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Formulas for Birkhoff-(Rota-Baxter) decompositions related to connected bialgebras.

Frédéric Menous*

Abstract

In recent years, The usual BPHZ algorithm for renormalization in quantum field theory has been interpreted, after dimensional regularization, as the Birkhoff-(Rota-Baxter) decomposition (BRB) of characters on the Hopf algebra of Feynman graphs, with values in a Rota-Baxter algebra.

We give in this paper formulas for the BRB decomposition in the group $\mathcal{C}(H, A)$ of characters on a connected Hopf algebra H , with values in a Rota-Baxter (commutative) algebra A .

To do so we first define the stuffle (or quasi-shuffle) Hopf algebra A^{st} associated to an algebra A . We prove then that for any connected Hopf algebra $H = k1_H \oplus H'$, there exists a canonical injective morphism from H to H'^{st} . This morphism induces an action of $\mathcal{C}(A^{\text{st}}, A)$ on $\mathcal{C}(H, A)$ so that the BRB decomposition in $\mathcal{C}(H, A)$ is determined by the action of a unique (universal) element of $\mathcal{C}(A^{\text{st}}, A)$.

1 Introduction.

In this paper we deal with connected bialgebras $H = k1_H \oplus H'$. As reminded in section 2, such bialgebras are Hopf algebras and the coalgebra structure on H induces a convolution product on the space $\mathcal{L}(H, A)$ of linear morphisms from H to an associative algebra A . If A is unital, then the subset $\mathcal{U}(H, A)$ of linear morphisms that send 1_H on 1_A is a group for the convolution and, if A is commutative the subset $\mathcal{C}(H, A)$ of characters (algebra morphisms) is a subgroup of $\mathcal{U}(H, A)$.

In section 3, the target unital algebra A is equipped with a Rota-Baxter operator p_+ :

1. $p_+^2 = p_+$.
2. $A = \text{Im } p_+ \oplus \text{Im } p_- = A_+ \oplus A_-$ ($p_- = \text{Id} - p_+$).
3. A_+ and A_- are subalgebras.

With this hypothesis, It is well-know that there exists a unique Birkhoff-(Rota-Baxter) (or BRB) decomposition of any morphism $\varphi \in \mathcal{U}(H, A)$

$$\varphi_- * \varphi = \varphi_+ \quad \varphi_+, \varphi_- \in \mathcal{U}(H, A)$$

where $\varphi_+(H') \subset A_+$ and $\varphi_-(H') \subset A_-$. Moreover, if A is commutative, this decomposition is defined in the subgroup $\mathcal{C}(H, A)$. The proof of this result is recursive, using the filtration on H . We propose to give explicit, and in some sense universal, formulas for φ_+ and φ_- .

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To do so, we define in section 4 the stuffle Hopf algebra A^{st} of an algebra A , that is, the standard tensor coalgebra of A equipped with the stuffle (or quasi-shuffle product). With give then the example of computation for the inverse and the BRB decomposition of a fundamental element $j \in \mathcal{U}(A^{\text{st}}, A)$ defined by

$$j(\emptyset) = 1_A, \quad j(a_1) = a_1, \quad j(a_1 \otimes \dots \otimes a_s) = 0 \quad \text{if } s \geq 2$$

Stuffle Hopf algebras play a central role since, as proved in section 5, For any given connected Hopf algebra $H = k1_H \oplus H'$ there exists a canonical Hopf morphism $\iota : H \rightarrow H'^{\text{st}}$ defined with the help of the reduced coproduct. We prove then in section 6 that this morphism induces an action of $\mathcal{U}(A^{\text{st}}, A)$ on $\mathcal{U}(H, A)$. More precisely we define a map $T : \mathcal{U}(A^{\text{st}}, A) \times \mathcal{U}(H, A) \rightarrow \mathcal{U}(H, A)$ such that

$$T(j, \varphi) = \varphi \quad \text{and} \quad T(f * g, \varphi) = T(f, \varphi) * T(g, \varphi)$$

and we obtain explicit formulas as follows :

1. If j^{*-1} is the inverse of j , then $\varphi^{*-1} = T(j^{*-1}, \varphi)$.
2. If $j_- * j = j_+$ (BRB decomposition), the $\varphi_- * \varphi = \varphi_+$ where $\varphi_{\pm} = T(j_{\pm}, \varphi)$.

Note that such formulas were already derived in the Hopf algebra of Feynman graphs using matrix calculus (see [5]) but the formulas presented here are intrinsic : no choice of a basis is needed since we don't have to deal with matrices.

2 Reminder about connected bialgebras.

We follow here the notations and definitions given in [8] (see also [11] and [14]). Let k be a commutative field.

2.1 Bialgebras and Hopf algebras.

In the sequel, we will work with bialgebras over k :

Definition 1 *A bialgebra H is a k -vector space equipped with four linear maps $m : H \otimes H \rightarrow H$ (product : $m(x \otimes y) = xy$), $u : k \rightarrow H$ (unit : $u(1_k) = 1_H$), $\Delta : H \rightarrow H \otimes H$ (coproduct) and $\eta : H \rightarrow k$ (counit) such that*

1. (H, m, u) is a unital associative algebra. This reads :
 - a) *Associativity* : $m \circ (m \otimes \text{Id}) = m \circ (\text{Id} \otimes m) : H \otimes H \rightarrow H$;
 - b) *Unit* : $m \circ (u \otimes \text{Id}) = m \circ (\text{Id} \otimes u) = \text{Id} : k \otimes H = H \otimes k = H \rightarrow H$.
2. (H, Δ, η) is a coassociative coalgebra with a counit :
 - a) *Coassociativity* : $(\Delta \otimes \text{Id}) \circ \Delta = (\text{Id} \otimes \Delta) \circ \Delta : H \rightarrow H \otimes H \otimes H$;
 - b) *Counit* : $(\eta \otimes \text{Id}) \circ \Delta = (\text{Id} \otimes \eta) \circ \Delta = \text{Id} : H \rightarrow H$.
3. *The following diagram commutes :*

$$\begin{array}{ccc} H \otimes H & \xrightarrow{m} & H \\ \downarrow \Delta \otimes \Delta & & \uparrow m \otimes m \\ H \otimes H \otimes H \otimes H & \xrightarrow{\text{Id} \otimes \tau \otimes \text{Id}} & H \otimes H \otimes H \otimes H \end{array}$$

where τ is the linear map defined by $\tau(h \otimes g) = g \otimes h$ and

$$\Delta(1_H) = 1_H \otimes 1_H, \quad \eta(hg) = \eta(h)\eta(g)$$

Note that $p = u \circ \eta$ is an idempotent ($p' = \text{Id} - p$) and $H = \text{Im } p \oplus \text{Ker } p = \text{Im } u \oplus \text{Ker } \eta = k1_H \oplus H'$.

