



**HAL**  
open science

# Derivation of asymptotic two-dimensional time-dependent equations for ocean wave propagation

David Lannes, Philippe Bonneton

► **To cite this version:**

David Lannes, Philippe Bonneton. Derivation of asymptotic two-dimensional time-dependent equations for ocean wave propagation. 2007. hal-00177251

**HAL Id: hal-00177251**

**<https://hal.science/hal-00177251>**

Preprint submitted on 6 Oct 2007

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Derivation of asymptotic two-dimensional time-dependent equations for ocean wave propagation

David Lannes<sup>(1)</sup> and Philippe Bonneton<sup>(2)</sup>

(1) Université Bordeaux 1; IMB and CNRS UMR 5251, Talence, F-33405 France

(2) Université Bordeaux 1; CNRS; UMR 5805-EPOC, Talence, F-33405 France

(e-mail: David.Lannes@math.u-bordeaux1.fr; p.bonneton@epoc.u-bordeaux1.fr )

## Abstract

A general method for the derivation of asymptotic nonlinear shallow water and deep water models is presented. Starting from a general dimensionless version of the water-wave equations, we reduce the problem to a system of two equations on the surface elevation and the velocity potential at the free surface. These equations involve a Dirichlet-Neumann operator and we show that all the asymptotic models can be recovered by a simple asymptotic expansion of this operator, in function of the shallowness parameter (shallow water limit) or the steepness parameter (deep water limit). Based on this method, a new two-dimensional fully dispersive model for small wave steepness is also derived, which extends to uneven bottom the approach developed by Matsuno [27] and Choi [28]. This model is still valid in shallow water but with less precision than what can be achieved with Green-Naghdi model, when fully nonlinear waves are considered. The combination, or the coupling, of the new fully dispersive equations with the fully nonlinear shallow water Green-Naghdi equations represents a relevant model for describing ocean wave propagation from deep to shallow waters.

# 1 Introduction

The propagation of surface waves through an incompressible homogenous inviscid fluid is described by the 3D Euler equations combined with nonlinear boundary conditions at the free-surface and at the bottom. This problem is extremely difficult to solve, in particular because the moving surface boundary is part of the solution. The complexity of this problem led physicists, oceanographers and mathematicians to derive simpler sets of equations in some specific physical regimes. Equations thus obtained may be divided into two groups, namely shallow-water models and deep-water models. One of the goals of this paper is to clarify the range of validity of these models and to show why at least two different asymptotic models are necessary for a correct description of ocean waves.

In shallow water conditions, the classical approach is based on a perturbation method with respect to a small parameter  $\mu = (h_0/L)^2$  ( $h_0$  the characteristic water depth,  $L$  the characteristic horizontal scale), in order to reduce the three-dimensional equation system to a two-dimensional one. This method, initially introduced by Boussinesq [1], allows to derive several shallow-water equations, which are named "Boussinesq-type equations". A large class of such equations can be expressed in the following 2D nondimensional form:

$$\begin{cases} \partial_t \zeta + \nabla \cdot (hV) = O(\mu^2) \\ \partial_t V + \varepsilon(V \cdot \nabla)V + \nabla \zeta = \mu \mathcal{D} + O(\mu^2) \end{cases} \quad (1)$$

where  $\varepsilon = a/h_0$  ( $a$  the order of free surface amplitude) is the nonlinearity parameter and the dimensionless flow variables are:  $h$  the water depth,  $\zeta$  the surface elevation and  $V$  the depth-averaged velocity.  $\mathcal{D}$  characterizes non-hydrostatic and dispersive effects and is a function of wave variables and their derivatives. Higher-order Boussinesq equations can be derived (e.g. Madsen et Schäffer [2]), but in this paper we restrict our analysis to  $O(\mu^2)$ , which is a good approximation for most of nearshore wave applications (e.g. Madsen et al. [3], [4], Cienfuegos et al. [5]).

In its classical form, Boussinesq wave theory is a one-dimensional approach based on the assumptions of weak dispersion, weak non-linearity and balance between dispersion and non-linearity:  $\varepsilon = O(\mu) \ll 1$ . The KdV [6] and BBM [7] equations can also be derived in the case of unidirectional waves; this approach is relevant to study fundamental wave dynamics problems, such as 1D solitary wave propagation on flat bottom (e.g. Boussinesq [1] and Benjamin [8]). However, the classical Boussinesq assumptions may severely restrict applicability to real world wave propagation problems. Applications to coastal zone, have motivated theoretical developments (see reviews by Dingemans [9], Madsen et Schäffer [2], Kirby [10] and Barthélemy [11]) for extending the range of applicability of Boussinesq-type equations in term of varying bottom, dispersive and nonlinearity effects, which play an important role in the nearshore wave dynamics.

The 2D Boussinesq equations ( $\varepsilon = O(\mu) \ll 1$ ) for non flat bottom were first derived by Peregrine [12]:

$$\mathcal{D} = \frac{h}{2} \nabla (\nabla \cdot (h \partial_t V)) - \frac{h^2}{6} \nabla^2 \partial_t V . \quad (2)$$

For many coastal applications the weak dispersion of these equations is a critical limitation. Witting [13] proposed a method based on Padé expansion of the exact

linear phase velocity to improve Boussinesq-type equations. From this method, several equations (order  $O(\mu^2)$ ) with improved dispersion characteristic have been derived (e.g. Madsen et al. [14], Nwogu [15], Schäffer and Madsen [16], Bona et al. [17, 18]).

In 1953, a breakthrough treating nonlinearity was made by Serre (see Barthélemy [11] for a review). He derived 1D fully nonlinear ( $\epsilon = O(1)$ ) weakly dispersive equations for horizontal bottom:

$$\mathcal{D} = \frac{1}{3h} \partial_x (h^3 (V_{xt} + VV_{xx} - (V_x)^2)). \quad (3)$$

The same system was obtained later by Su and Gardner [19]. Seabra-Santos et al. [20] have provided an extension of this model to non flat bottom:

$$\mathcal{D} = -\frac{1}{h} \left( \partial_x (h^2 (\frac{1}{3}P + \frac{1}{2}Q)) + b_x h (\frac{1}{2}P + Q) \right), \quad (4)$$

with  $P = -h(V_{xt} + VV_{xx} - (V_x)^2)$  and  $Q = b_x(V_t + VV_x + b_{xx}V^2)$ . Dingemans [9] expressed mistrust for the validity of these equations for uneven topographies, considering that the derivation required the assumption of vertical uniformity of the horizontal velocity. Cienfuegos et al. [21] have shown that Serre equations can be formally established without assumption on the velocity profile.

Finally, Green and Naghdi [22] derived 2D fully nonlinear weakly dispersive equations for uneven bottom which represents a two-dimensional extension of Serre equations. Except for being formulated in terms of the velocity vector at an arbitrary  $z$  level, the equations of Wei et al. [23] are basically equivalent to the 2D Serre or Green-Naghdi equations; this is also the case of the equations derived in [24] through Hamilton's principle.

