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# A SECOND ORDER ACCURACY IN TIME FOR A FULL DISCRETIZED TIME-DEPENDENT NAVIER-STOKES EQUATIONS BY A TWO-GRID SCHEME

HYAM ABOUD<sup>†</sup>, VIVETTE GIRAULT<sup>‡</sup> AND TONI SAYAH<sup>\*</sup>.

**ABSTRACT.** We study a second-order two-grid scheme fully discrete in time and space for solving the Navier-Stokes equations. The two-grid strategy consists in discretizing, in the first step, the fully non-linear problem, in space on a coarse grid with mesh-size  $H$  and time step  $\Delta t$  and, in the second step, in discretizing the linearized problem around the velocity  $u_H$  computed in the first step, in space on a fine grid with mesh-size  $h$  and the same time step. The two-grid method has been applied for an analysis of a first order fully-discrete in time and space algorithm and we extend the method to the second order algorithm.

**Keywords** Two-grid scheme, Non-linear problem, Incompressible flow, Time and Space discretizations, Taylor-Hood finite element, Duality argument, “superconvergence”.

## 1. INTRODUCTION.

Let  $\Omega$  be a bounded domain of  $\mathbb{R}^2$  with a polygonal boundary  $\partial\Omega$  and let  $]0, T[$  be a given time-interval. Consider the following Navier-Stokes problem for an incompressible fluid

$$\frac{\partial u}{\partial t}(x, t) - \nu \Delta u(x, t) + u(x, t) \cdot \nabla u(x, t) + \nabla p(x, t) = f(x, t) \text{ in } \Omega \times ]0, T[, \quad (1.1)$$

with the incompressibility condition

$$\operatorname{div} u(x, t) = 0 \text{ in } \Omega \times ]0, T[, \quad (1.2)$$

the homogeneous Dirichlet boundary condition

$$u(x, t) = 0 \text{ on } \partial\Omega \times ]0, T[, \quad (1.3)$$

and the initial condition

$$u(x, 0) = 0 \text{ in } \Omega, \quad (1.4)$$

where  $u$  and  $p$  represent respectively the velocity and the pressure of the fluid. All the quantities are taken at the point  $(x, t)$  where  $x = (x_i)_{1 \leq i \leq 2} \in \mathbb{R}^2$  denotes the position and  $t \in [0, T]$  the time. We suppose that the fluid density is a constant ( $\rho = 1$ );  $f$  denotes the external forces applied to the fluid and  $\nu$  is the viscosity. The notations  $u \cdot \nabla u$ ,  $\Delta u$  and  $\operatorname{div} u$  mean :

$$u \cdot \nabla u = \sum_{i=1}^2 u_i \frac{\partial u}{\partial x_i}, \Delta u = \sum_{i=1}^2 \frac{\partial^2 u}{\partial x_i^2} \text{ and } \operatorname{div} u = \sum_{i=1}^2 \frac{\partial u_i}{\partial x_i}.$$

The term  $u \cdot \nabla u$  is the convection term and  $\nu \Delta u$  is the diffusion one.

The purpose of this article is to solve by a second-order, in time and space, two-grid scheme, on a coarse grid and a fine grid, the non-stationary incompressible Navier-Stokes problem and to show that the two-grid algorithm's global error is similar to the error of the direct resolution of the non-linear problem on a fine grid. The two-grid strategy is a general method for solving a non-linear Partial Differential Equation (PDE), depending or not in time, with solution  $u$ . This technique consists on what follows : In a first step, we discretize the fully non-linear PDE on a coarse grid of mesh-size  $H$  and we compute an approximate solution  $u_H$ . Then, in a second step, we linearize the PDE around  $u_H$  and we discretize

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the linearized problem on a fine grid of mesh-size  $h$ ; let  $u_h^{lin}$  be the corresponding solution. Then, under suitable assumptions, we can prove that if  $h, H$  and the time step  $\Delta t$  are well-chosen, the global error of the two-grid algorithm  $\|u - u_h^{lin}\|$  has the same order as the error  $\|u - u_h\|$  that would have been obtained if the non-linear problem had been directly discretized on the fine grid.

Two-grid discretizations have been widely applied to linear and non-linear elliptic boundary value problems: J. Xu in [22], [23], [24] has pioneered their development. These methods have been extended to the steady Navier-Stokes equations, cf. for instance the work of W. Layton in [13], W. Layton & W. Lenferink in [14] and V. Girault & J.-L. Lions in [7]. Also, this method has been applied to the time-dependent Navier-Stokes problem, cf. V. Girault & J.-L. Lions [8] in which they analyze a semi-discrete algorithm, H. Abboud & T. Sayah in [2] and H. Abboud, V. Girault & T. Sayah in [3] for an analysis of a first order fully-discrete in time and space algorithm and in [1] for a numerical analysis of a second-order totally discrete in time and space scheme.

Setting  $L_0^2(\Omega) = \{q \in L^2(\Omega); \int_{\Omega} q \, dx = 0\}$  and assuming that  $f$  belongs to  $L^2(0, T; H^{-1}(\Omega)^2)$ , it is well-known that (1.1)–(1.2) has the following variational formulation in  $]0, T[$ : Find  $u(t) \in H_0^1(\Omega)^2$ , such that in the sense of distributions on  $]0, T[$ ,

$$\forall v \in H_0^1(\Omega)^2, \frac{d}{dt}(u(t), v) + \nu(\nabla u(t), \nabla v) + (u(t) \cdot \nabla u(t), v) - (p(t), \operatorname{div} v) = \langle f(t), v \rangle, \quad (1.5)$$

$$\forall q \in L_0^2(\Omega), (q, \operatorname{div} u(t)) = 0, \quad (1.6)$$

and

$$u(0) = 0, \quad (1.7)$$

where  $u(t) = u(x, t)$ .

Furthermore, this problem has one and only one solution  $u$  in  $L^\infty(0, T; L^2(\Omega)^2) \cap L^2(0, T; H^1(\Omega)^2)$  and  $p$  in the dual space of  $W_0^{1,1}(0, T; L_0^2(\Omega))$  (see e.g. J.-L. Lions in [15] and O.A. Ladyzenskaya in [12]).

In addition, we have the following regularity result:

**Theorem 1.1.** *If  $\Omega$  is convex and  $f \in L^2(0, T; L^2(\Omega)^2)$ , then*

$$u \in L^\infty(0, T; H^1(\Omega)^2) \cap L^2(0, T; H^2(\Omega)^2) \text{ and } p \in L^2(0, T; H^1(\Omega)). \quad (1.8)$$

For discretizing (1.5)–(1.7), let  $\eta > 0$  be a discretization parameter in space and for each  $\eta$ , let  $\mathcal{T}_\eta$  be a corresponding regular (or non-degenerate) family of triangulations of  $\bar{\Omega}$ , consisting of triangles such that any two triangles are either disjoint or share a vertex or an entire side. For an arbitrary triangle  $\kappa$ , we denote by  $\eta_\kappa$  the diameter of  $\kappa$  and by  $\rho_\kappa$  the diameter of the circle inscribed in  $\kappa$ . Then  $\eta$  denotes the maximum of  $\eta_\kappa$  and we assume that  $\mathcal{T}_\eta$  is regular in the sense of Ciarlet [6]: there exists a constant  $\sigma$  independent of  $\eta$  such that

$$\sup_{\kappa \in \mathcal{T}_\eta} \frac{\eta_\kappa}{\rho_\kappa} = \sigma_\kappa \leq \sigma. \quad (1.9)$$

Let  $X_\eta$  and  $M_\eta$  be a “stable” pair of finite-element spaces for discretizing the velocity  $u$  and the pressure  $p$ , stable in the sense that it satisfies a uniform discrete inf-sup condition: there exists a constant  $\beta^* \geq 0$ , independent of  $\eta$ , such that

$$\forall q_\eta \in M_\eta, \sup_{v_\eta \in X_\eta} \frac{1}{|v_\eta|_{H^1(\Omega)}} \int_{\Omega} q_\eta \operatorname{div} v_\eta \, dx \geq \beta^* \|q_\eta\|_{L^2(\Omega)}. \quad (1.10)$$

Let  $\mathbb{P}_\kappa$  denote the space of polynomials with total degree less than or equal to  $\kappa$ . For a second-order two-grid scheme, we choose the Taylor-Hood finite-element, where in each triangle  $\kappa$ , each component of the velocity is a polynomial of  $\mathbb{P}_2$  and the pressure  $p$  is a polynomial of  $\mathbb{P}_1$ . Therefore, the finite-element spaces are:

$$X_\eta = \{v_\eta \in C^0(\bar{\Omega})^2; \forall \kappa \in \mathcal{T}_\eta, v_{\eta|_\kappa} \in \mathbb{P}_2^2, v_{\eta|_{\partial\Omega}} = 0\}, \quad (1.11)$$

$$M_\eta = \left\{ q_\eta \in C^0(\bar{\Omega}); \forall \kappa \in \mathcal{T}_\eta, q_{\eta|_\kappa} \in \mathbb{P}_1, \int_{\Omega} q_\eta \, dx = 0 \right\}. \quad (1.12)$$

There exists an approximation operator  $P_\eta \in \mathcal{L}(H_0^1(\Omega)^2; X_\eta)$  such that (see V. Girault and P.-A. Raviart in [9]):

$$\forall v \in H_0^1(\Omega)^2, \quad \forall q_\eta \in M_\eta, \quad \int_\Omega q_\eta \operatorname{div}(P_\eta(v) - v) dx = 0, \quad (1.13)$$

and for  $k = 0, 1$  or  $2$ ,

$$\forall v \in [H^{1+k}(\Omega) \cap H_0^1(\Omega)]^2, \quad \|P_\eta(v) - v\|_{L^2(\Omega)} \leq C\eta^{1+k}|v|_{H^{1+k}(\Omega)}, \quad (1.14)$$

and for all  $r \geq 2, k = 0, 1$  or  $2$ ,

$$\forall v \in [W^{1+k,r}(\Omega) \cap H_0^1(\Omega)]^2, \quad |P_\eta(v) - v|_{W^{1,r}(\Omega)} \leq C_r \eta^k |v|_{W^{1+k,r}(\Omega)}. \quad (1.15)$$

In addition, as  $M_\eta$  contains all polynomials of degree one, there exists an operator  $r_\eta \in \mathcal{L}(L_0^2(\Omega); M_\eta)$ , such that for any real number  $s \in [0, 2]$ ,

$$\forall q \in H^s(\Omega) \cap L_0^2(\Omega), \quad \|r_\eta(q) - q\|_{L^2(\Omega)} \leq C\eta^s |q|_{H^s(\Omega)}. \quad (1.16)$$

To discretize in time, we divide the interval  $[0, T]$  into  $N$  subintervals of equal length  $k = \frac{T}{N}$ , with grid-points  $t^n = nk, 0 \leq n \leq N$ .

With these spaces, we propose the following two-grid scheme for discretizing (1.5)–(1.7). We use two regular triangulations of  $\bar{\Omega}$ : a coarse triangulation  $\mathcal{T}_H$  and a fine one  $\mathcal{T}_h$ , that for practical purposes, is a refinement of  $\mathcal{T}_H$ . On each of these, we define the same stable pair of finite-element spaces,  $(X_H, M_H)$  and  $(X_h, M_h)$  such that  $X_H \subset X_h$  and  $M_H \subset M_h$ . At each time step, we solve (1.17)–(1.18) and (1.19)–(1.20) below. The two-grid algorithm reads :

• **Step One** (non-linear problem on coarse grid): Knowing  $u_h^{n-1}$  and  $u_h^n$ , find  $(u_H^{n+1}, p_H^{n+1})$  with values in  $X_H \times M_H$ , solution of

$$\begin{aligned} \forall v_H \in X_H, \quad \frac{1}{2\Delta t} (3u_H^{n+1} - 4u_H^n + u_H^{n-1}, v_H) + \nu(\nabla u_H^{n+1}, \nabla v_H) + (u_H^{n+1} \cdot \nabla u_H^{n+1}, v_H) \\ + \frac{1}{2} (\operatorname{div} u_H^{n+1}, u_H^{n+1} \cdot v_H) - (p_H^{n+1}, \operatorname{div} v_H) = (f^{n+1}, v_H), \end{aligned} \quad (1.17)$$

$$\forall q_H \in M_H, \quad (q_H, \operatorname{div} u_H^{n+1}) = 0. \quad (1.18)$$

• **Step Two** (linearized problem on fine grid): Knowing  $(u_H^{n+1}, p_H^{n+1})$ , find  $(u_h^{n+1}, p_h^{n+1})$  with values in  $X_h \times M_h$  solution of

$$\begin{aligned} \forall v_h \in X_h, \quad \frac{1}{2\Delta t} (3u_h^{n+1} - 4u_h^n + u_h^{n-1}, v_h) + \nu(\nabla u_h^{n+1}, \nabla v_h) + (u_h^{n+1} \cdot \nabla u_h^{n+1}, v_h) \\ - (p_h^{n+1}, \operatorname{div} v_h) = (f^{n+1}, v_h), \end{aligned} \quad (1.19)$$

$$\forall q_h \in M_h, \quad (q_h, \operatorname{div} u_h^{n+1}) = 0. \quad (1.20)$$

By assumption,  $u_H^0 = u_h^0 = 0$  and  $u_H^1$  and  $u_h^1$  are computed by solving one iteration of an Euler scheme. In both (1.17) and (1.19),  $f^{n+1}$  is a suitable approximation of  $f$  at time  $t^{n+1}$ . The purpose of this two-grid algorithm is to reduce the time of computation for both velocity and pressure.

In the sequel, we shall take  $(\Delta t)^2$  of the order of  $H^3$ : there exist constants  $\alpha_1$  and  $\alpha_2$  that do not depend on  $H$  and  $\Delta t$  such that

$$\alpha_1 H^3 \leq (\Delta t)^2 \leq \alpha_2 H^3.$$

The remainder of this article is organized as follows : In Section 2, we present some conventions and notations that will be used throughout the article. In Section 3, we present a first error estimate for the fully-discrete Step One, then in section 4 we establish a duality argument based on the backward semi-discrete Stokes system and we derive some uniform bounds that allow us to prove the Stokes problem's error estimate in  $L^2(\Omega \times ]0, T])^2$ , then we apply it to the Navier-Stokes problem. We also prove a “superconvergence” result for the non-linear part. Finally, the pressure is estimated in section 5 and the error estimation for the solution of Step Two is studied in section 6.

## 2. PRELIMINARIES.

To begin with, we present some conventions and notations that will be used throughout the article. As usual, for handling time-dependent problems, it is convenient to consider functions defined on a time interval  $]a, b[$  with values in a functional space, say  $X$  (cf. Lions and Magenes [16]). More precisely, let  $\| \cdot \|_X$  denote the norm of  $X$ ; then for any  $r$ ,  $1 \leq r \leq \infty$ , we define

$$L^r(a, b; X) = \left\{ f \text{ measurable in } ]a, b[; \int_a^b \| f(t) \|_X^r dt < \infty \right\}$$

equipped with the norm

$$\| f \|_{L^r(a, b; X)} = \left( \int_a^b \| f(t) \|_X^r dt \right)^{1/r},$$

with the usual modifications if  $r = \infty$ . It is a Banach space if  $X$  is a Banach space.

