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# Wiener integrals with respect to the Hermite process and a Non-Central Limit Theorem

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## Abstract

We introduce Wiener integrals with respect to the Hermite process and we prove a Non-Central Limit Theorem in which this integral appears as limit. As an example, we study a generalization of the fractional Ornstein-Uhlenbeck process.

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## 1 Introduction

The selfsimilar processes have been widely studied due to their applications as models for various phenomena, like hydrology, network traffic analysis and mathematical finance.

An interesting class of selfsimilar processes is given as limits of the so called *Non-Central Limit Theorem* studied in [21] and [9]. We briefly recall the context. Let  $g$  be a function of Hermite rank  $k$  (see Section 5 for the definition) and let  $(\xi_n)_{n \in \mathbb{Z}}$  be a stationary Gaussian sequence with mean 0 and variance 1 which exhibits long range dependence in the sense that the correlation function satisfies

$$r(n) := E(\xi_0 \xi_n) = n^{\frac{2H-2}{k}} L(n)$$

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where  $H \in (\frac{1}{2}, 1)$ ,  $k \geq 1$  and  $L$  is a slowly varying function at infinity (see e.g. [10] for the definition). Then the following holds:

$$\frac{1}{n^H} \sum_{j=1}^{[nt]} g(\xi_j) \quad (1)$$

converges as  $n \rightarrow \infty$  in the sense of finite dimensional distributions, to the process

$$Z_H^k(t) = c(H, k) \int_{\mathbb{R}^k} \left( \int_0^t \left( \prod_{i=1}^k (s - y_i)_+^{-\left(\frac{1}{2} + \frac{1-H}{k}\right)} \right) ds \right) dB(y_1) \dots dB(y_k) \quad (2)$$

where the above integral is a Wiener-Itô multiple integral of order  $k$  with respect to the standard Brownian motion  $(B(y))_{y \in \mathbb{R}}$  and  $c(H, k)$  is a positive normalization constant depending only on  $H$  and  $k$ . The process  $(Z_H^k(t))_{t \geq 0}$  is called *the Hermite process* and it is  $H$ -selfsimilar in the sense that for any  $c > 0$ ,  $(Z_H^k(ct)) \stackrel{(d)}{=} (c^H Z_H^k(t))$ , where " $\stackrel{(d)}{=}$ " means equivalence of all finite dimensional distributions, and it has stationary increments.

The most studied Hermite process is of course *the fractional Brownian motion* (which is obtained in (2) for  $k = 1$ ) due to its large range of applications. Recently, a rich theory of stochastic integration with respect to this process has been introduced and stochastic differential equations driven by the fractional Brownian motion have been considered. We refer to [4], [8] or [11], to cite only a few. The process obtained in (2) for  $k = 2$  is known as *the Rosenblatt process*. It was introduced by Rosenblatt in [19] and it has been called in this way by M. Taqqu in [20]. The Rosenblatt process has also practical applications and different aspects of this process, like wavelet type expansion or extremal properties have been studied in [1], [2], [15] or [16].

The aim of this paper is to make a first step in the direction of a stochastic calculus driven by the Hermite process of order  $k$  by introducing Wiener integrals with respect to this process. The basic observation is the fact that the covariance structure of the Hermite process is similar to the one of the fractional Brownian motion and this allows the use of the same classes of deterministic integrands as in the fractional Brownian motion case whose properties are known. We will distinguish as in [18], [17], a time domain and a spectral domain of deterministic integrands. As an application, we discuss the existence and the properties of the *Hermite Ornstein-Uhlenbeck process* (the correspondent of the fractional Ornstein-Uhlenbeck process, see [6]) that appears as the unique solution of a Langevin type equation.

Of central interest is for us a Non-Central Limit Theorem that has as limit the Wiener integral with respect to the Hermite process  $(Z_H^k(t))_{t \geq 0}$  in (2). Recall that it has been proved in [17] that, if  $f$  is a deterministic function in a suitable space, then the sequence

$$\frac{1}{n^H} \sum_{j \in \mathbb{Z}} f\left(\frac{j}{n}\right) X_j, \quad (3)$$

where  $(X_j)_{j \in \mathbb{Z}}$  is in the domain of attraction of the fractional Brownian motion, converges weakly, as  $n \rightarrow \infty$ , to the Wiener integral  $\int_{\mathbb{R}} f(u) dB_H(u)$ , where  $B_H$  denotes the fractional Brownian motion. The case of stable random variables, where  $(X_j)_{j \in \mathbb{Z}}$  are i.i.d. belonging to