The range of validity of all the models introduced above may vary as far as the nonlinearity parameter  $\epsilon$  is concerned, but they all require that the shallowness parameter  $\mu$  is small. In deep water ( $\mu \ll 1$ ) the Boussinesq wave theory fails but it is yet possible to derive asymptotic expansions from the water-wave equations under the condition that the *steepness*  $\epsilon\sqrt{\mu} = a/\lambda$  is small. Such an approach was first made in 1D and flat bottoms by Matsuno [25] and extended to uneven bottoms [26], 2D weakly transverse waves [27] and higher order expansions [28]. Since one always has  $\epsilon \lesssim 1$ , the small steepness assumption  $\epsilon\sqrt{\mu} \ll 1$  is also satisfied in the shallow water regime  $\mu \ll 1$  discussed above and this is the reason why it is often claimed that the models derived in [25, 26, 27, 28] are valid in the whole range  $\mu \in (0, \infty)$ . However, as we show here, their precision is then far below the one of the Green-Naghdi equations.

In this paper we propose a systematic derivation of all the models evoked in this introduction (and of some new ones); to this end, we use the global method introduced in the recent mathematical work [29]. Starting from a general nondimensionalized version of the water-wave equations (which takes into accounts the different nondimensionalizations used in deep and shallow water), following [30, 31], we reduce the problem to a system of two equations on the surface elevation and the velocity potential at the free surface. These equations involve a Dirichlet-Neumann operator and we show that all the asymptotic models can be recovered by a simple asymptotic expansion of this operator. In the deep water setting ( $\mu \ll 1$ ) this expansion

is made with respect to  $\varepsilon$  (which is small, as a consequence of the small steepness assumption  $\varepsilon\sqrt{\mu} \ll 1$  and the deep water assumption  $\mu \ll 1$ ); in the shallow-water setting, we rather choose to expand the Dirichlet-Neumann operator in terms of  $\mu$  since  $\varepsilon$  is not always small (e.g.,  $\varepsilon \sim 1$  for Green-Naghdi).

As said above, all the asymptotic models are obtained in a systematic way with the present approach. In addition, we derive a new fully dispersive model for 3D waves over uneven bottom. Our method brings clarifications about the validity domain of the main asymptotic wave models. In particular,

- We can easily show that the shallow water limit of the Dirichlet-Neumann expansion corresponding to the deep-water models has a very poor precision in the fully nonlinear case  $\varepsilon \sim 1$ ; this proves that the deep-water models such as those of [25, 26, 27, 28] are not in general suitable for nonlinear shallow-water waves;
- We do not make any assumption on the velocity profile or on any related quantity. The only assumptions we make concern the value of the parameters  $\varepsilon$  and  $\mu$  (and a third parameter  $\beta$  linked to the amplitude of the bottom variations for uneven bottoms); the properties of the velocity profile (and in particular its vertical behavior) are then rigorously established; we thus believe that this article should clarify the discussions concerning the assumptions made on the velocity profile, and in particular the controversy raised in [9] about the Serre equations.

## 2 General nondimensionalized water-wave equations

Parameterizing the free surface by  $z = \zeta(t, X)$  (with  $X = (x, y) \in \mathbb{R}^2$ ) and the bottom by  $z = -h_0 + b(X)$  (with  $h_0 > 0$  constant), one can use the incompressibility and irrotationality conditions to write the water-wave equations under Bernoulli's formulation, in terms of a potential velocity  $\phi$  (i.e., the velocity field is given by  $\mathbf{v} = \nabla_{X,z}\phi$ ):

$$\left\{ \begin{array}{ll} \partial_x^2 \phi + \partial_y^2 \phi + \partial_z^2 \phi = 0, & -h_0 + b \leq z \leq \zeta, \\ \partial_n \phi = 0, & z = -h_0 + b, \\ \partial_t \zeta + \nabla \zeta \cdot \nabla \phi = \partial_z \phi, & z = \zeta, \\ \partial_t \phi + \frac{1}{2}(|\nabla \phi|^2 + (\partial_z \phi)^2) + g\zeta = 0, & z = \zeta, \end{array} \right. \quad (5)$$

where  $g$  is the gravitational acceleration,  $\nabla = (\partial_x, \partial_y)^T$  and  $\partial_n \phi$  is the outward normal derivative at the boundary of the fluid domain.

In deriving approximate equations by asymptotic methods it is necessary to introduce dimensionless quantities based on characteristic scales for the wave motion. Four main length scales are involved in this problem:  $h_0$  the characteristic water depth,  $L$  the characteristic horizontal scale,  $a$  the order of free surface amplitude and  $B$  the order of bottom topography variation. Three independent dimensionless

parameters can be formed from these four scales. We choose:

$$\frac{a}{h_0} = \epsilon, \quad \frac{h_0^2}{L^2} = \mu, \quad \frac{B}{h_0} = \beta, \quad (6)$$

where  $\epsilon$  is often called the nonlinearity parameter, while  $\mu$  is the shallowness parameter.

Commonly two distinct nondimensionalizations are used in oceanography (e.g. Dingemans (1997)) depending on the value of  $\mu$ : shallow water scaling ( $\mu \ll 1$ ) and Stokes wave scaling for intermediate to deep water ( $\mu \not\ll 1$ ). In this paper, we present a general nondimensionalization which applies to any wave regime. The order of magnitude of wave motion variables are given by the linear wave theory. In particular,  $(gh_0\nu)^{1/2}$  and  $\frac{aL}{h_0}(\frac{gh_0}{\nu})^{1/2}$  are the characteristic scales of respectively the wave celerity and the potential velocity, with  $\nu = \tanh(\mu^{1/2})/\mu^{1/2}$ .

Let us normalize all variables according to the scales anticipated on physical grounds:

$$\begin{aligned} x &= Lx', & y &= Ly', & z &= h_0\nu z', & t &= \frac{\lambda}{\sqrt{gh_0\nu}}t', \\ \zeta &= a\zeta', & \Phi &= \frac{a}{h_0}L\sqrt{\frac{gh_0}{\nu}}\Phi', & b &= Bb'. \end{aligned}$$

From this general nondimensionalization we can recover the classical scalings for shallow and deep water:  $\nu \sim 1$  when  $\mu \ll 1$  and  $\nu \sim \mu^{-1/2}$  when  $\mu \gg 1$ .

The equations of motion (5) then become (after dropping the primes for the sake of clarity):

$$\left\{ \begin{array}{l} \nu^2\mu\partial_x^2\Phi + \nu^2\mu\partial_y^2\Phi + \partial_z^2\Phi = 0, \quad \frac{1}{\nu}(-1 + \beta b) \leq z \leq \frac{\epsilon}{\nu}\zeta, \\ -\nu^2\mu\nabla\left(\frac{\beta}{\nu}b\right) \cdot \nabla\Phi + \partial_z\Phi = 0, \quad z = \frac{1}{\nu}(-1 + \beta b), \\ \partial_t\zeta - \frac{1}{\mu\nu^2}\left(-\nu^2\mu\nabla\left(\frac{\epsilon}{\nu}\zeta\right) \cdot \nabla\Phi + \partial_z\Phi\right) = 0, \quad z = \frac{\epsilon}{\nu}\zeta, \\ \partial_t\Phi + \frac{1}{2}\left(\frac{\epsilon}{\nu}|\nabla\Phi|^2 + \frac{\epsilon}{\mu\nu^3}(\partial_z\Phi)^2\right) + \zeta = 0, \quad z = \frac{\epsilon}{\nu}\zeta. \end{array} \right. \quad (7)$$