If $\Delta h = \sum_j h_{j(1)} \otimes h_{j(2)}$, we will sometimes write

$$\Delta h = \sum h_{(1)} \otimes h_{(2)} = h_{(1)} \otimes h_{(2)} \quad (1)$$

For example, coassociativity read

$$\sum h_{(1)(1)} \otimes h_{(1)(2)} \otimes h_{(2)} = \sum h_{(1)} \otimes h_{(2)(1)} \otimes h_{(2)(2)} = \sum h_{(1)} \otimes h_{(2)} \otimes h_{(3)}$$

and thanks to coassociativity, we can define recursively and without any ambiguity the linear morphisms $\Delta^{[n]} : H \rightarrow H^{\otimes n}$ ($n \geq 1$) by $\Delta^{[1]} = \text{Id}$ and, for $n \geq 1$,

$$\Delta^{[n+1]} = (\text{Id} \otimes \Delta^{[n]}) \circ \Delta = (\Delta^{[n]} \otimes \text{Id}) \circ \Delta = (\Delta^{[k]} \otimes \Delta^{[n+1-k]}) \circ \Delta \quad (1 \leq k \leq n) \quad (2)$$

and write

$$\Delta^{[n]} h = \sum h_{(1)} \otimes \dots \otimes h_{(n)} \quad (3)$$

On the same way, for $n \geq 1$, we define $m^{[n]} : H^{\otimes n} \rightarrow H$ by $m^{[1]} = \text{Id}$ and

$$m^{[n+1]} = m \circ (\text{Id} \otimes m^{[n]}) = m \circ (m^{[n]} \otimes \text{Id}) \quad (4)$$

Such a bialgebra is a Hopf algebra if there exists an antipode S , that is to say a linear map $S : H \rightarrow H$ such that :

$$m \circ (\text{Id} \otimes S) \circ \Delta = m \circ (S \otimes \text{Id}) \circ \Delta = u \circ \eta : H \rightarrow H \quad (5)$$

From now on, we should focus on connected bialgebras since they are automatically Hopf algebras.

2.2 Connected bialgebras.

Definition 2 A bialgebra \mathbb{N} -filtered as a vector space is called a filtered bialgebra when the filtering is compatible with both the algebra and the coalgebra structure; that is, there exist a sequence of subspaces $H_0 \subsetneq H_1 \subsetneq \dots$ such that $\bigcup_{n \geq 0} H_n = H$ and

$$\Delta H_n \subseteq \bigoplus_{k=0}^n H_k \otimes H_{n-k}; \quad H_n H_m \subseteq H_{n+m} \quad (6)$$

Connected bialgebras are those filtered bialgebras such that $H_0 = k1_H = \text{Im } u = \text{Im } p$.

We shall come back in the next section on the fact that the antipode automatically exists. For such a connected bialgebra, if, for all $n \geq 1$, $H'_n = H' \cap H_n$, then

$$\forall h \in H'_n, \quad \Delta h = 1 \otimes h + h \otimes 1 + y \quad \text{where} \quad y = (p' \otimes p') \circ \Delta h \in \bigoplus_{k=1}^{n-1} H'_k \otimes H'_{n-k} \quad (7)$$

and, it will be useful to define the reduced coproduct Δ' on H' defined by

$$\Delta' h = \Delta h - 1 \otimes h - h \otimes 1 \quad (8)$$

Then, recursively, for $n \geq 1$, we define $\Delta'^{[n]} = p'^{\otimes n} \circ \Delta^{[n]} : H' \rightarrow H'^{\otimes n}$. We note

$$\forall h \in H', \quad \Delta'^{[n]}h = \sum h'_{(1)} \otimes \dots \otimes h'_{(n)} \quad (9)$$

and we have, for h in H , $p'^{\otimes n} \circ \Delta^{[n]}(h) = \Delta'^{[n]} \circ p'(h)$.

Moreover, for $n \geq 1$ and $h \in H'$

$$\Delta'^{[n+1]}(h) = (\Delta'^{[k]} \otimes \Delta'^{[n+1-k]}) \circ \Delta'(h) \quad (1 \leq k \leq n) \quad (10)$$

and if $h \in H'_k$ ($k \geq 1$) then, for $n > k$, $\Delta'^{[n]}(h) = 0$.

Given a connected bialgebra H and an algebra (A, m_A, u_A) the coalgebra structure of H induces an associative convolution product on the vector space $\mathcal{L}(H, A)$ of k -linear maps :

$$\forall (f, g) \in \mathcal{L}(H, A) \times \mathcal{L}(H, A), \quad f * g = m_A \circ (f \otimes g) \circ \Delta \quad (11)$$

with a unit given by $u_A \circ \eta$, such that $(\mathcal{L}(H, A), *, u_A \circ \eta)$ is an associative unital algebra.

2.3 The group $(\mathcal{U}(H, A), *)$.

Proposition 1 *Let*

$$\mathcal{U}(H, A) = \{f \in \mathcal{L}(H, A) \ ; \ f(1_H) = 1_A\} \quad (12)$$

then $\mathcal{U}(H, A)$ is a group for the convolution product.

Proof $\mathcal{U}(H, A)$ is obviously stable for the convolution product and following [8] we should remind why any element $f \in \mathcal{U}(H, A)$ as a unique inverse f^{*-1} in $\mathcal{U}(H, A)$. There are two ways to define this inverse.

Since $H = H_0 \oplus H'$ it is sufficient to define recursively f^{*-1} on H'_n for $n \geq 1$ ($f^{-1}(1_H) = 1_A$). For $n \geq 1$ and $h \in H'_n$, we have

$$\Delta h = 1 \otimes h + h \otimes 1 + \sum h'_{(1)} \otimes h'_{(2)}$$

where

$$\sum h'_{(1)} \otimes h'_{(2)} \in \bigoplus_{k=1}^{n-1} H'_k \otimes H'_{n-k}$$

thus

$$f^{*-1} * f(h) = u_A(\eta(a)) = 0 = f^{*-1}(h) + f(h) + \sum f^{*-1}(h'_{(1)})f(h'_{(2)}) \quad (13)$$

and this defines f^{*-1} recursively. On the other hand, we can write

$$f^{*-1} = (u_A \circ \eta - (u_A \circ \eta - f))^{*-1} = u_A \circ \eta + \sum_{k \geq 1} (u_A \circ \eta - f)^{*k} \quad (14)$$

In fact, this series seems to be infinite but for $h \in H'_n$

$$(u_A \circ \eta - f)^{*k}(h) = (-1)^k \sum f(h'_{(1)}) \dots f(h'_{(k)}) = (-1)^k m_A^{[k]} \circ f^{\otimes k} \circ \Delta'^{[k]}(h) \quad (15)$$

vanishes as soon as $k > n$.

□

The principle of recursive computation will be useful when dealing with Birkhoff-Rota-Baxter decomposition and the main goal of this paper will be to find also formulas like 15.