Let  $(k_1, k_2)$  denote a pair of non-negative integers, set  $|k| = k_1 + k_2$  and define the partial derivative  $\partial^k$  by  $\partial^k v = \frac{\partial^{|k|} v}{\partial x_1^{k_1} \partial x_2^{k_2}}$ . Here  $X$  is usually a Sobolev space, such as (cf. Adams [4] or Nečas [17]): for any non-negative integer  $m$  and number  $r \geq 1$ ,

$$W^{m, r}(\Omega) = \{ v \in L^r(\Omega); \partial^k v \in L^r(\Omega), \forall |k| \leq m \}.$$

This space is equipped with the seminorm

$$|v|_{W^{m, r}(\Omega)} = \left[ \sum_{|k|=m} \int_{\Omega} |\partial^k v|^r dx \right]^{1/r},$$

and is a Banach space for the norm

$$\| v \|_{W^{m, r}(\Omega)} = \left[ \sum_{0 \leq |k| \leq m} |v|_{W^{k, r}(\Omega)}^r \right]^{1/r},$$

with the usual extension when  $r = \infty$ . When  $r = 2$ , this space is the Hilbert space  $H^m(\Omega)$ . In particular, the scalar product of  $L^2(\Omega)$  is denoted by  $(\cdot, \cdot)$ . Similarly,  $L^2(a, b; H^m(\Omega))$  is a Hilbert space and in particular  $L^2(a, b; L^2(\Omega))$  coincides with  $L^2(\Omega \times ]a, b[)$ .

For functions that vanish on the boundary, we recall Poincaré's inequality: there exists a constant  $\mathcal{P}$  such that

$$\forall v \in H_0^1(\Omega), \| v \|_{L^2(\Omega)} \leq \mathcal{P} |v|_{H^1(\Omega)}. \quad (2.1)$$

More generally, recall the inequalities of Sobolev imbeddings in two dimensions: for each  $r \in [2, \infty[$ , there exists a constant  $S_r$  such that

$$\forall v \in H_0^1(\Omega), \| v \|_{L^r(\Omega)} \leq S_r |v|_{H^1(\Omega)}, \quad (2.2)$$

where

$$|v|_{H^1(\Omega)} = \| \nabla v \|_{L^2(\Omega)}. \quad (2.3)$$

When  $r = 2$ , (2.2) reduces to Poincaré's inequality and  $S_2$  is Poincaré's constant. The case  $r = \infty$  is excluded and is replaced by: for any  $r > 2$ , there exists a constant  $M_r$  such that

$$\forall v \in W_0^{1, r}(\Omega), \| v \|_{L^\infty(\Omega)} \leq M_r |v|_{W^{1, r}(\Omega)}. \quad (2.4)$$

We also have in dimension 2,

$$\| g \|_{L^4(\Omega)} \leq 2^{1/4} \| g \|_{L^2(\Omega)}^{1/2} \| \nabla g \|_{L^2(\Omega)}^{1/2}. \quad (2.5)$$

Owing to (2.1), we use the seminorm  $|\cdot|_{H^1(\Omega)}$  as a norm on  $H_0^1(\Omega)$  and we use it to define the norm of the dual space  $H^{-1}(\Omega)$  of  $H_0^1(\Omega)$ :

$$\| f \|_{H^{-1}(\Omega)} = \sup_{v \in H_0^1(\Omega)} \frac{\langle f, v \rangle}{|v|_{H^1(\Omega)}},$$

where  $\langle \cdot, \cdot \rangle$  denotes the duality pairing between  $H^{-1}(\Omega)$  and  $H_0^1(\Omega)$ . Also, we recall the spaces we introduced at the beginning:

$$V = \{v \in H_0^1(\Omega)^2; \operatorname{div} v = 0 \text{ in } \Omega\} \quad \text{and} \quad L_0^2(\Omega) = \{q \in L^2(\Omega); \int_{\Omega} q \, dx = 0\},$$

and the orthogonal complement of  $V$  in  $H_0^1(\Omega)^2$ :

$$V^\perp = \{v \in H_0^1(\Omega)^2; \forall w \in V, (\nabla v, \nabla w) = 0\}.$$

The results of this article are based on the identity:

$$2(a^{n+1}, 3a^{n+1} - 4a^n + a^{n-1}) = |a^{n+1}|^2 + |2a^{n+1} - a^n|^2 + |\delta^2 a^n|^2 - |a^n|^2 - |2a^n - a^{n-1}|^2, \quad (2.6)$$

where

$$\delta^2 a^n = a^{n+1} - 2a^n + a^{n-1}. \quad (2.7)$$

### 3. ERROR ESTIMATES FOR THE SOLUTION OF STEP ONE

The results in this paragraph are written for the non-linear scheme (1.17)–(1.18).

To simplify, we denote by  $\eta$  the mesh parameter. First of all, we prove the existence and the uniqueness of the solution of (1.17)–(1.18).

**Lemma 3.1. (Stability)** *Let  $u_\eta^{n+1}$  be a solution of (1.17)–(1.18) with the initial datas  $u_\eta^0$  and  $u_\eta^1 \in V_\eta$ ; We have*

$$\begin{aligned} & \sup_{2 \leq n \leq N} \|u_\eta^n\|_{L^2(\Omega)} + \sup_{2 \leq n \leq N} \|2u_\eta^n - u_\eta^{n-1}\|_{L^2(\Omega)} + \sqrt{2\nu} \left( \sum_{n=2}^N \Delta t \|\nabla u_\eta^n\|_{L^2(\Omega)}^2 \right)^{1/2} \\ & + \left( \sum_{n=1}^{N-1} \|\delta^2 u_\eta^n\|_{L^2(\Omega)}^2 \right)^{1/2} \leq C \left( \frac{2S_2^2}{\nu} \sum_{n=2}^N \Delta t \|f^n\|_{L^2(\Omega)}^2 + \|u_\eta^1\|_{L^2(\Omega)}^2 + \|2u_\eta^1 - u_\eta^0\|_{L^2(\Omega)}^2 \right)^{1/2}. \end{aligned}$$

*Proof.* We take the scalar product of (1.17) by  $4\Delta t u_\eta^{n+1}$ , use (2.6) and sum the result over  $1 \leq n \leq m-1$ .  $\square$

The stability of (1.17)–(1.18) results from the following a priori estimation:

**Lemma 3.2. (Uniqueness)** *The scheme (1.17)–(1.18) has a solution for all  $\nu > 0$ , all initial datas  $u_\eta^0, u_\eta^1 \in V_\eta$  and for all data  $f \in \mathcal{C}^0([0, T]; L^2(\Omega)^2)$ . The solution is unique for  $\Delta t$  sufficiently small.*

*Proof.* For all  $1 \leq n \leq N-1$ , the problem (1.17)–(1.18) is a square system of algebraic non-linear equations in finite dimension. Due to the anti-symetrisation of the non-linear term, we prove, by the theorem of the saddle point of Brouwer and the inf-sup condition, that for all  $1 \leq n \leq N-1$ , the problem has at least a solution  $(u_\eta^n, p_\eta^n)$ . For the unicity, we consider two solutions  $(u_\eta^{(1)}, p_\eta^{(1)})$  and  $(u_\eta^{(2)}, p_\eta^{(2)})$ . Their difference  $(w_\eta^n, p_\eta^n)$  satisfies:

$$\begin{aligned} & \forall v_\eta \in V_\eta, \frac{1}{2\Delta t} (3w_\eta^{n+1} - 4w_\eta^n + w_\eta^{n-1}, v_\eta) + \nu (\nabla w_\eta^{n+1}, \nabla v_\eta) + (w_\eta^{n+1} \cdot \nabla u_\eta^{(1)n+1}, v_\eta) \\ & + (u_\eta^{(2)n+1} \cdot w_\eta^{n+1}, v_\eta) + \frac{1}{2} (\operatorname{div} w_\eta^{n+1}, u_\eta^{(1)n+1} \cdot v_\eta) + \frac{1}{2} (\operatorname{div} u_\eta^{(2)n+1}, w_\eta^{n+1} \cdot v_\eta) = 0. \end{aligned}$$

By using the identity (2.6) and choosing  $v_\eta = w_\eta^{n+1}$ , we obtain

$$\begin{aligned} & \frac{1}{4\Delta t} \left( \|w_\eta^{n+1}\|_{L^2(\Omega)}^2 + \|2w_\eta^{n+1} - w_\eta^n\|_{L^2(\Omega)}^2 + \|\delta^2 w_\eta^n\|_{L^2(\Omega)}^2 - \|w_\eta^n\|_{L^2(\Omega)}^2 - \|2w_\eta^n - w_\eta^{n-1}\|_{L^2(\Omega)}^2 \right) \\ & + \nu |w_\eta^{n+1}|_{H^1(\Omega)}^2 - \|w_\eta^{n+1}\|_{L^4(\Omega)}^2 |u_\eta^{(1)n+1}|_{H^1(\Omega)} - \frac{1}{2} |w_\eta^{n+1}|_{H^1(\Omega)} \|w_\eta^{n+1}\|_{L^4(\Omega)} \|u_\eta^{(1)n+1}\|_{L^4(\Omega)} \leq 0. \end{aligned}$$

Due to the fact that in finite dimension, all the norms are equivalent, summing the precedent inequality from  $n = 1$  to  $m - 1$ , and using Lemma 3.1 and  $w_\eta^0 = w_\eta^1 = 0$ , we obtain

$$\|w_\eta^m\|_{L^2(\Omega)}^2 + \nu \sum_{n=2}^m \Delta t |w_\eta^n|_{H^1(\Omega)}^2 + \sum_{n=1}^{m-1} \|\delta^2 w_\eta^n\|_{L^2(\Omega)}^2 + \|2w_\eta^m - w_\eta^{m-1}\|_{L^2(\Omega)}^2 \leq C \Delta t \sum_{n=1}^m \|w_\eta^n\|_{L^2(\Omega)}^2, \quad (3.1)$$

with a constant  $C$  that depends on  $\eta$  but does not depend on  $\Delta t$ . For the last term of the sum of the right-hand side, we write:

$$w_\eta^m = \delta^2 w_\eta^{m-1} + 2w_\eta^{m-1} - w_\eta^{m-2}.$$

Then

$$\|w_\eta^m\|_{L^2(\Omega)}^2 \leq 2 \left( \|\delta^2 w_\eta^{m-1}\|_{L^2(\Omega)}^2 + \|2w_\eta^{m-1} - w_\eta^{m-2}\|_{L^2(\Omega)}^2 \right),$$

and for  $\Delta t$  sufficiently small, the term

$$2C \Delta t \left( \|\delta^2 w_\eta^{m-1}\|_{L^2(\Omega)}^2 + \|2w_\eta^{m-1} - w_\eta^{m-2}\|_{L^2(\Omega)}^2 \right)$$

can be absorbed by the term in the left-hand side of the inequality. Applying Gronwall's lemma, we obtain  $w_\eta^n = \mathbf{0}$  then, the inf-sup condition implies  $p_\eta^n = 0$ ,  $2 \leq n \leq N$ .  $\square$

In the next proposition, we will establish the error estimate for the solution computed by one iteration of Euler's scheme  $(u_\eta^1 - u(\Delta t), p_\eta^1 - p(\Delta t))$ :

**Proposition 3.3.** *Suppose that  $u'' \in \mathcal{C}^0(0, T; L^2(\Omega)^2)$ ,  $u(\Delta t) \in H^3(\Omega)^2$  and  $p(\Delta t) \in H^2(\Omega)$ , the error of the solution computed by one iteration of Euler's scheme satisfies the following estimations, for  $\Delta t \leq k_0 > 0$  sufficiently small,*

$$\begin{aligned} & \frac{1}{2} \|u_\eta^1 - u(\Delta t)\|_{L^2(\Omega)}^2 + \frac{\nu \Delta t}{2} |u_\eta^1 - u(\Delta t)|_{H^1(\Omega)}^2 \\ & \leq \frac{(\Delta t)^4}{4} \|u''\|_{L^\infty(0, T; L^2(\Omega)^2)}^2 + C(\Delta t) \eta^4 \left( |u(\Delta t)|_{H^3(\Omega)}^2 + |p(\Delta t)|_{H^2(\Omega)}^2 \right) + C \eta^6 |u(\Delta t)|_{H^3(\Omega)}^2, \end{aligned} \quad (3.2)$$

and

$$(\Delta t)^{1/2} \|p(\Delta t) - p_\eta^1\|_{L^2(\Omega)} \leq C \left( (\Delta t)^{3/2} + \eta^2 + \frac{\eta^3}{\sqrt{\Delta t}} \right). \quad (3.3)$$

*Proof.* Due to the regularity assumption of  $u$ , there exists  $\theta \in ]0, 1[$  such that

$$0 = u_0 = u(\Delta t) - (\Delta t)u'(\Delta t) + \frac{1}{2}(\Delta t)^2 u''(\theta \Delta t),$$

and  $u_\eta^1$  satisfies the following error equation

$$\begin{aligned} \forall v_\eta \in V_\eta, \quad & \frac{1}{\Delta t} (u_\eta^1 - u(\Delta t), v_\eta) + \nu (\nabla(u_\eta^1 - u(\Delta t)), \nabla v_\eta) = \frac{\Delta t}{2} (u''(\theta \Delta t), v_\eta) \\ & - (p(\Delta t) - r_\eta p(\Delta t), \operatorname{div} v_\eta) + (u(\Delta t) \cdot \nabla u(\Delta t) - u_\eta^1 \cdot \nabla u_\eta^1, v_\eta) - \frac{1}{2} (\operatorname{div} u_\eta^1, u_\eta^1 \cdot v_\eta). \end{aligned} \quad (3.4)$$

Setting  $v_\eta = v_\eta^1 = u_\eta^1 - P_\eta u(\Delta t)$  and  $\varphi_\eta^1 = P_\eta u(\Delta t) - u(\Delta t)$ , we obtain

$$\begin{aligned} \frac{1}{\Delta t} \|v_\eta^1\|_{L^2(\Omega)}^2 + \nu |v_\eta^1|_{H^1(\Omega)}^2 & = \frac{\Delta t}{2} (u''(\theta \Delta t), v_\eta^1) + (r_\eta p(\Delta t) - p(\Delta t), \operatorname{div} v_\eta^1) - (v_\eta^1 \cdot \nabla u_\eta^1, v_\eta^1) \\ & \quad - \frac{1}{2} (\operatorname{div} v_\eta^1, u_\eta^1 \cdot v_\eta^1) - (\varphi_\eta^1 \cdot \nabla u_\eta^1, v_\eta^1) - \frac{1}{2} (\operatorname{div} \varphi_\eta^1, u_\eta^1 \cdot v_\eta^1) \\ & \quad - (u(\Delta t) \cdot \nabla \varphi_\eta^1, v_\eta^1) - \frac{1}{\Delta t} (\varphi_\eta^1, v_\eta^1) - \nu (\nabla \varphi_\eta^1, \nabla v_\eta^1). \end{aligned} \quad (3.5)$$

Then (3.2) follows readily by applying the error approximation of  $P_\eta$ .  
For the pressure, we have

$$\begin{aligned} (r_\eta p(\Delta t) - p(\Delta t), \operatorname{div} v_\eta) + (p_\eta^1 - r_\eta p(\Delta t), \operatorname{div} v_\eta) &= \frac{1}{\Delta t} (u_\eta^1 - u(\Delta t), v_\eta) + \nu (\nabla(u_\eta^1 - u(\Delta t)), \nabla v_\eta) \\ &\quad - \frac{\Delta t}{2} (u''(\theta \Delta t), v_\eta) - (u(\Delta t) \cdot \nabla u(\Delta t) - u_\eta^1 \cdot \nabla u_\eta^1, v_\eta) + \frac{1}{2} (\operatorname{div} u_\eta^1, u_\eta^1 \cdot v_\eta), \end{aligned} \quad (3.6)$$

and owing to the inf-sup condition (1.10), there exists  $v_\eta \in V_\eta^\perp$  such that

$$(p_\eta^1 - r_\eta p(\Delta t), \operatorname{div} v_\eta) = \|p_\eta^1 - r_\eta p(\Delta t)\|_{L^2(\Omega)}^2 \quad \text{and} \quad |v_\eta|_{H^1(\Omega)} \leq \frac{1}{\beta^*} \|p_\eta^1 - r_\eta p(\Delta t)\|_{L^2(\Omega)},$$

with  $\beta^* > 0$  independent of  $\eta$ . Then, by applying (3.2), we obtain (3.3).  $\square$

The next result, stated in Lemma 3.4, is a standard error estimate. We give the proof for the sake of completeness.