In order to reduce this set of equations into a system of two evolution equations, we introduce the trace of the velocity potential at the free surface, namely  $\psi = \Phi|_{z=\epsilon/\nu\zeta}$  and the Dirichlet-Neumann operator  $\mathcal{G}_\mu^\nu[\frac{\epsilon}{\nu}\zeta]$ : as

$$\mathcal{G}_\mu^\nu[\frac{\epsilon}{\nu}\zeta]\psi = -\nu^2\mu\nabla\left(\frac{\epsilon}{\nu}\zeta\right) \cdot \nabla\Phi|_{z=\frac{\epsilon}{\nu}\zeta} + \partial_z\Phi|_{z=\frac{\epsilon}{\nu}\zeta},$$

with  $\Phi$  solving the boundary value problem

$$\left\{ \begin{array}{l} \nu^2\mu\partial_x^2\Phi + \nu^2\mu\partial_y^2\Phi + \partial_z^2\Phi = 0, \quad \frac{1}{\nu}(-1 + \beta b) \leq z \leq \frac{\epsilon}{\nu}\zeta, \\ \Phi|_{z=\frac{\epsilon}{\nu}\zeta} = \psi, \quad \partial_n\Phi|_{z=\frac{1}{\nu}(-1+\beta b)} = 0, \end{array} \right.$$

(one can check that  $\mathcal{G}_\mu^\nu[\frac{\epsilon}{\nu}\zeta]\psi = \sqrt{1 + |\nabla(\frac{\epsilon}{\nu}\zeta)|^2}\partial_n\Phi|_{z=\frac{\epsilon}{\nu}\zeta}$ , where  $\partial_n\Phi$  stands for the upwards nondimensionalized normal derivative at the surface). As remarked in [30, 31], the equations (7) are equivalent to a set of two equations on the free surface

parameterization  $\zeta$  and the trace of the velocity potential at the surface  $\psi = \Phi|_{z=\varepsilon/\nu\zeta}$  involving the Dirichlet-Neumann operator. Namely,

$$\begin{cases} \partial_t \zeta - \frac{1}{\mu\nu^2} \mathcal{G}_\mu^\nu[\frac{\varepsilon}{\nu}\zeta]\psi = 0, \\ \partial_t \psi + \zeta + \frac{\varepsilon}{2\nu} |\nabla\psi|^2 - \frac{\varepsilon\mu}{\nu^3} \frac{(\frac{1}{\mu} \mathcal{G}_\mu^\nu[\frac{\varepsilon}{\nu}\zeta]\psi + \nu \nabla(\varepsilon\zeta) \cdot \nabla\psi)^2}{2(1 + \varepsilon^2\mu|\nabla\zeta|^2)} = 0. \end{cases} \quad (8)$$

In order to derive the system (9), let  $\mathcal{G}_\mu[\varepsilon\zeta, \beta b]$  be the Dirichlet-Neumann operator  $\mathcal{G}_\mu^\nu[\frac{\varepsilon}{\nu}\zeta]$  corresponding to the case  $\nu = 1$ . One will easily check that

$$\forall \nu > 0, \quad \mathcal{G}_\mu[\varepsilon\zeta, \beta b] = \frac{1}{\nu} \mathcal{G}_\mu^\nu[\frac{\varepsilon}{\nu}\zeta],$$

so that plugging this relation into (8) yields

$$\begin{cases} \partial_t \zeta - \frac{1}{\mu\nu} \mathcal{G}_\mu[\varepsilon\zeta, \beta b]\psi = 0, \\ \partial_t \psi + \zeta + \frac{\varepsilon}{2\nu} |\nabla\psi|^2 - \frac{\varepsilon\mu}{\nu} \frac{(\frac{1}{\mu} \mathcal{G}_\mu[\varepsilon\zeta, \beta b]\psi + \nabla(\varepsilon\zeta) \cdot \nabla\psi)^2}{2(1 + \varepsilon^2\mu|\nabla\zeta|^2)} = 0. \end{cases} \quad (9)$$

The system (9) recasts the water-wave equations in terms of the free surface elevation  $\zeta$  and the velocity potential  $\psi$  at the surface; this is the formulation which will serve as a basis for all the computations in this article.

### 3 Shallow water models

This section is devoted to the study of shallow water waves:  $\mu \ll 1$ . The most general situation (when no assumption is made on  $\varepsilon$  and  $\beta$ ) is addressed in §3.1, where the the Green-Naghdi equations (also called Serre [32] or *fully* nonlinear Boussinesq equations [23]) are derived. The key point is an asymptotical expansion of the Dirichlet-Neumann operator  $\mathcal{G}_\mu[\varepsilon\zeta]$  with respect to  $\mu$ .

We then briefly show how to recover simpler models under additional assumptions on  $\varepsilon$ ; the *moderately* nonlinear case  $\varepsilon = O(\sqrt{\mu})$  is addressed in §3.2 and the *weakly* nonlinear case  $\varepsilon = O(\mu)$  in §3.3. Some new models are then derived in the next sections.

#### 3.1 The fully nonlinear case: $\varepsilon \sim 1$ , $\beta \sim 1$ and $\mu \ll 1$

From the assumption  $\mu \ll 1$ , one gets  $\nu \sim 1$  and we take  $\nu = 1$  in (9) to simplify (this corresponds to the usual shallow-water nondimensionalization).

##### 3.1.1 Asymptotic expansion of the Dirichlet-Neumann operator

Since  $\mu \ll 1$ , we look for an asymptotic expansion of  $\Phi$  under the form

$$\Phi_{app} = \sum_{j=0}^N \mu^j \Phi_j. \quad (10)$$

Plugging this expression into the nondimensionalized water-wave equations one can cancel the residual up to the order  $O(\mu^{N+1})$  provided that

$$\forall j = 0, \dots, N, \quad \partial_z^2 \Phi_j = -\partial_x^2 \Phi_{j-1} - \partial_y^2 \Phi_{j-1} \quad (11)$$

(with the convention that  $\Phi_{-1} = 0$ ), together with the boundary conditions

$$\forall j = 0, \dots, N, \quad \begin{cases} \Phi_j|_{z=\varepsilon\zeta} = \delta_{0,j}\psi, \\ \partial_z \Phi_j = \beta \nabla b \cdot \nabla \Phi_{j-1}|_{z=-1+\beta b} \end{cases} \quad (12)$$

(where  $\delta_{0,j} = 1$  if  $j = 0$  and 0 otherwise).

Solving the ODE (11) with (12) is completely straightforward, and this procedure can be implemented on any symbolic computation software to compute the  $\Phi_j$  at any order. For our purposes here, we must take  $N = 1$  and thus need to compute  $\Phi_0$  and  $\Phi_1$ ; one finds

$$\Phi_0 = \psi, \quad (13)$$

$$\Phi_1 = (z - \varepsilon\zeta) \left( -\frac{1}{2}(z + \varepsilon\zeta) - 1 + \beta b \right) \Delta \psi + \beta(z - \varepsilon\zeta) \nabla b \cdot \nabla \psi. \quad (14)$$

**Remark 3.1** *This shows that at order  $O(\mu^2)$ , and without assumption on  $\varepsilon$ , the potential depends quadratically on  $z$ , as is well-known. When  $\mu$  is not small, the expansion (10) is not convergent and this is the reason why we use another technique to expand the Dirichlet-Neumann operator in §4.*