the domain of attraction of a stable law, has been studied in [12] and [13]. A natural extension of the convergence of sequences (1) and (3) is to show that

$$\frac{1}{n^H} \sum_{j \in \mathbb{Z}} f\left(\frac{j}{n}\right) g(\xi_j)$$

converges weakly when  $n \rightarrow \infty$ , to the Wiener integral  $\int_{\mathbb{R}} f(u) dZ_H^k(u)$  assuming that the sequence  $g(\xi_j), j \in \mathbb{Z}$  belongs to the domain of attraction of the Hermite process (see Theorem 3.4.1 in [10] for such processes or also [7]). We study the time domain approach in which we prove the general result. The spectral domain approach has been considered in [17]. We also mention that in order to not overcharge the paper, we decided to omit the slowly varying function  $L$ ; indeed, although from the mathematical point of view this function could be sometimes tedious to treat, from the philosophical point of view this is less significant.

We organize our paper as follows. Section 2 recalls several properties of the Hermite process. In Section 3 we construct Wiener integrals with respect to this process and in Section 4 we discuss, as an example, the Hermite Ornstein-Uhlenbeck process which is a generalization of the fractional Ornstein-Uhlenbeck process. Section 5 contains a Non-Central Limit Theorem which has as limit the Wiener integral with respect to the Hermite process.

## 2 The Hermite process

We will present in this section some basic properties of the Hermite process  $(Z_H^k(t))_{t \in \mathbb{R}}$  of order  $k \geq 1$ ,  $k \in \mathbb{Z}$  and with Hurst parameter  $H \in (\frac{1}{2}, 1)$ . This stochastic process is defined as a multiple Wiener-Itô integral of order  $k$  with respect to the standard Brownian motion  $B((t))_{t \in \mathbb{R}}$

$$Z_H^k(t) = c(H, k) \int_{\mathbb{R}^k} \int_0^t \left( \prod_{j=1}^k (s - y_j)_+^{-\left(\frac{1}{2} + \frac{1-H}{k}\right)} \right) ds dB(y_1) \dots dB(y_k), \quad (4)$$

where  $x_+ = \max(x, 0)$ . We refer to e.g. [14] for the properties of the multiple stochastic integrals. For  $k = 1$  the above process is nothing else than the fractional Brownian motion with Hurst parameter  $H \in (0, 1)$ . For  $k \geq 2$  the process  $Z_H^k$  is not Gaussian and if  $k = 2$  it is known (see [20]) as *the Rosenblatt process*.

Let us compute the covariance  $R(t, s) := E [Z_H^k(t) Z_H^k(s)]$  of the Hermite process. By Fubini and the isometry of multiple Wiener-Itô integrals, one has

$$\begin{aligned} R(t, s) &= 2c(H, k)^2 \int_{\mathbb{R}^k} \left( \int_0^t \int_0^s \prod_{j=1}^k (u - y_j)_+^{-\left(\frac{1}{2} + \frac{1-H}{k}\right)} (v - y_j)_+^{-\left(\frac{1}{2} + \frac{1-H}{k}\right)} dv du \right) dy_1 \dots dy_k \\ &= 2c(H, k)^2 \int_0^t \int_0^s \int_{\mathbb{R}^k} \left[ \prod_{j=1}^k (u - y_j)_+^{-\left(\frac{1}{2} + \frac{1-H}{k}\right)} (v - y_j)_+^{-\left(\frac{1}{2} + \frac{1-H}{k}\right)} dy_1 \dots dy_k \right] dv du \\ &= 2c(H, k)^2 \int_0^t \int_0^s \left[ \int_{\mathbb{R}} (u - y)_+^{-\left(\frac{1}{2} + \frac{1-H}{k}\right)} (v - y)_+^{-\left(\frac{1}{2} + \frac{1-H}{k}\right)} dy \right]^k dv du. \end{aligned}$$

Let  $\beta(p, q) = \int_0^1 z^{p-1}(1-z)^{q-1} dz$ ,  $p, q > 0$  be the beta function. By using the identity

$$\int_{\mathbb{R}} (u-y)_+^{a-1} (v-y)_+^{a-1} dy = \beta(a, 2a-1) |u-v|^{2a-1}$$

we get

$$\begin{aligned} R(t, s) &= 2c(H, k)^2 \beta\left(\frac{1}{2} - \frac{1-H}{k}, \frac{2H-2}{k}\right)^k \int_0^t \int_0^s (|u-v|^{\frac{2H-2}{k}})^k dv du \\ &= 2c(H, k)^2 \frac{\beta\left(\frac{1}{2} - \frac{1-H}{k}, \frac{2H-2}{k}\right)^k}{H(2H-1)} \frac{1}{2} (t^{2H} + s^{2H} - |t-s|^{2H}). \end{aligned}$$