**Lemma 3.4.** *Let  $X_\eta$  and  $M_\eta$  be defined by (1.11) and (1.12) and approximate  $f^{n+1}$  by  $f^{n+1} = f(t^{n+1})$ . At each time step, (1.17)–(1.18) has a solution  $u_\eta^{n+1}$  and this solution is unique if  $\Delta t$  is sufficiently small. Suppose that  $u \in L^2(0, T; H^3(\Omega)^2)$ ,  $u' \in L^2(0, T; H^2(\Omega)^2)$ ,  $u^{(3)} \in L^2(\Omega \times ]0, T])^2$  and  $p \in L^2(0, T; H^2(\Omega))$ , there exist a constant  $C$  that does not depend on  $\eta$  and  $\Delta t$  and a constant  $k_0 > 0$  that does not depend on  $\eta$  such that, for all  $\Delta t \leq k_0$ ,*

$$\begin{aligned} \sup_{1 \leq n \leq N} \|u_\eta^n - u(t^n)\|_{L^2(\Omega)} + \left( \sum_{n=1}^{N-1} \|\delta^2(u_\eta^n - u(t^n))\|_{L^2(\Omega)}^2 \right)^{1/2} + \sqrt{\nu} \left( \sum_{n=1}^{N-1} \Delta t |u_\eta^{n+1} - u(t^{n+1})|_{H^1(\Omega)}^2 \right)^{1/2} \\ \leq C(\eta^2 + (\Delta t)^2). \end{aligned} \quad (3.7)$$

*Proof.* Setting  $v_\eta^n = u_\eta^n - P_\eta u(t^n)$  and  $\varphi_\eta^n = P_\eta u(t^n) - u(t^n)$ ,  $0 \leq n \leq N$ , we subtract (1.17) and (1.1) taken at  $t = t^{n+1}$  and by using the following second-order backward finite difference scheme

$$\frac{\partial u}{\partial t}(t^{n+1}) = \frac{3u(t^{n+1}) - 4u(t^n) + u(t^{n-1}))}{2\Delta t} + \mathcal{O}((\Delta t)^2), \quad (3.8)$$

we have

$$\left| u'(t + \Delta t) - \frac{3u(t + \Delta t) - 4u(t) + u(t - \Delta t))}{2\Delta t} \right| \leq \frac{(\Delta t)^{3/2}}{2\sqrt{3}} \|u^{(3)}\|_{L^2(t - \Delta t; t + \Delta t)}, \quad (3.9)$$

and by summing the result over  $1 \leq n \leq m - 1$ , we obtain :

$$\begin{aligned} &\|v_\eta^m\|_{L^2(\Omega)}^2 + \|2v_\eta^m - v_\eta^{m-1}\|_{L^2(\Omega)}^2 + \sum_{n=1}^{m-1} \|\delta^2 v_\eta^n\|_{L^2(\Omega)}^2 + 4\nu \sum_{n=1}^{m-1} \Delta t |v_\eta^{n+1}|_{H^1(\Omega)}^2 \\ &\leq \left( \|v_\eta^1\|_{L^2(\Omega)}^2 + \|2v_\eta^1 - v_\eta^0\|_{L^2(\Omega)}^2 \right) + 2 \left| \sum_{n=1}^{m-1} (3\varphi_\eta^{n+1} - 4\varphi_\eta^n + \varphi_\eta^{n-1}, v_\eta^{n+1}) \right| + 4\nu \left| \sum_{n=1}^{m-1} \Delta t (\nabla \varphi_\eta^{n+1}, \nabla v_\eta^{n+1}) \right| \\ &\quad + 4 \left| \sum_{n=1}^{m-1} \Delta t (p(t^{n+1}) - r_\eta p(t^{n+1}), \operatorname{div} v_\eta^{n+1}) \right| + 4 \sum_{n=1}^{m-1} \Delta t \frac{(\Delta t)^{3/2}}{2\sqrt{3}} \|u^{(3)}\|_{L^2(t^{n-1}, t^{n+1})} \|v_\eta^{n+1}\|_{L^2(\Omega)} \\ &\quad + 4 \left| \sum_{n=1}^{m-1} \Delta t (u(t^{n+1}) \cdot \nabla u(t^{n+1}) - u_\eta^{n+1} \cdot \nabla u_\eta^{n+1}, v_\eta^{n+1}) - \frac{1}{2} \sum_{n=1}^{m-1} \Delta t (\operatorname{div} u_\eta^{n+1}, u_\eta^{n+1} \cdot v_\eta^{n+1}) \right|. \end{aligned} \quad (3.10)$$

Let us study the terms of the right hand side of (3.10) denoted by  $((t_{rhs})_i)_{1 \leq i \leq 7}$ . The first term  $(t_{rhs})_1$  is bounded as in Proposition 3.3.

To study the second term, we have

$$\frac{3\varphi_\eta^{n+1} - 4\varphi_\eta^n + \varphi_\eta^{n-1}}{2\Delta t} = P_\eta u'(t^{n+1}) - u'(t^{n+1}) + R_2,$$



with

$$|R_2| \leq \frac{(\Delta t)^{3/2}}{2\sqrt{3}} \|P_\eta u^{(3)} - u^{(3)}\|_{L^2(t^{n-1}; t^{n+1})}.$$

Hence, by assuming that  $P_\eta$  is stable in the norm  $L^2$  (cf. Girault and Lions [8]), we have

$$\begin{aligned} |(t_{rhs})_2| &= \left| 4 \sum_{n=1}^{m-1} \Delta t \left( \frac{3\varphi_\eta^{n+1} - 4\varphi_\eta^n + \varphi_\eta^{n-1}}{2\Delta t}, v_\eta^{n+1} \right) \right| \\ &\leq \frac{C\eta^4}{2\varepsilon_2} \|u'\|_{L^\infty(0,T;H^2(\Omega)^2)}^2 + \frac{C(\Delta t)^4}{2\varepsilon_2} \|u^{(3)}\|_{L^2(\Omega \times ]0,T])^2}^2 + \frac{\varepsilon_2}{2} \sum_{n=1}^{m-1} \Delta t |v_\eta^{n+1}|_{H^1(\Omega)}^2. \end{aligned}$$

The third term is bounded as follows :

$$|(t_{rhs})_3| = \left| 4\nu \sum_{n=1}^{m-1} \Delta t (\nabla \varphi_\eta^{n+1}, \nabla v_\eta^{n+1}) \right| \leq \frac{2C\nu\eta^4}{\varepsilon_3} \|u\|_{L^2(0,T;H^3(\Omega)^2)}^2 + 2\nu\varepsilon_3 \sum_{n=1}^{m-1} \Delta t |v_\eta^{n+1}|_{H^1(\Omega)}^2.$$

For the pressure contribution, we have :

$$|(t_{rhs})_4| = \left| 4 \sum_{n=1}^{m-1} \Delta t (p(t^{n+1}) - r_\eta p(t^{n+1}), \operatorname{div} v_\eta^{n+1}) \right| \leq \frac{2C\eta^4}{\varepsilon_4} \|p\|_{L^2(0,T;H^2(\Omega))}^2 + 2\varepsilon_4 \sum_{n=1}^{m-1} \Delta t |v_\eta^{n+1}|_{H^1(\Omega)}^2.$$

The fifth term is treated as follows :

$$\begin{aligned} |(t_{rhs})_5| &= 4 \sum_{n=1}^{m-1} \Delta t \frac{(\Delta t)^{3/2}}{2\sqrt{3}} \|u^{(3)}\|_{L^2(t^{n-1}; t^{n+1})} \|v_\eta^{n+1}\|_{L^2(\Omega)} \\ &\leq \frac{(\Delta t)^4 S_2^2}{3\varepsilon_5} \|u^{(3)}\|_{L^2(\Omega \times ]0,T])^2}^2 + \varepsilon_5 \sum_{n=1}^{m-1} \Delta t |v_\eta^{n+1}|_{H^1(\Omega)}^2. \end{aligned}$$

Let us consider now the non-linear terms,  $(t_{rhs})_6 + (t_{rhs})_7$ , which are treated like follows :

$$\begin{aligned} &(-u(t^{n+1}) \cdot \nabla u(t^{n+1}) + u_\eta^{n+1} \cdot \nabla u_\eta^{n+1}, v_\eta^{n+1}) + \frac{1}{2} (\operatorname{div} u_\eta^{n+1}, u_\eta^{n+1} \cdot v_\eta^{n+1}) \\ &= -(v_\eta^{n+1} \cdot \nabla v_\eta^{n+1}, P_\eta u(t^{n+1})) - \frac{1}{2} (\operatorname{div} v_\eta^{n+1}, P_\eta u(t^{n+1}) \cdot v_\eta^{n+1}) - (\varphi_\eta^{n+1} \cdot \nabla v_\eta^{n+1}, P_\eta u(t^{n+1})) \\ &\quad - \frac{1}{2} (\operatorname{div} \varphi_\eta^{n+1}, P_\eta u(t^{n+1}) \cdot v_\eta^{n+1}) - (u(t^{n+1}) \cdot \nabla v_\eta^{n+1}, \varphi_\eta^{n+1}). \end{aligned} \quad (3.11)$$

The study of the three terms in the right-hand side of (3.11), denoted by  $((t_{rhs})_{67}).j$ ,  $j = 1, 2, 3$ , will end the proof. Setting

$$C_1 = \sup_n |u(t^n)|_{H^1(\Omega)},$$

applying on one hand

$$\begin{aligned} \int_\Omega \operatorname{div}(v_\eta^{n+1} - u(t^{n+1})) u(t^{n+1}) \cdot \varphi_\eta^{n+1} dx &= - \int_\Omega (v_\eta^{n+1} - u(t^{n+1})) \cdot \nabla u(t^{n+1}) \cdot \varphi_\eta^{n+1} dx \\ &\quad - \int_\Omega (v_\eta^{n+1} - u(t^{n+1})) \cdot \nabla \varphi_\eta^{n+1} \cdot u(t^{n+1}) dx, \end{aligned} \quad (3.12)$$

and on the other hand

$$ab \leq \frac{a^p}{p} + \frac{b^{p'}}{p'}, \text{ avec } \frac{1}{p} + \frac{1}{p'} = 1, \quad (3.13)$$

we have

$$\begin{aligned} |((t_{rhs})_{67}).1| &= \left| 4 \sum_{n=1}^{m-1} \Delta t \left( (v_\eta^{n+1} \cdot \nabla v_\eta^{n+1}, P_\eta u(t^{n+1})) + \frac{1}{2} (\operatorname{div} v_\eta^{n+1}, P_\eta u(t^{n+1}) \cdot v_\eta^{n+1}) \right) \right| \\ &\leq \frac{3S_4 C_1 \sqrt{2}}{2\varepsilon_6^4} \sum_{n=1}^{m-1} \Delta t \|v_\eta^{n+1}\|_{L^2(\Omega)}^2 + \frac{9S_4 C_1 \varepsilon_6^{4/3}}{2\sqrt{2}} \sum_{n=1}^{m-1} \Delta t |v_\eta^{n+1}|_{H^1(\Omega)}^2, \end{aligned}$$

$$\begin{aligned} |((t_{rhs})_{67}).2| &= \left| 4 \sum_{n=1}^{m-1} \Delta t \left( (\varphi_\eta^{n+1} \cdot \nabla v_\eta^{n+1}, P_\eta u(t^{n+1})) + \frac{1}{2} (\operatorname{div} \varphi_\eta^{n+1}, P_\eta u(t^{n+1}) \cdot v_\eta^{n+1}) \right) \right| \\ &\leq 3S_4^2 C C_1 \left\{ \frac{\eta^4}{\varepsilon_7} \|u\|_{L^2(0,T;H^3(\Omega)^2)}^2 + \varepsilon_7 \sum_{n=1}^{m-1} \Delta t |v_\eta^{n+1}|_{H^1(\Omega)}^2 \right\}, \end{aligned}$$

and

$$\begin{aligned} |((t_{rhs})_{67}).3| &= \left| 4 \sum_{n=1}^{m-1} \Delta t \left( u(t^{n+1}) \cdot \nabla v_\eta^{n+1}, \varphi_\eta^{n+1} \right) \right| \\ &\leq \frac{S_4^2 C_1}{2} \left\{ \frac{C^2 \eta^4}{\varepsilon_8} \|u\|_{L^2(0,T;H^3(\Omega)^2)}^2 + \varepsilon_8 \sum_{n=1}^{m-1} \Delta t |v_\eta^{n+1}|_{H^1(\Omega)}^2 \right\}. \end{aligned}$$

After a suitable choice of  $\varepsilon_i$ , (3.10) becomes

$$\begin{aligned} \|v_\eta^m\|_{L^2(\Omega)}^2 + \|2v_\eta^m - v_\eta^{m-1}\|_{L^2(\Omega)}^2 + \sum_{n=1}^{m-1} \|\delta^2 v_\eta^n\|_{L^2(\Omega)}^2 + \nu \sum_{n=1}^{m-1} \Delta t |v_\eta^{n+1}|_{H^1(\Omega)}^2 \\ \leq \alpha(\Delta t)^4 + \beta \eta^4 + \xi \sum_{n=1}^{m-1} \Delta t \|v_\eta^{n+1}\|_{L^2(\Omega)}^2, \end{aligned} \quad (3.14)$$

where  $\alpha, \beta$  and  $\xi$  are constants that do not depend on  $\eta$  and  $\Delta t$ .

Then after applying Gronwall's lemma and for  $\Delta t$  sufficiently small, the result follows from this inequality:

$$\begin{aligned} \sup_{1 \leq n \leq N} \|v_\eta^n\|_{L^2(\Omega)} + \sup_{1 \leq n \leq N} \|2v_\eta^n - v_\eta^{n-1}\|_{L^2(\Omega)} + \left( \sum_{n=1}^{N-1} \|\delta^2 v_\eta^n\|_{L^2(\Omega)}^2 \right)^{1/2} \\ + \sqrt{\nu} \left( \sum_{n=1}^{N-1} \Delta t |v_\eta^{n+1}|_{H^1(\Omega)}^2 \right)^{1/2} \leq \alpha'(\Delta t)^2 + \beta' \eta^2. \end{aligned} \quad (3.15)$$

Finally, (3.7) follows by applying a triangular inequality and the  $P_\eta$ 's properties.  $\square$

**Remark 3.5.** We suppose that there exist two constants  $\alpha'$  and  $\gamma' > 0$  that do not depend on  $\eta$  and  $\Delta t$  such that

$$\alpha' \eta^3 \leq (\Delta t)^2 \leq \gamma' \eta^3, \quad (3.16)$$

which means that  $(\Delta t)^2$  is of the same order of  $\eta^3$ .

