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Dumping influence on a non iterative dynamics

Cécile Hardouin¹

SAMOS-Matisse - CES
Université Paris 1
90 rue de Tolbiac
75634 Paris Cedex 13, France
(e-mail: hardouin@univ-paris1.fr)

Abstract. We consider n agents displayed on S choosing one by one a standard A or B according to a local assignment rule. There is no asymptotics in space or in time, since the scan of the network is unique. We study the final behaviour by simulations. The main goal of this work is to evaluate the effect of an initial dumping on the final configuration.

Keywords: Adoption dynamics, Cooperative systems, Dumping.

1 Introduction

This paper explores the diffusion of technological innovations, under a simple and real framework: agents choosing between competitive technologies. Many empirical or theoretical works study the agents' behaviour in the process of standard's adoption. Several modellings are proposed: Markov chain, cellular automata, Gibbs fields... We consider here a unique and non reversible choice for each agent, but we examine various choice procedures, in which the previous decisions are of more or less significance. More precisely, let S be a spatial finite set, $S = \{1, 2, \dots, N\}^2$, with $n = N^2$ sites; we can choose for S the bidimensional torus, and we assume that the neighbourhood system is the four nearest neighbours system. Other sets S and other neighbourhood systems can be conceived, but radically, this will not change the qualitative nature of our results. If A is a subset of S , we denote $\partial A = \{i \in S, i \notin A \text{ and } \exists j \in A \text{ s.t. } i \text{ and } j \text{ are neighbours}\}$ the neighbourhood of A , and $\partial i = \partial\{i\}$. For all the dynamics we study in this work, the agents make a choice between two standards A and B . The choice is made individually, one by one, according to a *sequential* assignment rule. When this choice depends of the local context, we say that there is spatial coordination, the spatial dependency being positive if there is cooperation between the agents, and negative in case of competition. A *scan* of S is a tour of all the sites. When the scans are indefinitely repeated, we get the well known Gibbs sampler and it is possible to characterize the probability distribution of limits configurations. When the dynamics is synchronous (all the agents make their decision simultaneously), there is still ergodicity but it is difficult to explicit the limit distribution (See [10], and [9] for a full description).

Our context here is different: we consider a non iterative dynamics with a unique scan of S . In this case, we don't know the final configuration, since there is no asymptotics in space nor in time; besides, obviously, the final configuration is linked to the initial one. We propose to study empirically this situation, in the case of an initial occurrence of standards A at a given rate τ .

2 Non-iterative dynamics and dumping

Let us note $y_i \in E = \{-1, +1, 0\}$ the state of site i , where $+1$ states for A is chosen, -1 for B, and 0 denotes that the choice has not yet been made. We want to study the effect of an initial contamination or dumping on the final configuration. An initial rate of contamination τ ($\tau \in [0, 1]$) means that $[n\tau]$ agents have A ($[r]$ denotes the integer part of r): the initial layout is therefore composed of $[n\tau]$ sites $+1$ randomly distributed on S , the other sites being "non occupied", with assignment 0 . Then, these sites are visited one by one, in a random arrangement, and the new visited site is credited with $+1$ or -1 according to a local assignment rule. This rule is the same rule for all the agents and it is associated with the possibly previous choices of the neighbour sites. Obviously, the final configuration depends on the initial dumping rate τ .

We consider the three following assignment rules:

1. *The strong majority choice*: the agent chooses the majority standard adopted by his neighbours. In case of equality, or if there are no occupied neighbour sites, he chooses A (resp. B) with probability π (resp. $1-\pi$).
2. *The weak majority choice*: if the number of occupied sites is less or equal 2, then the agent chooses A (resp. B) with probability π (resp. $1-\pi$). If the number of occupied neighbour sites is more or equal 3, the agent follows the strong majority rule.
3. *The probabilistic Ising type choice*: if the 4 neighbour sites are non occupied, the agent chooses A (resp. B) with probability π (resp. $1-\pi$). On the contrary, noting $y_{\partial i}$ the configuration of the 4 nearest neighbours of site i , he chooses A with probability

$$\pi_i(A|y_{\partial i}) = \frac{\exp \left\{ \beta \sum_{j \in \partial i} y_j \right\}}{\exp \left\{ \beta \sum_{j \in \partial i} y_j \right\} + \exp \left\{ -\beta \sum_{j \in \partial i} y_j \right\}} = 1 - \pi_i(B|y_{\partial i})$$

β is a parameter of spatial coordination; there is cooperation if $\beta > 0$, while $\beta < 0$ leads to competition. When $\beta \rightarrow +\infty$, the Ising rule is similar to the strong majority rule. For simplicity, we fix $\pi = 0.5$.