In order to obtain  $E(Z_H^k(t))^2 = 1$ , we will choose

$$c(H, k)^2 = \left( \frac{\beta\left(\frac{1}{2} - \frac{1-H}{k}, \frac{2H-2}{k}\right)^k}{2H(2H-1)} \right)^{-1}$$

and we will have

$$R(t, s) = \frac{1}{2} (t^{2H} + s^{2H} - |t-s|^{2H}).$$

We also recall the following properties of the Hermite process:

- The process  $Z_H^k$  is  $H$ -selfsimilar with stationary increments and all moments are finite.
- From the stationarity of increments and the selfsimilarity, it follows that, for any  $p \geq 1$

$$E \left[ \left| Z_H^k(t) - Z_H^k(s) \right|^p \right] = c(p, H, k) |t-s|^{pH}.$$

As a consequence the Hermite process has Hölder continuous paths of order  $\delta < H$ .

We mention that different expressions of the exponent in (4) are used in the literature, but we choose this one in order to have the order of similarity equal to  $H$ .

### 3 Wiener integrals with respect to the Hermite process

In this paragraph we introduce Wiener integrals with respect to the Hermite process.

Let us denote by  $\mathcal{E}$  the class of elementary functions on  $\mathbb{R}$  of the form

$$f(u) = \sum_{l=1}^n a_l 1_{(t_l, t_{l+1}]}(u), \quad t_l < t_{l+1}, \quad a_l \in \mathbb{R}, \quad l = 1, \dots, n. \quad (5)$$

For  $f \in \mathcal{E}$  as above it is natural to define its Wiener integral with respect to the Hermite process  $Z_H^k$  by

$$\int_{\mathbb{R}} f(u) dZ_H^k(u) = \sum_{l=1}^n a_l \left( Z_H^k(t_{l+1}) - Z_H^k(t_l) \right). \quad (6)$$

In order to extend the definition (6) to a larger class of integrands, let us make first some observations. By formula (4) we can write

$$Z_H^k(t) = \int_{\mathbb{R}^k} I(1_{[0,t]}) (y_1, \dots, y_k) dB(y_1) \dots dB(y_k) \quad (7)$$

where by  $I$  we denote the mapping on the set of functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  to the set of functions  $f : \mathbb{R}^k \rightarrow \mathbb{R}$

$$I(f)(y_1, \dots, y_k) = c(H, k) \int_{\mathbb{R}} f(u) \prod_{j=1}^k (u - y_j)_+^{-\left(\frac{1}{2} + \frac{1-H}{k}\right)} du.$$

Note that for  $k = 1$  this operator can be expressed in terms of fractional integrals and derivatives (see [18], [3]). Thus the definition (6) can be also written in the form, due to the obvious linearity of  $I$

$$\begin{aligned} \int_{\mathbb{R}} f(u) dZ_H^k(u) &= \sum_{l=1}^n a_l \left( Z_H^k(t_{l+1}) - Z_H^k(t_l) \right) \\ &= \sum_{l=1}^n a_l \int_{\mathbb{R}^k} I(1_{(t_l, t_{l+1}]}) (y_1, \dots, y_k) dB(y_1) \dots dB(y_k) \\ &= \int_{\mathbb{R}^k} I(f)(y_1, \dots, y_k) dB(y_1) \dots dB(y_k). \end{aligned} \quad (8)$$

We introduce now the following space

$$\mathcal{H} = \left\{ f : \mathbb{R} \rightarrow \mathbb{R}; \int_{\mathbb{R}^k} (I(f)(y_1, \dots, y_k))^2 dy_1 \dots dy_k < \infty \right\}$$

endowed with the norm

$$\|f\|_{\mathcal{H}}^2 = \int_{\mathbb{R}^k} (I(f)(y_1, \dots, y_k))^2 dy_1 \dots dy_k.$$