Notation 1 If $B \subset A$ is a subalgebra of A which is not unital, then we write

$$\mathcal{U}(H, B) = \{f \in \mathcal{L}(H, A) \ ; \ f(1_H) = 1_A \ \text{and} \ f(H') \subset B\}$$

This is a subgroup of $\mathcal{U}(H, A)$.

If this result is applied to $\text{Id} : H \rightarrow H \in \mathcal{U}(H, H)$, then its convolution inverse is the antipode S and this proves that any connected bialgebra is a Hopf algebra. Moreover S is an antialgebra morphism :

$$S(gh) = S(h)S(g) \tag{16}$$

2.4 Algebra morphisms or characters.

Let $\mathcal{C}(H, A)$ the subset of $\mathcal{L}(H, A)$ whose elements are algebra morphisms (also called characters over A). Of course,

$$\mathcal{C}(H, A) \subset \mathcal{U}(H, A)$$

but this shall not be a subgroup. If A is not commutative, there is no reason why this should be stable for the convolution product. Moreover if $f \in \mathcal{U}(H, A)$ is an algebra map, then its inverse f^{*-1} in $\mathcal{U}(H, A)$ is an antialgebra map. In fact, if f is an algebra map then $f^{*-1} = f \circ S$:

$$\begin{aligned} f * f \circ S &= m_A \circ (f \otimes f \circ S) \circ \Delta \\ &= m_A \circ (f \otimes f) \circ (\text{Id} \otimes S) \circ \Delta \\ &= f \circ m \circ (\text{Id} \otimes S) \circ \Delta \\ &= f \circ u \circ \eta \\ &= u_A \circ \eta \end{aligned} \tag{17}$$

Nonetheless if A is commutative, then $\mathcal{C}(H, A)$ is a subgroup of $\mathcal{U}(H, A)$.

3 Rota-Baxter algebras and Birkhoff-type decomposition in $\mathcal{U}(H, A)$.

Following [6], let p_+ an idempotent of $\mathcal{L}(A, A)$ where A is a unital algebra. If we have for x, y in A :

$$p_+(x)p_+(y) + p_+(xy) = p_+(xp_+(y)) + p_+(p_+(x)y) \tag{18}$$

Then p_+ is a Rota-Baxter operator, (A, p_+) is a Rota-Baxter algebra and if $p_- = \text{Id} - p_+$, $A_+ = \text{Im } p_+$ and $A_- = \text{Im } p_-$ then

- $A = A_+ \oplus A_-$.
- p_- satisfies the same relation.
- A_+ and A_- are subalgebras.

Conversely if $A = A_+ \oplus A_-$ and A_+ and A_- are subalgebras, then the projection p_+ on A_+ parallel to A_- defines a Rota-Baxter algebra (A, p_+) .

The principle of renormalization in physics can be formulated in the following way

Proposition 2 *Let H be a connected bialgebra and (A, p_+) a Rota-Baxter algebra then for any $\varphi \in \mathcal{U}(H, A)$ there exists a unique pair $(\varphi_+, \varphi_-) \in \mathcal{U}(H, A_+) \times \mathcal{U}(H, A_-)$ such that*

$$\varphi_- * \varphi = \varphi_+ \quad (19)$$

Moreover, if A is commutative and φ is a character over A , then φ_+ and φ_- are also characters. This factorization will be called the Birkhoff-Rota-Baxter (or BRB) decomposition of φ .

Proof We will give the proof for characters later. As A_+ and A_- are subalgebras of A , $\mathcal{U}(H, A_+)$ and $\mathcal{U}(H, A_-)$ are subgroups of $\mathcal{U}(H, A)$.

If such a factorization exists, then it is unique : If $\varphi = \varphi_-^{*-1} * \varphi_+ = \psi_-^{*-1} * \psi_+$, then

$$\phi = \psi_+ * \varphi_+^{*-1} = \psi_- * \varphi_-^{*-1} \in \mathcal{U}(H, A_+) \cap \mathcal{U}(H, A_-)$$

thus for $h \in H'$, $\phi(h) \in A_+ \cap A_- = 0$. We finally get that

$$\psi_+ * \varphi_+^{*-1} = \psi_- * \varphi_-^{*-1} = u_A \circ \eta$$

and $\varphi_+ = \psi_+$, $\varphi_- = \psi_-$.

Let us prove now that such a factorization exists. Let $\varphi \in \mathcal{U}(H, A)$, we must have $\varphi_+(1_H) = \varphi_-(1_H) = 1_A$. Let $\bar{\varphi} \in \mathcal{U}(H, A)$ the Bogoliubov preparation map defined recursively on vector spaces H'_n ($n \geq 1$) by

$$\bar{\varphi}(h) = \varphi(h) - m_A \circ (p_- \otimes \text{Id}) \circ (\bar{\varphi} \otimes \varphi) \circ \Delta'(h) \quad (20)$$

Now if φ_+ and φ_- are the elements of $\mathcal{U}(H, A)$ defined on H' by

$$\varphi_+(h) = p_+ \circ \bar{\varphi}(h) \quad , \quad \varphi_-(h) = -p_- \circ \bar{\varphi}(h) \quad (\bar{\varphi}(h) = \varphi_+(h) - \varphi_-(h))$$

Then it is clear that

$$\varphi_+ \in \mathcal{U}(H, A_+) \quad , \quad \varphi_- \in \mathcal{U}(H, A_-) \quad , \quad \varphi_- * \varphi = \varphi_+$$

□

4 The stuffle Hopf algebra A^{st} of an algebra A .

For details on the stuffle (or quasi-shuffle) product, the reader can refer to [9].

4.1 Definition and properties.

Let A be an associative algebra. A^{st} is the graded vector space $A^{\text{st}} = \bigoplus_{n \geq 0} A_{(n)}^{\text{st}}$ where, for $n \geq 1$, $A_{(n)}^{\text{st}} = A^{\otimes n}$ and $A_{(0)}^{\text{st}} = k\emptyset$ where \emptyset is a symbol for the empty tensor product. It is obviously graded and we note $l(\mathbf{a}) = n$ the length of an element \mathbf{a} of $A_{(n)}^{\text{st}}$. For convenience, an element $\mathbf{a} = a_1 \otimes \dots \otimes a_r$ of A^{st} we be called a tuple or a word and if \mathbf{a} and \mathbf{b} are two words, then $\mathbf{a} \otimes \mathbf{b}$ is the concatenation of the words. Note also that, as \otimes already denotes the tensor product in A , when there may be some ambiguity, we use \otimes_{st} for the tensor product of elements of A^{st} .