An approximation of order  $O(\mu^{N+1})$  of  $\mathcal{G}_\mu[\varepsilon\zeta, \beta b]\psi$  is then given by

$$\mathcal{G}_\mu[\varepsilon\zeta, \beta b]\psi = -\mu \nabla(\varepsilon\zeta) \cdot \nabla \Phi_{app}|_{z=\varepsilon\zeta} + \partial_z \Phi_{app}|_{z=\varepsilon\zeta} + O(\mu^{N+2}); \quad (15)$$

for  $N = 1$ , we thus obtain from (10), (13) and (14),

$$\mathcal{G}_\mu[\varepsilon\zeta, \beta b]\psi = -\mu \nabla \cdot (h \nabla \psi) + \mu^2 \nabla \cdot \mathcal{T}[h, \beta b] \cdot \nabla \psi + O(\mu^3) \quad (16)$$

where  $h = 1 + \varepsilon\zeta - \beta b$  and where the linear operator  $\mathcal{T}[h, \beta b]$  is defined as

$$\mathcal{T}[h, \beta b] \cdot W = \mathcal{T}_1^* h \mathcal{T}_1 \cdot W + \mathcal{T}_2^* h \mathcal{T}_2 \cdot W. \quad (17)$$

with  $\mathcal{T}_j^*$  ( $j = 1, 2$ ) denoting the adjoint of the operators  $\mathcal{T}_j$  given by

$$\mathcal{T}_1 \cdot W := \frac{h}{\sqrt{3}} \nabla \cdot W - \frac{\sqrt{3}}{2} \beta \nabla b \cdot W, \quad \text{and} \quad \mathcal{T}_2 \cdot W := \frac{1}{2} \beta \nabla b \cdot W. \quad (18)$$

### 3.1.2 Derivation of the Green-Naghdi equations

We use here the asymptotic expansion (16) to derive the Green-Naghdi equations (also called Serre equations) from the full water waves equations (9).

One can equivalently write (16) under the form

$$\mathcal{G}_\mu[\varepsilon\zeta, \beta b]\psi = -\mu \nabla \cdot (h \mathbf{v}) + O(\mu^3), \quad (19)$$

where  $\mathbf{v} = \nabla \psi - \mu \frac{1}{h} \mathcal{T}[h, \beta b] \nabla \psi$ .



**Remark 3.2** According to formulae (13)-(14), the horizontal component of the velocity in the fluid domain is given by  $V(z) = \nabla\Phi_0 + \nabla\Phi_1(z) + O(\mu^2)$ , and one can check that  $\mathbf{v} = \frac{1}{h} \int_{-1+\beta b}^{\varepsilon\zeta} V(z) dz + O(\mu^2)$ , so that, in accordance with the classical derivations of the Green-Naghdi equations,  $\mathbf{v}$  is the vertically averaged horizontal component of the velocity (up to  $O(\mu^2)$  terms).

Recalling that  $\nu = 1$  here, it follows from (19) that the first equation of (9) can be written  $\partial_t\zeta + \nabla \cdot (h\mathbf{v}) = O(\mu^2)$ .

From the definition of  $\mathbf{v}$  given in (19), one gets easily

$$\nabla\psi = \mathbf{v} + \mu \frac{1}{h} \mathcal{T}[h, \beta b] \mathbf{v} + O(\mu^2);$$

taking the gradient of the second equation of (9), replacing  $\nabla\psi$  in the resulting equation by the above formula, using (19), and neglecting the  $O(\mu^2)$  quantities gives the Green-Naghdi equations (recall that  $h = 1 + \varepsilon\zeta - \beta b$ ):

$$\begin{cases} \partial_t\zeta + \nabla \cdot (h\mathbf{v}) = 0, \\ (1 + \frac{\mu}{h} \mathcal{T}[h, \beta b]) \partial_t \mathbf{v} + \nabla\zeta + \varepsilon(\mathbf{v} \cdot \nabla)\mathbf{v} \\ + \mu\varepsilon \left\{ -\frac{1}{3h} \nabla[h^3((\mathbf{v} \cdot \nabla)(\nabla \cdot \mathbf{v}) - (\nabla \cdot \mathbf{v})^2)] + \mathcal{Q}[h, \beta b](\mathbf{v}) \right\} = 0, \end{cases} \quad (20)$$

where

$$\begin{aligned} \mathcal{T}[h, \beta b] \cdot W &= \mathcal{T}_1^* h \mathcal{T}_1 \cdot W + \mathcal{T}_2^* h \mathcal{T}_2 \cdot W \\ &= -\frac{1}{3} \nabla(h^3 \nabla \cdot W) + \beta \frac{1}{2} [\nabla(h^2 \nabla b \cdot W) - h^2 \nabla b \nabla \cdot W] + \beta^2 h \nabla b \nabla b \cdot W \end{aligned} \quad (21)$$

and the purely topographical term  $\mathcal{Q}[h, \beta b](\mathbf{v})$  (which is quadratic in  $\mathbf{v}$ ) is defined as

$$\begin{aligned} \mathcal{Q}[h, \beta b](\mathbf{v}) &= \frac{\beta}{2h} [\nabla(h^2(\mathbf{v} \cdot \nabla)^2 b) - h^2((\mathbf{v} \cdot \nabla)(\nabla \cdot \mathbf{v}) - (\nabla \cdot \mathbf{v})^2) \nabla b] \\ &\quad + \beta^2((\mathbf{v} \cdot \nabla)^2 b) \nabla b. \end{aligned} \quad (22)$$

**Remark 3.3** The formulation (20) of the Green-Naghdi equation is not at first sight the same as usual (see [24] and [23]). A closer look shows however that they are exactly the same, as expected. The interest of the present form is that the second equation gives a straightforward control of the quantities  $(h\mathbf{v}, \mathbf{v})$ ,  $(h\mathcal{T}_1\mathbf{v}, \mathcal{T}_1\mathbf{v})$  and  $(h\mathcal{T}_2\mathbf{v}, \mathcal{T}_2\mathbf{v})$  (with  $\mathcal{T}_j$  given by (18)). This control yields regularizing effects of the same kind as those of the BBM equation compared to KdV, and is thus expected to ease numerical computations (work in progress).

In [23], Wei et al. derived some Green-Naghdi equations (fully nonlinear Boussinesq models in that reference) with improved frequency dispersion by replacing the vertically averaged horizontal velocity  $\mathbf{v}$  by the velocity  $\mathbf{v}_\alpha$  taken at some intermediate depth  $z_\alpha(x, y)$  (thus following the approach of Nwogu [15] for weakly nonlinear Boussinesq systems). Such systems could of course be derived similarly from (20).

### 3.2 The moderately nonlinear case: $\mu \ll 1$ and $\varepsilon = O(\sqrt{\mu})$

The simplifications that can be made on the Green-Naghdi equations (20) under the assumption  $\varepsilon = O(\sqrt{\mu})$  depend on the topography. As for the surface variations, we distinguish three different regimes:

(a) *Fully nonlinear topography:*  $\beta = O(1)$ . In this case, no significative simplification can be made, and the full equations must be kept.

(b) *Moderately nonlinear topography:*  $\beta = O(\sqrt{\mu})$ . In this regime, the last term of the second equation of (20) can be written

$$-\frac{\mu\varepsilon}{3}\nabla[((\mathbf{v} \cdot \nabla)(\nabla \cdot \mathbf{v}) - (\nabla \cdot \mathbf{v})^2)] + O(\mu^2),$$

so that one can replace (20) by

$$\begin{cases} \partial_t \zeta + \nabla \cdot (h\mathbf{v}) = 0, \\ (1 + \frac{\mu}{h}\mathcal{T}[h, \beta b])\partial_t \mathbf{v} + \nabla \zeta + \varepsilon(\mathbf{v} \cdot \nabla)\mathbf{v} - \frac{\mu\varepsilon}{3}\nabla[((\mathbf{v} \cdot \nabla)(\nabla \cdot \mathbf{v}) - (\nabla \cdot \mathbf{v})^2)] = 0, \end{cases}$$

(note that some simplifications could also be made in the term  $\frac{\mu}{h}\mathcal{T}[h, \beta b]\partial_t \mathbf{v}$  but they are not interesting because they would partially destroy the regularizing effects evoked in Remark 3.3).