It holds that

$$\begin{aligned} \|f\|_{\mathcal{H}}^2 &= c(H, k)^2 \int_{\mathbb{R}^k} \left( \int_{\mathbb{R}} \int_{\mathbb{R}} f(u) f(v) \prod_{j=1}^k (u - y_j)_+^{-\left(\frac{1}{2} + \frac{1-H}{k}\right)} (v - y_j)_+^{-\left(\frac{1}{2} + \frac{1-H}{k}\right)} dv du \right) dy_1 \dots dy_k \\ &= c(H, k)^2 \int_{\mathbb{R}} \int_{\mathbb{R}} f(u) f(v) \left( \int_{\mathbb{R}} (u - y)_+^{-\left(\frac{1}{2} + \frac{1-H}{k}\right)} (v - y)_+^{-\left(\frac{1}{2} + \frac{1-H}{k}\right)} dy \right)^k dv du \\ &= H(2H - 1) \int_{\mathbb{R}} \int_{\mathbb{R}} f(u) f(v) |u - v|^{2H-2} dv du. \end{aligned}$$

Hence we have

$$\mathcal{H} = \left\{ f : \mathbb{R} \rightarrow \mathbb{R} \mid \int_{\mathbb{R}} \int_{\mathbb{R}} f(u) f(v) |u - v|^{2H-2} dv du < \infty \right\}$$

and

$$\|f\|_{\mathcal{H}}^2 = H(2H-1) \int_{\mathbb{R}} \int_{\mathbb{R}} f(u)f(v)|u-v|^{2H-2} dvdu. \quad (9)$$

Let us observe that the mapping

$$f \mapsto \int_{\mathbb{R}} f(u) dZ_H^k(u) \quad (10)$$

provides an isometry from  $\mathcal{E}$  to  $L^2(\Omega)$ . Indeed, for  $f$  of the form (5), it holds that

$$\begin{aligned} E [I(f)^2] &= \sum_{i,j=0}^{n-1} a_i a_j E [Z_H(t_{i+1}) - Z_H(t_i)] (Z_H(t_{j+1}) - Z_H(t_j)) \\ &= \sum_{i,j=0}^{n-1} a_i a_j (R(t_{i+1}, t_{j+1}) - R(t_{i+1}, t_j) - R(t_i, t_{j+1}) + R(t_i, t_j)) \\ &= \sum_{i,j=0}^{n-1} a_i a_j H(2H-1) \int_{t_i}^{t_{i+1}} \int_{t_j}^{t_{j+1}} |u-v|^{2H-2} dvdu \\ &= \sum_{i,j=0}^{n-1} a_i a_j \langle 1_{(t_i, t_{i+1}]}, 1_{(t_j, t_{j+1}]} \rangle_{\mathcal{H}} = \|f\|_{\mathcal{H}}^2. \end{aligned}$$

On the other hand, it has been proved in [18] that the set of elementary functions  $\mathcal{E}$  is dense in  $\mathcal{H}$ . As a consequence the mapping (10) can be extended to an isometry from  $\mathcal{H}$  to  $L^2(\Omega)$  and relation (8) still holds.

The followings facts has also been proved in [18]:

- The elements of  $\mathcal{H}$  may be not functions but distributions; it is therefore more practical to work with subspaces of  $\mathcal{H}$  that are sets of functions. Such a subspace is

$$|\mathcal{H}| = \{f : \mathbb{R} \rightarrow \mathbb{R} \mid \int_{\mathbb{R}} \int_{\mathbb{R}} |f(u)||f(v)||u-v|^{2H-2} dvdu < \infty\}.$$

Then  $|\mathcal{H}|$  is a strict subspace of  $\mathcal{H}$  and we actually have the inclusions

$$L^2(\mathbb{R}) \cap L^1(\mathbb{R}) \subset L^{\frac{1}{H}}(\mathbb{R}) \subset |\mathcal{H}| \subset \mathcal{H}. \quad (11)$$

- The space  $|\mathcal{H}|$  is not complete with respect to the norm  $\|\cdot\|_{\mathcal{H}}$  but it is a Banach space with respect to the norm

$$\|f\|_{|\mathcal{H}|}^2 = \int_{\mathbb{R}} \int_{\mathbb{R}} |f(u)||f(v)||u-v|^{2H-2} dvdu.$$

- A "spectral domain" included in  $\mathcal{H}$  can also be defined as

$$\widehat{\mathcal{H}} = \{f \in L^2(\mathbb{R}) \mid \int_{\mathbb{R}} |\widehat{f}(x)|^2 |x|^{-2H+1} dx < \infty\}, \quad (12)$$

where  $\widehat{f}$  denotes the Fourier transform of  $f$ . We have again that  $\widehat{\mathcal{H}}$  is a strict subspace of  $\mathcal{H}$  and the inclusion

$$L^2(\mathbb{R}) \cap L^1(\mathbb{R}) \subset L^{\frac{1}{H}}(\mathbb{R}) \subset \widehat{\mathcal{H}} \subset \mathcal{H}$$

is true. We define

$$\|f\|_{\widehat{\mathcal{H}}}^2 = \int_{\mathbb{R}} \left| \widehat{f}(x) \right|^2 |x|^{-2H+1} dx.$$

- There are elements in  $|\mathcal{H}|$  that are not in  $\widehat{\mathcal{H}}$  and viceversa.