Our experimental study is the following: we start with an initial rate $\tau=2\%$ and we increase τ until 99%. For each value, and for each assignment rule we simulate 400 realizations on the square torus of size $N = 64$ ($n = 4096$). For the Ising rule, we have chosen $\beta = 0.5$ (weak spatial coordination), $\beta = 1$ which corresponds to a beginning of aggregation, and $\beta = 3$ which leads to clusters. On the basis of the simulations, we present some estimations of important characteristics of the resulting configurations: the final proportion of standards A, spatial correlations, clustering indexes, connectedness measurements. These features show the influences of both the assignment rule and the initial rate on the final layouts.

Finally we give analytic results for the distribution of the number of occupied neighbour sites under the null hypothesis H_0 that there is neither cooperation nor competition. These results allow testing H_0 .

3 Empirical study of the final layout

3.1 The final frequency of standard A

For each value of τ and each assignment rule, we get a sample of 400 final frequencies of standard A. The final frequency is estimated by the mean $\hat{\pi}_A(\tau)$.

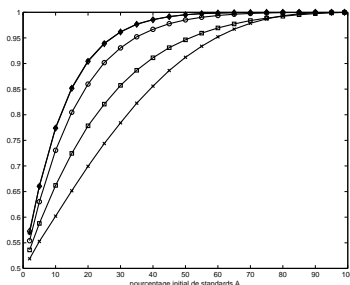


Fig. 1. Final frequency of standards A function of the assignment rule : x : weak ; \square : Ising $\beta = 0.5$; o : Ising $\beta = 1$; \diamond : Ising $\beta = 3$; * : strong.

Figure 1 shows the evolution $\tau \mapsto \hat{\pi}_A(\tau)$ for five assignment rules. As expected, the increase is stronger when the rule strengthens local cooperation. Thus, for the strong majority rule with an initial dumping rate of 20%, the standard A will occupy 90% of the sites; moreover, if 80% of the sites initially non occupied would have been set equally to A and B, the final rate of A would have been $\frac{80}{2}\% + 20\% = 60\%$; the difference between the two cases is

30%. We can see that the dumping amplifies the initial bias, intensifying the A choice along the adoption process.

In the case of a 10% rate of dumping, the difference is equal to 23% (the final rate of A being 78%). we can see that the dumping influence is more important for little values of τ .

We also can compare the different rules; the curves for $\beta = 3$ and the strong majority rule coincide, while the increase for the weak rule is slower than for the Ising $\beta = 0.5$ case.

Finally, we plotted the histograms of $\hat{\pi}_A(\tau)$ against τ and the assignment rule. When τ is small, we observe a Gaussian shape which is confirmed by a test. On the contrary, when the rate is greater than a critical rate τ_c , the gaussian hypothesis is rejected. In fact, the final proportion of A is very close to 1, with a quasi null dispersion. The threshold τ_c depends on the rule; it is about 40% for the strong majority rule and Ising $\beta = 3$, and 50% for Ising $\beta = 1$.

The main interest of a Gaussian feature is to build a confidence interval for the final proportion of standards A.

3.2 Spatial correlations

The analytic expression of the spatial correlation is not explicit but we can easily obtain its characteristics via a Monte Carlo method. Thus, we calculate in this way the correlations at distance one, distance 2 and $\sqrt{2}$, and finally the one based on the 8 nearest neighbours, which we will denote respectively ρ_1 , $\rho_{\sqrt{2}}$, ρ_2 , and ρ_{8v} , and this is done for each rule.

Comparing the different correlations for a same rule, we observe $\rho_1 \geq \rho_{8v} \geq \rho_{\sqrt{2}} \geq \rho_2$; besides the correlations all decay towards zero but more quickly when the neighbourhood is close; then they are all equal from $\tau = 85\%$.

Comparing the rules, all the different correlations have similar behaviours, so we present only ρ_1 .

The main aspect is that the dumping's influence is more important for weak levels of initial rates, producing high correlation.

We present on Figure 2 the evolution of $\rho_1 = \frac{(2n)^{-1} \sum_{i=1,n} y_i y_{\partial i} - \bar{y}^2}{1 - \bar{y}^2}$ in function of τ for the different assignment rules. We take the agreement $\rho_1 = 0$ when the field is constant with zero variance.