One can define recursively the stuffle or quasi shuffle product $m_{\text{st}} : A^{\text{st}} \otimes_{\text{st}} A^{\text{st}} \rightarrow A^{\text{st}}$ on A^{st} :

1. For any $\mathbf{a} \in A^{\text{st}}$, $m_{\text{st}}(\emptyset \otimes_{\text{st}} \mathbf{a}) = m_{\text{st}}(\mathbf{a} \otimes_{\text{st}} \emptyset) = \mathbf{a}$

2. Let $\mathbf{a} = a_1 \otimes \dots \otimes a_r \in A_{(r)}^{\text{st}}$ and $\mathbf{b} = b_1 \otimes \dots \otimes b_s \in A_{(s)}^{\text{st}}$ with $r \geq 1$ and $s \geq 1$. If $\tilde{\mathbf{a}} = a_1 \otimes \dots \otimes a_{r-1} \in A_{(r-1)}^{\text{st}}$ ($\tilde{\mathbf{a}} = \emptyset$ if $r = 1$) and $\tilde{\mathbf{b}} = b_1 \otimes \dots \otimes b_{s-1} \in A_{(s-1)}^{\text{st}}$ ($\tilde{\mathbf{b}} = \emptyset$ if $r = 1$), then :

$$m_{\text{st}}(\mathbf{a} \otimes_{\text{st}} \mathbf{b}) = m_{\text{st}}(\tilde{\mathbf{a}} \otimes_{\text{st}} \mathbf{b}) \otimes a_r + m_{\text{st}}(\mathbf{a} \otimes_{\text{st}} \tilde{\mathbf{b}}) \otimes b_s + m_{\text{st}}(\tilde{\mathbf{a}} \otimes_{\text{st}} \tilde{\mathbf{b}}) \otimes a_r b_s \quad (21)$$

where $a_r b_s$ is the product in A of a_r and b_s .

For example :

$$m_{\text{st}}((a_1 \otimes a_2) \otimes_{\text{st}} b_1) = a_1 \otimes a_2 \otimes b_1 + a_1 \otimes b_1 \otimes a_2 + b_1 \otimes a_1 \otimes a_2 + a_1 \otimes a_2 b_1 \quad (22)$$

With this product, A^{st} is a unital algebra (unit \emptyset) and if A is commutative, then A^{st} is commutative. Moreover

$$\pi_{\text{st}}(A_{(r)}^{\text{st}} \otimes_{\text{st}} A_{(s)}^{\text{st}}) \subset \bigoplus_{t=\max(r,s)}^{r+s} A_{(t)}^{\text{st}} \quad (23)$$

On the same way one can define :

- a counit $\eta_{\text{st}} : A^{\text{st}} \rightarrow k$ by $\eta_{\text{st}}(\emptyset) = 1_k$ and for $s \geq 1$, $\eta_{\text{st}}(a_1 \otimes \dots \otimes a_s) = 0$,
- a coproduct $\Delta_{\text{st}} : A^{\text{st}} \rightarrow A^{\text{st}} \otimes_{\text{st}} A^{\text{st}}$ such that $\Delta_{\text{st}}(\emptyset) = \emptyset \otimes_{\text{st}} \emptyset$ and for $s \geq 1$ and $\mathbf{a} = a_1 \otimes \dots \otimes a_s \in A_{(s)}^{\text{st}}$,

$$\Delta_{\text{st}}(\mathbf{a}) = \mathbf{a} \otimes_{\text{st}} \emptyset + \emptyset \otimes_{\text{st}} \mathbf{a} + \sum_{r=1}^{s-1} (a_1 \otimes \dots \otimes a_r) \otimes_{\text{st}} (a_{r+1} \otimes \dots \otimes a_s) \quad (24)$$

such that A^{st} is a graded coalgebra.

It is a matter of fact to check that A^{st} is a connected bialgebra (and thus a Hopf algebra) for the filtration :

$$A_n^{\text{st}} = \bigoplus_{k=0}^n A_{(k)}^{\text{st}} \quad (25)$$

which is called the stuffle Hopf algebra on A and $A_0^{\text{st}} = k\emptyset$, $A^{\text{st}'} = \bigoplus_{n=1}^{+\infty} A_{(n)}^{\text{st}}$. We also have, for a sequence $\mathbf{a} \in A^{\text{st}'}$ and $n \geq 1$,

$$\Delta_{\text{st}}^{[n]}(\mathbf{a}) = \sum_{\mathbf{a}^1 \otimes \dots \otimes \mathbf{a}^n = \mathbf{a}} \mathbf{a}^1 \otimes_{\text{st}} \dots \otimes_{\text{st}} \mathbf{a}^n \quad (26)$$

where the sum is over n -tuple of non-empty words $(\mathbf{a}^1, \dots, \mathbf{a}^n)$ such that the concatenation of these sequences gives \mathbf{a} . In particular, for the antipode, $S(\emptyset) = \emptyset$ and if $\mathbf{a} = a_1 \otimes \dots \otimes a_s \in A^{\text{st}'}$,

$$\begin{aligned} S(\mathbf{a}) &= \sum_{k \geq 1} (-1)^k m_{\text{st}}^{[k]} \circ \Delta_{\text{st}}^{[k]}(\mathbf{a}) \\ &= \sum_{k \geq 1} (-1)^k \sum_{\mathbf{a}^1 \otimes \dots \otimes \mathbf{a}^k = \mathbf{a}} m_{\text{st}}^{[k]}(\mathbf{a}^1 \otimes_{\text{st}} \dots \otimes_{\text{st}} \mathbf{a}^k) \end{aligned} \quad (27)$$

If B is an algebra, then once again, there is a convolution on $\mathcal{L}(A^{\text{st}}, B)$:

$$\varphi * \psi = m_B \circ (\varphi \otimes \psi) \circ \Delta_{\text{st}}$$

and, if B is unital, $(\mathcal{U}(A^{\text{st}}, B), *)$ is a group. Moreover if B is commutative then $\mathcal{C}(A^{\text{st}}, B)$ is a subgroup.

Finally a map $l \in \mathcal{L}(A, B)$ induces a map $l^{\text{st}} \in \mathcal{U}(A^{\text{st}}, B^{\text{st}})$ defined by

$$l^{\text{st}}(\emptyset) = \emptyset \quad \text{and} \quad l^{\text{st}}(a_1 \otimes \dots \otimes a_r) = l(a_1) \otimes \dots \otimes l(a_r) \quad (r \geq 1)$$

and $\Delta_{\text{st}} \circ l^{\text{st}} = (l^{\text{st}} \otimes_{\text{st}} l^{\text{st}}) \circ \Delta_{\text{st}}$. Moreover, if l is an algebra map, then $l^{\text{st}} \circ m_{\text{st}} = m_{\text{st}} \circ (l^{\text{st}} \otimes_{\text{st}} l^{\text{st}})$, thus l^{st} is a Hopf morphism.