## 4 Hermite Ornstein-Uhlenbeck processes

Introducing Wiener integrals with respect to the Hermite process allows us to consider a first example of stochastic differential equation with the Hermite process as driving noise. We will consider the Langevin type stochastic equation

$$X_t = \xi - \lambda \int_0^t X_s ds + \sigma Z_H^k(t), \quad t \geq 0, \quad (13)$$

where  $\sigma, \lambda > 0$ , the Hurst parameter  $H$  belongs to  $(\frac{1}{2}, 1)$  and  $k \geq 1$  is an integer. The initial condition  $\xi$  is a random variable in  $L^0(\Omega)$ . The case  $k = 1$  has been considered in [6]. Recall that in this case (when  $Z_H^k = B_H$ , a fractional Brownian motion) the unique solution of the equation (13), that is understood in a Riemann-Stieltjes sense, has the Wiener integral representation

$$Y_H^\xi(t) = e^{-\lambda t} \left( \xi + \sigma \int_0^t e^{\lambda u} dB_H(u) \right), \quad t \geq 0.$$

When the initial condition is  $\xi = \sigma \int_{-\infty}^0 e^{\lambda u} dB_H(u)$ , the solution of (13) can be written as

$$Y_H(t) = \sigma \int_{-\infty}^t e^{-\lambda(t-u)} dB_H(u) \quad (14)$$

and it is called the *stationary fractional Ornstein-Uhlenbeck process*. It follows from [6] that the process (14) is a stationary centered Gaussian process,  $H$ -selfsimilar and it has stationary increments. Moreover, it is ergodic and exhibits long range dependence for  $H \in (\frac{1}{2}, 1)$ . Its covariance function behaves as, when  $t \in \mathbb{R}$ ,  $N = 1, 2, \dots$  and  $s \rightarrow \infty$ ,

$$E[Y_H(t)Y_H(t+s)] = \frac{1}{2}\sigma^2 \sum_{m=1}^N \lambda^{-2n} \left( \prod_{j=0}^{2n-1} (2H-j) \right) s^{2H-2n} + O(s^{2H-2N-2}). \quad (15)$$

When  $H = \frac{1}{2}$  the process  $Y_{\frac{1}{2}}$  can be also defined by using the Lamperti transform

$$Y_H(t) = e^{-\lambda t} B_H \left( \alpha \exp\left(\frac{\lambda}{H}t\right) \right) \quad (16)$$

and both definitions coincide. When  $H \neq \frac{1}{2}$ , processes obtained by (14) and (16) are different.

The above context can be easily generalized to the case of the Hermite process. More precisely, we have the following result.

**Proposition 1** Let  $(Z_H^k(t))_{t \in \mathbb{R}}$  be a Hermite process of order  $k$  and let  $\xi \in L^0(\mathbb{R})$ . The following are true for almost all  $\omega$  and for every  $\lambda, \sigma > 0$ .

- a. For all  $t > a$ , the integral  $\int_a^t e^{\lambda u} dZ_H^k(u, \omega)$  exists in the Riemann-Stieljes sense and it is equal to

$$e^{\lambda t} Z_H^k(t, \omega) - e^{\lambda a} Z_H^k(a, \omega) - \lambda \int_a^t Z_H^k(u, \omega) e^{\lambda u} du.$$

Moreover, the function  $t \rightarrow \int_a^t e^{\lambda u} dZ_H^k(u, \omega)$  is continuous.

