

An Alternative Method for Numerical Inversion of Laplace Transforms

P. Bréhonnet, N. Tanguy, P. Vilbé, and L.C. Calvez

Abstract— Based on least-squares approximation of the rectangular pulse [1] by exponential functions, this paper presents an alternative method for performing numerical inversion of the Laplace transform. It compares favourably with the celebrated Vlach's method.

Index Terms— Approximation, Laplace transform, numerical inversion, rectangular pulse

I. INTRODUCTION

VARIOUS methods have been proposed for numerical inversion of the Laplace transform. Among them, methods based on approximation of delta or exponential functions [2]-[4], on Fourier series [5] and on orthogonal expansions [6] have been considered. In this paper, we propose a particularly easy to use and robust method as an alternative to the well-known Vlach's method [3].

II. OUTLINE OF VLACH'S METHOD

It is based upon rational approximation of the exponential function in the complex inversion formula

$$f(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} F(s) e^{st} ds = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} F\left(\frac{z}{t}\right) e^z dz,$$

where $F(s)$ denotes the Laplace transform of the time function $f(t)$. Replacing e^z by some rational approximant $E(z)$ yields the following approximate inverse Laplace transform

$$\tilde{f}(t) = \frac{1}{t} \sum_{i=1}^N a_i F\left(\frac{\gamma_i}{t}\right), \quad (1)$$

where a_i and γ_i are respectively the residues and the poles of the approximant $E(z)$, either real or occurring in complex-conjugate pairs. In each new problem, only the evaluation of

the Laplace transform has to be recalculated. A number of methods for obtaining optimal sets of constants a_i and γ_i have been investigated. The accuracy of the results depends on both the approximant of the exponential function e^z and of the function under consideration. There are many known methods for finding a rational function which is an approximant to e^z . First, Vlach [3] used a Padé approximation with the same degree of numerator and denominator. A numerator lower by one degree has also been used. A drawback of Padé approximation has been instability [4], especially for low-order approximations. In [4], the Padé approximant to e^z is written $E(z) = \frac{P_M(z)}{Q_N(z)}$. The main

difficulty lies in the choice of the polynomial degrees M and N . For an increasing N , we find an increasingly more accurate approximant to the exponential function. But in this case, residues are increasing too, generating numerical inaccuracies. For each N -order model, it is possible to have different solutions by varying the degree of numerator. Best results are generally obtained when the numerator degree is small, provided the denominator is stable. For each particular Laplace transform to be inverted, it is possible to determine the optimal degree M to be used with a given denominator $Q_N(z)$, but this is a relatively long and tedious work.

To apply Vlach's method, the following steps must be respected [4]:

- 1) Choose a sufficiently high number of poles N and adjust M in such a way that $F\left(\frac{z}{t}\right) \frac{P_M(z)}{Q_N(z)}$ has at least two zeros at infinity. Compute the corresponding approximant $\tilde{f}(t)$ to $f(t)$ via equation (1) and tabulated values of a_i and γ_i [3].
- 2) Solve other approximant $\tilde{f}(t)$ for same N and different M . If the solution differs considerably from the previous solution, keep changing M until the results stabilise for different M .

In practice, Vlach's method is often carried out with $M \in \{N, N-1, N-2\}$ which does not necessarily correspond to the best choice. It is worth noting that with increasing order N , numerical problems appear because the residues are too

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great. So, Vlach's method is not applied beyond order 30, with calculators using double precision. The problem of the choice of degrees of numerator and denominator is a great drawback because it depends on the Laplace transform under consideration. Moreover, to prove the efficiency of a method, examples are tested for some Laplace transforms for which the exact inverses are known but this is not the case in common engineering practise so, it is very interesting to dispose of several methods to ratify the results. The method proposed in this correspondence has proved its efficiency. It can be considered as an alternative to Vlach's method, without the above-mentioned drawbacks.