Whatever the rule is, the correlation is positive and decreases to zero. We can observe a kind of hierarchy between the rules: for a dumping rate less than 50%, the spatial correlation is more important for the rules which enforces the choice of A. Then the decrease is faster for "strong" rules: the correlation equals 0 for $\tau = 75\%$ in the case of the strong majority rule while 0 is reached for $\tau = 0.95$ in the case of the weak rule.

We also see that when the dumping rate is 50%, all the correlations are equal.

We now compare the obtained features issued from configurations (C) with the case of layouts resulting from a random uniform distribution (C_0) of the same final rates of sites A and B .

The spatial characteristics of this new field (C_0) are of course different and we compare them with those obtained from (C). For instance, Figure 2 show the different behaviours of ρ_1 for fields (C) and (C_0). In the uniform case, the correlation is always close to zero (about 10^{-3}), can be negative. So it is clear that spatial correlation is a good criterion to distinguish the fields, a positive correlation more than 0.002 corresponding to type (C).

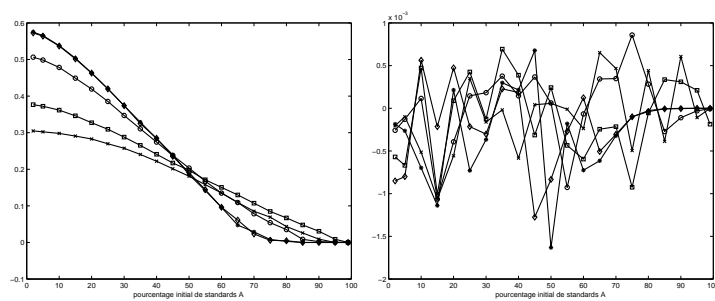


Fig. 2. left: correlation ρ_1 for fields (C); right: correlation ρ_1 for fields (C_0) associated to fields (C); rules: x : weak ; \square : Ising $\beta = 0.5$; \circ : Ising $\beta = 1$; \diamond : Ising $\beta = 3$; * : strong.

3.3 Spatial clustering measurements

Figure 3 below present an example of a realization of fields (C) and (C_0). As expected we observe specific cooperative textures in the images (C). We propose here several indexes evaluating this spatial feature.

Two clustering indexes Let us define the *absolute cluster index* IA as the number of edges joining neighbour sites which are together A, normalized by the total number of sites A.

$$IA = \frac{\sum_{i \in S} x_i x_{\partial i}}{2 \sum_{i \in S} x_i},$$

where $x_i = \frac{y_i + 1}{2}$ is equal to 1 if the site i has been assigned by A, and 0 else.



Fig. 3. Ising assignment rule $\beta = 1$, with initial rate $\tau = 0.05$; final A proportion: 63.04; $IR = 1.2856$

Left: $\rho_1 = 0.4829$, $IA = 1.6177$, $ncc = 9$, $mcc = 286.89$, $maxcc = 2553$.

Right: $\rho_1 = -0.0033$, $IA = 1.2583$, $ncc = 61$, $mcc = 42.33$, $maxcc = 2465$.

The *relative cluster index* IR is defined as the ratio of the absolute cluster indexes of fields (C) and associated (C_0).

$$IR = \frac{IA(C)}{IA(C_0)}.$$

Figure 4 show these indexes' evolution according to the initial rate τ of standards A , for the different assignment rules and for the two types of fields.

Concerning the absolute cluster index, its evolution is similar for fields (C) and associated (C_0). It is the initial rate which allows to distinguish the fields. When it is more than 50%, the curves are identical, while the values $IA(C)$ are much more important than those of $IA(C_0)$ in case of small rates τ . The smaller τ is, the more important is the difference between the two fields, and this whatever the assignment rule. The threshold rate categorizing the fields is varying with the rule. For instance, it is $\tau = 20\%$ for the strong majority rule and 30% for the Ising $\beta = 0.5$ rule.

The graph of the relative cluster index (Figure 4, Right) confirms the previous remark. The decrease of IR is faster for a rule enforcing standard A .

Finally, we conclude that the absolute cluster index is a good criterion to determine fields issued from a choice procedure if we know that the initial dumping rate is low valued.