- b. The unique continuous solution of the equation

$$y(t) = \xi(\omega) - \lambda \int_0^t y(s) ds + \sigma Z_H^k(t, \omega), \quad t \geq 0$$

is given by

$$y(t) = e^{-\lambda t} \left( \xi(\omega) + \sigma \int_0^t e^{\lambda u} dZ_H^k(u, \omega) \right), \quad t \geq 0.$$

In particular, if  $\xi = \sigma \int_{-\infty}^0 e^{\lambda u} dZ_H^k(u, \omega)$ , then

$$y(t) = \sigma \int_{-\infty}^t e^{-\lambda(t-u)} dZ_H^k(u, \omega), \quad t \geq 0.$$

**Proof:** The proof of Proposition A.1 in [6] and the fact that the process  $Z_H^k$  is Hölder continuous of order  $\delta < H$  imply the conclusion. ■

As a consequence, the unique solution of (13) with initial condition  $\sigma \int_{-\infty}^0 e^{\lambda u} dZ_H^k(u)$  is given by

$$Y_H^k(t) = \sigma \int_{-\infty}^t e^{-\lambda(t-u)} dZ_H^k(u), \quad t \geq 0. \quad (17)$$

The process given by (17) will be called *Hermite Ornstein-Uhlenbeck process of order  $k$* . Because the covariance structure of the Hermite process is the same as the fractional Brownian motion's one, it is easy to see that the above integral coincides with the Wiener integral defined in Section 3. Hence the process  $Y_H^k$  is a centered Gaussian process with covariance

$$E \left[ Y_H^k(t) Y_H^k(s) \right] = \sigma^2 \int_{-\infty}^t \int_{-\infty}^s e^{-\lambda(t-u)} e^{-\lambda(s-v)} |u - v|^{2H-2} dv du < \infty.$$

Clearly, relation (15) still holds for the Hermite Ornstein-Uhlenbeck process.

## 5 Non-Central Limit Theorem

In this section we extend the results in [9] and [21] by proving a Non-Central Limit Theorem which has as limit the Wiener integral with respect to the Hermite process introduced in Section 2.

We will consider a sequence of centered stationary Gaussian random variables  $(\xi_j)_{j \in \mathbb{Z}}$  with  $E\xi_j^2 = 1$  and the correlation function

$$r(n) := E(\xi_0 \xi_n) = n^{-\frac{2H-2}{k}}. \quad (18)$$

Let us recall the notion of Hermite rank. Denote by  $H_m(x)$  the Hermite polynomial of degree  $m$  given by  $H_m(x) = (-1)^m e^{\frac{x^2}{2}} \frac{d^m}{dx^m} e^{-\frac{x^2}{2}}$  and let  $g$  be a function on  $\mathbb{R}$  such that  $E[g(\xi_0)] = 0$  and  $E[g(\xi_0)^2] < \infty$ . Assume that  $g$  has the following expansion in Hermite polynomials

$$g(x) = \sum_{j=0}^{\infty} c_j H_j(x)$$

where  $c_j = \frac{1}{j!} E[g(\xi_0) H_j(\xi_0)]$ . The Hermite rank of  $g$  is defined by

$$k = \min\{j; c_j \neq 0\}.$$

Since  $E[g(\xi_0)] = 0$ , we have  $k \geq 1$ .

We also introduce the sequence of stochastic processes  $Z_H^{k,n}$  given by

$$Z_H^{k,n}(u) = \frac{1}{n^H} \sum_{l=1}^{[nu]} g(\xi_l) \quad (u \geq 0) \quad \text{and} \quad Z_H^{k,n}(u) = \frac{1}{n^H} \sum_{j=-[nu]-1}^0 g(\xi_j) \quad (u < 0), \quad (19)$$

where  $g$  is a function of Hermite rank  $k$ . By the results in [9] and [21] (see also Theorem 3.4.1 in [10]), it holds that

$$Z_H^{k,n} \xrightarrow{(d)} c_k Z_H^k, \quad n \rightarrow \infty.$$

Here " $\xrightarrow{(d)}$ " means the convergence in the sense of finite dimensional distributions. We also use the notation, if  $f$  is a function on  $\mathbb{R}$ ,

$$f_n = \sum_{j=-\infty}^{\infty} f\left(\frac{j}{n}\right) 1_{\left(\frac{j}{n}, \frac{j+1}{n}\right]}, \quad f_{n,T}^+ = \sum_{j=0}^T f\left(\frac{j}{n}\right) 1_{\left(\frac{j}{n}, \frac{j+1}{n}\right]}, \quad f_{n,T}^- = \sum_{j=-T}^{-1} f\left(\frac{j}{n}\right) 1_{\left(\frac{j}{n}, \frac{j+1}{n}\right]} \quad (20)$$

and also  $f_n^+ = f_{n,\infty}^+$ ,  $f_n^- = f_{n,\infty}^-$ .