4.2 The map $j \in \mathcal{U}(A^{\text{st}}, A)$ where A is unital.

We shall now illustrate the computations of the previous section on the following map $j \in \mathcal{C}(A^{\text{st}}, A)$ defined by $j(\emptyset) = 1_A$, $j(a_1) = a_1$ and $j(a_1 \otimes \dots \otimes a_r) = 0$ if $r \geq 2$. In a sense, this will be the only computation of inverse and of Birkhoff-Rota-Baxter decomposition we will need.

For the inverse, we get the antialgebra morphism j^{*-1} :

$$j^{*-1} = u_A \circ \eta_{\text{st}} + \sum_{k \geq 1} (u_A \circ \eta_{\text{st}} - j)^{*k}$$

Which means that $j^{*-1}(\emptyset) = 1_A$ and for a sequence $\mathbf{a} = a_1 \otimes \dots \otimes a_s \in A^{\text{st}'}$,

$$\begin{aligned} j^{*-1}(\mathbf{a}) &= \sum_{k \geq 1} (-1)^k m_A^{[k]} \circ j^{\otimes k} \circ \Delta'_{\text{st}}^{[k]}(\mathbf{a}) \\ &= \sum_{k \geq 1} (-1)^k \sum_{\mathbf{a}^1 \otimes \dots \otimes \mathbf{a}^k = \mathbf{a}} m_A^{[k]} \circ j^{\otimes k}(\mathbf{a}^1 \otimes_{\text{st}} \dots \otimes_{\text{st}} \mathbf{a}^k) \\ &= \sum_{k \geq 1} (-1)^k \sum_{\mathbf{a}^1 \otimes \dots \otimes \mathbf{a}^k = \mathbf{a}} j \circ m_{\text{st}}^{[k]}(\mathbf{a}^1 \otimes_{\text{st}} \dots \otimes_{\text{st}} \mathbf{a}^k) \\ &= (-1)^s a_1 \dots a_s \\ &= j \circ S(\mathbf{a}) \end{aligned} \tag{28}$$

If (A, p_+) is a Rota-Baxter algebra then the Bogoliubov preparation map \bar{j} associated to j is such that $\bar{j}(\emptyset) = 1_A$ and is defined recursively on vector spaces $A_n^{\text{st}'}$ ($n \geq 1$) by

$$\bar{j}(h) = j(h) - m_A \circ (p_- \otimes \text{Id}) \circ (\bar{j} \otimes j) \circ \Delta'_{\text{st}}(h) \tag{29}$$

Let us begin the recursion on the length of the sequence. If $\mathbf{a} = a_1$ then $\bar{j}(a_1) = j(a_1) = a_1$. Now

$$\bar{j}(a_1 \otimes a_2) = j(a_1 \otimes a_2) - m_A \circ (p_- \otimes \text{Id}) \circ (\bar{j} \otimes j)((a_1) \otimes_{\text{st}} (a_2)) = -p_-(a_1)a_2 \tag{30}$$

and

$$\begin{aligned} \bar{j}(a_1, a_2, a_3) &= -m_A \circ (p_- \otimes \text{Id}) \circ (\bar{j} \otimes j)((a_1 \otimes a_2) \otimes_{\text{st}} (a_3)) \\ &= -p_-(p_-(a_1)a_2)a_3 \end{aligned} \tag{31}$$

Thus, for $r \geq 2$,

$$\bar{j}(a_1 \otimes \dots \otimes a_r) = -p_-(\bar{j}(a_1, \dots, a_{r-1}))a_r \tag{32}$$

It is then easy to prove that

Proposition 3 *The Birkhoff-Rota-Baxter decomposition $(j_+, j_-) \in \mathcal{U}(A^{\text{st}}, A_+) \times \mathcal{U}(A^{\text{st}}, A_-)$ such that*

$$j_- * j = j_+$$

is given by the formula : for $r \geq 1$ and $\mathbf{a} = a_1 \otimes \dots \otimes a_r \in A^{\text{st}'}$,

$$\begin{cases} j_+(\mathbf{a}) = p_+(\bar{j}(\mathbf{a})) = (-1)^{r-1} p_+(p_-(\dots(p_-(a_1)a_2)\dots a_{r-1})a_r) \\ j_-(\mathbf{a}) = -p_-(\bar{j}(\mathbf{a})) = (-1)^r p_-(p_-(\dots(p_-(a_1)a_2)\dots a_{r-1})a_r) \end{cases} \tag{33}$$

Moreover, if A is commutative then $\mathcal{C}(A^{\text{st}}, A)$ is a group and j_+ and j_- are characters.

Proof It remains to prove the last assumption, when A is commutative. Since j is a character it is sufficient to prove that j_- is a character. By induction on $t \in \mathbb{N}$ we will show that for two sequences \mathbf{a} and \mathbf{b} in A^{st} , if $l(\mathbf{a}) + l(\mathbf{b}) = t$, then

$$j_-(m_{\text{st}}(\mathbf{a} \otimes_{\text{st}} \mathbf{b})) = j_-(\mathbf{a})j_-(\mathbf{b}) \quad (34)$$

This identity is trivial for $t = 0$ and $t = 1$ since at least one of the sequences is the empty sequence. This also trivial for any t if one of the sequence is empty. Now suppose that $t \geq 2$ and that $\mathbf{a} = a_1 \otimes \dots \otimes a_r \in A_{(r)}^{\text{st}}$ and $\mathbf{b} = b_1 \otimes \dots \otimes b_s \in A_{(s)}^{\text{st}}$ with $r \geq 1$, $s \geq 1$ and $r + s = t$. Let $\tilde{\mathbf{a}} = a_1 \otimes \dots \otimes a_{r-1} \in A_{(r-1)}^{\text{st}}$ ($\tilde{\mathbf{a}} = \emptyset$ if $r = 1$) and $\tilde{\mathbf{b}} = b_1 \otimes \dots \otimes b_{s-1} \in A_{(s-1)}^{\text{st}}$ ($\tilde{\mathbf{b}} = \emptyset$ if $s = 1$), then :

$$m_{\text{st}}(\mathbf{a} \otimes_{\text{st}} \mathbf{b}) = m_{\text{st}}(\tilde{\mathbf{a}} \otimes_{\text{st}} \mathbf{b}) \otimes a_r + m_{\text{st}}(\mathbf{a} \otimes_{\text{st}} \tilde{\mathbf{b}}) \otimes b_s + m_{\text{st}}(\tilde{\mathbf{a}} \otimes_{\text{st}} \tilde{\mathbf{b}}) \otimes a_r b_s$$

Now we have

$$j_-(\mathbf{a}) = -p_-(j_-(\tilde{\mathbf{a}})a_r) = -p_-(x) \quad \text{and} \quad j_-(\mathbf{b}) = -p_-(j_-(\tilde{\mathbf{b}})b_s) = -p_-(y)$$

