lambda^*}$ to specify that $\zeta_{\pm}|_{L_{\lambda^*} \cap L'_{\lambda_{\diamond}}} \in L'_{\lambda^*}$
- $\zeta \in K_{\lambda}$ to specify that $\zeta_{\pm}|_{L_{\lambda^*} \cap L'_{\lambda_{\diamond}}} \in L_{\lambda_{\pm}}$
- $\zeta \in K^s_{\lambda^*}$ to specify that $\zeta_{\pm}|_{L_{\lambda^*} \cap L'_{\lambda_{\diamond}}} \in L^s_{\lambda^*}$
- $\zeta \in K^{s'}_{\lambda}$ to specify that $\zeta_{\pm}|_{L_{\lambda^*} \cap L'_{\lambda_{\diamond}}} \in L^{s'}_{\lambda_{\pm}}$

where λ_{\pm} are defined at (4.1) and $\zeta_{\pm}|_{L_{\pm} \cap L'_{\lambda_{\diamond}}} \in L'_{\pm}$ means that the restriction of ζ_{\pm} to $L_{\pm} \cap L'_{\lambda_{\diamond}}$ is continuous with respect to relative topology generated by the strong topology of L_{\pm} on $L_{\pm} \cap L'_{\lambda_{\diamond}}$.

- (1) The sets $K''_{\lambda}, K'_{\lambda^*}, K_{\lambda}, K_{\lambda^*}^s$ and $K_{\lambda}^{s'}$ are defined to be the corresponding subsets of $L_{\lambda_{\diamond}}^b$. They are not vector spaces in general but convex cones with vertex 0.
- (2) The $\sigma(K''_{\lambda}, K'_{\lambda})$ -closure \overline{A} of a set A is defined as follows: $\zeta \in L_{\lambda_{\diamond}}^b$ is in \overline{A} if $\zeta_{\pm}|_{L'_{\lambda_{\pm}} \cap L'_{\lambda_{\diamond}}}$ is in the $\sigma(L''_{\lambda_{\pm}} \cap L_{\lambda_{\diamond}}^b, L'_{\lambda_{\pm}} \cap L'_{\lambda_{\diamond}})$ -closure of $A_{\pm} = \{\zeta_{\pm}; \zeta \in A\}$. Clearly, $\overline{A}_{\pm} = \{\zeta_{\pm}; \zeta \in \overline{A}\}$.
One defines similarly the $\sigma(K'_{\lambda^*}, K_{\lambda^*})$, $\sigma(K_{\lambda}, K'_{\lambda})$, $\sigma(K_{\lambda^*}^s, K_{\lambda^*})$ and $\sigma(K_{\lambda}^{s'}, K_{\lambda}^s)$ -closures.
- (3) Let A be a subset of $L_{\lambda_{\diamond}}$. Its strong closure $s\text{-cl } A$ in K_{λ} is the set of all measurable functions u such that u_{\pm} is in the $\|\cdot\|_{\lambda_{\pm}}$ -closure of $A_{\pm} = \{v_{\pm}; v \in A\}$.

Let ρ be a Young function. By Theorem 2.3, we have $L''_{\rho} = [L_{\rho} \cdot R \oplus L_{\rho}^s] \oplus L_{\rho}^{s'}$. For any $\zeta \in L''_{\rho} = (L_{\rho} \cdot R \oplus L_{\rho}^s)'$, let us denote the restrictions $\zeta_1 = \zeta|_{L_{\rho^*} \cdot R}$ and $\zeta_2 = \zeta|_{L_{\rho}^s}$. Since, $(L_{\rho} \cdot R)' \simeq L_{\rho} \oplus L_{\rho^*}^s$, one sees that any $\zeta \in L''_{\rho}$ is uniquely decomposed into

$$\zeta = \zeta_1^a + \zeta_1^s + \zeta_2 \quad (4.3)$$

with $\zeta_1 = \zeta_1^a + \zeta_1^s \in L'_{\rho^*}$, $\zeta_1^a \in L_{\rho}$, $\zeta_1^s \in L_{\rho^*}^s$ and $\zeta_2 \in L_{\rho}^{s'}$. With our definitions, $K''_{\lambda} = [K_{\lambda} \oplus K_{\lambda^*}^s] \oplus K_{\lambda}^{s'}$ and the decomposition (4.3) holds for any $\zeta \in K''_{\lambda}$ with

$$\begin{cases} \zeta_1 = \zeta_1^a + \zeta_1^s \in K_{\lambda} \oplus K_{\lambda^*}^s = K'_{\lambda^*}, \\ \zeta_2 \in K_{\lambda}^{s'}. \end{cases}$$

4.2. The ingredients of the saddle-point method. One applies the abstract results of Section 3.1 with

$$\Phi(u) = I_{\lambda}(u) := \int_{\mathcal{Z}} \lambda(u) dR, \quad u \in \mathcal{U}_o := \mathcal{L}_{\lambda_o} \quad (4.4)$$

This gives $\mathcal{U} = L_{\lambda_o}$ with the Orlicz norm $|u|_{\Phi} = \|u\|_{\lambda_o}$ and $\mathcal{L} = L'_{\lambda_o} = L_{\lambda_o^*} \cdot R \oplus L_{\lambda_o}^s$, by Theorem 2.3. The space \mathcal{Y} is the completion of \mathcal{Y}_o endowed with the norm $|y|_{\Lambda} = \|\langle y, \theta \rangle\|_{\lambda_o}$. One denotes $\mathcal{Y} = \mathcal{Y}_L$. It is isomorphic to the closure of the subspace $\{\langle y, \theta \rangle; y \in \mathcal{Y}_o\}$ in L_{λ_o} , see assumption (A_{θ}^{\exists}) . With some abuse of notation, one still denotes $T^*y = \langle y, \theta \rangle$ for $y \in \mathcal{Y}_L$. Remark that this can be interpreted as a dual bracket between \mathcal{X}_o^* and \mathcal{X}_o since $T^*y = \langle \tilde{y}, \theta \rangle$ R -a.e. for some $\tilde{y} \in \mathcal{X}_o^*$. The topological dual space $\mathcal{X}_L = \mathcal{Y}'_L$ is identified with $L'_{\lambda_o}/\ker T$ and its norm is given by $|x|_{\Lambda}^* = \inf\{\|\ell\|_{\lambda_o}^*; \ell \in L'_{\lambda_o} : T(\ell) = x\}$. This last identity is a dual equality as in Theorem 3.2-b with $\Phi = \iota_B$ where B is the unit ball of L_{λ_o} and $C = \{x\}$.