(c) *Weakly nonlinear topography:*  $\beta = O(\varepsilon)$ . This stronger assumption allows a simplification of the term  $\frac{\mu}{h}\mathcal{T}[h, \beta b]\partial_t \mathbf{v}$ , and one gets

$$\begin{cases} \partial_t \zeta + \nabla \cdot (h\mathbf{v}) = 0, \\ (1 - \frac{\mu}{3h}\nabla(h^3\nabla \cdot))\partial_t \mathbf{v} + \nabla \zeta + \varepsilon(\mathbf{v} \cdot \nabla)\mathbf{v} - \frac{\mu\varepsilon}{3}\nabla[((\mathbf{v} \cdot \nabla)(\nabla \cdot \mathbf{v}) - (\nabla \cdot \mathbf{v})^2)] = 0; \end{cases}$$

the main interest of this model is that in the case of flat bottoms ( $b = 0$ ), its unidirectional limit gives the Cammassa-Holm equation ([33, 34, 35, 36]).

### 3.3 The weakly nonlinear case: $\mu \ll 1$ and $\varepsilon = O(\mu)$

The assumption  $\varepsilon = O(\mu)$  is the classical long waves assumption which yields the usual Boussinesq models. Here again, we briefly show how to recover these models in different topographic regimes:

(a) *Fully nonlinear topography:*  $\beta = O(1)$ . Neglecting in (20) the terms which are of order  $O(\mu^2)$  under the assumption  $\varepsilon = O(\mu)$  yields the equations

$$\begin{cases} \partial_t \zeta + \nabla \cdot (h\mathbf{v}) = 0, \\ (1 + \frac{\mu}{h}\mathcal{T}[h, \beta b])\partial_t \mathbf{v} + \nabla \zeta + \varepsilon(\mathbf{v} \cdot \nabla)\mathbf{v} = 0, \end{cases}$$

equivalent to equations (1) and (2) and which correspond to the Boussinesq model for nonflat bottoms derived by Peregrine [12].

For many coastal applications the weak dispersion of these equations is a critical limitation. Several alternative equations, with the same approximations  $\mu \ll 1$  and  $\varepsilon = O(\mu)$ , have been developed to improve dispersion properties (e.g. Madsen et al. [14], Nwogu [15], Schäffer and Madsen [16] or Bona et al. [17, 18]).

(b) *Weakly nonlinear topography:*  $\beta = O(\mu)$ . In this case, the equations simplify further into

$$\begin{cases} \partial_t \zeta + \nabla \cdot (h\mathbf{v}) = 0, \\ (1 - \frac{\mu}{3}\Delta)\partial_t \mathbf{v} + \nabla \zeta + \varepsilon(\mathbf{v} \cdot \nabla)\mathbf{v} = 0, \end{cases} \quad (23)$$

for which topographic effects play a role only through the presence of  $h = 1 + \varepsilon\zeta - \beta b$  in the first equation.

**Remark 3.4** *This latter Boussinesq system corresponds to the following approximation of the Dirichlet-Neumann operator:*

$$\mathcal{G}_\mu[\varepsilon\zeta, \beta b]\psi = -\mu\nabla \cdot (h\nabla\psi) - \frac{\mu^3}{3}\nabla \cdot \Delta\nabla\psi + O(\mu^3),$$

which is deduced from (19) by neglecting the terms which are  $O(\mu^3)$  in the present scaling.

## 4 Fully dispersive models: $\varepsilon\sqrt{\mu} \ll 1$ and $\varepsilon \ll 1$

Small steepness asymptotics ( $\varepsilon\sqrt{\mu} \ll 1$ ) have been introduced by Matsuno [25] and generalized in [26, 27, 28]. It is often claimed that the derived equations are valid under the condition  $\varepsilon\sqrt{\mu} \ll 1$  only, and that the other asymptotic models (which satisfy this condition) can be deduced from them. We show here that this is not always the case. The unified framework used in this article allows us to check for instance that the Matsuno equations do not degenerate into the Green-Naghdi equations in shallow-water and that their precision is much smaller.

After making a new asymptotic expansion of the Dirichlet-Neumann operator in this physical regime, we also derive a new generalization of the Matsuno equations for 3D flows with uneven bottom.

### 4.1 Asymptotic expansion of the Dirichlet-Neumann operator

When  $\varepsilon$  and  $\beta$  are small, it is possible to make a Taylor expansion of  $\mathcal{G}_\mu[\varepsilon\zeta, \beta b]\psi$  with respect to  $\varepsilon$  and  $\beta$ ; such an expansion has been derived for one dimensional surfaces in [37]; this method could be generalized to two dimensional surfaces, but we chose here to use another technique based on the following formulas

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon}(\mathcal{G}_\mu[\varepsilon\zeta, \beta b]\psi - \mathcal{G}_\mu[0, \beta b]\psi) &= -\mathcal{G}_\mu[0, \beta b](\zeta\mathcal{G}_\mu[0, \beta b]\psi) - \mu\nabla \cdot (\zeta(\nabla\psi)) \\ \lim_{\beta \rightarrow 0} \frac{1}{\beta}(\mathcal{G}_\mu[0, \beta b]\psi - \mathcal{G}_\mu[0, 0]\psi) &= \mu \operatorname{sech}(\sqrt{\mu}|D|)[\nabla \cdot (b(\operatorname{sech}(\sqrt{\mu}|D|)\nabla\psi))], \end{aligned}$$

where we used the Fourier multiplier notation: given two functions  $f$  and  $u$ , and denoting by  $\widehat{\cdot}$  the Fourier transform,  $f(D)u$  is defined as:

$$\forall \xi \in \mathbb{R}^2, \quad \widehat{f(D)u}(\xi) := f(\xi)\widehat{u}(\xi).$$

We only prove the second of these formulas because the first one can be established with the same techniques and can also be found in the literature (e.g. [38], and also

Th. 3.20 of [39] and Th. 3.1 of [29] where a similar formula is established for non necessarily flat surfaces).