We have the following Non-Central Limit Theorem.

**Theorem 1** *Let  $f \in |\mathcal{H}|$  such that  $f_n^\pm \in |\mathcal{H}|$  for every  $n \geq 1$  and assume that*

$$|f_n - f|_{|\mathcal{H}|} \xrightarrow{n \rightarrow \infty} 0$$

and for every  $n$

$$|f_{n,T}^\pm - f_n^\pm|_{|\mathcal{H}|} \xrightarrow{T \rightarrow \infty} 0.$$

Then, as  $n \rightarrow \infty$ ,

$$\frac{1}{n^H} \sum_{j \in \mathbb{Z}} f\left(\frac{j}{n}\right) g(\xi_j) \xrightarrow{(d)} c_k \int_{\mathbb{R}} f(u) dZ_H^k(u) \quad (21)$$

where  $g$  is a function of Hermite rank  $k$  of the form  $g(x) = \sum_{l=k}^{\infty} c_l H_l(x)$ .

**Proof:** Let us prove first that the sum

$$S(n) := \frac{1}{n^H} \sum_{j \in \mathbb{Z}} f\left(\frac{j}{n}\right) g(\xi_j)$$

is convergent in  $L^2(\Omega)$ . We regard only the right tail part, the left part being similar. Using the relation

$$E[H_{k_1}(\xi_i)H_{k_2}(\xi_j)] = \delta_{k_1, k_2} k_1! r(i, j)^{k_1}$$

we get

$$\begin{aligned} E[S(n)^2] &= \frac{1}{n^{2H}} \sum_{j_1, j_2 \in \mathbb{Z}} f\left(\frac{j_1}{n}\right) f\left(\frac{j_2}{n}\right) \sum_{l=k}^{\infty} c_l^2 E[H_l(\xi_{j_1})H_l(\xi_{j_2})] \\ &= \frac{1}{n^{2H}} \sum_{j_1, j_2 \in \mathbb{Z}} f\left(\frac{j_1}{n}\right) f\left(\frac{j_2}{n}\right) \sum_{l=k}^{\infty} c_l^2 l! r(j_1 - j_2)^l \\ &= \frac{1}{n^{2H}} \sum_{j_1 \neq j_2} f\left(\frac{j_1}{n}\right) f\left(\frac{j_2}{n}\right) \sum_{l=k}^{\infty} c_l^2 l! |j_1 - j_2|^{\frac{(2H-2)l}{k}} + \frac{1}{n^{2H}} \sum_{j \in \mathbb{Z}} f\left(\frac{j}{n}\right)^2 \sum_{l=k}^{\infty} l! c_l^2 \end{aligned}$$

Now, since for  $|j_1 - j_2| \geq 1$  and for  $l \geq k$  one has  $|j_1 - j_2|^{\frac{(2H-2)l}{k}} \leq |j_1 - j_2|^{2H-2}$ , and since the sum  $C_0 := \sum_{j \in \mathbb{N}} c_l^2 l!$  is convergent, we obtain

$$\begin{aligned} ES(n)^2 &\leq \frac{C_0}{n^2} \sum_{j_1 \neq j_2} \left| f\left(\frac{j_1}{n}\right) \right| \left| f\left(\frac{j_2}{n}\right) \right| \left| \frac{j_1}{n} - \frac{j_2}{n} \right|^{2H-2} + \frac{C_0}{n^{2H}} \sum_{j \in \mathbb{Z}} f\left(\frac{j}{n}\right)^2 \\ &\leq C_0 \sum_{j_1 \neq j_2} \left| f\left(\frac{j_1}{n}\right) \right| \left| f\left(\frac{j_2}{n}\right) \right| \int_{\frac{j_1}{n}}^{\frac{j_1+1}{n}} \int_{\frac{j_2}{n}}^{\frac{j_2+1}{n}} |u - v|^{2H-2} dudv + \frac{C_0}{n^{2H}} \sum_{j \in \mathbb{Z}} f\left(\frac{j}{n}\right)^2 \\ &= C_0 \|f_n\|_{\mathcal{H}}^2. \end{aligned} \tag{22}$$

In the same way we will obtain

$$E \left[ \left| \frac{1}{n^H} \sum_{j=p_1+1}^{p_2} f\left(\frac{j}{n}\right) g(\xi_j) \right|^2 \right] \leq C_0 \|f_{n, p_2}^+ - f_{n, p_1}^+\|_{\mathcal{H}}^2$$

and this tend to zero as  $p_1, p_2 \rightarrow \infty$  by assumption.