III. BASIS OF PROPOSED METHOD

It is based on the approximation of the rectangular pulse by a linear combination of damped exponentials. Let $\Pi(t)$ denote the causal rectangular function defined as

$$\Pi(t) = \begin{cases} 1 & \text{if } t \in [0,1] \\ 0 & \text{if } t \notin [0,1] \end{cases} \quad (2)$$

Assuming $f \in C^1[0,\infty[$ has a continuous derivative f' on $[0,\infty[$, we can write

$$\int_0^\infty \Pi\left(\frac{x}{t}\right) f'(x) dx = \int_0^t f'(\tau) d\tau = f(t) - f(+0) \quad (3)$$

which may be rewritten as

$$f(t) = f(+0) + \int_0^\infty \Pi\left(\frac{x}{t}\right) f'(x) dx.$$

Replacing the rectangular function by some approximant

$$\tilde{\Pi}(t) = \sum_{k=1}^N a_k e^{-\gamma_k t},$$

with the real part $\Re(\gamma_k) > 0$, and integrating by parts yield the following approximant to $f(t)$

$$\tilde{f}(t) = f(+0) + \sum_{k=1}^N a_k \left(\frac{\gamma_k}{t} F\left(\frac{\gamma_k}{t}\right) - f(+0) \right) \quad (4)$$

provided $\sigma_f t < \min_k \{\Re(\gamma_k)\}$, where σ_f denotes the abscissa of convergence of the Laplace integral. Notice that $\tilde{f}(+0) = f(+0)$ is guaranteed by (4), on account of the initial value theorem for the Laplace transform. To preserve the final value when it exists, the constraint $\sum_{k=1}^N a_k = 1$ has to be imposed. In this case, equation (4) reduces to

$$\tilde{f}(t) = \sum_{k=1}^N a_k \left(\frac{\gamma_k}{t} \right) F\left(\frac{\gamma_k}{t}\right) \quad (5)$$

in which only the knowledge of the constants a_k, γ_k relative to an approximant to the rectangular pulse are required to compute $\tilde{f}(t)$ from $F(s)$. Like in Vlach's method, determination of the poles of the Laplace transform to be inverted has been avoided. Residues a_k [Table I] are significantly smaller than in Vlach's method [3] for a same order (the ratio which is about 10^5 for $N = 10$ increases with N) which makes our method less sensitive to numerical inaccuracies (see Fig. 3 and Fig. 4).

The preciseness of formula (5) depends on the accuracy of the approximation of the function $\Pi(t)$. In [1], an optimization technique is provided for the least-squares approximation of the rectangular pulse, owing to a near optimal starting point, the integrated squared error is reduced through few iterations of a Gauss Newton process, beyond all usual orders (up to order 200). This great accuracy is particularly useful for determining the required constants (a_k, γ_k) which must be computed only once for a given order N and can be stored for permanent use.

IV. ILLUSTRATION

We have checked many examples to illustrate the applicability of the suggested numerical inversion technique. For some of them, Vlach's method gives also good results but usually after several tests. First consider the irrational Laplace transform

$$F(s) = \frac{1}{1 + \frac{sh(\sqrt{s})}{\sqrt{s}}},$$

which is the normalised transfer function of a RC distributed circuit. Fig. 1-2 represent the time signal obtained by Vlach's method and by our method; in both cases, the accuracy is acceptable.

Consider the irrational Laplace transform

$$F(s) = \frac{23e^{-\sqrt{s}}}{1 + 23e^{-\sqrt{s}}}$$

with a view to computing the time signal $f(t)$, which is a very severe test. The numerical differences between the methods are shown in Table II and Fig. 3-6. In Vlach's

TABLE I

Negatives of Poles – Present method	Residues – Present method
3.65623977941246 ± 2.33980675356003 j	-2.02699639954484 ± 6.5556240442333 j
3.04259335096752 ± 7.47702946613745 j	2.21838424179506 ± 0.27882389516450 j
2.47879760012144 ± 12.94030319544558 j	0.48583161398352 ± 0.62262273410175 j
1.94081484142228 ± 18.38586389094272 j	-0.07636822980337 ± 0.30776792623453 j
1.22236607997850 ± 23.52676351409633 j	-0.10085122644020 ± 0.02574141374327 j

method, for an N -order approximant to e^z , M represents the most appropriate degree of the numerator obtained after several tests. On the other hand, for a given N order, our method yields directly the approximate time signal. In both cases, t_{\max} denotes the maximal time for which accuracy is acceptable. In practice, we used

$$t_{\max} \approx \frac{|f - \tilde{f}|_{\text{acceptable}}^2}{\|\Pi - \tilde{\Pi}\|^2 \|f'\|^2}.$$