Connectedness indexes A topological parameter which well characterizes clustering is a connectedness measure of standards A in the final configuration. The images obtained from simulations show that, for fields (C) issued from a choice procedure, a clustering organization of sites A appears. Moreover, the clusters become less numerous but wider when τ increases. For the corresponding fields (C) with same final number of sites A but randomly uniformly dispatched, we get many and small clusters.

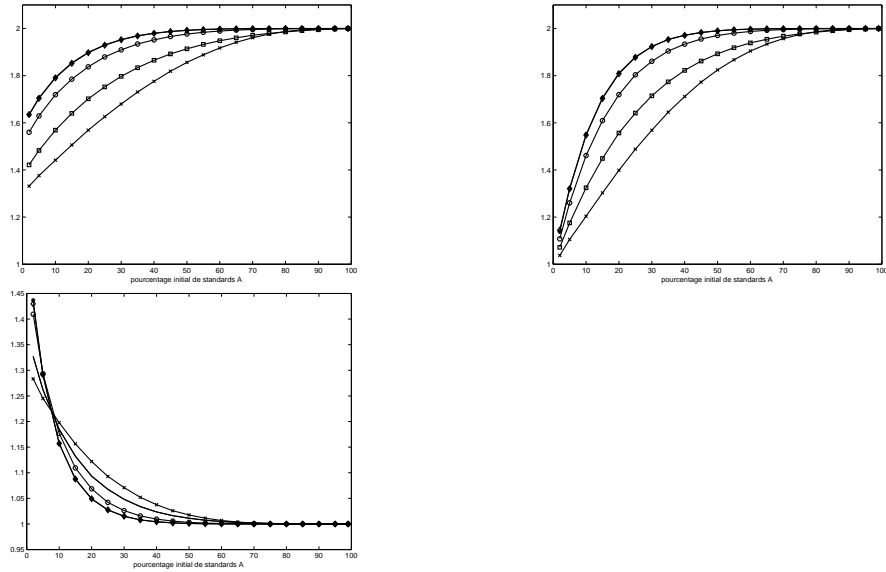


Fig. 4. Left: absolute cluster index for field (C);
 Right: absolute cluster index for field (C_0) associated;
 Below: relative cluster index;
 Rules: x : weak ; \square : Ising $\beta = 0.5$; o : Ising $\beta = 1$; \diamond : Ising $\beta = 3$; * : strong.

We propose to calculate three connectedness indexes, for fields (C) and associated (C_0); ncc is the number of connected components (of sites A); mcc is the mean size of these components, and $\max cc$ is the size of the largest one.

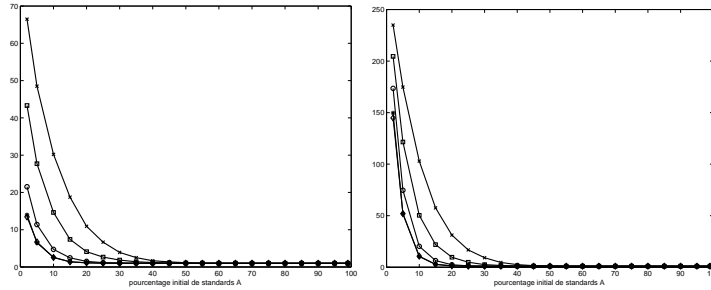


Fig. 5. mean number of connected components for fields (C)(left) and (C_0) (right):
 x : weak ; \square : Ising $\beta = 0.5$; o : Ising $\beta = 1$; \diamond : Ising $\beta = 3$; * : strong

We show in Figure 5 the evolution of ncc according to the different assignment rules for fields (C) and corresponding (C_0). Once again we observe a hierarchy between the rules. More interesting is the comparison of fields (C)

and (C_0) ; their behaviour seems to be similar but the scale is different and when the initial contamination rate is low, we can clearly distinguish fields of type (C) and (C_0) . For each rule, the number of connected components is much more important for fields (C_0) with random spreading. From an initial threshold rate which depends on the rules but not exceeding 30%, then the number ncc is similar for both fields.