Thanks to the Rota-Baxter identity

$$\begin{aligned} j_-(\mathbf{a})j_-(\mathbf{b}) &= p_-(x)p_-(y) \\ &= p_-(xp_-(y)) + p_-(p_-(x)y) - p_-(xy) \\ &= p_-(j_-(\tilde{\mathbf{a}})a_r p_-(j_-(\tilde{\mathbf{b}})b_s)) + p_-(p_-(j_-(\tilde{\mathbf{a}})a_r)j_-(\tilde{\mathbf{b}})b_s) - p_-(j_-(\tilde{\mathbf{a}})a_r j_-(\tilde{\mathbf{b}})b_s) \end{aligned}$$

but as A is commutative, by induction we get

$$\begin{aligned} j_-(\mathbf{a})j_-(\mathbf{b}) &= -p_-(j_-(\tilde{\mathbf{a}})j_-(\mathbf{b})a_r) - p_-(j_-(\mathbf{a})j_-(\tilde{\mathbf{b}})b_s) - p_-(j_-(\tilde{\mathbf{a}})j_-(\tilde{\mathbf{b}})a_r b_s) \\ &= -p_-(j_-(m_{\text{st}}(\tilde{\mathbf{a}} \otimes \mathbf{b})a_r)) - p_-(j_-(m_{\text{st}}(\mathbf{a} \otimes \tilde{\mathbf{b}})b_s)) - p_-(j_-(m_{\text{st}}(\tilde{\mathbf{a}} \otimes \tilde{\mathbf{b}})a_r b_s)) \\ &= j_-(m_{\text{st}}(\tilde{\mathbf{a}} \otimes \mathbf{b}) \otimes a_r) + j_-(m_{\text{st}}(\mathbf{a} \otimes \tilde{\mathbf{b}}) \otimes b_s) + j_-(m_{\text{st}}(\tilde{\mathbf{a}} \otimes \tilde{\mathbf{b}}) \otimes a_r b_s) \\ &= j_-(m_{\text{st}}(\tilde{\mathbf{a}} \otimes \mathbf{b}) \otimes a_r + m_{\text{st}}(\mathbf{a} \otimes \tilde{\mathbf{b}}) \otimes b_s + m_{\text{st}}(\tilde{\mathbf{a}} \otimes \tilde{\mathbf{b}}) \otimes a_r b_s) \\ &= j_-(m_{\text{st}}(\mathbf{a} \otimes \mathbf{b})) \end{aligned}$$

□

As we will see these formulas are almost sufficient to compute the Birkhoff decomposition in any connected bialgebra.

5 The Hopf morphism $\iota : H \rightarrow H'^{\text{st}}$.

Theorem 1 Let $H = H_0 \oplus H'$ be a connected bialgebra, then the map $\iota : H \rightarrow H'^{\text{st}}$ defined by $\iota(1_H) = \emptyset$ and

$$\forall h \in H', \quad \iota(h) = \sum_{k \geq 1} \Delta'^{[k]}(h) \in H'^{\text{st}'} \quad (35)$$

defines an injective Hopf morphism.

Proof This map is well defined since, if $k > n \geq 1$,

$$\forall h \in H'_n, \quad \Delta'^{[k]}(h) = 0$$

It is obviously linear and injective : For $h_1 = \alpha_1 1_H + p'(h_1)$ and $h_2 = \alpha_2 1_H + p'(h_2)$ then, thanks to the graduation of the vector space H'^{st} , if $\iota(h_1) = i(h_2)$ then $\alpha_1 = \alpha_2$ and $p'(h_1) = p'(h_2)$ thus $h_1 = h_2$. This is a coalgebra map since

$$\Delta_{\text{st}}(\iota(1_H)) = \Delta_{\text{st}}(\emptyset) = \emptyset \otimes_{\text{st}} \emptyset = (\iota \otimes \iota) \circ \Delta(1_H)$$

where, to avoid ambiguity, we noted \otimes_{st} the tensor product of two elements of H'^{st} . For $h \in H'$,

$$\begin{aligned}
(\iota \otimes_{\text{st}} \iota) \circ \Delta(h) &= (\iota \otimes_{\text{st}} \iota)(1 \otimes h + h \otimes 1 + \Delta'(h)) \\
&= \emptyset \otimes_{\text{st}} \iota(h) + \iota(h) \otimes_{\text{st}} \emptyset + \left(\sum_{k \geq 1} \sum_{l \geq 1} \Delta'^{[k]} \otimes_{\text{st}} \Delta'^{[l]} \right) \circ \Delta'(h) \\
&= \emptyset \otimes_{\text{st}} \iota(h) + \iota(h) \otimes_{\text{st}} \emptyset + \sum_{k \geq 1} \sum_{l \geq 1} (\text{Id}^{\otimes k} \otimes_{\text{st}} \text{Id}^{\otimes l}) \circ (\Delta'^{[k]} \otimes_{\text{st}} \Delta'^{[l]}) \circ \Delta'(h) \\
&= \emptyset \otimes_{\text{st}} \iota(h) + \iota(h) \otimes_{\text{st}} \emptyset + \sum_{n \geq 1} \sum_{k=1}^{n-1} (\text{Id}^{\otimes k} \otimes_{\text{st}} \text{Id}^{\otimes l}) \circ \Delta'^{[n]}(h) \\
&= \emptyset \otimes_{\text{st}} \iota(h) + \iota(h) \otimes_{\text{st}} \emptyset + \Delta'_{\text{st}}(\iota(h)) \\
&= \Delta'_{\text{st}}(\iota(h))
\end{aligned}$$

But ι is also an algebra map. Let g and h be two elements of H . If g or h is in H_0 then we get trivially that

$$\iota(gh) = m_{\text{st}}(\iota(g) \otimes_{\text{st}} \iota(h))$$

As in the previous section we will prove by induction on $t \geq 2$ that for any positive integer r and s such that $r + s = t$, then

$$\forall (g, h) \in H'_r \times H'_s, \quad \iota(gh) = m_{\text{st}}(\iota(g) \otimes_{\text{st}} \iota(h))$$

Note that

$$\begin{aligned}
\iota(h) &= \sum_{k \geq 1} \Delta'^{[k]}(h) \\
&= h + \sum_{k \geq 1} \Delta'^{[k+1]}(h) \\
&= h + \sum_{k \geq 1} (\Delta'^{[k]} \otimes \text{Id}) \circ \Delta'(h) \\
&= h + \sum_{k \geq 1} (\Delta'^{[k]} \otimes \text{Id})(h'_{(1)} \otimes h'_{(2)}) \\
&= h + \iota(h'_{(1)}) \otimes h'_{(2)}
\end{aligned} \tag{36}$$

For $t = 2$ ($r = s = 1$) then $\iota(g) = g$, $\iota(h) = h$ and

$$\Delta'^{[2]}(gh) = h \otimes g + g \otimes h$$

thus

$$\iota(gh) = gh + h \otimes g + g \otimes h = m_{\text{st}}((g) \otimes_{\text{st}} (h))$$