First recall that by definition,  $\mathcal{G}_\mu[0, \beta b]\psi = \partial_z \Phi[\beta b]|_{z=0}$ , where  $\Phi[\beta b]$  solves

$$\begin{cases} \mu \Delta \Phi[\beta b] + \partial_z^2 \Phi[\beta b] = 0, & \text{in } -1 + \beta b < z < 0, \\ \Phi[\beta b]|_{z=0} = \psi, & (\partial_z \Phi[\beta b] - \mu \beta \nabla \Phi[\beta b] \cdot \nabla b)|_{z=-1+\beta b} = 0 \end{cases} \quad (24)$$

(we denoted  $\Phi[\beta b]$  instead of  $\Phi$  to enhance the dependance on the bottom topography). We thus have, for all smooth function  $\varphi$  compactly supported in  $\bar{\Omega}$  (with  $\bar{\Omega} = \{(X, z), -1 + \beta b(X) \leq z \leq 0\}$ ),

$$\begin{aligned} \int_{\Omega} (\mu \Delta \Phi[\beta b] + \partial_z^2 \Phi[\beta b]) \varphi dX dz &+ \int_{z=0} (\Phi[\beta b] - \psi) \varphi dX \\ &+ \int_{z=-1+\beta b} (\partial_z \Phi[\beta b] - \mu \beta \nabla b \cdot \nabla \Phi[\beta b]) \varphi dX = 0. \end{aligned}$$

Denoting  $\chi = \lim_{\beta \rightarrow 0} \frac{1}{\beta} (\Phi[\beta b] - \Phi[0])$ , one can differentiate the above identity to obtain

$$\begin{aligned} \int_{\Omega} (\mu \Delta \chi + \partial_z^2 \chi) \varphi dX dz &+ \int_{z=0} \chi \varphi dX \\ &+ \int_{z=-1} (\partial_z \chi - \mu \nabla b \cdot \nabla \Phi[0] + b \partial_z^2 \Phi[0]) \varphi dX = 0. \end{aligned}$$

Since this identity holds for all test function  $\varphi$ , and since  $\partial_z^2 \Phi[0] = -\mu \Delta \Phi[0]$ , we deduce that  $\chi$  solves the boundary value problem

$$\begin{cases} \mu \Delta \chi + \partial_z^2 \chi = 0, & \text{in } -1 < z < 0 \\ \chi|_{z=0} = 0, & \partial_z \chi|_{z=-1} = \mu \nabla \cdot (b \nabla \Phi[0]|_{z=-1}). \end{cases}$$

This problem can be solved explicitly:

$$\chi(\cdot, z) = \sqrt{\mu} \frac{\sinh(\sqrt{\mu} z |D|)}{\cosh(\sqrt{\mu} |D|)} \frac{\nabla}{|D|} \cdot (b \nabla \Phi[0]|_{z=-1});$$

since moreover one has  $\lim_{\beta \rightarrow 0} \frac{1}{\beta} (\mathcal{G}_\mu[0, \beta b]\psi - \mathcal{G}_\mu[0, 0]\psi) = \partial_z \chi|_{z=0}$  and that  $\Phi[0](\cdot, z) = \frac{\cosh(\sqrt{\mu}(z+1)|D|)}{\cosh(\sqrt{\mu}|D|)} \psi$ , we get the formula.

A first order Taylor expansion of  $\mathcal{G}_\mu[\varepsilon \zeta, \beta b]\psi$  with respect to  $\varepsilon$ , together with the first formula, shows therefore that

$$\mathcal{G}_\mu[\varepsilon \zeta, \beta b]\psi = \mathcal{G}_\mu[0, \beta b]\psi - \varepsilon \mathcal{G}_\mu[0, \beta b](\zeta \mathcal{G}_\mu[0, \beta b]\psi) - \varepsilon \mu \nabla \cdot (\zeta \nabla \psi) + O(\sqrt{\mu}(\varepsilon \sqrt{\mu})^2); \quad (25)$$

a first order Taylor expansion of  $\mathcal{G}_\mu[0, \beta b]\psi$  with respect to  $b$ , together with the second formula, gives also

$$\mathcal{G}_\mu[0, \beta b]\psi = \mathcal{G}_\mu[0, 0]\psi + \mu \beta \operatorname{sech}(\sqrt{\mu} |D|) [\nabla \cdot (b(\operatorname{sech}(\sqrt{\mu} |D|) \nabla \psi))] + O(\sqrt{\mu}(\beta \sqrt{\mu})^2). \quad (26)$$

Let us now define the operator  $\mathbf{T}_\mu$  and  $B_\mu$  as

$$\mathbf{T}_\mu = -\frac{\tanh(\sqrt{\mu}|D|)}{|D|}\nabla \quad \text{and} \quad B_\mu = \operatorname{sech}(\sqrt{\mu}|D|)[b(\operatorname{sech}(\sqrt{\mu}|D|)\cdot)] \quad (27)$$

We thus have  $\mathcal{G}_\mu[0,0]\psi = \sqrt{\mu}\mathbf{T}_\mu \cdot \nabla\psi$ , and (25) and (26) show that

$$\begin{aligned} \mathcal{G}_\mu[\varepsilon\zeta, \beta b]\psi &= \sqrt{\mu}\mathbf{T}_\mu \cdot \nabla\psi + \mu\beta\nabla \cdot (B_\mu\nabla\psi) - \varepsilon\mu\mathbf{T}_\mu \cdot \nabla(\zeta\mathbf{T}_\mu \cdot \nabla\psi) \\ &\quad - \varepsilon\mu\nabla \cdot (\zeta\nabla\psi) + O(\sqrt{\mu}(\varepsilon\sqrt{\mu})^2, \sqrt{\mu}(\beta\sqrt{\mu})^2). \end{aligned} \quad (28)$$

**Remark 4.1** *In the shallow water, weakly nonlinear regime, and with a weakly nonlinear topography (that is,  $\mu \ll 1$ ,  $\varepsilon = O(\mu)$ ,  $\beta = O(\mu)$ ), one can deduce from (28) that*

$$\mathcal{G}_\mu[\varepsilon\zeta, \beta b]\psi = -\mu\nabla \cdot (h\nabla\psi) - \frac{\mu^2}{3}\nabla \cdot \Delta\nabla\psi + O(\mu^3),$$

and we thus recover the approximation derived with another technique in §3.3 (see Remark 3.4).

## 4.2 Derivation of a fully dispersive model for 3D flows over uneven bottoms

We derive here a new system of fully dispersive equations which generalizes to the case of 2D surfaces and nonflat bottoms the systems derived by Matsuno (1D surfaces, flat [25] and uneven [26] bottoms), Choi [28] and Smith [40] (2D surfaces, flat bottoms).

We first define the horizontal velocity at the surface as  $\mathbf{v}_S = (\nabla\Phi)|_{z=\varepsilon\zeta}$ , where  $\Phi$  is the velocity potential given by (24). By definition of  $\psi$  and  $\mathcal{G}_\mu[\varepsilon\zeta, \beta b]\psi$ , we get

$$\begin{aligned} \nabla\psi &= \mathbf{v}_S + \varepsilon\nabla\zeta(\partial_z\Phi)|_{z=\varepsilon\zeta} \\ &= \mathbf{v}_S + \varepsilon\frac{\mathcal{G}_\mu[\varepsilon\zeta, \beta b]\psi + \varepsilon\mu\nabla\zeta \cdot \nabla\psi}{1 + \varepsilon\mu|\nabla\zeta|^2}\nabla\zeta \\ &= \mathbf{v}_S + \varepsilon\sqrt{\mu}(\mathbf{T}_\mu \cdot \mathbf{v}_S)\nabla\zeta + O((\varepsilon\sqrt{\mu})^2), \end{aligned}$$

where we used (28) and  $\nabla\psi = \mathbf{v}_S + O(\varepsilon\sqrt{\mu})$  for the last relation. Plugging this relation into (28), one gets similarly

$$\begin{aligned} \frac{1}{\sqrt{\mu}}\mathcal{G}_\mu[\varepsilon\zeta, \beta b]\psi &= \mathbf{T}_\mu \cdot \mathbf{v}_S + \sqrt{\mu}\beta\nabla \cdot (B_\mu\mathbf{v}_S) - \varepsilon\sqrt{\mu}\mathbf{T}_\mu \cdot (\zeta\nabla\mathbf{T}_\mu \cdot \mathbf{v}_S) \\ &\quad - \varepsilon\sqrt{\mu}\nabla \cdot (\zeta\mathbf{v}_S) + O(\sqrt{\mu}(\varepsilon\sqrt{\mu})^2, \sqrt{\mu}(\beta\sqrt{\mu})^2). \end{aligned} \quad (29)$$