The assumption (H_C) that C is $\sigma(\mathcal{X}_L, \mathcal{Y}_L)$ -closed convex is equivalent to

$$T_o^{-1}C \cap L'_{\lambda_o} = \bigcap_{y \in Y} \{\ell \in L'_{\lambda_o}; \langle \langle y, \theta \rangle, \ell \rangle \geq a_y\} \quad (4.5)$$

for some subset $Y \subset \mathcal{Y}_L$ and some functions $y \in Y \mapsto a_y \in \mathbb{R}$. For comparison, note that if C is only supposed to be convex, $\bigcap_{(y,a) \in A} \{\ell \in L'_{\lambda_o}; \langle \langle y, \theta \rangle, \ell \rangle > a\}$ with $A \subset \mathcal{Y} \times \mathbb{R}$ is the general shape of $T^{-1}C$.

4.3. The main result. Let us define

$$\Gamma^*(x) = \sup_{y \in \mathcal{Y}_o} \{\langle y, x \rangle - I_{\gamma}(\langle y, \theta \rangle)\}, \quad x \in \mathcal{X}_o$$

which is the convex conjugate of $\Gamma(y) = I_\gamma(\langle y, \theta \rangle)$, $y \in \mathcal{Y}_o$. The dual problem (\mathcal{D}) associated with (P_C) and (\overline{P}_C) is

$$\text{maximize} \quad \inf_{x \in C \cap \mathcal{X}} \langle y, x \rangle - I_\gamma(\langle y, \theta \rangle), \quad y \in \mathcal{Y} \quad (\mathcal{D}_C)$$

The extended dual problem is

$$\text{maximize} \quad \inf_{x \in C \cap \mathcal{X}_L} \langle \omega, x \rangle - I_\lambda([T^* \omega]_1^a) + \iota_{\text{dom } I_\lambda^*}^*([T^* \omega]_1^s) + \iota_D([T^* \omega]_2), \quad \omega \in \overline{\mathcal{Y}} \quad (\overline{\mathcal{D}}_C)$$

where

- $T^* : \mathcal{X}_L^* \rightarrow L_{\lambda_o}^*$ is the extension of T_o^* which is defined at Section 3.1,
- D is the $\sigma(K_\lambda^{s'}, K_\lambda^s)$ -closure of $\text{dom } I_\lambda$ and
- $\overline{\mathcal{Y}}$ is the cone of all $\omega \in \mathcal{X}_L^*$ such that $T^* \omega \in K_\lambda''$.

Clearly, $\iota_{\text{dom } I_\lambda^*}^*(\zeta_1^s) = \iota_{\text{dom } I_{\lambda_+}^*}^*(\zeta_{1+}^s) + \iota_{\text{dom } I_{\lambda_-}^*}^*(\zeta_{1-}^s)$ and $\iota_D(\zeta_2) = \iota_{D_+}(\zeta_{2+}) + \iota_{D_-}(\zeta_{2-})$ where D_\pm is the $\sigma(L_{\lambda_\pm}^{s'} \cap L_{\lambda_o}^{t_b}, L_{\lambda_\pm}^s \cap L_{\lambda_o}^{t_b})$ -closure of $\text{dom } I_{\lambda_\pm}$.

As R is assumed to be σ -finite, there exists a measurable partition $(\mathcal{Z}_k)_{k \geq 1}$ of $\mathcal{Z} : \bigsqcup_k \mathcal{Z}_k = \mathcal{Z}$, such that $R(\mathcal{Z}_k) < \infty$ for each $k \geq 1$.

Theorem 4.6. *Suppose that*

- (1) *the assumptions (A) are satisfied;*
- (2) *for each $k \geq 1$, $L_{\lambda_o}(\mathcal{Z}_k, R|_{\mathcal{Z}_k})$ is dense in $L_{\lambda_+}(\mathcal{Z}_k, R|_{\mathcal{Z}_k})$ and $L_{\lambda_-}(\mathcal{Z}_k, R|_{\mathcal{Z}_k})$ with respect to the topologies associated with $\|\cdot\|_{\lambda_+}$ and $\|\cdot\|_{\lambda_-}$;*
- (3) *C satisfies (4.5) with $\langle y, \theta \rangle \in L_{\lambda_o}$ for all $y \in Y$.*

Then:

- (a) *The dual equality for (\overline{P}_C) is*

$$\inf(\overline{P}_C) = \inf_{x \in C} \Gamma^*(x) = \sup(\mathcal{D}_C) = \sup(\overline{\mathcal{D}}_C) \in [0, \infty].$$

- (b) *If $C \cap \text{dom } \Gamma^* \neq \emptyset$ or equivalently $C \cap T_o \text{dom } \bar{I} \neq \emptyset$, then (\overline{P}_C) admits solutions in L_{λ_o}' , any minimizing sequence admits $\sigma(L_{\lambda_o}', L_{\lambda_o})$ -cluster points and every such point is a solution to (\overline{P}_C) .*

Suppose that in addition we have

$$C \cap \text{icordom } \Gamma^* \neq \emptyset \quad (4.7)$$

or equivalently $C \cap \text{icor}(T_o \text{dom } \bar{I}) \neq \emptyset$. Then:

- (c) *Let us denote $\hat{x} \triangleq T^* \hat{\ell}$. There exists $\bar{\omega} \in \overline{\mathcal{Y}}$ such that*

$$\begin{cases} (a) & \hat{x} \in C \cap \text{dom } \Gamma^* \\ (b) & \langle \bar{\omega}, \hat{x} \rangle_{\mathcal{X}_L^*, \mathcal{X}_L} \leq \langle \bar{\omega}, x \rangle_{\mathcal{X}_L^*, \mathcal{X}_L}, \forall x \in C \cap \text{dom } \Gamma^* \\ (c) & \hat{\ell} \in \gamma_z'([T^* \bar{\omega}]_1^a) R + D^\perp([T^* \bar{\omega}]_2) \end{cases} \quad (4.8)$$

where

$$D^\perp(\eta) = \{k \in L_{\lambda_o}^s; \forall h \in L_{\lambda_o}, \eta + h \in D \Rightarrow \langle h, k \rangle \leq 0\}$$

is the outer normal cone of D at η .