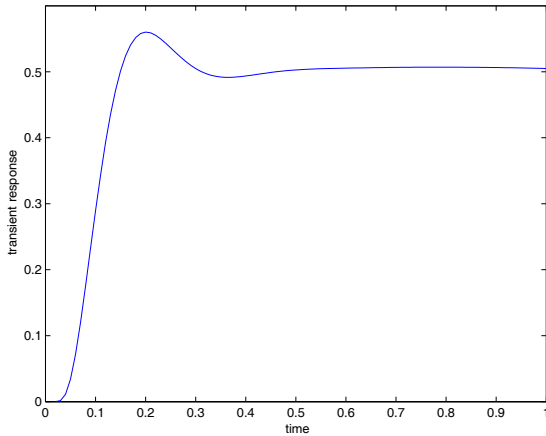


Fig. 1 : Vlach's method, order10

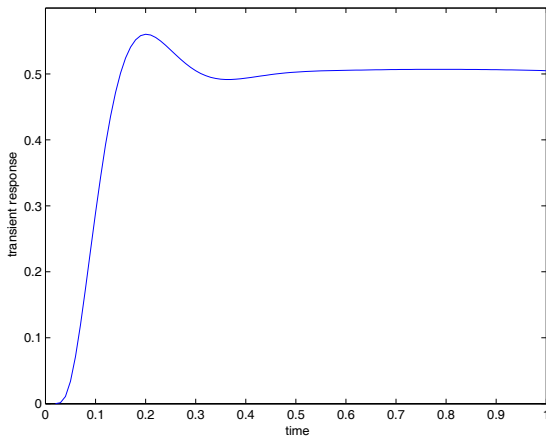


Fig. 2 : Present method, order 10

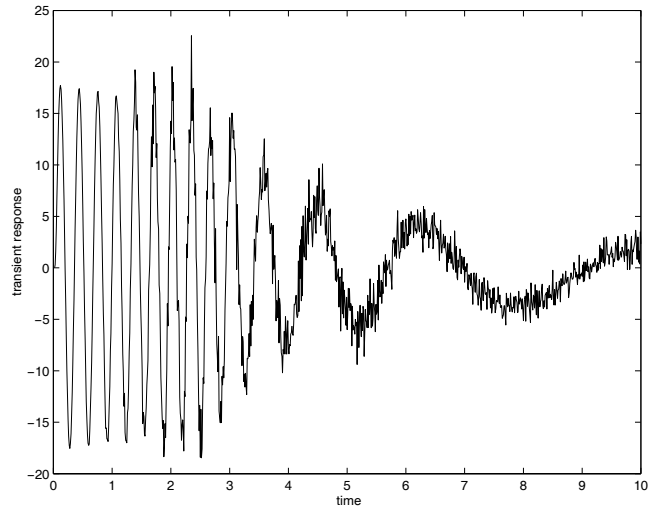


Fig. 3 : Vlach's method, order 30, $M = 29$

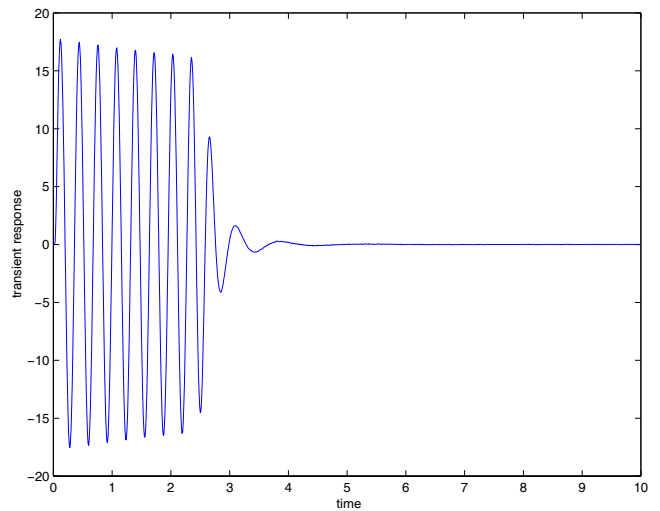


Fig. 4 : Vlach's method, order 30, $M = 25$

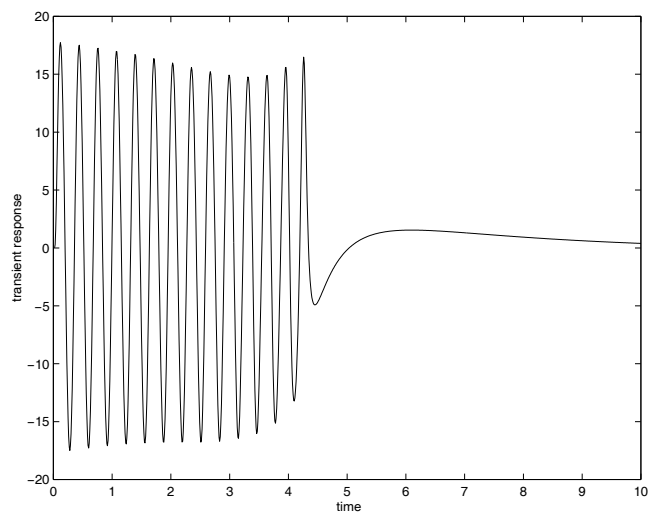


Fig. 5 : Present method, order 30

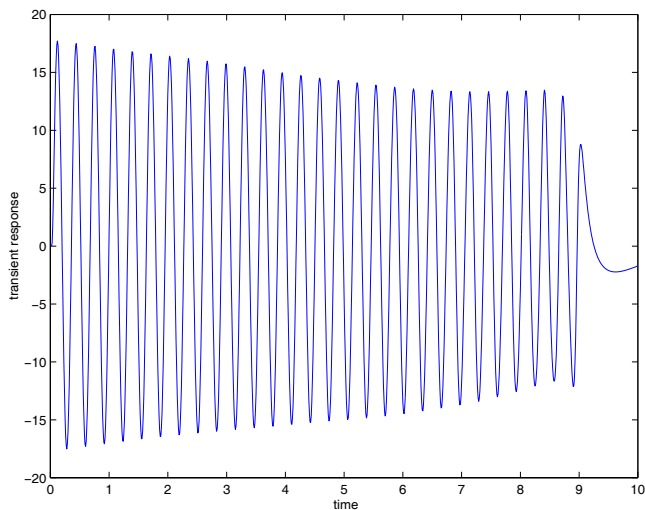


Fig. 6 : Present method, order 60

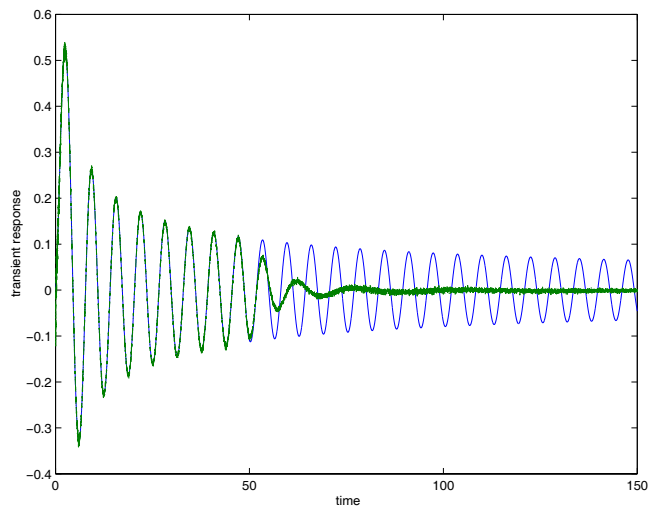
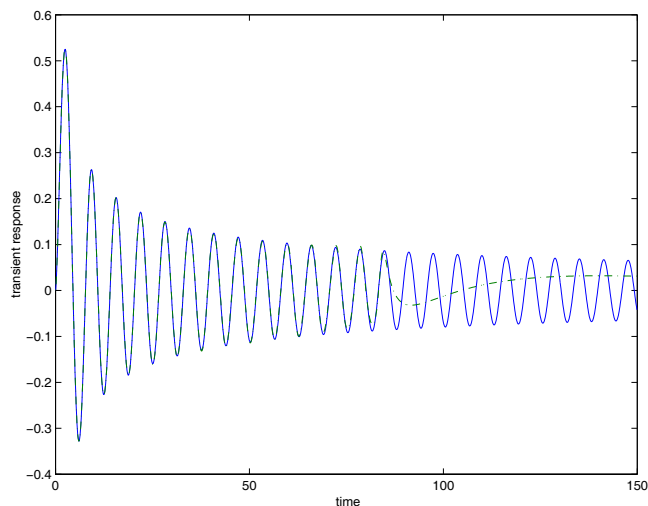
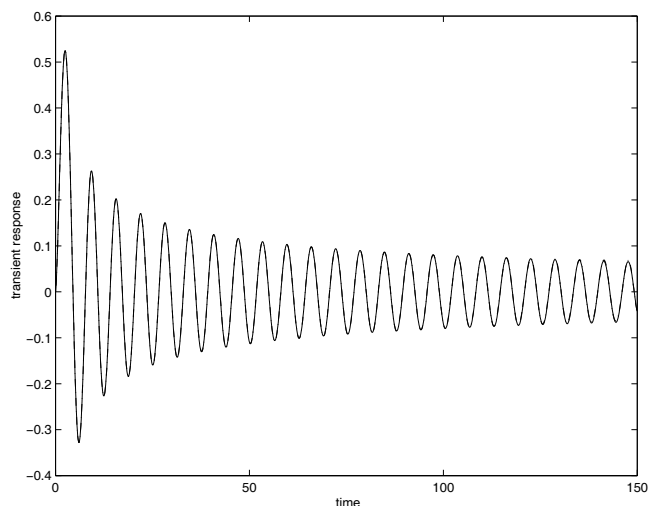
TABLE II

Method	order	t_{\max}
Vlach	30, $M = 29$	1.5
Vlach	30, $M = 25$	2.5
Present method	30	3.5
Present method	60	8.5

The third example deals with the Laplace transform $F(s)$ of an oscillating signal. For this example, the exact inverse is known to be the Bessel function $J_{3/2}$, so the quadratic error can be calculated. The normalised quadratic error criterion is defined as $Q_n \triangleq \|e\|/\|f\|$ where $e \triangleq \tilde{f} - f$ and