Taking the gradient of the second equation of (9) and using the above two identities gives therefore the following set of deep-water equations:

$$\left\{ \begin{aligned} &\partial_t\zeta - \frac{1}{\sqrt{\mu\nu}}\mathbf{T}_\mu \cdot \mathbf{v}_S + \frac{\varepsilon\sqrt{\mu}}{\sqrt{\mu\nu}}(\mathbf{T}_\mu \cdot (\zeta\nabla\mathbf{T}_\mu \cdot \mathbf{v}_S) + \nabla \cdot (\zeta\mathbf{v}_S)) \\ &= \frac{\beta\sqrt{\mu}}{\sqrt{\mu\nu}}\nabla \cdot (B_\mu\mathbf{v}_S) \\ &\partial_t\mathbf{v}_S + \nabla\zeta + \varepsilon\sqrt{\mu}\left(\frac{1}{2\sqrt{\mu\nu}}\nabla|\mathbf{v}_S|^2 - \nabla\zeta\mathbf{T}_\mu \cdot \nabla\zeta\right) = 0, \end{aligned} \right. \quad (30)$$

where we recall that  $\mathbf{T}_\mu$  and  $B_\mu$  are defined in (27), and that  $\nu = \tanh(\sqrt{\mu})/\sqrt{\mu}$  (so that one can replace  $\sqrt{\mu}\nu$  by 1 in (30) in deep water).

**Remark 4.2** *If we remove the topography term  $B_\mu$  from these equations, we recover the two-dimensional equations (3.25)-(3.26) derived by Choi [28]. If we take the one dimensional version of (30) we recover the equations derived by Matsuno (Eqs (19)-(20) of [25] for flat bottoms and (4.28)-(4.29) of [27] for nonflat bottoms).*

**Remark 4.3** *The equations (30) are precise up to order  $O(\varepsilon\sqrt{\mu}, \varepsilon\sqrt{\beta})$  in deep water (since  $\nu \sim \mu^{-1/2}$ ); one could also use them in shallow water, but they are then precise up to order  $O(\varepsilon, \beta)$  only (since  $\nu \sim 1$  in shallow water). This is the same accuracy as the one provided by the weakly nonlinear shallow water models (with a weakly nonlinear topography) of Section 3.3; it is therefore not surprising to check that one recovers the Boussinesq system (23) from (30) by a simple Taylor expansion of the operators  $\mathbf{T}_\mu$  and  $B_\mu$  and by observing that  $\mathbf{v} = (1 + \mu\frac{1}{3}\Delta)\mathbf{v}_s + O(\mu^2)$ .*

*In the fully nonlinear regime (with fully nonlinear topography) studied in Section 3.1, the equations (30) are precise up to order  $O(\varepsilon, \beta) = O(1)$  while we saw that the Green-Naghdi equations (20) are precise up to order  $O(\mu)$ . It is therefore not surprising to check that the shallow water limit of (30) does NOT give the Green-Naghdi equations (20).*

**Remark 4.4** *The system (30) is “fully dispersive” in the sense that its dispersion relation is the same as for the full water-wave equations. This is the case because the expansion (29) keeps the nonlocal effects of the Dirichlet-Neumann operator  $G_\mu[\varepsilon\zeta, \beta b]\psi$ . In [41, 42, 43], the authors make a differential approximation of shallow water type based on Padé approximants; they show that the dispersive properties remain good far beyond the shallow water regime when bathymetric changes are not too strong (in this latter case, the model is more complex and its range of validity much narrower [44]).*

## 5 Conclusion

In this paper we have presented a systematic derivation of the main 2D asymptotic models for shallow and deep water waves, which allows to clarify their validity domain. We have also derived a new 2D fully dispersive model, system (30), for small wave steepness which extends to uneven bottom the approach developed by Matsuno [27] and Choi [28]. We have shown that even though these models remain valid in shallow water, their precision is then far below what can be achieved with the Green-Naghdi or Serre model when fully nonlinear waves are considered ( $\epsilon \sim 1$ ). Hence, contrary to what it is generally thought [25, 26, 27, 28] this approach cannot in practical be considered as a unified theory of nonlinear waves because it is not accurate enough for nonlinear shallow water waves ( $\epsilon \sim 1$ ,  $\mu \ll 1$ ). In particular, we have shown that system (30) in the shallow water limit ( $\mu \ll 1$ ) does not correspond to the correct fully nonlinear equations, namely the Green-Naghdi or Serre equations. Another reason why these fully dispersive water models are not likely to furnish interesting models in shallow water is that there is no obvious shoreline boundary condition for them.

The Green-Naghdi equations represent the appropriate model to describe nonlinear shallow water wave propagation and wave oscillations at the shoreline. For coastal applications, the Green-Naghdi equations can be easily extended to include accurate linear dispersive effects, which allow to describe shoaling processes in intermediate water depth (see Wei et al. [23] and Cienfuegos et al. [21, 5]). Another interest of these equations is that there is a natural shoreline boundary condition given by the flux conservation equation (the flux  $h\mathbf{v}$  vanishes at the shoreline in the first equation of 20). However, for deep water wave propagation the Green-Naghdi model is no more valid and the fully dispersive model (30) must therefore be used.

It follows from these considerations that the asymptotic description of coastal flows requires at least the use of 2 different models: one for shallow-water (e.g. the Green-Naghdi equations (20)) and another one for deep water (e.g. (30)). The numerical coupling of these two models is therefore a natural perspective for further works.

### ACKNOWLEDGMENT

This work has been supported by ACI Jeunes Chercheurs "Dispersion et non-linéarités" and was also performed within the framework of the LEFE-IDAO program (Interactions et Dynamique de l'Atmosphère et de l'Océan) sponsored by the CNRS/INSU.

## References

- [1] Boussinesq, J., 1872 Théorie des ondes et des remous qui se propagent le long d'un canal rectangulaire horizontal, en communiquant au liquide contenu dans ce canal des vitesses sensiblement pareilles de la surface au fond. *J. Math. Pures Appl.*, **17**(2), 55-108.
- [2] Madsen, P.A. and Schäffer, H.A. 1999 A review of Boussinesq-type equations for gravity waves. In *Advances in Coastal and Ocean Engineering*, World Scientific

Publishing Co. Pte. Ltd., edited by Philip L.-F. Liu, 5,1-94.