$T^ \bar{\omega}$ is in the $\sigma(K_\lambda'', K_\lambda')$ -closure of $T^*(\text{dom } \Lambda)$ and there exists some $\tilde{\omega} \in \mathcal{X}_o^*$ such that*

$$[T^* \bar{\omega}]_1^a = \langle \tilde{\omega}, \theta(\cdot) \rangle_{\mathcal{X}_o^*, \mathcal{X}_o}$$

is a measurable function in the strong closure of $T^(\text{dom } \Lambda)$ in K_λ .*

Furthermore, $\hat{\ell} \in L_{\lambda_o}'$ and $\bar{\omega} \in \overline{\mathcal{Y}}$ satisfy (4.8) if and only if $\hat{\ell}$ solves (\overline{P}_C) and $\bar{\omega}$ solves $(\overline{\mathcal{D}}_C)$.

- (d) *Of course, (4.8-c) implies $\hat{x} = \int_{\mathcal{Z}} \theta \gamma'(\langle \tilde{\omega}, \theta \rangle) dR + \langle \theta, \hat{\ell}^s \rangle$. Moreover,*

1. \hat{x} minimizes Γ^* on C ,
2. $\bar{I}(\hat{\ell}) = \Gamma^*(\hat{x}) = \int_{\mathcal{Z}} \gamma^* \circ \gamma'(\langle \tilde{\omega}, \theta \rangle) dR + \sup\{\langle u, \hat{\ell}^s \rangle; u \in \text{dom } I_\gamma\} < \infty$ and
3. $\bar{I}(\hat{\ell}) + \int_{\mathcal{Z}} \gamma(\langle \tilde{\omega}, \theta \rangle) dR = \int_{\mathcal{Z}} \langle \tilde{\omega}, \theta \rangle d\hat{\ell}^a + \langle [T^* \tilde{\omega}]_2, \hat{\ell}^s \rangle_{K_{\lambda}^s, K_{\lambda}^s}$.

Proposition 4.9. *For the assumption (2) of Theorem 4.6 to be satisfied, it is enough that one of these conditions holds*

- (i) λ is even or more generally $0 < \liminf_{t \rightarrow \infty} \frac{\lambda_+}{\lambda_-}(t) \leq \limsup_{t \rightarrow \infty} \frac{\lambda_+}{\lambda_-}(t) < +\infty$;
- (ii) $\lim_{t \rightarrow \infty} \frac{\lambda_+}{\lambda_-}(t) = +\infty$ and λ_- satisfies the Δ_2 -condition (2.6).

Proof. It is enough to work with a bounded measure R .

Condition (i) is equivalent to $L_{\lambda_+} = L_{\lambda_-} = L_{\lambda_\diamond}$ and the result follows immediately.

Condition (ii) says that $\lambda_+ = \lambda_\diamond$ and $L_{\lambda_-} = E_{\lambda_-}$. As γ^* is assumed to be strictly convex, zero is in the interior of $\text{dom } \lambda$ and L_{λ_\diamond} contains the space B of all bounded measurable functions. But B is dense in E_{λ_-} and the result follows. \square

Remarks 4.10. General remarks about Theorem 4.6.

- (a) The assumption (3) is equivalent to C is $\sigma(\mathcal{X}_L, \mathcal{Y}_L)$ -closed convex.
- (b) The dual equality with $C = \{x\}$ gives for all $x \in \mathcal{X}_o$

$$\Gamma^*(x) = \inf \{ \bar{I}(\ell); \ell \in L'_{\lambda_\diamond}, \langle \theta, \ell \rangle = x \}.$$

- (c) Note that $\bar{\omega}$ does not necessarily belong to \mathcal{Y}_o . Therefore, the Young equality $\langle \bar{\omega}, \hat{x} \rangle = \Gamma^*(\hat{x}) + \Gamma(\bar{\omega})$ is meaningless. Nevertheless, there exists a natural extension $\bar{\Gamma}$ of Γ such that $\langle \hat{x}, \bar{\omega} \rangle = \Gamma^*(\hat{x}) + \bar{\Gamma}(\bar{\omega})$ holds, see (3.5). This gives the statement (d-3).
- (d) Removing the assumption $(A_{\gamma^*}^2)$: $m \in L_{\lambda_\diamond}^*$, one can still consider the minimization problem

$$\text{minimize } \bar{I}(\ell) \text{ subject to } \langle \theta, \ell - mR \rangle \in C_o, \quad \ell \in mR + L'_{\lambda_\diamond} \quad (\bar{P}_{C_o})$$

instead of (\bar{P}_C) . The transcription of Theorem 4.6 is as follows. Denote

$$\Lambda^*(x) = \sup_{y \in \mathcal{Y}_o} \left\{ \langle y, x \rangle - \int_{\mathcal{Z}} \lambda(\langle y, \theta \rangle) dR \right\}, \quad x \in \mathcal{X}_o$$

and replace respectively (\bar{P}_C) , C , Γ^* , \hat{x} and γ by (\bar{P}_{C_o}) , C_o , Λ^* , \tilde{x} and λ where $\tilde{x} = \langle \theta, \hat{\ell} - mR \rangle$ is well-defined.

The statement (b) must be replaced by the following one: *If $C_o \cap \text{dom } \Lambda^* \neq \emptyset$, then (\bar{P}_{C_o}) admits solutions in $mR + L'_{\lambda_\diamond}$, any minimizing sequence $(\ell_n)_{n \geq 1}$ is such that $(\ell_n - mR)_{n \geq 1}$ admits cluster points $\hat{\ell} - mR$ in L'_{λ_\diamond} with respect to the topology $\sigma(L'_{\lambda_\diamond}, L_{\lambda_\diamond})$ and $\hat{\ell}$ is a solution of (\bar{P}_{C_o}) .*

Proof of Theorem 4.6. It is an application of Theorems 3.2 and 3.3. We use the notation and framework of Section 3.1.