We turn to the average size of connected components, defining the size of a component by the number of sites which lay inside. The evolution of mcc is given in Figure 6 for each rule and for fields (C) and (C_0) . On the contrary of the previous index ncc , it is difficult to visually distinguish between the two behaviours, since the ordinates scale is very large. Therefore we plot for a single rule the evolution of ncc for the two fields; we give for example the case of the Ising rule with $\beta = 1$ in Figure 6, Right. We get the same behaviour for the other rules. The curves join and overlay for $\tau \geq 30\%$. For small values of τ , the two curves appear to be not so different and we could think that mcc is a bad criterion to distinguish the fields; in fact, the ordinate scale is still large and for instance, $\tau=2\%$ corresponds to 144 for the field (C) and 13.8 for the associated (C_0) , that is $mcc(C_0)$ is more than 8 times $mcc(C)$. We conclude that mcc is useful to determine the types of final configurations in the case of small rates of initial contamination. Besides, the effect of dumping is more important for small values of τ .

Finally, we have calculated the size of the largest connected component $maxcc$. The previous comments apply again, even more significantly, since the increase of $maxcc$ is faster than the one of mcc . However, the evolution curves are quickly identical, and we can clearly see the dumping effect only for the smallest values of the initial rate of standards A; for instance $\tau < 10\%$ in the case of the Ising rule with $\beta = 1$, see Figure 7.

4 Distribution of the number of occupied neighbour sites

We can achieve some probabilities calculus on this non asymptotical framework. We consider a fix site visited on the scan tour. Whatever the assignment rule is, the choice of the agent depends on the number of his neighbours who have already make their decision. We give here the distribution of this number of “occupied” neighbours at the moment.

We consider the lattice S with n sites such that each site has the same number of neighbours ν . We assume that at $k = 0$, n_τ sites are occupied by standard A ($n_\tau = n\tau$); then at each time, a free site is randomly visited and becomes occupied. There are $n - n_\tau$ successive settings. For an arbitrary but fix site, we define the random variable Y_k by the number of occupied

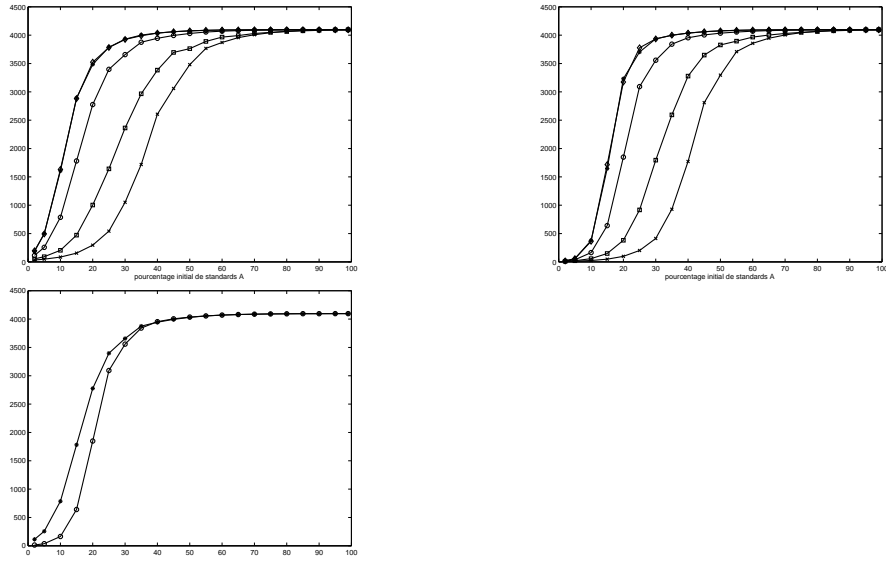


Fig. 6. Average size of the connected components for the fields (C)(left) and (C_0) (right); rules: x : weak ; \square : Ising $\beta = 0.5$; o : Ising $\beta = 1$; \diamond : Ising $\beta = 3$; * : strong

Below: Average size of the connected components for the Ising rule with $\beta = 1$: (C) : * ; (C_0) : o

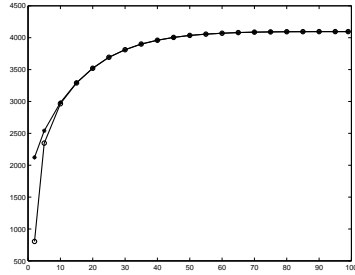


Fig. 7. Ising rule with $\beta = 1$; $\max cc(C)$: * ; $\max cc(C_0)$: o

neighbours at time k . We suppose $n_\tau < n$, $n \gg 2\nu + 1$; Y_k takes its values in the set $\{\max(0, n_\tau + k - n + \nu), 1, \dots, \min(\nu, k - 1 + n_\tau)\}$. We get

$$P(Y_k = l) = \frac{C_\nu^l C_{n-1-\nu}^{k-1+n_\tau-l}}{C_{n-1}^{k-1+n_\tau}}$$

Then we recognize that Y_k follows the hypergeometric distribution.