#### 4. SOME ERROR ESTIMATES FOR THE STOKES PROBLEM

The error estimate of order two in  $L^2(\Omega \times ]0, T])^2$ , that will be established in the next section, is based on a duality argument for the transient Stokes problem:

$$\frac{\partial v}{\partial t}(x, t) - \nu \Delta v(x, t) + \nabla q(x, t) = g(x, t) \text{ in } \Omega \times ]0, T[, \quad (4.1)$$

$$\operatorname{div} v(x, t) = 0 \text{ in } \Omega \times ]0, T[, \quad v(x, t) = 0 \text{ on } \partial\Omega \times ]0, T[, \quad v(x, 0) = 0 \text{ in } \Omega. \quad (4.2)$$

The fully-discrete scheme for (4.1)–(4.2) is: Find  $(v_\eta^{n+1}, q_\eta^{n+1})$  with values in  $X_\eta \times M_\eta$ , for each  $1 \leq n \leq N-1$ , solution of:

$$\forall z_\eta \in X_\eta, \quad \frac{1}{2\Delta t} (3v_\eta^{n+1} - 4v_\eta^n + v_\eta^{n-1}, z_\eta) + \nu (\nabla v_\eta^{n+1}, \nabla z_\eta) - (q_\eta^{n+1}, \operatorname{div} z_\eta) = (g^{n+1}, z_\eta), \quad (4.3)$$

$$\forall \lambda_\eta \in M_\eta, \quad (\lambda_\eta, \operatorname{div} v_\eta^{n+1}) = 0. \quad (4.4)$$

These equations are completed by initial conditions similar to the Navier-Stokes problem's ones.

This linear problem (4.3)–(4.4) has a unique solution, owing to the inf-sup condition (1.10), without

any restriction on  $\Delta t$ . This solution satisfies the following error estimates in norm  $L^\infty(0, T; L^2(\Omega)^2)$  and  $L^2(0, T; H^1(\Omega)^2)$ : We prove, first of all, that the initial value  $v_\eta^1$ , as in the Navier-Stokes problem, satisfies:

If  $v(\Delta t) \in H^3(\Omega)^2$ ,  $q(\Delta t) \in H^2(\Omega)$  and  $v'' \in \mathcal{C}^0([0, T]; L^2(\Omega)^2)$ , then

$$\begin{aligned} & \|v_\eta^1 - v(\Delta t)\|_{L^2(\Omega)}^2 + \nu \Delta t |v_\eta^1 - v(\Delta t)|_{H^1(\Omega)}^2 \\ & \leq \frac{(\Delta t)^4}{2} \|v''\|_{L^\infty(0, T; L^2(\Omega)^2)}^2 + C(\Delta t) \eta^4 \left( |v(\Delta t)|_{H^3(\Omega)}^2 + |q(\Delta t)|_{H^2(\Omega)}^2 \right) + C\eta^6 |v(\Delta t)|_{H^3(\Omega)}^2. \end{aligned} \quad (4.5)$$

Secondly, in the general case, we have the following result (the proof is similar to the one of Lemma 3.4, but simpler because of the absence of the convection term).

**Lemma 4.1.** *Let  $(v, q)$  and  $(v_\eta^n, q_\eta^n)$  be the respective solution of (4.1)–(4.2) and (4.3)–(4.4). In addition to the precedent hypotheses, we suppose that  $g$  is regular enough in space and in time,  $v \in L^2(0, T; H^3(\Omega)^2)$ ,  $v' \in L^2(0, T; H^2(\Omega)^2)$ ,  $v^{(3)} \in L^2(\Omega \times ]0, T])^2$  and  $q \in L^2(0, T; H^2(\Omega))$ . There exists a constant  $C$  that does not depend on  $\eta$  and  $\Delta t$  such that*

$$\begin{aligned} & \sup_{1 \leq n \leq N} \|v_\eta^n - v(t^n)\|_{L^2(\Omega)} + \sup_{1 \leq n \leq N} \|2(v_\eta^n - v(t^n)) - (v_\eta^{n-1} - v(t^{n-1}))\|_{L^2(\Omega)} \\ & + \left( \sum_{n=1}^{N-1} \|\delta^2(v_\eta^n - v(t^n))\|_{L^2(\Omega)}^2 \right)^{1/2} + \sqrt{\nu} \left( \sum_{n=1}^N \Delta t |v_\eta^n - v(t^n)|_{H^1(\Omega)}^2 \right)^{1/2} \leq C(\eta^2 + (\Delta t)^2). \end{aligned} \quad (4.6)$$

In addition, the solution  $(v_\eta^{n+1}, q_\eta^{n+1})$  of (4.3)–(4.4) satisfies an error estimate in  $L^\infty(0, T; H^1(\Omega)^2)$ . To simplify, we introduce the following notation

$$\delta^1 a^n = \frac{3a^{n+1} - 4a^n + a^{n-1}}{2\Delta t}. \quad (4.7)$$

The proof is based on the following Stokes projection:  $\forall (u, p) \in V \times L_0^2(\Omega)$ ,  $S_\eta(u) \in V_\eta$  satisfies

$$\forall v_\eta \in V_\eta, \quad \nu(\nabla(S_\eta(u) - u), \nabla v_\eta) = -(p, \operatorname{div} v_\eta). \quad (4.8)$$

The operator  $S_\eta$  satisfies the following inequalities:

**Lemma 4.2.** *Let  $(u, p) \in V \times L_0^2(\Omega)$ .  $S_\eta(u)$  defined by (4.8) satisfies*

$$|S_\eta(u) - u|_{H^1(\Omega)} \leq 2|P_\eta(u) - u|_{H^1(\Omega)} + \frac{1}{\nu} \|r_\eta(p) - p\|_{L^2(\Omega)}. \quad (4.9)$$

*If in addition  $\Omega$  is convex, there exists a constant  $C$  that does not depend on  $\eta$  such that*

$$\|S_\eta(u) - u\|_{L^2(\Omega)} \leq C\eta(|S_\eta(u) - u|_{H^1(\Omega)} + \|r_\eta(p) - p\|_{L^2(\Omega)}). \quad (4.10)$$

**Lemma 4.3.** *In addition to the hypotheses of Lemma 4.1, suppose that  $v' \in \mathcal{C}^0(0, T; H^2(\Omega)^2)$ ,  $v'' \in L^2(0, T; H^1(\Omega)^2)$ ,  $v^{(3)} \in L^2(\Omega \times ]0, T])^2$ ,  $q' \in \mathcal{C}^0(0, T; H^1(\Omega))$  and  $q'' \in L^2(\Omega \times ]0, T])$ . Then, if  $\Omega$  is convex, there exists a constant  $C$  that does not depend on  $\eta$  and  $\Delta t$  such that*

$$\begin{aligned} & \sup_{1 \leq n \leq N} |v_\eta^n - v(t^n)|_{H^1(\Omega)} + \sup_{1 \leq n \leq N-1} |2(v_\eta^{n+1} - v(t^{n+1})) - (v_\eta^n - v(t^n))|_{H^1(\Omega)} \\ & + \left( \sum_{n=1}^{N-1} \Delta t \|\delta^1(v_\eta^n - v(t^n))\|_{L^2(\Omega)}^2 \right)^{1/2} + \left( \sum_{n=1}^{N-1} |\delta^2(v_\eta^n - v(t^n))|_{H^1(\Omega)}^2 \right)^{1/2} \leq C(\eta^2 + (\Delta t)^{3/2} + \frac{\eta^3}{\sqrt{\Delta t}}). \end{aligned} \quad (4.11)$$

*Proof.* Setting  $\varphi(t) = v(t) - S_\eta v(t)$ ,  $\varphi_\eta^i = \varphi(t^i)$  and  $e_\eta^i = v_\eta^i - S_\eta v(t^i)$  and applying (3.9) to (4.3), we obtain

$$\forall z_\eta \in V_\eta, \quad (\delta^1 e_\eta^n, z_\eta) + \nu(\nabla e_\eta^{n+1}, \nabla z_\eta) = (\delta^1 \varphi_\eta^n, z_\eta) + R_3, \quad (4.12)$$

where

$$|R_3| \leq \frac{(\Delta t)^{3/2}}{2\sqrt{3}} \|v^{(3)}\|_{L^2(t^{n-1}, t^{n+1}; L^2(\Omega)^2)} \|z_\eta\|_{L^2(\Omega)}.$$

Taking the scalar product by  $z_\eta = z_\eta^{n+1} = \frac{3e_\eta^{n+1} - 4e_\eta^n + e_\eta^{n-1}}{2\Delta t}$ , summing over  $1 \leq n \leq m-1$ , and applying Jensen's inequality, (4.12) becomes

$$\begin{aligned} & \frac{1}{2} \sum_{n=1}^{m-1} \Delta t \|z_\eta^{n+1}\|_{L^2(\Omega)}^2 + \frac{\nu}{4} \left( |e_\eta^m|_{H^1(\Omega)}^2 + |2e_\eta^m - e_\eta^{m-1}|_{H^1(\Omega)}^2 + \sum_{n=1}^{m-1} |\delta^2 e_\eta^n|_{H^1(\Omega)}^2 \right) \\ & \leq \frac{5\nu}{4} |e_\eta^1|_{H^1(\Omega)}^2 + \frac{(\Delta t)^4}{3} \|v^{(3)}\|_{L^2(0, T; L^2(\Omega)^2)}^2 + \sum_{n=1}^{m-1} \Delta t \|\delta^1 \varphi_\eta^n\|_{L^2(\Omega)}^2. \end{aligned} \quad (4.13)$$

Then (4.11) follows readily by applying (4.5), (4.9) and (4.10).  $\square$

The parabolic duality argument (cf. [21]) consists in defining the solution  $(w^{n-1}, \lambda^{n-1})$  of the backward semi-discrete Stokes system of second order in time :

$$-\frac{w^{n+1} - 4w^n + 3w^{n-1}}{2\Delta t} + \nu \Delta w^{n-1} - \nabla \lambda^{n-1} = v_\eta^{n-1} - v(t^{n-1}) \text{ in } \Omega, \quad 1 \leq n \leq N+1, \quad (4.14)$$

$$\operatorname{div} w^{n-1} = 0 \text{ in } \Omega, \quad 1 \leq n \leq N+1, \quad (4.15)$$

$$w^{n-1}|_{\partial\Omega} = 0, \quad 1 \leq n \leq N+1, \quad (4.16)$$

$$w^{N+2} = 0, \quad w^{N+1} = 0 \text{ in } \Omega. \quad (4.17)$$

For each  $n, 0 \leq n \leq N$ , the Stokes problem (4.14)–(4.17) has a unique solution  $w^n \in H_0^1(\Omega)^2, \lambda^n \in L_0^2(\Omega)$ , (cf. [9], [20]).

The next lemma establishes basic estimates for the velocity  $w^n$  of the backward semi-discrete Stokes problem (4.14)–(4.17).

**Lemma 4.4.** *Standard arguments give the uniform bounds:*

$$\begin{aligned} & \sup_{0 \leq n \leq N} \|w^n\|_{L^2(\Omega)} + \sup_{1 \leq n \leq N+1} \|2w^{n-1} - w^n\|_{L^2(\Omega)} + \sqrt{2\nu} \left( \sum_{n=0}^N \Delta t |w^n|_{H^1(\Omega)}^2 \right)^{1/2} \\ & + \sum_{n=1}^{N+1} \|\delta^2 w^n\|_{L^2(\Omega)} \leq \sqrt{\frac{2S_2}{\nu}} \left( \sum_{n=0}^N \Delta t \|v(t^n) - v_\eta^n\|_{L^2(\Omega)}^2 \right)^{1/2}, \end{aligned} \quad (4.18)$$

where  $S_2$  is the constant of Poincars inequality, and

$$\begin{aligned} & \sqrt{\frac{\nu}{2}} \sup_{0 \leq n \leq N} |w^n|_{H^1(\Omega)} + \sqrt{\frac{\nu}{2}} \left( \sum_{n=1}^{N+1} |\delta^2 w^n|_{H^1(\Omega)}^2 \right)^{1/2} + \sqrt{\frac{\nu}{2}} \sup_{0 \leq n \leq N} |2w^n - w^{n+1}|_{H^1(\Omega)} \\ & + \left( \sum_{n=1}^{N+1} \Delta t \left\| \frac{w^{n+1} - 4w^n + 3w^{n-1}}{2\Delta t} \right\|_{L^2(\Omega)}^2 \right)^{1/2} \leq \left( \sum_{n=0}^N \Delta t \|v(t^n) - v_\eta^n\|_{L^2(\Omega)}^2 \right)^{1/2}. \end{aligned} \quad (4.19)$$

If  $\Omega$  is convex, then  $\forall 0 \leq n \leq N, w^n \in H^2(\Omega)^2, \lambda^n \in H^1(\Omega)$  and (4.19) implies the uniform bound

$$\left( \sum_{n=0}^N \Delta t \left( |w^n|_{H^2(\Omega)}^2 + |\lambda^n|_{H^1(\Omega)}^2 \right) \right)^{1/2} \leq C \left( \sum_{n=0}^N \Delta t \|v(t^n) - v_\eta^n\|_{L^2(\Omega)}^2 \right)^{1/2} \quad (4.20)$$

with a constant  $C$  independent of  $\Delta t$  and  $\eta$ .

*Proof.* For the first inequality, we take the scalar product of (4.14) with  $z = 4\Delta t w^{n-1}$ , and we use the incompressibility condition. Multiplying the result by  $\Delta t$ , summing it over  $n$  from  $m+1$  to  $N+1$ , and applying the Poincars inequality, we obtain for any  $\varepsilon > 0$

$$\begin{aligned} \|w^m\|_{L^2(\Omega)}^2 + \|2w^m - w^{m+1}\|_{L^2(\Omega)}^2 + 4\nu \sum_{n=m}^N \Delta t |w^n|_{H^1(\Omega)}^2 + \sum_{n=m+1}^{N+1} \|\delta^2 w^n\|_{L^2(\Omega)}^2 \\ \leq \frac{2}{\varepsilon} \sum_{n=m}^N \Delta t \|v(t^n) - v_\eta^n\|_{L^2(\Omega)}^2 + 2\varepsilon S_2 \sum_{n=m}^N \Delta t |w^n|_{H^1(\Omega)}^2, \end{aligned}$$

where  $S_2$  is Poincars constant. Then (4.18) follows after the suitable choice of  $\varepsilon = \frac{\nu}{S_2}$ .

Similarly, for the second inequality, we take the scalar product of (4.14) with  $z = \frac{w^{n+1} - 4w^n + 3w^{n-1}}{2\Delta t}$ , we multiply the equation by  $\Delta t$  and sum it over  $n$ . We obtain (4.19) by choosing  $\varepsilon = \frac{1}{2\Delta t}$ .

Now, we assume that  $\Omega$  is convex. Since (4.14)–(4.17) is a steady Stokes problem with right-hand side  $-\frac{w^{n+1} - 4w^n + 3w^{n-1}}{2\Delta t} + (v(t^{n-1}) - v_\eta^{n-1})$ , we have  $w^n \in H^2(\Omega)^2, \lambda^n \in H^1(\Omega)$  (cf. [10]) and (4.19) implies also the uniform bound (4.20).  $\square$

From now on, we assume that  $\Omega$  is convex. Using these inequalities, the next theorem establishes that the error satisfies an estimate of order two in  $L^2(\Omega \times ]0, T])^2$ .