With (4.4) and Theorem 3.2-a, $\text{dom } \Phi^* \subset \mathcal{L} = L'_{\lambda_\diamond}$. For all $\ell \in L'_{\lambda_\diamond}$,

$$\begin{aligned} \Phi^*(\ell) &\stackrel{(a)}{=} \Phi_+^*(\ell_+) + \Phi_-^*(\ell_-) \\ &\stackrel{(b)}{=} \inf \{ I_{\lambda_+}^*(k^a) + \iota_{\text{dom } \lambda_+}^*(k^s); k \in L'_{\lambda_+} : k \geq 0, k|_{L_{\lambda_\diamond}} = \ell_+ \} \\ &\quad + \inf \{ I_{\lambda_-}^*(k^a) + \iota_{\text{dom } \lambda_-}^*(k^s); k \in L'_{\lambda_-} : k \geq 0, k|_{L_{\lambda_\diamond}} = \ell_- \} \end{aligned}$$

Equality (a) comes from Proposition 3.10-a and equality (b) is a dual equality of the type of Theorem 3.2-b applied with

$$I_\rho^*(k) = I_\rho^*(k^a) + \iota_{\text{dom } \rho}^*(k^s) \quad k \in L'_\rho \quad (4.11)$$

which holds for any Young function ρ . This identity is proved by Fougères, Giner, Kozek and Rockafellar [11, 13, 21] under the assumptions (A_R) and $(A_{\gamma^*}^1)$. The function I_ρ^* is strongly continuous on $\text{icordom } I_\rho^* \subset L'_\rho$, see [14, Lemma 2.1]. Hence, under the assumption (2), we obtain that

$$\bar{I}(\ell) = \Phi^*(\ell - mR), \quad \ell \in L'_{\lambda_\diamond} \quad (4.12)$$

taking advantage of the direct sum $\ell = \oplus_k \ell|_{\mathcal{Z}_k}$ acting on $u = (u|_{\mathcal{Z}_k})_{k \geq 1}$ which lead to the nonnegative series $\Phi(u) = \oplus_k \Phi(u|_{\mathcal{Z}_k})$ and $\Phi^*(\ell) = \sum_k \Phi^*(\ell|_{\mathcal{Z}_k})$.

• *Reduction to $m = 0$.* We have seen at (2.10) that the transformation $Q \rightsquigarrow \ell = Q - mR$ corresponds to the transformations $\gamma \rightsquigarrow \lambda$ and $(P_C) \rightsquigarrow (2.10)$. This still works with (\bar{P}_C) and one can assume from now on without loss of generality that $m = 0$ and $\gamma = \lambda$.

The assumption $(A_{\gamma^*}^2)$ will not be used during the rest of the proof. This allows Remark 4.10-d.

• *Verification of (H_Φ) and (H_T) .* Suppose that $W = \{z \in \mathcal{Z}; \lambda(z, s) = 0, \forall s \in \mathbb{R}\}$ is such that $R(W) > 0$. Then, any ℓ such that $\langle u \mathbf{1}_W, \ell \rangle > 0$ for some $u \in L_{\lambda_\diamond}$ satisfies $\Phi^*(\ell) = +\infty$. Therefore, one can remove W from \mathcal{Z} without loss of generality. Once, this is done, the hypothesis (H_Φ) is satisfied under the assumption $(A_{\gamma^*}^1)$. The hypothesis (H_{T1}) is (A_θ^\exists) while (H_{T2}) is (A_θ^2) .

• *The computation of $\bar{\Phi}$ in the case where λ is even.* Since Φ is even, Theorem 3.6 tells us that $\text{dom } \bar{\Phi}$ is included in the $\sigma(L''_\lambda, L'_\lambda)$ -closure of $\text{dom } \Phi$. Thanks to (4.11) and the decomposition (4.3), the extension $\bar{\Phi}$ is given for each $\zeta \in L''_\lambda$ by

$$\begin{aligned} \bar{\Phi}(\zeta) &= (\bar{I}_{\lambda^*})^*(\zeta_1, \zeta_2) \\ &= \sup_{f \in L_{\lambda_\diamond}^*, k \in L_{\lambda_\diamond}^s} \{ \langle \zeta_1, fR \rangle + \langle \zeta_2, k \rangle - I_{\lambda^*}(fR) - \iota_{\text{dom } I_\lambda}^*(k) \} \\ &= I_{\lambda^*}^*(\zeta_1) + \iota_{\text{dom } I_\lambda}^{**}(\zeta_2) \\ &= \bar{I}_\lambda(\zeta_1) + \iota_D(\zeta_2) \\ &= I_\lambda(\zeta_1^a) + \iota_{\text{dom } I_{\lambda^*}}^*(\zeta_1^s) + \iota_D(\zeta_2) \end{aligned}$$

where D is the $\sigma(L_\lambda^s, L_\lambda^s)$ -closure of $\text{dom } I_\lambda$ and we dropped the restrictions $\zeta|_L$ for simplicity.

• *Extension to the case where λ is not even.* By Proposition 3.10-b, we have $\bar{\Phi}(\zeta) = \bar{\Phi}_+(\zeta_{+|L'_{\lambda_+} \cap L'_{\lambda_\diamond}}) + \bar{\Phi}_-(\zeta_{-|L'_{\lambda_-} \cap L'_{\lambda_\diamond}})$ if $\zeta \in L_{\lambda_\diamond}^b$ and $+\infty$ otherwise. It follows that

$$\bar{\Phi}(\zeta) = \bar{I}_\lambda(\zeta) = I_\lambda(\zeta_1^a) + \iota_{\text{dom } I_{\lambda^*}}^*(\zeta_1^s) + \iota_D(\zeta_2) \quad (4.13)$$

if $\zeta \in K''_\lambda$ and $+\infty$ otherwise. In particular, we have

$$\begin{aligned} \Lambda(y) &= I_\lambda(\langle y, \theta \rangle), \quad y \in \mathcal{Y} \\ \bar{\Lambda}(\omega) &= \begin{cases} I_\lambda([T^*\omega]_1^a) + \iota_{\text{dom } I_{\lambda^*}}^*([T^*\omega]_1^s) + \iota_D([T^*\omega]_2) & \text{if } \omega \in \bar{\mathcal{Y}} \\ +\infty & \text{otherwise} \end{cases}, \quad \omega \in \mathcal{X}_L^*. \end{aligned}$$

This provides us with the dual problems (D_C) and (\bar{D}_C) .