More generally

$$\begin{aligned}
\Delta'(gh) &= h \otimes g + g \otimes h + gh'_{(1)} \otimes h'_{(2)} + h'_{(1)} \otimes gh'_{(2)} \\
&\quad + g'_{(1)} h \otimes g'_{(2)} + g'_{(1)} \otimes g'_{(2)} h + g'_{(1)} h'_{(1)} \otimes g'_{(2)} h'_{(2)}
\end{aligned}$$

and if $f = gh$

$$\iota(gh) = f + \iota(f'_{(1)}) \otimes f'_{(2)}$$

Now

$$\begin{aligned}
\iota(gh) &= (gh) + \iota(h) \otimes g + \iota(g) \otimes h + \iota(gh'_{(1)}) \otimes h'_{(2)} + \iota(h'_{(1)}) \otimes gh'_{(2)} \\
&\quad + \iota(g'_{(1)}h) \otimes g'_{(2)} + \iota(g'_{(1)}) \otimes g'_{(2)}h + \iota(g'_{(1)}h'_{(1)}) \otimes g'_{(2)}h'_{(2)} \\
&= (gh) + h \otimes g + \iota(h'_{(1)}) \otimes h'_{(2)} \otimes g + g \otimes h + \iota(g'_{(1)}) \otimes g'_{(2)} \otimes h + \iota(gh'_{(1)}) \otimes h'_{(2)} \\
&\quad + \iota(h'_{(1)}) \otimes gh'_{(2)} + \iota(g'_{(1)}h) \otimes g'_{(2)} + \iota(g'_{(1)}) \otimes g'_{(2)}h + \iota(g'_{(1)}h'_{(1)}) \otimes g'_{(2)}h'_{(2)} \\
&= (gh) + h \otimes g + g \otimes h \\
&\quad + \iota(h'_{(1)}) \otimes h'_{(2)} \otimes g + \iota(gh'_{(1)}) \otimes h'_{(2)} + \iota(h'_{(1)}) \otimes gh'_{(2)} \\
&\quad + \iota(g'_{(1)}) \otimes g'_{(2)} \otimes h + \iota(g'_{(1)}h) \otimes g'_{(2)} + \iota(g'_{(1)}) \otimes g'_{(2)}h \\
&\quad + \iota(g'_{(1)}h'_{(1)}) \otimes g'_{(2)}h'_{(2)}
\end{aligned}$$

By the induction we get

$$\begin{aligned}
\iota(gh) &= m_{\text{st}}((g) \otimes_{\text{st}} (h)) \\
&\quad + \iota(h'_{(1)}) \otimes h'_{(2)} \otimes g + m_{\text{st}}(\iota(g) \otimes_{\text{st}} \iota(h'_{(1)})) \otimes h'_{(2)} + \iota(h'_{(1)}) \otimes gh'_{(2)} \\
&\quad + \iota(g'_{(1)}) \otimes g'_{(2)} \otimes h + m_{\text{st}}(\iota(g'_{(1)}) \otimes_{\text{st}} \iota(h)) \otimes g'_{(2)} + \iota(g'_{(1)}) \otimes g'_{(2)}h \\
&\quad + m_{\text{st}}(\iota(g'_{(1)}) \otimes_{\text{st}} \iota(h'_{(1)})) \otimes g'_{(2)}h'_{(2)}
\end{aligned}$$

As $\iota(g) = g + \iota(g'_{(1)}) \otimes g'_{(2)}$ and $\iota(h) = h + \iota(h'_{(1)}) \otimes h'_{(2)}$, we get

$$\begin{aligned}
\iota(gh) &= m_{\text{st}}((g) \otimes_{\text{st}} (h)) \\
&\quad + \iota(h'_{(1)}) \otimes h'_{(2)} \otimes g + m_{\text{st}}((g) \otimes_{\text{st}} \iota(h'_{(1)})) \otimes h'_{(2)} + \iota(h'_{(1)}) \otimes gh'_{(2)} \\
&\quad + \iota(g'_{(1)}) \otimes g'_{(2)} \otimes h + m_{\text{st}}(\iota(g'_{(1)}) \otimes_{\text{st}} (h)) \otimes g'_{(2)} + \iota(g'_{(1)}) \otimes g'_{(2)}h \\
&\quad + m_{\text{st}}(\iota(g'_{(1)}) \otimes_{\text{st}} \iota(h'_{(1)})) \otimes g'_{(2)}h'_{(2)} \\
&\quad + m_{\text{st}}((\iota(g'_{(1)}) \otimes g'_{(2)}) \otimes_{\text{st}} \iota(h'_{(1)})) \otimes h'_{(2)} + m_{\text{st}}(\iota(g'_{(1)}) \otimes_{\text{st}} (\iota(h'_{(1)}) \otimes h'_{(2)})) \otimes g'_{(2)} \\
&= m_{\text{st}}((g) \otimes_{\text{st}} (h)) \\
&\quad + m_{\text{st}}((g) \otimes_{\text{st}} (\iota(h'_{(1)}) \otimes h'_{(2)})) \\
&\quad + m_{\text{st}}((\iota(g'_{(1)}) \otimes g'_{(2)}) \otimes_{\text{st}} (h)) \\
&\quad + m_{\text{st}}((\iota(g'_{(1)}) \otimes g'_{(2)}) \otimes_{\text{st}} (\iota(h'_{(1)}) \otimes h'_{(2)})) \\
&= m_{\text{st}}(\iota(g) \otimes_{\text{st}} \iota(h))
\end{aligned}$$

□

This morphisms shows that any connected bialgebra can be canonically identified to a sub-algebra of a stuffle algebra. This will help us to define $\mathcal{U}(A^{\text{st}}, B)$ as a group of function from $\mathcal{U}(H, A)$ to $\mathcal{U}(H, B)$ where H (resp. A, B) is a connected bialgebra (resp. unital algebras).

6 The map $T : \mathcal{U}(A^{\text{st}}, B) \times \mathcal{U}(H, A) \rightarrow \mathcal{U}(H, B)$ and associated formulas.

6.1 Definition and properties.

Let H be a connected bialgebra and A, B two unital algebras. For $\varphi \in \mathcal{U}(H, A)$ and $f \in \mathcal{U}(A^{\text{st}}, B)$ we define

$$T(f, \varphi) = f \circ \varphi^{\text{st}} \circ \iota$$

It is clear that $T(f, \varphi)$ is a linear morphism from H to B and

$$T(f, \varphi)(1_H) = f \circ \varphi^{\text{st}} \circ \iota(1_H) = f \circ \varphi^{\text{st}}(\emptyset) = f(\emptyset) = 1_B$$

thus $T(f, \varphi) \in \mathcal{U}(H, B)$. Moreover if $\varphi \in \mathcal{C}(H, A)$ and $f \in \mathcal{C}(A^{\text{st}}, B)$ it is clear, by composition of algebra morphisms that $T(f, \varphi) \in \mathcal{C}(H, B)$.