**Theorem 4.5.** *If  $g \in L^2(\Omega \times ]0, T])^2, v \in L^2(0, T; H^3(\Omega)^2), q \in L^2(0, T; H^2(\Omega)), v' \in L^2(0, T; H^2(\Omega)^2)$  and  $v^{(3)} \in L^2(\Omega \times ]0, T])^2$ , then there exists a constant  $C$ , that depends on the norm of  $v, v', v^{(3)}$  and  $q$ , but not on  $\eta$  and  $\Delta t$  such that*

$$\left( \sum_{n=0}^N \Delta t \|v_\eta^n - v(t^n)\|_{L^2(\Omega)}^2 \right)^{1/2} \leq C(\eta^3 + (\Delta t)^2 + \eta(\Delta t)^2). \quad (4.21)$$

In particular, if (3.16) holds, then

$$\left( \sum_{n=0}^N \Delta t \|v_\eta^n - v(t^n)\|_{L^2(\Omega)}^2 \right)^{1/2} \leq C\eta^3. \quad (4.22)$$

*Proof.* Let  $e^{n-1} = v_\eta^{n-1} - v(t^{n-1})$ . Taking the scalar product of (4.14) by  $e^{n-1}$ , summing over  $n$  from 1 to  $N+1$  and applying a discrete integration by parts, we obtain

$$\begin{aligned} \sum_{n=0}^N \Delta t \|e^n\|_{L^2(\Omega)}^2 &= \sum_{n=1}^{N-1} \left\{ -\frac{1}{2}(3e^{n+1} - 4e^n + e^{n-1}, P_\eta w^{n+1}) - \nu \Delta t (\nabla P_\eta w^{n+1}, \nabla e^{n-1}) \right\} \\ &\quad - \frac{1}{2} \sum_{n=1}^{N+1} (3e^{n+1} - 4e^n + e^{n-1}, w^{n+1} - P_\eta w^{n+1}) - \nu \sum_{n=1}^{N+1} \Delta t (\nabla(w^{n-1} - P_\eta w^{n-1}), \nabla e^{n-1}) \quad (4.23) \\ &\quad + \sum_{n=1}^{N+1} \Delta t (\lambda^{n-1} - r_\eta \lambda^{n-1}, \operatorname{div} e^{n-1}) - \left\{ \frac{3}{2}(w^1, e^1) + \nu \Delta t (\nabla e^1, \nabla P_\eta w^1) \right\}. \end{aligned}$$

Denote the terms in the right-hand side of (4.23) by  $(W_{RH})_j, j = 1, \dots, 5$ . The first term is treated as

follows :

$$\begin{aligned}
 |(W_{RH})_1| &\leq \left| \sum_{n=1}^N \Delta t \left( q(t^{n+1}) - r_\eta q(t^{n+1}), \operatorname{div}(P_\eta w^{n+1} - w^{n+1}) \right) \right| \\
 &\quad + \frac{P}{\sqrt{3}} (\Delta t)^2 \|v^{(3)}\|_{L^2(\Omega \times ]0, T])^2} \left( \sum_{n=1}^{N-1} \Delta t |P_\eta w^{n+1}|_{H^1(\Omega)}^2 \right)^{1/2} \\
 &\leq \left[ C\eta^3 \|q\|_{L^2(0, T; H^1(\Omega))} + \frac{P}{\sqrt{3}} (\Delta t)^2 \|v^{(3)}\|_{L^2(\Omega \times ]0, T])^2} \right] \left( \sum_{n=0}^N \Delta t \|e^n\|_{L^2(\Omega)}^2 \right)^{1/2}.
 \end{aligned}$$

The second term is bounded as follows :

$$\begin{aligned}
 |(W_{RH})_2| &\leq \left( \sum_{n=1}^{N-1} \Delta t \|\delta^1 e^n\|_{L^2(\Omega)}^2 \right)^{1/2} \left( \sum_{n=1}^{N-1} \Delta t \|w^{n+1} - P_\eta w^{n+1}\|_{L^2(\Omega)}^2 \right)^{1/2} \\
 &\leq C\eta^2 ((\Delta t)^{3/2} + \eta^2) \left( \sum_{n=0}^N \Delta t \|e^n\|_{L^2(\Omega)}^2 \right)^{1/2}.
 \end{aligned}$$

Owing to Lemma 4.1, the third and fourth terms can be bounded by:

$$\begin{aligned}
 |(W_{RH})_3| &\leq C\eta \left( \sum_{n=0}^N \Delta t |e^n|_{H^1(\Omega)}^2 \right)^{1/2} \left( \sum_{n=0}^N \Delta t |w^n|_{H^2(\Omega)}^2 \right)^{1/2} \\
 &\leq C\eta ((\Delta t)^2 + \eta^2) \left( \sum_{n=0}^N \Delta t \|e^n\|_{L^2(\Omega)}^2 \right)^{1/2},
 \end{aligned}$$

and

$$|(W_{RH})_4| \leq C\eta ((\Delta t)^2 + \eta^2) \left( \sum_{n=0}^N \Delta t \|e^n\|_{L^2(\Omega)}^2 \right)^{1/2}.$$

Finally, the last term can be written as follows :

$$|(W_{RH})_5| = \frac{3}{2} (w^1 - P_\eta w^1, e^1) - \frac{3}{2} \left[ (P_\eta w^1, e^1) + \nu \Delta t (\nabla e^1, \nabla P_\eta w^1) \right] + \frac{\nu}{2} \Delta t (\nabla e^1, \nabla P_\eta w^1).$$

Let us consider the terms between square brackets and write the error equation at time  $t^1$  : there exists  $\theta \in ]0, 1[$  such that

$$(e^1, P_\eta w^1) + \nu \Delta t (\nabla e^1, \nabla P_\eta w^1) = \Delta t (r_\eta q(\Delta t) - q(\Delta t), \operatorname{div} P_\eta w^1) - \frac{(\Delta t)^2}{2} (v''(\theta \Delta t), P_\eta w^1),$$

then

$$\begin{aligned}
 \left| (e^1, P_\eta w^1) + \nu \Delta t (\nabla e^1, \nabla P_\eta w^1) \right| &\leq C \left[ (\Delta t) \eta^2 |q(\Delta t)|_{H^2(\Omega)} + \frac{(\Delta t)^2}{2} \|v''\|_{L^\infty(0, T; L^2(\Omega)^2)} \right] \\
 &\quad \left( \sum_{n=0}^N \Delta t \|e^n\|_{L^2(\Omega)}^2 \right)^{1/2}.
 \end{aligned}$$

The first and last parts of  $(W_{RH})_5$  are bounded by using (4.5).

Substituting these inequalities into (4.23), we obtain (4.21). In addition, if (3.16) holds, then (4.21) implies (4.22).  $\square$

Now, we split  $u_\eta^n - u(t^n)$  into a linear contribution,  $v_\eta^n - u(t^n)$ , and a non-linear one  $u_\eta^n - v_\eta^n$ . Here  $v_\eta^{n+1}$  is the solution of the Stokes problem (4.3)–(4.4) with  $g = f - u \cdot \nabla u$ . Therefore,  $v = u$  and  $v_\eta^{n+1}$  solves the discrete problem  $\forall w_\eta \in V_\eta$ ,

$$\frac{(3v_\eta^{n+1} - 4v_\eta^n + v_\eta^{n-1}, w_\eta)}{2\Delta t} + \nu (\nabla v_\eta^{n+1}, \nabla w_\eta) - (q_\eta^{n+1}, \operatorname{div} w_\eta) = (f(t^{n+1}) - u(t^{n+1}) \cdot \nabla u(t^{n+1}), w_\eta). \quad (4.24)$$

Therefore, Theorem 4.5 gives

**Corollary 4.6.** *Suppose that  $u$  satisfies the hypotheses on  $v$  in Theorem 4.5 and that  $f \in C^0([0, T]; L^2(\Omega)^2)$ , then*

$$\left( \sum_{n=0}^N \Delta t \| v_\eta^n - u(t^n) \|_{L^2(\Omega)}^2 \right)^{1/2} \leq C(\eta^3 + (\Delta t)^2 + \eta(\Delta t)^2), \quad (4.25)$$

with another constant  $C(f, u, p, \nu, T)$  that does not depend on  $\eta$  nor on  $\Delta t$ .

Furthermore, if  $p'$  belongs to  $L^2(0, T; H^1(\Omega))$ , Lemma 4.3 implies that

$$\sup_{0 \leq n \leq N} |v_\eta^n - u(t^n)|_{H^1(\Omega)} \leq C(\eta^2 + (\Delta t)^{3/2} + \frac{\eta^3}{\sqrt{\Delta t}}). \quad (4.26)$$

On the other hand, we prove the following ‘‘superconvergence’’ result for the non-linear part.

**Theorem 4.7.** *Under the assumptions of Corollary 4.6 and if  $p' \in L^2(0, T; H^1(\Omega))$ ,  $u' \in L^2(0, T; H^1(\Omega)^2)$  and  $u \in L^\infty(0, T; W^{1,4}(\Omega)^2)$  then there exists a constant  $C$  that does not depend on  $\eta$  and  $\Delta t$ , such that*

$$\begin{aligned} & \sup_{0 \leq n \leq N} \| v_\eta^n - u_\eta^n \|_{L^2(\Omega)} + \sup_{1 \leq n \leq N} \| 2(v_\eta^n - u_\eta^n) - (v_\eta^{n-1} - u_\eta^{n-1}) \|_{L^2(\Omega)} \\ & + \left( \sum_{n=1}^{N-1} \| \delta^2(v_\eta^n - u_\eta^n) \|_{L^2(\Omega)}^2 \right)^{1/2} + \left( \sum_{n=0}^{N-1} \Delta t |v_\eta^{n+1} - u_\eta^{n+1}|_{H^1(\Omega)}^2 \right)^{1/2} \leq C(\eta^3 + (\Delta t)^2). \end{aligned} \quad (4.27)$$

*Proof.* In one hand, we take the difference between (4.24) and (1.17). We split the non-linear term as follows :

$$\begin{aligned} & u_\eta^{n+1} \cdot \nabla u_\eta^{n+1} + \frac{1}{2} \operatorname{div} u_\eta^{n+1} u_\eta^{n+1} - u(t^{n+1}) \cdot \nabla u(t^{n+1}) \\ & = -\varphi_\eta^{n+1} \cdot \nabla u_\eta^{n+1} - \frac{1}{2} \operatorname{div} \varphi_\eta^{n+1} \cdot u_\eta^{n+1} - v_\eta^{n+1} \cdot \nabla \varphi_\eta^{n+1} - \frac{1}{2} \operatorname{div} v_\eta^{n+1} \varphi_\eta^{n+1} \\ & \quad + (v_\eta^{n+1} - u(t^{n+1})) \cdot \nabla (v_\eta^{n+1} - u(t^{n+1})) + \frac{1}{2} \operatorname{div} (v_\eta^{n+1} - u(t^{n+1})) (v_\eta^{n+1} - u(t^{n+1})) \\ & \quad + (v_\eta^{n+1} - u(t^{n+1})) \cdot \nabla u(t^{n+1}) + \frac{1}{2} \operatorname{div} (v_\eta^{n+1} - u(t^{n+1})) u(t^{n+1}) + u(t^{n+1}) \cdot \nabla (v_\eta^{n+1} - u(t^{n+1})). \end{aligned}$$

On the other hand, we multiply the resultant equation by  $\varphi_\eta^{n+1}$  and sum it over  $n = 1, \dots, m-1$ . We obtain:

$$\begin{aligned} & \frac{1}{2} \sum_{n=1}^{m-1} (\varphi_\eta^{n+1}, 3\varphi_\eta^{n+1} - 4\varphi_\eta^n + \varphi_\eta^{n-1}) + \nu \sum_{n=1}^{m-1} \Delta t |\varphi_\eta^{n+1}|_{H^1(\Omega)}^2 \\ & = \sum_{n=1}^{m-1} \Delta t \left\{ (-\varphi_\eta^{n+1} \cdot \nabla u_\eta^{n+1}, \varphi_\eta^{n+1}) - \frac{1}{2} (\operatorname{div} \varphi_\eta^{n+1}, u_\eta^{n+1} \cdot \varphi_\eta^{n+1}) \right\} + \sum_{n=1}^{m-1} \Delta t (u(t^{n+1}) \cdot \nabla (v_\eta^{n+1} - u(t^{n+1})), \varphi_\eta^{n+1}) \\ & \quad + \sum_{n=1}^{m-1} \Delta t \left\{ ((v_\eta^{n+1} - u(t^{n+1})) \cdot \nabla (v_\eta^{n+1} - u(t^{n+1})), \varphi_\eta^{n+1}) + \frac{1}{2} (\operatorname{div} (v_\eta^{n+1} - u(t^{n+1})), (v_\eta^{n+1} - u(t^{n+1})) \cdot \varphi_\eta^{n+1}) \right\} \\ & \quad + \sum_{n=1}^{m-1} \Delta t \left\{ ((v_\eta^{n+1} - u(t^{n+1})) \cdot \nabla u(t^{n+1}), \varphi_\eta^{n+1}) + \frac{1}{2} (\operatorname{div} (v_\eta^{n+1} - u(t^{n+1})), u(t^{n+1}) \cdot \varphi_\eta^{n+1}) \right\}. \end{aligned} \quad (4.28)$$

The left-hand side of (4.28) can be written as follows :

$$\begin{aligned} & \frac{1}{4} \| \varphi_\eta^m \|_{L^2(\Omega)}^2 - \frac{1}{4} \| \varphi_\eta^1 \|_{L^2(\Omega)}^2 + \frac{1}{4} \| 2\varphi_\eta^m - \varphi_\eta^{m-1} \|_{L^2(\Omega)}^2 - \frac{1}{4} \| 2\varphi_\eta^1 - \varphi_\eta^0 \|_{L^2(\Omega)}^2 \\ & \quad + \frac{1}{4} \sum_{n=1}^{m-1} \| \delta^2 \varphi_\eta^n \|_{L^2(\Omega)}^2 + \nu \sum_{n=1}^{m-1} \Delta t |\varphi_\eta^{n+1}|_{H^1(\Omega)}^2. \end{aligned}$$

We note  $(U_{RH})_i, i = 1, \dots, 4$ , the terms in the right-hand side of (4.28). For the first term, setting  $C_1 = \sup_n |u_\eta^n|_{H^1(\Omega)}$ , we can write

$$|(U_{RH})_1| \leq \frac{C_1}{2} \left\{ (\sqrt{2}\varepsilon_1 + \frac{2^{1/4} 3 S_4 \varepsilon_2^{4/3}}{8}) \sum_{n=1}^{m-1} \Delta t |\varphi_\eta^{n+1}|_{H^1(\Omega)}^2 + \left( \frac{\sqrt{2}}{\varepsilon_1} + \frac{2^{1/4} S_4}{8\varepsilon_2^4} \right) \sum_{n=1}^{m-1} \Delta t \| \varphi_\eta^{n+1} \|_{L^2(\Omega)}^2 \right\}.$$

Setting  $C_3 = \sup_n \|u(t^{n+1})\|_{L^\infty(\Omega)}$  and due to Corollary 4.6, the second term is bounded as follows :

$$|(U_{RH})_2| \leq \frac{CC_3}{2\varepsilon_3}(\eta^6 + (\Delta t)^4 + (\Delta t)^4\eta^2) + \frac{C_3\varepsilon_3}{2} \sum_{n=1}^{m-1} \Delta t |\varphi_\eta^{n+1}|_{H^1(\Omega)}^2.$$

For the third term, we use Lemma 4.1 and (4.26) and we obtain :

$$|(U_{RH})_3| \leq \frac{CS_4^2}{2\varepsilon_4}(\eta^8 + (\Delta t)^7 + (\Delta t)^3\eta^4) + \frac{3S_4^2\varepsilon_4}{4} \sum_{n=1}^{m-1} \Delta t |\varphi_\eta^{n+1}|_{H^1(\Omega)}^2.$$