• *Proof of (a) and (b).* Apply Theorem 3.2. \square

Let us go on with the proof of (c). By Theorem 3.3, (\bar{P}_C, \bar{D}_C) admits a solution in $L'_{\lambda_\diamond} \times \bar{\mathcal{Y}}$ and $(\hat{\ell}, \bar{\omega}) \in L'_{\lambda_\diamond} \times \bar{\mathcal{Y}}$ solves (\bar{P}_C, \bar{D}_C) if and only if

$$\begin{cases} (a) & \hat{x} \in C \cap \text{dom } \Gamma^* \\ (b) & \langle \bar{\omega}, \hat{x} \rangle \leq \langle \bar{\omega}, x \rangle, \forall x \in C \cap \text{dom } \Gamma^* \\ (c) & \hat{\ell} \in \partial_{L'_{\lambda_\diamond}} \bar{\Phi}(T^*\bar{\omega}) \end{cases} \quad (4.14)$$

where $\hat{x} \triangleq T\hat{\ell}$ is defined in the weak sense with respect to the duality $\langle \mathcal{Y}_L, \mathcal{X}_L \rangle$. Since $\text{dom } \Gamma^* \subset \mathcal{X}_L$, the above dual brackets are meaningful.

• *The computation of $\partial_{L'_\lambda} \bar{\Phi}(\zeta)$.* Let us first assume that λ is even. For all $u \in L_\lambda$, $u_1^a = u_2 = u$ and $u_1^s = 0$. This gives $\bar{\Phi}(\zeta + u) - \bar{\Phi}(\zeta) = I_\lambda(\zeta_1^a + u_1) - I_\lambda(\zeta_1^a) + \iota_D(\zeta_2 + u_2) - \iota_D(\zeta_2)$ where $u_1 = u$ and $u_2 = u$ act respectively on $L_{\lambda^*}R$ and L_λ^s . This direct sum structure leads us to

$$\partial_{L'_\lambda} \bar{\Phi}(\zeta) = \partial_{L_{\lambda^*}R} I_\lambda(\zeta_1^a) + \partial_{L_\lambda^s} \iota_D(\zeta_2). \quad (4.15)$$

which again is the direct sum of the absolutely continuous and singular components of $\partial_{L'_\lambda} \bar{\Phi}(\zeta)$. Differentiating in the directions of $\mathcal{U} = L_\lambda$, one obtains $\partial_{L_{\lambda^*}R} I_\lambda(\zeta_1^a) = \{\lambda'(\zeta_1^a)R\}$. The computation of $\partial_{L_\lambda^s} \iota_D(\zeta_2)$ is standard: $\partial_{L_\lambda^s} \iota_D(\zeta_2) = D^\perp(\zeta_2)$ is the outer normal cone of D at ζ_2 .

Now, consider a general λ . By Proposition 3.10-a, $\hat{\ell}_+ \in \partial_{L'_{\lambda_+} \cap L'_{\lambda_o}} \bar{\Phi}_+([T^*\bar{\omega}]_+)$ and $\hat{\ell}_- \in \partial_{L'_{\lambda_-} \cap L'_{\lambda_o}} \bar{\Phi}_-([T^*\bar{\omega}]_-)$. Therefore, (4.15) becomes

$$\partial_{L'_{\lambda_o}} \bar{\Phi}(\zeta) = \partial_{K_{\lambda^*}R \cap L_{\lambda_o^*}R} I_\lambda(\zeta_1^a) + \partial_{K_{\lambda^s} \cap L_{\lambda_o^s}} \iota_D(\zeta_2).$$

• *Representation of $[T^*\bar{\omega}]_1^a$.* One still has to prove that

$$[T^*\bar{\omega}]_1^a(z) = \langle \theta(z), \tilde{\omega} \rangle \quad (4.16)$$

for R -a.e. $z \in \mathcal{Z}$ and some linear form $\tilde{\omega}$ on \mathcal{X}_o .

If $W_- := \{z \in \mathcal{Z}; \lambda(z, s) = 0, \forall s \leq 0\}$ satisfies $R(W_-) > 0$, $\text{dom } \bar{I}$ is a set of linear forms which are nonnegative on W and $\gamma'_z(s) = 0$ for all $s \leq 0, z \in W$. Hence, one can take any function for the restriction to W_- of $[T^*\bar{\omega}]_1^a$ without modifying (4.14)-c. As a symmetric remark holds for $W_+ = \{z \in \mathcal{Z}; \lambda(z, s) = 0, \forall s \geq 0\}$, it remains to consider the situation where for R -a.e. z , there are $s_-(z) < 0 < s_+(z)$ such that $\lambda(z, s_\pm(z)) > 0$. This implies that $\lim_{s \rightarrow \pm\infty} \lambda(z, s)/s > 0$.

By Theorem 3.3, $T^*\bar{\omega}$ is in the $\sigma(K''_\lambda, K'_\lambda)$ -closure of $T^*(\text{dom } \Lambda)$. Therefore, $[T^*\bar{\omega}]_1^a$ is in the $\sigma(K_\lambda, K'_\lambda)$ -closure of $T^*(\text{dom } \Lambda)$. As $T^*(\text{dom } \Lambda)$ is convex, this closure is its strong closure in K_λ . Since there exists a finite measurable function $c(z)$ such that $0 < c(z) \leq \lim_{s \rightarrow \infty} \lambda(z, s)/s$, one can consider the nontrivial Young function $\rho(z, s) = c(z)|s|$ and the corresponding Orlicz spaces L_ρ and $L'_\rho = L_{\rho^*}$. If R is a bounded measure, we have $L_{\lambda_o} \subset L_\rho$ and $L_{\rho^*} \subset L_{\lambda_o^*}$, so that $[T^*\bar{\omega}]_1^a$ is in the strong closure of $T^*(\text{dom } \Lambda)$ in L_ρ .