We can add two results. Let us define the events:

A_k : “the site j is occupied at exactly time k ”

B_k : “The site j is already occupied at time k ”

The index j does not appear in these probabilities since the scan is random. We get

$$P(A_k) = \begin{cases} (1 - \frac{n_\tau}{n})(1 - \frac{1}{n-n_\tau})(1 - \frac{1}{n-n_\tau-1})\dots(1 - \frac{1}{n-n_\tau-k+2})\frac{1}{n-n_\tau-k+1} = \frac{1}{n} & \text{if } k \geq 1 \\ \frac{n_\tau}{n} & \text{if } k = 0 \end{cases}.$$

We deduce $P(B_k) = \frac{n_\tau+k-1}{n}$.

Let us then explicit the mean probability P_l of the occupied neighbours during the course.

$$\begin{aligned} P_l &= \frac{1}{n-n_\tau} \sum_{k=1}^{n-n_\tau} P(Y_k = l) = \frac{1}{n-n_\tau} \sum_{k=1}^{n-n_\tau} \frac{C_\nu^l C_{n-1-\nu}^{k-1+n_\tau-l}}{C_{n-1}^{k-1+n_\tau}} \\ &= \frac{C_\nu^l (n-1-\nu)!}{(n-n_\tau) \cdot (n-1)!} \sum_{k=1}^{n-n_\tau} \prod_{s=1}^l (k+n_\tau-s) \prod_{u=0}^{\nu-l-1} (n-k-n_\tau-u) \end{aligned}$$

Let us denote $N_\tau = n - n_\tau = n(1 - \tau)$ the number of non initialized sites.

$$P_l = \frac{C_\nu^l N_\tau^\nu}{(n-1)(n-2)\dots(n-\nu)} \cdot \frac{1}{N_\tau} \sum_{k=1}^{N_\tau} \prod_{s=1}^l \left(\frac{k}{N_\tau} + \frac{\tau}{1-\tau} - \frac{s}{N_\tau} \right) \prod_{u=0}^{\nu-l-1} \left(1 - \frac{k}{N_\tau} - \frac{u}{N_\tau} \right)$$

l and $\nu - l$ being fix, we get:

$$P \xrightarrow{n \rightarrow \infty} C_\nu^l (1-\tau)^\nu \int_0^1 \left(x + \frac{\tau}{1-\tau} \right)^l (1-x)^{\nu-l} dx.$$

If the initial rate $\tau = 0$, then P_0 's limit is $\frac{1}{\nu+1}$. Else, we obtain by successive integrating $P_l \xrightarrow{n \rightarrow \infty} C_\nu^l (1-\tau)^\nu \sum_{k=0}^l \frac{l!}{(l-k)!} \left(\frac{\tau}{1-\tau} \right)^{l-k} \frac{(\nu-l)!}{(\nu-l+1+k)!}$ which is again

$$P_l \xrightarrow{n \rightarrow \infty} \frac{1}{\nu+1} Z_{l,\nu,\tau} \text{ where } Z_{l,\nu,\tau} = \sum_{j=0}^l C_{\nu+1}^j \tau^j (1-\tau)^{\nu-j}$$

This formula is still valid for $\tau = 0$. It is interesting to know if $Z_{l,\nu,\tau} > 1$ to see the dumping's effect against non initial contamination. Without loss of generality, we can assume that ν is even. If $\tau \geq \frac{1}{2}$ and $l \leq \frac{\nu}{2}$, then $Z_{l,\nu,\tau} \leq 1$. In fact $(1-\tau)^\nu \left(\frac{\tau}{1-\tau} \right)^j \leq [\tau(1-\tau)]^{\nu/2} \leq (1/4)^{\nu/2}$ which implies $Z_{l,\nu,\tau} \leq \left(\frac{1}{2} \right)^\nu \sum_{j=0}^l C_{\nu+1}^j \leq \left(\frac{1}{2} \right)^\nu \sum_{j=0}^{\nu/2} C_{\nu+1}^j \leq 1$ since $\sum_{j=0}^{\nu/2} C_{\nu+1}^j = 2^\nu$. Else, $Z_{l,\nu,\tau}$ can take values less or more 1 (but $\leq \nu+1$).

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