In order to bound the last term, we use the well-known formula (3.12) :

$$\begin{aligned} & ((v_\eta^{n+1} - u(t^{n+1})) \cdot \nabla u(t^{n+1}), \varphi_\eta^{n+1}) + \frac{1}{2}(\operatorname{div}(v_\eta^{n+1} - u(t^{n+1})), u(t^{n+1}) \cdot \varphi_\eta^{n+1}) \\ &= \frac{1}{2}((v_\eta^{n+1} - u(t^{n+1})) \cdot \nabla u(t^{n+1}), \varphi_\eta^{n+1}) - \frac{1}{2}((v_\eta^{n+1} - u(t^{n+1})) \cdot \nabla \varphi_\eta^{n+1}, u(t^{n+1})), \end{aligned} \quad (4.29)$$

we set  $C_2 = \sup_{1 \leq n \leq N} |u(t^{n+1})|_{W^{1,4}(\Omega)}$  and we obtain :

$$|(U_{RH})_4| \leq \frac{C(C_2S_4 + C_3)}{4\varepsilon_5}(\eta^6 + (\Delta t)^4 + \eta^2(\Delta t)^4) + \frac{(S_4C_2 + C_3)\varepsilon_5}{4} \sum_{n=1}^{m-1} \Delta t |\varphi_\eta^{n+1}|_{H^1(\Omega)}^2.$$

Finally, we still have to estimate  $\varphi_\eta^1$  :

$$\|\varphi_\eta^1\|_{L^2(\Omega)}^2 + \nu \Delta t |\varphi_\eta^1|_{H^1(\Omega)}^2 = \Delta t |(u_\eta^1 \cdot \nabla u_\eta^1 - u(\Delta t) \cdot u(\Delta t), \varphi_\eta^1)|.$$

The non-linear term is splitted as the general one. The first part is bounded by :

$$\frac{C_1}{2} \left\{ (\sqrt{2}\varepsilon_6 + \frac{2^{1/4}3S_4\varepsilon_7^{4/3}}{8}) \Delta t |\varphi_\eta^1|_{H^1(\Omega)}^2 + (\frac{\sqrt{2}}{\varepsilon_6} + \frac{2^{1/4}S_4}{8\varepsilon_7^4}) \Delta t \|\varphi_\eta^1\|_{L^2(\Omega)}^2 \right\},$$

and if  $\Delta t$  is sufficiently small, these terms are absorbed by the left-hand side of (4.28). In the second part, we obtain :

$$\Delta t \|v_\eta^1 - u(t^1)\|_{L^2(\Omega)}^2 \leq C(\eta^6 + (\Delta t)^4),$$

and in the third one :

$$\begin{aligned} & \Delta t |v_\eta^1 - u(\Delta t)|_{H^1(\Omega)} \|v_\eta^1 - u(\Delta t)\|_{L^4(\Omega)} |\varphi_\eta^1|_{H^1(\Omega)} \\ & \leq \frac{1}{2}(\varepsilon_8 \Delta t |\varphi_\eta^1|_{H^1(\Omega)}^2 + \frac{1}{\varepsilon_8} C(\eta^8 + \eta^6(\Delta t)^{3/2} + \frac{\eta^9}{\sqrt{\Delta t}} + \eta^4(\Delta t)^{7/2})). \end{aligned}$$

In the last part, we obtain

$$\Delta t \|v_\eta^1 - u(t^1)\|_{L^2(\Omega)}^2 \leq C(\eta^6 + (\Delta t)^4).$$

Then (4.27) follows readily by applying these results.  $\square$

Combining Corollary 4.6 and Theorem 4.7, we obtain :

**Corollary 4.8.** *Under the assumptions of Theorem 4.7, there exists a constant  $C$  that does not depend on  $\eta$  and  $\Delta t$ , such that*

$$\left( \sum_{n=0}^N \Delta t \|u(t^n) - u_\eta^n\|_{L^2(\Omega)}^2 \right)^{1/2} \leq C(\eta^3 + (\Delta t)^2), \quad (4.30)$$

In particular, if (3.16) holds, then

$$\left( \sum_{n=1}^{N-1} \Delta t \|u(t^{n+1}) - u_\eta^{n+1}\|_{L^2(\Omega)}^2 \right)^{1/2} \leq C\eta^3. \quad (4.31)$$



## 5. AN ESTIMATE FOR THE PRESSURE

The results of the preceding section allow one to establish an error estimate for the pressure. We start with a general bound.

**Lemma 5.1.** *Under the assumptions of Lemma 3.4, let  $(u(t^{n+1}), p(t^{n+1}))$  and  $(u_\eta^{n+1}, p_\eta^{n+1})$  be the respective solution of (1.1)–(1.4) and (1.17)–(1.18). We have*

$$\begin{aligned} \left( \sum_{n=1}^{N-1} \Delta t \| p_\eta^{n+1} - r_\eta p(t^{n+1}) \|_{L^2(\Omega)}^2 \right)^{1/2} &\leq \frac{1}{\beta^*} \left\{ C_1(\eta^2 + (\Delta t)^2) + C_2(\Delta t)^2 \| u^{(3)} \|_{L^2(\Omega \times ]0, T])^2 \right. \\ &\quad \left. + C_3 \eta^2 \| p \|_{L^2(0, T; H^2(\Omega))} + S_2 \left( \sum_{n=1}^N \Delta t \| \delta^1(u_\eta^n - u(t^n)) \|_{L^2(\Omega)}^2 \right)^{1/2} \right\}, \end{aligned} \quad (5.1)$$

where  $\beta^*$  is the constant of the inf-sup condition (1.10) and the coefficients  $C_i, 1 \leq i \leq 3$ , are independent of  $\eta$  and  $\Delta t$ .

*Proof.* Let us subtract the non-linear terms and set  $e_\eta^i = u_\eta^i - u(t^i)$ . We obtain

$$u(t^{n+1}) \cdot \nabla u(t^{n+1}) - u_\eta^{n+1} \cdot \nabla u_\eta^{n+1} - \frac{1}{2} \operatorname{div} u_\eta^{n+1} u_\eta^{n+1} = -u(t^{n+1}) \cdot \nabla e_\eta^{n+1} - e_\eta^{n+1} \cdot \nabla u_\eta^{n+1} - \frac{1}{2} \operatorname{div} e_\eta^{n+1} u_\eta^{n+1}. \quad \blacksquare$$

Then, for all  $w_\eta^n \in X_\eta$  and due to (3.9), we have

$$\begin{aligned} \sum_{n=1}^{N-1} \Delta t (p_\eta^{n+1} - r_\eta p(t^{n+1}), \operatorname{div} w_\eta^{n+1}) &= \sum_{n=1}^{N-1} \Delta t \left( \frac{3e_\eta^{n+1} - 4e_\eta^n + e_\eta^{n-1}}{2\Delta t}, w_\eta^{n+1} \right) + \nu \sum_{n=1}^{N-1} \Delta t (\nabla e_\eta^{n+1}, \nabla w_\eta^{n+1}) \\ &+ \sum_{n=1}^{N-1} R_1 + \sum_{n=1}^{N-1} \Delta t (u(t^{n+1}) \cdot \nabla e_\eta^{n+1}, w_\eta^{n+1}) + \sum_{n=1}^{N-1} \Delta t \left\{ (e_\eta^{n+1} \cdot \nabla u_\eta^{n+1}, w_\eta^{n+1}) + \frac{1}{2} (\operatorname{div} e_\eta^{n+1} u_\eta^{n+1}, w_\eta^{n+1}) \right\} \\ &+ \sum_{n=1}^{N-1} \Delta t (p(t^{n+1}) - r_\eta p(t^{n+1}), \operatorname{div} w_\eta^{n+1}). \end{aligned} \quad (5.2) \quad \blacksquare$$

Owing to the inf-sup condition (1.10), there exists a function  $w_\eta \in V_\eta^\perp$  such that

$$(\operatorname{div} w_\eta, p_\eta^{n+1} - r_\eta p(t^{n+1})) = \| p_\eta^{n+1} - r_\eta p(t^{n+1}) \|_{L^2(\Omega)}^2 \quad \text{and} \quad |w_\eta|_{H^1(\Omega)} \leq \frac{1}{\beta^*} \| p_\eta^{n+1} - r_\eta p(t^{n+1}) \|_{L^2(\Omega)}.$$

Let  $(P_{RH})_i, i = 1, \dots, 6$ , denote the terms of the right-hand side of (5.2).

We deduce by standard arguments:

$$\begin{aligned} |(P_{RH})_1| &\leq S_2 \left( \sum_{n=1}^{N-1} \Delta t \| \delta^1 e_\eta^n \|_{L^2(\Omega)}^2 \right)^{1/2} \left( \sum_{n=1}^{N-1} \Delta t |w_\eta^{n+1}|_{H^1(\Omega)}^2 \right)^{1/2}, \\ |(P_{RH})_2| &\leq \nu \left( \sum_{n=1}^{N-1} \Delta t |P_\eta u(t^{n+1}) - u(t^{n+1})|_{H^1(\Omega)}^2 \right)^{1/2} \left( \sum_{n=1}^{N-1} \Delta t |w_\eta^{n+1}|_{H^1(\Omega)}^2 \right)^{1/2} \\ &\leq C_1(\eta^2 + (\Delta t)^2) \left( \sum_{n=1}^{N-1} \Delta t |w_\eta^{n+1}|_{H^1(\Omega)}^2 \right)^{1/2}, \\ |(P_{RH})_3| &\leq C_2(\Delta t)^2 \| u^{(3)} \|_{L^2(\Omega \times ]0, T])^2 \left( \sum_{n=1}^{N-1} \Delta t |w_\eta^{n+1}|_{H^1(\Omega)}^2 \right)^{1/2}. \end{aligned}$$

The fourth and fifth terms  $(P_{RH})_4, (P_{RH})_5$  are bounded as follows :

$$\begin{aligned}
 |(P_{RH})_4| &\leq S_4^2 \left( \sup_t |u(t)|_{H^1(\Omega)} \left( \sum_{n=1}^{N-1} \Delta t |e_\eta^{n+1}|_{H^1(\Omega)}^2 \right)^{1/2} \left( \sum_{n=1}^{N-1} \Delta t |w_\eta^{n+1}|_{H^1(\Omega)}^2 \right)^{1/2} \right. \\
 &\quad \left. \leq C_3 (\eta^2 + (\Delta t)^2) \left( \sum_{n=1}^{N-1} \Delta t |w_\eta^{n+1}|_{H^1(\Omega)}^2 \right)^{1/2} . \right. \\
 |(P_{RH})_5| &= \frac{1}{2} \left| \sum_{n=0}^{N-1} \Delta t \left\{ (e_\eta^{n+1} \cdot \nabla u_\eta^{n+1}, w_\eta^{n+1}) - (e_\eta^{n+1} \cdot \nabla w_\eta^{n+1}, u_\eta^{n+1}) \right\} \right| \\
 &\leq S_4^2 \left( \sup |u_\eta^{n+1}|_{H^1(\Omega)} \left( \sum_{n=1}^{N-1} \Delta t |e_\eta^{n+1}|_{H^1(\Omega)}^2 \right)^{1/2} \left( \sum_{n=1}^{N-1} \Delta t |w_\eta^{n+1}|_{H^1(\Omega)}^2 \right)^{1/2} \right. \\
 &\quad \left. \leq C_4 (\eta^2 + (\Delta t)^2) \left( \sum_{n=1}^{N-1} \Delta t |w_\eta^{n+1}|_{H^1(\Omega)}^2 \right)^{1/2} , \right.
 \end{aligned}$$

and the last term is bounded as follows :

$$|(P_{RH})_6| \leq C_5 \eta^2 \|p\|_{L^2(0,T;H^2(\Omega))} \left( \sum_{n=1}^{N-1} \Delta t |w_\eta^{n+1}|_{H^1(\Omega)}^2 \right)^{1/2} .$$

Then (5.1) follows easily by substituting these inequalities into (5.2).  $\square$

We have to estimate  $\left( \sum_{n=1}^{N-1} \Delta t \|\delta^1(u_\eta^n - u(t^n))\|_{L^2(\Omega)}^2 \right)^{1/2}$ . This estimate is proven assuming the triangulation satisfies a milder regularity property than uniform regularity (1.9): there exists a constant  $\tilde{\tau}$  that does not depend on  $\eta$  or  $\Delta t$  such that

$$\rho_{\min} \geq \tilde{\tau} \eta^5, \quad \text{where } \rho_{\min} = \inf_{\kappa \in \mathcal{T}_\eta} \rho_\kappa. \quad (5.3)$$

More precisely, this assumption is used in proving that  $u_\eta^n$  is bounded in  $L^\infty(0, T; W^{1,5/2}(\Omega)^2)$  :

**Lemma 5.2.** *Under the assumptions of Theorem 4.7 and if  $\mathcal{T}_\eta$  satisfies (5.3), there exists a constant  $C$  that depends neither on  $\eta$  nor on  $\Delta t$ , such that*

$$\sup_n |u_\eta^n|_{W^{1,5/2}(\Omega)} \leq C. \quad (5.4)$$

*Proof.* We refer to [2] for the sketch of this proof.  $\square$

**Lemma 5.3.** *Under the assumptions of Theorem 4.7, there exists a constant  $C = C(u, u', u^{(3)})$  that does not depend on  $\eta$  and  $\Delta t$ , such that*

$$\begin{aligned}
 &\left( \sum_{n=1}^{N-1} \Delta t \|\delta^1(u_\eta^n - u(t^n))\|_{L^2(\Omega)}^2 \right)^{1/2} + \sqrt{\bar{\nu}} \sup_{1 \leq n \leq N} |u_\eta^n - u(t^n)|_{H^1(\Omega)} \\
 &+ \sqrt{\bar{\nu}} \sup_{1 \leq n \leq N} |2(u_\eta^n - u(t^n)) - (u_\eta^{n-1} - u(t^{n-1}))|_{H^1(\Omega)} + \sqrt{\bar{\nu}} \left( \sum_{n=1}^{N-1} |\delta^2(u_\eta^n - u(t^n))|_{H^1(\Omega)}^2 \right)^{1/2} \\
 &\leq C(\eta^2 + (\Delta t)^{3/2} + \frac{\eta^3}{\sqrt{\Delta t}}). \quad (5.5)
 \end{aligned}$$

*Proof.* The proof is similar to that of Lemma 5.1. By taking  $e_\eta^i = u_\eta^i - S_\eta u(t^i)$ ,  $\varphi_\eta^i = u(t^i) - S_\eta u(t^i)$  and the test function  $w_\eta = w_\eta^{n+1} = \delta^1 e_\eta^n$  :

$$\begin{aligned} & \sum_{n=1}^{m-1} \Delta t \|\delta^1 e_\eta^n\|_{L^2(\Omega)}^2 + \frac{\nu}{2} (|e_\eta^m|_{H^1(\Omega)}^2 - |e_\eta^1|_{H^1(\Omega)}^2 + |2e_\eta^m - e_\eta^{m-1}|_{H^1(\Omega)}^2 - |2e_\eta^1 - e_\eta^0|_{H^1(\Omega)}^2 + \sum_{n=1}^{m-1} |\delta^2 e_\eta^n|_{H^1(\Omega)}^2) \\ & \leq \nu \left| \sum_{n=1}^{m-1} \Delta t (\nabla \varphi_\eta^{n+1}, \nabla \delta^1 e_\eta^n) \right| + \left| \sum_{n=1}^{m-1} \Delta t (p(t^{n+1}), \operatorname{div} \delta^1 e_\eta^n) \right| + \sum_{n=1}^{m-1} \Delta t (\delta^1 \varphi_\eta^n, \delta^1 e_\eta^n) + \sum_{n=1}^{m-1} R_1 \\ & + \sum_{n=1}^{m-1} \Delta t \left\{ (u(t^{n+1}) \cdot \nabla u(t^{n+1}) - u_\eta^{n+1} \cdot \nabla u_\eta^{n+1}, \delta^1 e_\eta^n) - \frac{1}{2} (\operatorname{div} u_\eta^{n+1} u_\eta^{n+1}, \delta^1 e_\eta^n) \right\}, \end{aligned} \quad (5.6)$$

with

$$\begin{aligned} & u(t^{n+1}) \cdot \nabla u(t^{n+1}) - u_\eta^{n+1} \cdot \nabla u_\eta^{n+1} - \frac{1}{2} \operatorname{div} u_\eta^{n+1} u_\eta^{n+1} \\ & = u(t^{n+1}) \cdot \nabla (u(t^{n+1}) - u_\eta^{n+1}) + (u(t^{n+1}) - u_\eta^{n+1}) \cdot \nabla u_\eta^{n+1} - \frac{1}{2} \operatorname{div} (u(t^{n+1}) - u_\eta^{n+1}) u_\eta^{n+1}. \end{aligned}$$

Due to the definition of the operator  $S_\eta$ , we only have to estimate the three last terms  $(V_{RH})_i, i = 1, \dots, 3$ , in the right-hand side of (5.6).