As a consequence, $[T^*\bar{\omega}]_1^a$ is the pointwise limit of a sequence $(T^*y_n)_{n \geq 1}$ with $y_n \in \mathcal{Y}$. As $T^*y_n(z) = \langle y_n, \theta(z) \rangle$, we see that $[T^*\bar{\omega}]_1^a(z) = \langle \theta(z), \tilde{\omega} \rangle$ for some linear form $\tilde{\omega}$ on \mathcal{X}_o . If R is unbounded, it is still assumed to be σ -finite: there exists a sequence (\mathcal{Z}_k) of measurable subsets of \mathcal{Z} such that $\cup_k \mathcal{Z}_k = \mathcal{Z}$ and $R(\mathcal{Z}_k) < \infty$ for each k . Hence, for each k and all $z \in \mathcal{Z}_k$, $(T^*\bar{\omega})^a(z) = \langle \theta(z), \tilde{\omega}^k \rangle$ for some linear form $\tilde{\omega}^k$ on \mathcal{X}_o , from which (4.16) follows.

• *Proof of (c).* It follows from the previous considerations and Theorem 3.3.

• *Proof of (d).* Statement (d)-1 follows from Theorem 3.2. Statement (d)-2 is immediately deduced from (c). Finally, (d)-3 is (3.5). \square

5. SOLVING (P_C)

The general assumptions (A) are imposed and we study (P_C) under the additional good constraint assumption (A_θ[∇]) which imposes that the convex set C is such that

$$T_o^{-1}C \cap L_{\lambda_o^*}R = \bigcap_{y \in Y} \left\{ fR \in L_{\lambda_o^*}R; \int_{\mathcal{Z}} \langle y, \theta \rangle f dR \geq a_y \right\} \quad (5.1)$$

for some subset $Y \in \mathcal{X}_o^*$ such that $\langle y, \theta \rangle \in E_{\lambda_o}$ for all $y \in Y$ and some function $y \in Y \mapsto a_y \in \mathbb{R}$.

The dual problem (\mathcal{D}) associated with (P_C) is (D_C) and the extended dual problem is

$$\text{maximize } \inf_{x \in C} \langle \omega, x \rangle - I_\gamma(\langle \omega, \theta \rangle), \quad \omega \in \tilde{\mathcal{Y}} \quad (\tilde{D}_C)$$

where $\tilde{\mathcal{Y}}$ is the convex cone of all linear forms ω on \mathcal{X}_o which are such that

- the function $\langle \omega, \theta(\cdot) \rangle_{\mathcal{X}_o^*, \mathcal{X}_o}$ is measurable;
- $\int_{\mathcal{Z}} \lambda(t \langle \omega, \theta(\cdot) \rangle) dR < \infty$ for some $t > 0$;
- $\langle \omega, \theta(\cdot) \rangle$ is in the $\sigma(K_\lambda, K_{\lambda^*})$ -closure of $\{\langle y, \theta \rangle; y \in \mathcal{Y}_o\}$.

Theorem 5.2. *Suppose that*

- (1) *the assumptions (A) and (A_θ^\forall) are satisfied;*
- (2) *for R -almost every $z \in \mathcal{Z}$, $\lim_{t \rightarrow \pm\infty} \gamma_z^*(t)/t = +\infty$;*
- (3) *C satisfies (5.1) with $\langle y, \theta \rangle \in E_{\lambda_o}$ for all $y \in Y$.*

Then:

- (a) *The dual equality for (P_C) is*

$$\inf(P_C) = \sup(D_C) = \sup(\tilde{D}_C) = \inf_{x \in C} \Gamma^*(x) \in [0, \infty].$$

- (b) *If $C \cap \text{dom } \Gamma^* \neq \emptyset$ or equivalently $C \cap T_o \text{dom } I \neq \emptyset$, then (P_C) admits a unique solution \hat{Q} in $L_{\lambda_o^*} R$ and any minimizing sequence $(Q_n)_{n \geq 1}$ converges to \hat{Q} with respect to the topology $\sigma(L_{\lambda_o^*} R, E_{\lambda_o})$.*

Suppose that in addition $C \cap \text{icor} \text{dom } \Gamma^ \neq \emptyset$ or equivalently $C \cap \text{icor}(T_o \text{dom } I) \neq \emptyset$.*

- (c) *Let us define $\hat{x} \triangleq \int_{\mathcal{Z}} \theta d\hat{Q}$ in the weak sense with respect to the duality $\langle \mathcal{Y}_o, \mathcal{X}_o \rangle$. There exists $\tilde{\omega} \in \tilde{\mathcal{Y}}$ such that*

$$\begin{cases} (a) & \hat{x} \in C \cap \text{dom } \Gamma^* \\ (b) & \langle \tilde{\omega}, \hat{x} \rangle_{\mathcal{X}_o^*, \mathcal{X}_o} \leq \langle \tilde{\omega}, x \rangle_{\mathcal{X}_o^*, \mathcal{X}_o}, \forall x \in C \cap \text{dom } \Gamma^* \\ (c) & \hat{Q}(dz) = \gamma'_z(\langle \tilde{\omega}, \theta(z) \rangle) R(dz). \end{cases} \quad (5.3)$$

Furthermore, $\hat{Q} \in L_{\lambda_o^} R$ and $\tilde{\omega} \in \tilde{\mathcal{Y}}$ satisfy (5.3) if and only if \hat{Q} solves (P_C) and $\tilde{\omega}$ solves (\tilde{D}_C) .*

- (d) *Of course, (5.3-c) implies $\hat{x} = \int_{\mathcal{Z}} \theta \gamma'(\langle \tilde{\omega}, \theta \rangle) dR$ in the weak sense. Moreover,*

1. \hat{x} *minimizes Γ^* on C ,*
2. $I(\hat{Q}) = \Gamma^*(\hat{x}) = \int_{\mathcal{Z}} \gamma^* \circ \gamma'(\langle \tilde{\omega}, \theta \rangle) dR < \infty$ *and*
3. $I(\hat{Q}) + \int_{\mathcal{Z}} \gamma(\langle \tilde{\omega}, \theta \rangle) dR = \int_{\mathcal{Z}} \langle \tilde{\omega}, \theta \rangle d\hat{Q}.$

Proof. It is a corollary of the proof of Theorem 4.6. One applies the abstract results of Section 3.1 with

$$\Phi(u) = I_\lambda(u) := \int_{\mathcal{Z}} \lambda(u) dR, \quad u \in \mathcal{U}_o := \mathcal{E}_{\lambda_o} \quad (5.4)$$