$$\|f\|^2 \triangleq \sum_{k=1}^n |f(t_k)|^2.$$

The time function is calculated up to $t = 150s$ with a sampling period $T = 0.05s$, i.e. $n = 3000$. The best result obtained with Vlach's method corresponds to $(M, N) = (26, 30)$ and $Q_n = 0.4396$; in this case, the original signal can be built up to the 8th oscillation (Fig. 7). Our technique used with a 30-order model, yields directly an acceptable signal up to the 14th oscillation with $Q_n = 0.3569$ (Fig. 8). Furthermore the time interval can be significantly extended using our method and a 60-order approximant to the rectangular pulse (Fig. 9). In this case, we obtain a quadratic error $Q_n = 0.01058$.

Fig. 7 : Vlach's method, order 30, $M = 26$, $Q_n = 0.4396$ Fig. 8 : Present method, order 30, $Q_n = 0.3569$ Fig. 9 : Present method, order 60, $Q_n = 0.01058$

V. CONCLUSION

A numerical method particularly useful for time responses

of high or infinite dimension systems is given in this paper. It is based on the approximation of the rectangular pulse. The poles and residues of the approximant were computed with high accuracy. In practice, a 30-order approximation is usually sufficiently accurate for most engineering applications but higher approximations are easily available if necessary. The technique does not require the determination of poles of the Laplace transform to be inverted which represents a great advantage, and only one iteration is necessary. So, the method can be implemented very easily in a program. For a given precision of computer, the residues [Table I] being significantly smaller than with Vlach's method [4], the time signal can be built on a larger interval. This conclusion is supported by many numerical experiments demonstrating that our method is simple and efficient. As is well known, every Laplace transform numerical inversion method breaks down for some functions and therefore checking by different methods can greatly increase confidence in the results achieved.

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