The first one is bounded as precedently as follows :

$$\begin{aligned} |(V_{RH})_1| & = \left| \sum_{n=1}^m \Delta t (\delta^1 \varphi_\eta^n, \delta^1 e_\eta^n) \right| \leq \frac{C}{2\varepsilon_1} \left\{ \eta^4 (\|u'\|_{L^\infty(0,T;H^2(\Omega)^2)}^2 + \|p'\|_{L^\infty(0,T;H^1(\Omega))}^2) \right. \\ & \quad \left. + (\Delta t)^2 \eta^2 (\|u''\|_{L^2(0,T;H^1(\Omega)^2)}^2 + \|p''\|_{L^2(\Omega \times ]0,T[)}) \right\} + \frac{\varepsilon_1}{2} \sum_{n=1}^{m-1} \Delta t \|\delta^1 e_\eta^n\|_{L^2(\Omega)}^2. \end{aligned}$$

The second term is bounded as follows:

$$|(V_{RH})_2| = \left| \sum_{n=1}^{m-1} R_1 \right| \leq \frac{C(\Delta t)^4}{2\varepsilon_2} \|u^{(3)}\|_{L^2(\Omega \times ]0,T[)}^2 + \frac{\varepsilon_2}{2} \sum_{n=1}^{m-1} \Delta t \|\delta^1 e_\eta^n\|_{L^2(\Omega)}^2.$$

For the last term, it is splitted into two parts that we treat successively. The first part is treated as follows :

$$\begin{aligned} & \left| \sum_{n=1}^{m-1} \Delta t (u(t^{n+1}) \cdot \nabla (u(t^{n+1}) - u_\eta^{n+1}), \delta^1 e_\eta^n) \right| \\ & \leq \frac{C \|u\|_{L^\infty(\Omega \times ]0,T[)}}{2} \left( \frac{C'}{\varepsilon_3} (\eta^4 + (\Delta t)^4) + \varepsilon_3 \sum_{n=1}^{m-1} \Delta t \|\delta^1 e_\eta^n\|_{L^2(\Omega)}^2 \right), \end{aligned}$$

and for the second part, we notice that :

$$\begin{aligned} \|(u(t^{n+1}) - u_\eta^{n+1}) \cdot \nabla u_\eta^{n+1}\|_{L^2(\Omega)} & \leq |u_\eta^{n+1}|_{W^{1,5/2}(\Omega)} \|u(t^{n+1}) - u_\eta^{n+1}\|_{L^{10}(\Omega)} \\ & \leq S_{10} |u_\eta^{n+1}|_{W^{1,5/2}(\Omega)} \|u(t^{n+1}) - u_\eta^{n+1}\|_{H^1(\Omega)}, \end{aligned}$$

then it is bounded as follows :

$$\begin{aligned} & \left| \sum_{n=1}^{m-1} \Delta t \left( (u(t^{n+1}) - u_\eta^{n+1}) \cdot \nabla u_\eta^{n+1}, \delta^1 e_\eta^n \right) + \frac{1}{2} \left( \operatorname{div} (u(t^{n+1}) - u_\eta^{n+1}) u_\eta^{n+1}, \delta^1 e_\eta^n \right) \right| \\ & \leq \left( \frac{C}{2} + S_{10} \right) \sup_n |u_\eta^n|_{W^{1,5/2}(\Omega)} \sum_{n=1}^{m-1} \Delta t \|u(t^{n+1}) - u_\eta^{n+1}\|_{H^1(\Omega)} \|\delta^1 e_\eta^n\|_{L^2(\Omega)} \\ & \leq C'' \left( \frac{C'}{\varepsilon_4} (\eta^4 + (\Delta t)^4) + \varepsilon_4 \sum_{n=1}^{m-1} \Delta t \|\delta^1 e_\eta^n\|_{L^2(\Omega)}^2 \right). \end{aligned}$$

Then by setting  $C_1 = \|u\|_{L^\infty(\Omega \times ]0, T])^2}$ , the last term of the right-hand side of (5.6) is bounded by :

$$\left(\frac{C_1 C'}{2\varepsilon_3} + \frac{C'' C'}{\varepsilon_4}\right)(\eta^4 + (\Delta t)^4) + \left(\frac{C_1 \varepsilon_3}{2} + C'' \varepsilon_4\right) \sum_{n=1}^{m-1} \Delta t \|\delta^1 e_\eta^n\|_{L^2(\Omega)}^2.$$

Finally the initial datas are bounded due to Proposition 3.3. Then, choosing suitably the parameters  $\varepsilon_i$ , the equation (5.6) becomes

$$\begin{aligned} & \left(\sum_{n=1}^{m-1} \Delta t \|\delta^1(u_\eta^n - S_\eta u(t^n))\|_{L^2(\Omega)}^2\right)^{1/2} + \sqrt{\nu} \sup_{1 \leq n \leq N} |u_\eta^n - S_\eta u(t^n)|_{H^1(\Omega)} \\ & + \sqrt{\nu} \sup_{1 \leq n \leq N} |2(u_\eta^n - S_\eta u(t^n)) - (u_\eta^{n-1} - S_\eta u(t^{n-1}))|_{H^1(\Omega)} + \sqrt{\nu} \left(\sum_{n=1}^{N-1} |\delta^2(u_\eta^n - S_\eta u(t^n))|_{H^1(\Omega)}^2\right)^{1/2} \\ & \leq C(\eta^2 + (\Delta t)^{3/2} + \frac{\eta^3}{\sqrt{\Delta t}}). \end{aligned}$$

Finally (5.5) follows readily from this result and by applying a triangular inequality and  $S_\eta$ 's properties.  $\square$

From these three lemmas, we easily derive an estimate of the pressure.

**Theorem 5.4.** *Under the assumptions of Lemma 5.1, there exists a constant  $C$  that does not depend on  $\eta$  nor on  $\Delta t$ , such that*

$$\left(\sum_{n=1}^N \Delta t \|p(t^n) - p_\eta^n\|_{L^2(\Omega)}^2\right)^{1/2} \leq C(\eta^2 + (\Delta t)^{3/2} + \frac{\eta^3}{\sqrt{\Delta t}}). \quad (5.7)$$

## 6. ERROR ESTIMATE FOR THE SOLUTION OF STEP TWO

We assume at this stage that we know the solution  $u_H^{n+1}$  of the first step. Then at each time step, the second step (1.19)–(1.20) is a square system of linear equations in finite dimension, and if  $\Delta t$  is small enough, it has a unique solution. First, we will establish the error estimate for the solution computed by one step of Euler's scheme  $(u_h^1 - u(\Delta t), p_h^1 - p(\Delta t))$  :

**Proposition 6.1.** *The error of the solution computed by one iteration of Euler's scheme satisfies the following estimations, for  $\Delta t \leq k_0 > 0$  sufficiently small,*

$$\frac{1}{2} \|u_h^1 - u(\Delta t)\|_{L^2(\Omega)}^2 + \frac{\nu \Delta t}{2} |u_h^1 - u(\Delta t)|_{H^1(\Omega)}^2 \leq C(H^6 + h^4 + (\Delta t)^4), \quad (6.1)$$

and

$$(\Delta t)^{1/2} \|p(\Delta t) - p_h^1\|_{L^2(\Omega)} \leq C(h^2 + H^3 + (\Delta t)^{3/2}). \quad (6.2)$$

*Proof.* The error's equation is similar to (3.2).

$$\begin{aligned} \forall v_h \in V_h, (u_h^1 - u(\Delta t), v_h) + \nu \Delta t (\nabla(u_h^1 - u(\Delta t)), \nabla v_h) &= \frac{(\Delta t)^2}{2} (u''(\theta \Delta t), v_h) \\ &\quad - \Delta t (p(\Delta t) - r_h p(\Delta t), \operatorname{div} v_h) + \Delta t (u(\Delta t) \cdot \nabla u(\Delta t) - u_H^1 \cdot \nabla u_h^1, v_h). \end{aligned} \quad (6.3)$$

By setting  $v_h = u_h^1 - P_h u(\Delta t) - u_H^1$  and  $\varphi_h^1 = P_h u(\Delta t) - u(\Delta t)$ , the non-linear term can be written as follows :

$$\begin{aligned} (u(\Delta t) \cdot \nabla u(\Delta t) - u_H^1 \cdot \nabla u_h^1, v_h) &= ((u(\Delta t) - u_H^1) \cdot \nabla u(\Delta t), v_h) + (u_H^1 \cdot \nabla (u(\Delta t) - P_h u(\Delta t)), v_h) \\ &\quad + ((u(\Delta t) - u_H^1) \cdot \nabla (u_h^1 - P_h u(\Delta t)), v_h) + (u(\Delta t) \cdot \nabla (P_h u(\Delta t) - u_h^1), v_h) \\ &= ((u(\Delta t) - u_H^1) \cdot \nabla u(\Delta t), v_h) - (u_H^1 \cdot \nabla \varphi_h^1, v_h) - ((u(\Delta t) - u_H^1) \cdot \nabla v_h^1, v_h). \end{aligned}$$

Then, we have three contributions of the non-linear term. For the first part, we write :

$$\begin{aligned} \Delta t \left| ((u(\Delta t) - u_H^1) \cdot \nabla u(\Delta t), v_h^1) \right| &\leq S_4 |v_h^1|_{H^1(\Omega)} \sup_t |u(\Delta t)|_{W^{1,4}(\Omega)} \Delta t \| u(\Delta t) - u_H^{n+1} \|_{L^2(\Omega)} \\ &\leq \frac{1}{2} \left( \varepsilon_1 \Delta t |v_h^1|_{H^1(\Omega)}^2 + \frac{1}{\varepsilon_1} C^2 \Delta t (H^6 + (\Delta t)^4 + (\Delta t) H^4) \right). \end{aligned}$$

For the second part, we know that  $\| u_H^1 \|_{L^4(\Omega)}$  is bounded and we write :

$$\begin{aligned} \Delta t \left| (u_H^1 \cdot \nabla \varphi_h^1, v_h^1) \right| &\leq S_4 \Delta t |v_h^1|_{H^1(\Omega)} \| u_H^1 \|_{L^4(\Omega)} |u(\Delta t) - P_H u(\Delta t)|_{H^1(\Omega)} \\ &\leq \frac{1}{2} \left( \varepsilon_2 \Delta t |v_h^1|_{H^1(\Omega)}^2 + \frac{C}{\varepsilon_2} (\Delta t) h^4 \right). \end{aligned}$$

Finally, the last term can be written as :

$$\Delta t \left| ((u(\Delta t) - u_H^1) \cdot \nabla v_h^1, v_h^1) \right| \leq \Delta t \widehat{C} H^{1-\varepsilon} |v_h^1|_{H^1(\Omega)}^2 |u_H^1 - u(\Delta t)|_{H^1(\Omega)},$$

with

$$|u_H^1 - u(\Delta t)|_{H^1(\Omega)} \leq C((\Delta t)^{3/2} + H^2 + \frac{H^3}{\sqrt{\Delta t}}).$$

In that case, for  $H$  (and  $\Delta t$ ) sufficiently smooth, this term is absorbed by the left-hand side of the equation. And for the linear terms, we introduce  $P_h u(\Delta t)$  in (6.3) and we obtain:

$$\begin{aligned} \| v_h^1 \|_{L^2(\Omega)}^2 + \nu \Delta t |v_h^1|_{H^1(\Omega)}^2 &\leq \left| (\varphi_h^1, v_h^1) \right| + \nu \Delta t \left| (\nabla \varphi_h^1, \nabla v_h^1) \right| + \frac{(\Delta t)^2}{2} \sup \| u'' \|_{L^2(\Omega)} \| v_h^1 \|_{L^2(\Omega)} \\ &\quad + \Delta t \| p(\Delta t) - r_h p(\Delta t) \|_{L^2(\Omega)} |v_h^1|_{H^1(\Omega)} + \text{non-linear term}. \end{aligned} \quad (6.4)$$

For the pressure, we obtain :

$$\begin{aligned} &\Delta t (r_h p(\Delta t) - p(\Delta t), \operatorname{div} v_h) + \Delta t (p_h^1 - r_h p(\Delta t), \operatorname{div} v_h) \\ &= (u_H^1 - u(\Delta t), v_h) + \nu \Delta t (\nabla (u_H^1 - u(\Delta t)), \nabla v_h) - \frac{(\Delta t)^2}{2} (u''(\theta \Delta t), v_h) - \Delta t (u(\Delta t) \cdot \nabla u(\Delta t) - u_H^1 \cdot \nabla u_H^1, v_h). \end{aligned} \quad (6.5)$$

We choose  $v_h \in V_h^\perp$  such that

$$(p_h^1 - r_h p(\Delta t), \operatorname{div} v_h) = \| p_h^1 - r_h p(\Delta t) \|_{L^2(\Omega)}^2 \quad \text{and} \quad |v_h|_{H^1(\Omega)} \leq \frac{1}{\beta^*} \| p_h^1 - r_h p(\Delta t) \|_{L^2(\Omega)},$$

with  $\beta^* > 0$  that does not depend on  $h$ . Thus

$$\begin{aligned} (\Delta t)^{1/2} \| p_h^1 - r_h p(\Delta t) \|_{L^2(\Omega)} &\leq \frac{(\Delta t)^{1/2}}{\beta^*} \left( \| r_h p(\Delta t) - p(\Delta t) \|_{L^2(\Omega)} + \frac{S_2}{\Delta t} \| u_H^1 - u(\Delta t) \|_{L^2(\Omega)} \right) \\ &\quad + \nu |P_h u(\Delta t) - u(\Delta t)|_{H^1(\Omega)} + \frac{S_2}{2} (\Delta t) \| u''(\theta \Delta t) \|_{L^2(\Omega)} + S_4^2 (|u(\Delta t)|_{H^1(\Omega)} + |u_H^1|_{H^1(\Omega)}) |u_H^1 - u(\Delta t)|_{H^1(\Omega)} \\ &\leq C(h^2 + H^3 + (\Delta t)^{3/2}). \end{aligned}$$

□

The fine velocity satisfies the following error estimate:

**Theorem 6.2.** *Under the hypotheses of Theorem 4.7, the solution of Step 2,  $(u_h^{n+1}, p_h^{n+1})$ , satisfies the following error estimate*

$$\begin{aligned} &\sup_{1 \leq n \leq N} \| u_h^n - u(t^n) \|_{L^2(\Omega)} + \sup_{1 \leq n \leq N} \| 2(u_h^n - u(t^n)) - (u_h^{n-1} - u(t^{n-1})) \|_{L^2(\Omega)} \\ &+ \left( \sum_{n=1}^{N-1} \| \delta^2(u_h^n - u(t^n)) \|_{L^2(\Omega)}^2 \right)^{1/2} + \sqrt{\nu} \left( \sum_{n=1}^N \Delta t |u_h^n - u(t^n)|_{H^1(\Omega)}^2 \right)^{1/2} \\ &\leq C(H^3 + h^2 + (\Delta t)^2 + H(\Delta t)^2), \end{aligned} \quad (6.6)$$

with a constant  $C$  that does not depend on  $h, H$  and  $\Delta t$ .