There are two fundamental properties :

1. Let f and g in $\mathcal{U}(A^{\text{st}}, B)$ and φ in $\mathcal{U}(H, A)$, then

$$\begin{aligned} T(f * g, \varphi) &= m_B \circ (f \otimes g) \circ \Delta_{\text{st}} \circ \varphi^{\text{st}} \circ \iota \\ &= m_B \circ (f \otimes g) \circ (\varphi^{\text{st}} \otimes \varphi^{\text{st}}) \circ \Delta_{\text{st}} \circ \iota \\ &= m_B \circ (f \otimes g) \circ (\varphi^{\text{st}} \otimes \varphi^{\text{st}}) \circ (\iota \otimes \iota) \circ \Delta \\ &= m_B(T(f, \varphi) \otimes T(g, \varphi)) \circ \Delta \\ T(f * g, \varphi) &= T(f, \varphi) * T(g, \varphi) \end{aligned} \quad (37)$$

2. If $A = B$, then $T(j, \varphi) = \varphi$ because if $h \in H'$,

$$\begin{aligned} T(j, \varphi)(h) &= j \circ \varphi^{\text{st}} \left(h + \sum_{k \geq 2} \sum h'_{(1)} \otimes \dots \otimes h'_{(k)} \right) \\ &= j \left(\varphi(h) + \sum_{k \geq 2} \varphi(h'_{(1)}) \otimes \dots \otimes \varphi(h'_{(k)}) \right) \\ &= \varphi(h) \end{aligned} \quad (38)$$

6.2 The semigroup $(\mathcal{U}(A^{\text{st}}, A), \odot)$.

We leave the details to the reader but, or f and g in $\mathcal{U}(A^{\text{st}}, A)$ let us define

$$f \odot g = T(f, g)$$

This is a binary operator on $\mathcal{U}(A^{\text{st}}, A)$ which is associative (but non-commutative) and

$$T(f, j) = f \circ j^{\text{st}} \circ \iota = f$$

thus $(\mathcal{U}(A^{\text{st}}, A), \odot)$ is semigroup.

6.3 The inverse in $\mathcal{U}(H, A)$.

Let $\varphi \in \mathcal{U}(H, A)$. Since $T(j, \varphi) = \varphi$, if $\psi = T(j^{*-1}, \varphi)$, then

$$\psi * \varphi = T(j^{*-1}, \varphi) * T(j, \varphi) = T(j^{*-1} * j, \varphi) = T(u_A \circ \eta_{\text{st}}, \varphi) = u_A \circ \eta$$

and the inverse of φ is ψ . For example, if $h \in H'_3$, then

$$\iota(h) = h + h'_{(1)} \otimes h'_{(2)} + h'_{(1)} \otimes h'_{(2)} \otimes h'_{(3)}$$

so,

$$\varphi^{\text{st}} \circ \iota(h) = \varphi(h) + \varphi(h'_{(1)}) \otimes \varphi(h'_{(2)}) + \varphi(h'_{(1)}) \otimes \varphi(h'_{(2)}) \otimes \varphi(h'_{(3)})$$

and finally

$$\varphi^{*-1}(h) = j^{*-1} \circ \varphi^{\text{st}} \circ \iota = -\varphi(h) + \varphi(h'_{(1)})\varphi(h'_{(2)}) - \varphi(h'_{(1)})\varphi(h'_{(2)})\varphi(h'_{(3)})$$

We recover the usual formula for the inverse.

6.4 The BRB decomposition in $\mathcal{U}(H, A)$.

Finally, let $\varphi \in \mathcal{U}(H, A)$. Since $T(j, \varphi) = \varphi$, if $\varphi_- = T(j_-, \varphi)$ and $\varphi_+ = T(j_+, \varphi)$, then

$$\varphi_- * \varphi = T(j_-, \varphi) * T(j, \varphi) = T(j_- * j, \varphi) = T(j_+, \varphi) = \varphi_+$$

and, of course, $\varphi_{\pm} \in \mathcal{U}(H, A_{\pm})$. For example, if $h \in H'_3$, then

$$\begin{aligned}\varphi_+(h) &= p_+(\varphi(h)) - p_+(p_-(\varphi(h'_1))\varphi(h'_2)) + p_+(p_-(p_-(\varphi(h'_1))\varphi(h'_2))\varphi(h'_3)) \\ \varphi_-(h) &= -p_-(\varphi(h)) + p_-(p_-(\varphi(h'_1))\varphi(h'_2)) - p_-(p_-(p_-(\varphi(h'_1))\varphi(h'_2))\varphi(h'_3))\end{aligned}$$

Needless to say that if A is commutative, these computations works in the subgroup $\mathcal{C}(H, A)$.

7 Conclusion.

Once these formulas are given, we get formulas in the different contexts where renormalization, or rather BRB decomposition is needed :

- Renormalization in quantum field theory : the connected Hopf algebra is the connected graded Hopf algebra of Feynman graphs. The character is given, after dimensional regularization, by Feynman integrals with values in a commutative algebra of Laurent series in a parameter ε : $A = \mathcal{A}[[\varepsilon]][\varepsilon^{-1}]$ with $A_+ = \mathcal{A}[[\varepsilon]]$ and $A_- = \varepsilon^{-1}\mathcal{A}[\varepsilon^{-1}]$ (see for example [3], [4], [6]).
- Chen's iterated integrals : this kind of integrals (including multizeta values) define characters on a connected graded Hopf algebras of trees or ladders (see for example [10], [12]).
- The Birkhoff decomposition in the group of formal identity-tangent diffeomorphism with coefficients in $A = \mathbb{R}[[\varepsilon]][\varepsilon^{-1}]$. Any element

$$f(x) = x + \sum_{n \geq 2} f_n(\varepsilon)x^n \quad , \quad f_n(\varepsilon) \in \mathbb{R}[[\varepsilon]][\varepsilon^{-1}]$$

can be decomposed : $f_- \circ f = f_+$ with

$$\begin{aligned}f_-(x) &= x + \sum_{n \geq 2} f_{-,n}(\varepsilon)x^n & f_{-,n}(\varepsilon) &\in \varepsilon^{-1}\mathbb{R}[\varepsilon^{-1}] \\ f_+(x) &= x + \sum_{n \geq 2} f_{+,n}(\varepsilon)x^n & f_{+,n}(\varepsilon) &\in \mathbb{R}[[\varepsilon]][\varepsilon^{-1}]\end{aligned}$$

This factorization corresponds here to the BRB decomposition in the Faà di Bruno Hopf algebra (see [8],[13]).

The same ideas were also used for the the even-odd factorization of characters in combinatorial Hopf algebras (see [1], [2] and [7]).

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