Let us prove now that the sequence  $S(n)$  converges to the Wiener integral  $\int_{\mathbb{R}} f(u) dZ_H^k(u)$  when  $n$  tends to infinity. Here we follow the arguments [17]. Let us choose a sequence  $f^r, r \geq 1$  of elementary functions such that  $f^r \rightarrow_{r \rightarrow \infty} f$  with respect to norm  $|\mathcal{H}|$  (this is possible because this space is complete) and denote by

$$f_n^r = \sum_{j \in \mathbb{Z}} f^r\left(\frac{j}{n}\right) 1_{\left(\frac{j}{n}, \frac{j+1}{n}\right]}, \quad r, n \geq 1.$$

Note first that from a result in Bilingsley [5], it suffices to show that:

$$\int_{\mathbb{R}} f^r(u) dZ_H^k(u) \xrightarrow{(d)} \int_{\mathbb{R}} f(u) dZ_H^k(u), \quad r \rightarrow \infty, \quad (23)$$

$$S^r(n) := \frac{1}{n^H} \sum_{j=-\infty}^{\infty} f^r\left(\frac{j}{n}\right) g(\xi_j) \xrightarrow{(d)} \int_{\mathbb{R}} f^r(u) dZ_H^k(u), \quad n \rightarrow \infty, \text{ for every } r \geq 1, \quad (24)$$

and

$$\overline{\lim}_r \overline{\lim}_n E \left[ |S^r(n) - S(n)|^2 \right] = 0. \quad (25)$$

Concerning (23), we have by the construction of the Wiener integral and the choice of the sequence  $f^r$ ,

$$E \left[ \left| \int_{\mathbb{R}} f^r(u) dZ_H^k(u) - \int_{\mathbb{R}} f(u) dZ_H^k(u) \right|^2 \right] = \|f^r - f\|_{\mathcal{H}}^2 \leq \|f^r - f\|_{|\mathcal{H}|}^2 \xrightarrow{r \rightarrow \infty} 0.$$

The convergence (24) follows from the convergence of the sequence  $Z_H^{k,n}$  given by (19) to the Hermite process  $Z_H^k$  because (since  $f^r$  is an elementary function)  $S^r(n)$  constitutes a finite linear combination of instants of the process  $Z_H^{k,n}$  and  $\int_{\mathbb{R}} f^r(u) dZ_H^k(u)$  represents (the same) finite linear combination of instants of  $Z_H^k$ . For (25), proceeding as for the proof of (22), we obtain

$$E |S^r(n) - S(n)|^2 \leq \text{const.} \|f_n^r - f_n\|_{|\mathcal{H}|}^2$$

and thus by a dominated convergence argument (see [17], Proposition 3.1) we get

$$\overline{\lim}_r \overline{\lim}_n E |S^r(n) - S(n)|^2 \leq \text{const.} \overline{\lim}_r \overline{\lim}_n \|f_n^r - f_n\|_{|\mathcal{H}|}^2 = \text{const.} \lim_r \|f^r - f\|_{|\mathcal{H}|}^2 = 0. \quad \blacksquare$$

**Example 1** *By using relation (11) and the fact that  $\|f\|_{|\mathcal{H}|} \leq \text{const.} \|f\|_{L^{\frac{1}{H}}(\mathbb{R})}$  one can see that for every  $t > 0$  the function  $h(x) = 1_{(-\infty, t]} e^{\lambda x}$  belongs to  $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$  and thus to  $|\mathcal{H}|$  and the same holds for  $h_n$  given by (20). Using the continuity of the function  $h$  and comments after Theorem 3.2 in [17], we can see that the hypothesis of Theorem 1 are satisfied in the case of the function  $h$ . As a consequence, at each  $t$ , the Hermite Ornstein-Uhlenbeck process can be approximated in law by the sequence in the left side of (21).*

Let us briefly recall the spectral domain approach; by this, we mean the use of the space  $\widehat{\mathcal{H}}$  of deterministic integrands given by (12) which involves Fourier transforms. In [17] it has been proved that, if  $k = 1$  and the assumptions of Theorem 1 are satisfied with  $\widehat{\mathcal{H}}$  instead of  $|\mathcal{H}|$ , then the limit result (21) holds. It is not difficult to see that the fractional Ornstein-Uhlenbeck process satisfies the hypothesis in Theorem 3.2 in [17]. It would be interesting to generalize it to the case of functions of Hermite rank  $k$ .

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