- [3] Madsen, P.A., Sorensen, O.R. and Schäffer, H.A. 1997 Surf zone dynamics simulated by a Boussinesq type model. Part I. Model description and cross-shore motion of regular waves. *Coastal Eng.*, **32**, 255-287.
- [4] Madsen, P.A., Sorensen, O.R. and Schäffer, H.A. 1997 Surf zone dynamics simulated by a Boussinesq type model. Part II. Surf beat and swash oscillations for wave groups and irregular waves. *Coastal Eng.*, **32**, 289-319.
- [5] Cienfuegos, R., Barthelemy, E. and Bonneton, P. 2007 A fourth-order compact finite volume scheme for fully nonlinear and weakly dispersive Boussinesq-type equations. Part II: Boundary conditions and model validation. *Int. J. Numer. Meth. Fluids*, **53** (9), 1423-1455.
- [6] Korteweg, D. G. and de Vries, G. 1895 On the change of form of long waves advancing in the rectangular canal and a new type of long stationary waves, *Phil. Mag.*, **39**, 422-443.
- [7] Benjamin, T. B. and Bona, J. L. and Mahony, J. J. 1972, Model equations for long waves in nonlinear dispersive systems, *Philos. Trans. Roy. Soc. London Ser. A*, **272**, 47-78.
- [8] Benjamin, T. B. 1972, The stability of solitary waves. *Proc. Roy. Soc. (London) Ser. A*, **328**, 153-183.
- [9] Dingemans, M.W. 1997 Water wave propagation over uneven bottoms. Part1 : Linear wave propagation ; Part2 : Non-linear wave propagation. *World Scientific*.
- [10] Kirby, J. T., 2003 Boussinesq models and applications to nearshore wave propagation, surfzone processes and wave-induced currents, in *Advances in Coastal Modeling*, V. C. Lakhan (ed), Elsevier, 1-41.
- [11] Barthelemy, E. 2004 Nonlinear shallow water theories for coastal waves. *Surveys in Geophysics*, **25**, 315-337.
- [12] Peregrine, D. H. 1967 Long waves on a beach. *Journal of Fluid Mechanics*, **27**, 815-827.
- [13] Witting, J.M. 1984 A unified model for the evolution of nonlinear water waves. *Journal of Computational Physics*, **56**(2), 203-236.
- [14] Madsen, P.A., Murray, R., Sorensen, O.R. 1991 A new form of the Boussinesq equations with improved linear dispersion characteristics. *Coastal Engineering*, **15**, 371-388.
- [15] Nwogu, O. 1993 Alternative form of boussinesq equations for nearshore wave propagations. *Journal of Waterway, Port, Coastal and Ocean Engineering*, **119**(6), 618-638.
- [16] Schäffer, H.A., Madsen, P.A. 1995 Further enhancements of Boussinesq-type equations. *Coastal Engineering*, **26**, 1-14.



- [17] Bona, J. L., Chen, M. and Saut, J.-C. 2002 Boussinesq Equations and Other Systems for Small-Amplitude Long Waves in Nonlinear Dispersive Media. I: Derivation and Linear Theory, *J. Nonlinear Sci.*, **12**, 283-318.
- [18] Bona, J.L., Colin, T. and Lannes, D. 2005 Long Waves Approximations for Water Waves, *Arch. Rational Mech. Anal.* **178**, 373-410.
- [19] Su, C.H. and Gardner, C.S. 1969 Korteweg-de Vries equation and generalizations III. Derivation of the Korteweg-de Vries equation and Burgers equation. *J. Math. Phys.* **10** (3), 536-539.
- [20] Seabra-Santos, F.J., Renouard, D.P., Temperville, A.M. 1987 Numerical and experimental study of the transformation of a solitary wave over a shelf or isolated obstacle. *Journal of Fluid Mechanics*, **176**, 117-134.
- [21] Cienfuegos, R., Barthelemy, E. and Bonneton, P. 2006 A fourth-order compact finite volume scheme for fully nonlinear and weakly dispersive Boussinesq-type equations. Part I: Model development and analysis. *Int. J. Numer. Meth. Fluids*, **56**, 1217-1253.
- [22] Green, A.E. and Naghdi, P.M. 1976 A derivation of equations for wave propagation in water of variable depth. *Journal of Fluid Mechanics*, **78**(2), 237-246.
- [23] Wei, G., Kirby, J.T, Grilli, S.T. and Subramanya, R. 1995 A fully nonlinear Boussinesq model for surface waves. Part 1. Highly nonlinear unsteady waves. *Journal of Fluid Mechanics*, **294**, 71-92.
- [24] Miles, J. and Salmon, R. 1985 Weakly dispersive nonlinear gravity waves. *Journal of Fluid Mechanics*, **157**, 519-531.
- [25] Matsuno, Y. 1992 Nonlinear evolutions of surface gravity waves on fluid of finite depth, *Phys. Rev. Lett.*, **69**, 609-611.
- [26] Matsuno, Y. 1993 Nonlinear evolution of surface gravity waves over an uneven bottom, *J. Fluid Mech.* **249**, 121-133.
- [27] Matsuno, Y. 1993 Two-dimensional evolution of surface gravity waves on a fluid of arbitrary depth, *Phys. Rev. E*, **47**, 4593-4596.
- [28] Choi, W. 1995 Nonlinear evolution equations for two-dimensional surface waves in a fluid of finite depth, *J. Fluid Mech.*, **295**, 381-394.
- [29] Alvarez-Sameniego, B. and Lannes, D. 2007, Large time existence for 3D water waves and asymptotics, preprint.
- [30] Zakharov, V. E. (1968) Stability of periodic waves of finite amplitude on the surface of a deep fluid (1968) *J. Appl. Mech. Tech. Phys.*, **2**, 190-194.
- [31] Craig, W. and Sulem, C. and Sulem, P.-L. 1992 Nonlinear modulation of gravity waves: a rigorous approach, *Nonlinearity* **5**(2), 497-522.

- [32] Serre, F. 1953 Contribution à l'étude des écoulements permanents et variables dans les canaux. *Howille Blanche*, **8**, 374-388.
- [33] Camassa, R. and Holm, D. D. 1993 An integrable shallow water equation with peaked solitons *Phys. Rev. Lett.*, **71**, 1661-1664.
- [34] Camassa, R., Holm, D. D. and Hyman, J. M. 1994 A new integrable shallow water equation. *Adv. Appl. Mech.* **31**, 1-33.
- [35] Johnson, R. S. 2002 Camassa-Holm, Korteweg-de Vries and related models for water waves. *J. Fluid Mech.*, **455**, 63-82.
- [36] Constantin, A. and Lannes, D. 2007 The hydrodynamical relevance of the Camassa-Holm and Degasperis-Procesi equations, submitted.
- [37] Craig, W. and Guyenne, P. and Nicholls, D. and Sulem, C. (2005) Hamiltonian long-wave expansions for water waves over a rough bottom. *Proc. Royal Society A*, **461**, 1-35.
- [38] Craig, W. and Schanz, U. and Sulem, C. (1997) The modulational regime of three-dimensional water waves and the Davey-Stewartson system. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **14** (5), 615-667.
- [39] Lannes, D. 2005 Well-posedness of the water-wave equations. *J. Am. Math. Soc.* **18**, 605-654.
- [40] Smith, R. A. (1998) An operator expansion formalism for nonlinear surface waves over variable depth. *J. Fluid. Mech.*, **363**, 333-347.
- [41] Agnon, Y. and Madsen, P.A. and Schäffer, H.A., 1999. A new approach to high order Boussinesq models. *J. Fluid Mech.* **399**, 319-333.
- [42] Madsen, P.A. and Bingham, H.B. and Schäffer, H.A., 2003. Boussinesq-type formulations for fully nonlinear and extremely dispersive water waves: derivation and analysis. *Proc. R. Soc. Lond., A* **459**, 1075-1104.
- [43] Madsen, P.A. and Bingham, H.B. and Liu, H., 2002. A new Boussinesq method for fully nonlinear waves from shallow to deep water. *J. Fluid Mech.* **462**, 1-30.
- [44] Madsen, P. A. and Fuhrman, D. R. and Wang, B. (2006) A Boussinesq-type method for fully nonlinear waves interacting with a rapidly varying bathymetry. *Coastal Engineering*, **53** (5-6), 487-504.