This gives $\mathcal{U} = E_{\lambda_o}$ with the Orlicz norm $|u|_\Phi = \|u\|_{\lambda_o}$ and $\mathcal{L} = L_{\lambda_o^*} R$. The space $\mathcal{Y} := \mathcal{Y}_E$ is the completion of \mathcal{Y}_o endowed with the norm $|y|_\Lambda = \|\langle y, \theta \rangle\|_{\lambda_o}$. It is isomorphic to the closure of the subspace $\{\langle y, \theta \rangle; y \in \mathcal{Y}_o\}$ in E_{λ_o} , see assumption (A_θ^\forall) . The topological dual space $\mathcal{X}_E = \mathcal{Y}'_E$ is identified with $L_{\lambda_o^*} R / \ker T$ and its norm is given by $|x|_\Lambda^* = \inf\{\|f\|_{\lambda_o^*}; f \in L_{\lambda_o^*} : T(fR) = x\}$.

The assumption (3) is: C is a convex $\sigma(\mathcal{X}_E, \mathcal{Y}_E)$ -closed set.

As in the proof of Theorem 4.6, one reduces to the case where $m = 0$ without loss of generality.

The assumption (2) implies that λ is a finite function. It follows that $E'_{\lambda_\circ} = L_{\lambda_\circ^*}$, the convex conjugate Φ^* of Φ with respect to the duality $\langle E_{\lambda_\circ}, L_{\lambda_\circ^*} \rangle$ is

$$\Phi^* = I_{\lambda^*}$$

(see [20]) and the corresponding extended function $\bar{\Phi}$ is

$$\bar{\Phi}(\zeta) = I_\lambda(\zeta^a) + \iota_{\text{dom } I_{\lambda^*}}^*(\zeta^s)$$

if ζ is in $K_\lambda.R \oplus K_\lambda^s$ and $+\infty$ otherwise.

With these correspondences, the proof of the theorem is an immediate translation of the proof of Theorem 4.6. \square

Remarks 5.5.

- (a) The assumption (2) implies that λ is a finite function. Note that otherwise one would get $E_{\lambda_\circ} = \{0\}$.
- (b) As in Remark 4.10-d, removing the assumption $(A_{\gamma^*}^2)$: $m \in L_{\lambda_\circ^*}$, one can still consider the minimization problem (P_{C_o}) instead of (P_C) . The transcription of Theorem 5.2 is as follows. Replace respectively (P_C) , C , Γ^* , \hat{x} and γ by (P_{C_o}) , C_o , Λ^* , \tilde{x} and λ where $\tilde{x} = \int_{\mathcal{Z}} \theta d(\hat{Q} - mR)$ is well-defined. The statement (b) must be replaced by the following one: *If $C_o \cap \text{dom } \Lambda^* \neq \emptyset$, then (P_{C_o}) admits a unique solution \hat{Q} in $mR + L_{\lambda_\circ^*}R$ and any minimizing sequence $(Q_n)_{n \geq 1}$ is such that $(Q_n - mR)_{n \geq 1}$ converges in $L_{\lambda_\circ^*}R$ to $\hat{Q} - mR$ with respect to the topology $\sigma(L_{\lambda_\circ^*}R, E_{\lambda_\circ})$.*
- (c) Seeing Theorem 5.2 as a direct corollary of Theorem 4.6 would have been possible since Proposition 2.5 insures that $T_o^{-1}C \cap L'_{\lambda_\circ} = \bigcap_{y \in Y} \{\ell \in L'_{\lambda_\circ}; \langle \theta, \ell \rangle \geq a_y\} = \bigcap_{y \in Y} \{fR \in L_{\lambda_\circ^*}R; \int_{\mathcal{Z}} \langle y, \theta \rangle f dR \geq a_y\}$ whenever (A_θ^\forall) holds. But, the drawback is that the unnecessary assumption (2) of Theorem 4.6 has to be kept.

6. EXAMPLES

Standard examples of entropy minimization problems are presented.

6.1. Some examples of entropies. The entropies defined below occur naturally in statistical physics, probability theory, mathematical statistics and information theory.

Boltzmann entropy. The Boltzmann entropy with respect to the positive measure R is defined by $H_B(Q|R) = \begin{cases} \int_{\mathcal{Z}} \log\left(\frac{dQ}{dR}\right) dQ & \text{if } 0 \leq Q \prec R \\ +\infty, & \text{otherwise} \end{cases}$ for each $Q \in M_{\mathcal{Z}}$. It corresponds

to $\gamma_z^*(t) = \begin{cases} t \log t & \text{if } t > 0 \\ 0 & \text{if } t = 0 \\ +\infty & \text{if } t < 0 \end{cases}$. But this γ^* takes negative values and is ruled out by our

assumptions. A way to circumvent this problem is to consider the variant below.

A variant of the Boltzmann entropy. Let $m : \mathcal{Z} \rightarrow (0, \infty)$ be a positive measurable function. Considering

$$\gamma_z^*(t) = t \log t - [1 + \log m(z)]t + m(z), \quad t > 0,$$

one sees that it is nonnegative and that $\gamma_z^*(t) = 0$ if and only if $t = m(z)$. Hence γ^* enters the framework of this paper and

$$\lambda_z(s) = m(z)[e^s - s - 1], \quad s \in \mathbb{R}. \quad (6.1)$$

It is easily seen that

$$H_B(Q|R) = I_{\gamma^*}(Q) + \int_{\mathcal{Z}} (1 + \log m) dQ - \int_{\mathcal{Z}} m dR$$

which is meaningful if Q integrates $1 + \log m$ where $m \in L^1(R)$.

As an application, let R be the Lebesgue measure on $\mathcal{Z} = \mathbb{R}^d$ and minimize $H_B(Q|R)$ on the set $\mathcal{C} = \{Q \in P_{\mathcal{Z}}; \int_{\mathcal{Z}} |z|^2 Q(dz) = E\} \cap \mathcal{C}_0$. Taking $m(z) = e^{-|z|^2}$, one is led to minimizing I_{γ^*} on \mathcal{C} .

A special case. It is defined by

$$H(Q|R) = \begin{cases} \int_{\mathcal{Z}} \left[\frac{dQ}{dR} \log \left(\frac{dQ}{dR} \right) - \frac{dQ}{dR} + 1 \right] dR & \text{if } 0 \leq Q \prec R \\ +\infty & \text{otherwise} \end{cases}, \quad Q \in M_{\mathcal{Z}} \quad (6.2)$$

It corresponds to $\gamma_z^*(t) = \begin{cases} t \log t - t + 1 & \text{if } t > 0 \\ 1 & \text{if } t = 0 \\ +\infty & \text{if } t < 0 \end{cases}$, $m(z) = 1$ and $\lambda_z(s) = e^s - s - 1$,

$s \in \mathbb{R}$ for all $z \in \mathcal{Z}$. Note that $H(Q|R) < \infty$ implies that Q is nonnegative.

Relative entropy. The reference measure R is assumed to be a probability measure and one denotes $P_{\mathcal{Z}}$ the set of all probability measures on \mathcal{Z} . The relative entropy of $Q \in M_{\mathcal{Z}}$ with respect to $R \in P_{\mathcal{Z}}$ is the following variant of the Boltzmann entropy:

$$I(Q|R) = \begin{cases} \int_{\mathcal{Z}} \log \left(\frac{dQ}{dR} \right) dQ & \text{if } Q \prec R \text{ and } Q \in P_{\mathcal{Z}} \\ +\infty & \text{otherwise} \end{cases}, \quad Q \in M_{\mathcal{Z}}.$$

It is (6.2) with the additional constraint that $Q(\mathcal{Z}) = 1$:

$$I(Q|R) = H(Q|R) + \iota_{\{Q(\mathcal{Z})=1\}}$$

When minimizing the Boltzmann entropy $Q \mapsto H_B(Q|R)$ on a constraint set which is included in $P_{\mathcal{Z}}$, we have for all $P, Q \in P_{\mathcal{Z}}$,

$$H_B(Q|R) = I(Q|P) + \int_{\mathcal{Z}} \log \left(\frac{dP}{dR} \right) dQ$$

which is meaningful for each $Q \in P_{\mathcal{Z}}$ which integrates $\frac{dP}{dR}$.

Reverse relative entropy. The reference measure R is assumed to be a probability measure. The reverse relative entropy is

$$Q \in M_{\mathcal{Z}} \mapsto \begin{cases} I(R|Q) & \text{if } Q \in P_{\mathcal{Z}} \\ +\infty & \text{otherwise} \end{cases} \in [0, \infty].$$

It corresponds to $\gamma_z^*(t) = \begin{cases} -\log t + t - 1 & \text{if } t > 0 \\ +\infty & \text{if } t \leq 0 \end{cases}$, $m(z) = 1$ and

$$\lambda_z(s) = \begin{cases} -\log(1-s) - s & \text{if } s < 1 \\ +\infty & \text{if } s \geq 1 \end{cases}, \quad (6.3)$$

for all $z \in \mathcal{Z}$, with the additional constraint that $Q(\mathcal{Z}) = 1$.

6.2. Some examples of constraints. Let us consider two standard constraints which are the moment constraints and the marginal constraints.

Moment constraints. Let $\theta = (\theta_k)_{1 \leq k \leq K}$ be a measurable function from \mathcal{Z} to $\mathcal{X}_o = \mathbb{R}^K$. The moment constraint is specified by the operator

$$T_o \ell = \int_{\mathcal{Z}} \theta d\ell = \left(\int_{\mathcal{Z}} \theta_k d\ell \right)_{1 \leq k \leq K} \in \mathbb{R}^K,$$

which is defined for each $\ell \in M_{\mathcal{Z}}$ which integrates all the real valued measurable functions θ_k . The adjoint operator is

$$T_o^* y(z) = \sum_{1 \leq k \leq K} y_k \theta_k(z), \quad y = (y_1, \dots, y_K) \in \mathbb{R}^K, z \in \mathcal{Z}.$$

Marginal constraints. Let $\mathcal{Z} = A \times B$ be a product space, M_{AB} be the space of all *bounded* signed measures on $A \times B$ and U_{AB} be the space of all measurable bounded functions u on $A \times B$. Denote $\ell_A = \ell(\cdot \times B)$ and $\ell_B = \ell(A \times \cdot)$ the marginal measures of $\ell \in M_{AB}$. The constraint of prescribed marginal measures is specified by

$$\int_{A \times B} \theta d\ell = (\ell_A, \ell_B) \in M_A \times M_B, \quad \ell \in M_{AB}$$

where M_A and M_B are the spaces of all bounded signed measures on A and B . The function θ which gives the marginal constraint is

$$\theta(a, b) = (\delta_a, \delta_b), \quad a \in A, b \in B$$

where δ_a is the Dirac measure at a . Indeed, $(\ell_A, \ell_B) = \int_{A \times B} (\delta_a, \delta_b) \ell(dadb)$.

More precisely, let U_A, U_B be the spaces of measurable functions on A and B and take $\mathcal{Y}_o = U_A \times U_B$ and $\mathcal{X}_o = U_A^* \times U_B^*$. Then, θ is a measurable function from $\mathcal{Z} = A \times B$ to $\mathcal{X}_o = U_A^* \times U_B^*$. It is easy to see that the adjoint of the marginal operator

$$T_o \ell = (\ell_A, \ell_B) \in U_A^* \times U_B^*, \quad \ell \in \mathcal{L}_o = U_{AB}^*$$

where $\langle f, \ell_A \rangle := \langle f \otimes 1, \ell \rangle$ and $\langle g, \ell_B \rangle := \langle 1 \otimes g, \ell \rangle$ for all $f \in U_A$ and $g \in U_B$, is given by

$$T_o^*(f, g) = f \oplus g \in U_{AB}, \quad f \in U_A, g \in U_B \quad (6.4)$$

where $f \oplus g(a, b) := f(a) + g(b)$, $a \in A, b \in B$.

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MODAL-X, UNIVERSITÉ PARIS 10 & BÂT. G, 200 AV. DE LA RÉPUBLIQUE. 92001 NANTERRE CEDEX, FRANCE

CMAF, ÉCOLE POLYTECHNIQUE. 91128 PALAISEAU CEDEX, FRANCE
E-mail address: christian.leonard@polytechnique.fr