*Proof.* By subtracting the equations (1.19) and (1.17), by setting  $v_h^i = P_h u(t^i) - u_h^i$ ,  $\varphi_h^i = P_h u(t^i) - u(t^i)$ , by taking the test function  $v_h = v_h^{n+1}$  and by summing the result from  $n = 1$  to  $n = m - 1$ , we obtain

$$\begin{aligned} & \nu \sum_{n=1}^{m-1} \Delta t |v_h^{n+1}|_{H^1(\Omega)}^2 + \frac{1}{4} \left( \|v_h^m\|_{L^2(\Omega)}^2 - \|v_h^1\|_{L^2(\Omega)}^2 + \|2v_h^m - v_h^{m-1}\|_{L^2(\Omega)}^2 - \|2v_h^1 - v_h^0\|_{L^2(\Omega)}^2 \right. \\ & \left. + \sum_{n=1}^{m-1} \|\delta^2 v_h^n\|_{L^2(\Omega)}^2 \right) \leq \left| \sum_{n=1}^{m-1} R_1 \right| + \left| \nu \sum_{n=1}^{m-1} \Delta t (\nabla \varphi_h^{n+1}, \nabla v_h^{n+1}) \right| + \left| \sum_{n=1}^{m-1} \Delta t (p(t^{n+1}) - r_h p(t^{n+1}), \operatorname{div} v_h^{n+1}) \right| \\ & \quad + \left| \sum_{n=1}^{m-1} \Delta t (\delta^1 \varphi_h^n, v_h^{n+1}) \right| + \left| \sum_{n=1}^m \Delta t (u_H^{n+1} \cdot \nabla u_h^{n+1} - u(t^{n+1}) \cdot \nabla u(t^{n+1}), v_h^{n+1}) \right|. \end{aligned} \quad (6.7)$$

Let us estimate the terms  $(TG_{RH})_i, i = 1, \dots, 4$  in the right-hand side of (6.7). The first term is bounded as follows :

$$|(TG_{RH})_1| \leq \frac{C(\Delta t)^4}{2\varepsilon_1} \|u^{(3)}\|_{L^2(\Omega \times ]0, T])}^2 + \frac{\varepsilon_1}{2} \sum_{n=1}^{m-1} \Delta t \|v_h^{n+1}\|_{L^2(\Omega)}^2.$$

The second term and third terms are bounded respectively as follows :

$$|(TG_{RH})_2| \leq \frac{C\nu h^4}{2\varepsilon_2} \|u\|_{L^2(0, T; H^3(\Omega)^2)}^2 + \frac{\nu \varepsilon_2}{2} \sum_{n=1}^{m-1} \Delta t |v_h^{n+1}|_{H^1(\Omega)}^2.$$

and

$$|(TG_{RH})_3| \leq \frac{Ch^4}{2\varepsilon_3} \|p\|_{L^2(0, T; H^2(\Omega))}^2 + \frac{\varepsilon_3}{2} \sum_{n=1}^{m-1} \Delta t |v_h^{n+1}|_{H^1(\Omega)}^2,$$

and the fourth term is as follows :

$$\left| \sum_{n=1}^{m-1} \Delta t (\delta^1 \varphi_h^n, v_h^{n+1}) \right| \leq \frac{C(\Delta t)^4}{2\varepsilon_4} \|u^{(3)}\|_{L^2(\Omega \times ]0, T])}^2 + \frac{Ch^4}{2\varepsilon_4} \|u'\|_{L^\infty(0, T; H^2(\Omega)^2)}^2 + \frac{\varepsilon_4}{2} \sum_{n=1}^{m-1} \Delta t \|v_h^{n+1}\|_{L^2(\Omega)}^2.$$

The non-linear term in the right-hand side can be written as follows:

$$\begin{aligned} u_H^{n+1} \cdot \nabla u_h^{n+1} - u(t^{n+1}) \cdot \nabla u(t^{n+1}) &= (u_H^{n+1} - u(t^{n+1})) \cdot \nabla u(t^{n+1}) + u_H^{n+1} \cdot \nabla (P_h u(t^{n+1}) - u(t^{n+1})) \\ &\quad - (u(t^{n+1}) - u_H^{n+1}) \cdot \nabla (u_h^{n+1} - P_h u(t^{n+1})) - u(t^{n+1}) \cdot \nabla (P_h u(t^{n+1}) - u_h^{n+1}). \end{aligned}$$

We study the four parts  $(NL)_i, i = 1, \dots, 4$ , of the non-linear term separately. Setting  $C_{\infty,1} = \sup |u|_{W^{1,4}(\Omega)}$ , the first part is treated as follows :

$$\left| \sum_{n=0}^{m-1} \Delta t ((NL)_1, v_h^{n+1}) \right| \leq \frac{C_{\infty,1}}{2\varepsilon_{5,1}} C(H^6 + (\Delta t)^4) + \frac{S_4^2 C_{\infty,1} \varepsilon_{5,1}}{2} \sum_{n=1}^{m-1} \Delta t |v_h^{n+1}|_{H^1(\Omega)}^2.$$

Setting  $C_{\infty,2} = \sup \|u_H^{n+1}\|_{L^4(\Omega)}$ , the second part is treated as follows :

$$\left| \sum_{n=0}^{m-1} \Delta t ((NL)_2, v_h^{n+1}) \right| \leq \frac{CC_{\infty,2} h^4}{2\varepsilon_{5,2}} \|u\|_{L^2(0, T; H^3(\Omega)^2)}^2 + \frac{S_4^2 C_{\infty,2} \varepsilon_{5,2}}{2} \sum_{n=1}^{m-1} \Delta t |v_h^{n+1}|_{H^1(\Omega)}^2.$$

For the third part, we use the following estimation (cf. [7]): there exists a constant  $\widehat{C}$ , that does not depend on  $\eta$  such that, for all  $u_\eta \in V_\eta$ ,

$$\forall w_\eta \in X_\eta, |(u_\eta \cdot \nabla w_\eta, w_\eta)| \leq \widehat{C} \eta^{1-\varepsilon} \|\operatorname{div} u_\eta\|_{L^2(\Omega)} |w_\eta|_{H^1(\Omega)}^2, \quad (6.8)$$

we have

$$\left| \sum_{n=0}^{m-1} \Delta t ((NL)_3, v_h^{n+1}) \right| \leq \widehat{C} H^{1-\varepsilon} (H^2 + (\Delta t)^{3/2} + \frac{H^3}{\sqrt{\Delta t}}) \sum_{n=1}^{m-1} \Delta t |v_h^{n+1}|_{H^1(\Omega)}^2,$$

And the last part is bounded as follows :

$$\left| \sum_{n=0}^{m-1} \Delta t ((NL)_4, v_h^{n+1}) \right| = 0.$$

Then, collecting these inequalities, choosing suitably the parameters  $\varepsilon_i$  and  $\delta$  and applying Gronwall's Lemma, we get

$$\begin{aligned} & \sup_{1 \leq n \leq N} \|u_h^n - P_h u(t^n)\|_{L^2(\Omega)} + \sup_{1 \leq n \leq N} \|2(u_h^n - P_h u(t^n)) - (u_h^{n-1} - P_h u(t^{n-1}))\|_{L^2(\Omega)} \\ & + \left( \sum_{n=1}^{N-1} \|\delta^2(u_h^n - P_h u(t^n))\|_{L^2(\Omega)}^2 \right)^{1/2} + \sqrt{\nu} \left( \sum_{n=1}^N \Delta t |u_h^n - P_h u(t^n)|_{H^1(\Omega)}^2 \right)^{1/2} \\ & \leq C(H^3 + h^2 + (\Delta t)^2). \end{aligned}$$

Then, (6.6) follows readily from the above result and the  $P_h$ 's properties.  $\square$

Finally, we consider the error of the pressure. As in Section 5, the pressure satisfies the following bound.

**Lemma 6.3.** *Let  $(u(t^{n+1}), p(t^{n+1}))$  and  $(u_h^{n+1}, p_h^{n+1})$  be the respective solution of (1.1)–(1.4) and (1.19)–(1.20). We have*

$$\begin{aligned} \left( \sum_{n=1}^{N-1} \Delta t \|p_h^{n+1} - r_h p(t^{n+1})\|_{L^2(\Omega)}^2 \right)^{1/2} & \leq \frac{1}{\beta^*} \left\{ C_1 h^2 \|p\|_{L^2(0,T;H^2(\Omega))} + C_2 (\Delta t)^2 \|u^{(3)}\|_{L^2(\Omega \times ]0,T])^2} \right. \\ & \left. + C_3 (H^3 + (\Delta t)^2) + C_4 h^2 + S_2 \left( \sum_{n=1}^{N-1} \Delta t \|\delta^1(u_h^n - u(t^n))\|_{L^2(\Omega)}^2 \right)^{1/2} \right\}, \end{aligned} \quad (6.9)$$

where  $\beta^*$  is the constant of the inf-sup condition (1.10) and the coefficients  $C_i, i = 1, \dots, 4$ , do not depend on  $H, h$  and  $\Delta t$ .

*Proof.* The steps of this proof are similar to those of the proof of Lemma 5.1 and the only difference between these proofs concerns the non-linear term. Here we write

$$\begin{aligned} u(t^{n+1}) \cdot \nabla u(t^{n+1}) - u_H^{n+1} \cdot \nabla u_H^{n+1} & = (u(t^{n+1}) - u_H^{n+1}) \cdot \nabla u(t^{n+1}) + (u_H^{n+1} - u(t^{n+1})) \cdot \nabla (u(t^{n+1}) - u_H^{n+1}) \\ & \quad + u(t^{n+1}) \cdot \nabla (u(t^{n+1}) - u_H^{n+1}). \end{aligned}$$

Then, let us estimate the terms that compose the non-linear term.

$$\begin{aligned} & \left| \sum_{n=1}^{N-1} \Delta t (u(t^{n+1}) \cdot \nabla u(t^{n+1}) - u_H^{n+1} \cdot \nabla u_H^{n+1}, w_h^{n+1}) \right| \\ & \leq S_4 \left( \sum_{n=1}^{N-1} \Delta t |w_h^{n+1}|_{H^1(\Omega)}^2 \right)^{1/2} \left\{ \left( \sup_n |u(t^n)|_{W^{1,4}(\Omega)} \right) \left( \sum_{n=1}^{N-1} \Delta t \|u(t^{n+1}) - u_H^{n+1}\|_{L^2(\Omega)}^2 \right)^{1/2} \right. \\ & \quad \left. + S_4 \left( \sup_n |u(t^n) - u_H^n|_{H^1(\Omega)} + \sup_n |u(t^n)|_{H^1(\Omega)} \right) \left( \sum_{n=1}^{N-1} \Delta t |u(t^{n+1}) - u_H^{n+1}|_{H^1(\Omega)}^2 \right)^{1/2} \right\} \\ & \leq \left( C(H^3 + (\Delta t)^2) \right) \left( \sum_{n=1}^N \Delta t |w_h^{n+1}|_{H^1(\Omega)}^2 \right)^{1/2}. \end{aligned}$$

Then, (6.9) follows readily from these bounds and from the inf-sup condition (1.10).  $\square$

Therefore, here again, we must derive an estimate for  $\left( \sum_{n=1}^{N-1} \Delta t \|\delta^1(u_h^n - u(t^n))\|_{L^2(\Omega)}^2 \right)^{1/2}$ .

**Lemma 6.4.** *Under the assumptions of Theorem 4.7 and Corollary 3.4, there exists a constant  $C$  that does not depend on  $H, h$  and  $\Delta t$  such that :*

$$\begin{aligned} & \left( \sum_{n=1}^{N-1} \Delta t \|\delta^1(u_h^n - u(t^n))\|_{L^2(\Omega)}^2 \right)^{1/2} + \sqrt{\nu} \sup_{1 \leq n \leq N} |u_h^n - u(t^n)|_{H^1(\Omega)} \\ & + \sqrt{\nu} \sup_{1 \leq n \leq N} |2(u_h^n - u(t^n)) - (u_h^{n-1} - u(t^{n-1}))|_{H^1(\Omega)} + \sqrt{\nu} \left( \sum_{n=1}^{N-1} |\delta^2(u_h^n - u(t^n))|_{H^1(\Omega)}^2 \right)^{1/2} \\ & \leq C(h^2 + H^3 + (\Delta t)^2). \end{aligned} \quad (6.10)$$

*Proof.* We subtract the equations (1.17) and (1.19), we set  $e_h^i = u_h^i - S_h u(t^i)$  and  $\varphi_h^i = u(t^i) - S_h u(t^i)$  and we take the function test  $w_h = \delta^1 e_h^n$ . Due to the definition of the Stokes operator  $S_h$ , we have

$$\begin{aligned} \sum_{n=1}^{m-1} \Delta t \|\delta^1 e_h^n\|_{L^2(\Omega)}^2 + \frac{\nu}{2} \sum_{n=1}^{m-1} \Delta t (\nabla e_h^{n+1}, \nabla \delta^1 e_h^n) &= \sum_{n=1}^{m-1} \Delta t (\delta^1 \varphi_h^n, \delta^1 e_h^n) + \sum_{n=1}^{m-1} R_1 \\ &+ \sum_{n=1}^{m-1} \Delta t (u(t^{n+1}) \cdot \nabla u(t^{n+1}) - u_H^{n+1} \cdot \nabla u_h^{n+1}, \delta^1 e_h^n). \end{aligned}$$

The first term of the right-hand side is bounded as follows :

$$\begin{aligned} \left| \sum_{n=1}^{m-1} \Delta t (\delta^1 \varphi_h^n, \delta^1 e_h^n) \right| &\leq \frac{C}{2\varepsilon_1} \left\{ h^4 (\|u'\|_{L^\infty(0,T;H^2(\Omega)^2)}^2 + \|p'\|_{L^\infty(0,T;H^1(\Omega))}^2) \right. \\ &\quad \left. + (\Delta t)^2 h^2 (\|u''\|_{L^2(0,T;H^1(\Omega)^2)}^2 + \|p''\|_{L^2(\Omega \times ]0,T])}^2) \right\} + \frac{\varepsilon_1}{2} \sum_{n=1}^{m-1} \Delta t \|\delta^1 e_h^n\|_{L^2(\Omega)}^2. \end{aligned}$$

The second term is bounded as follows :

$$\left| \sum_{n=1}^{m-1} R_1 \right| \leq \frac{C(\Delta t)^4}{2\varepsilon_2} \|u^{(3)}\|_{L^2(\Omega \times ]0,T])}^2 + \frac{\varepsilon_2}{2} \sum_{n=1}^{m-1} \Delta t \|\delta^1 e_h^n\|_{L^2(\Omega)}^2.$$

Setting  $C_{\infty\infty} = \sup_n \|u(t^{n+1})\|_{L^\infty(\Omega)}$ , the third term is bounded as follows :

$$\left| \sum_{n=1}^{m-1} \Delta t ((u(t^{n+1}) - u_H^{n+1}) \cdot \nabla u(t^{n+1}), \delta^1 e_h^n) \right| \leq \frac{C_{\infty\infty}}{2\varepsilon_3} (H^6 + (\Delta t)^4) + \frac{C_{\infty\infty}\varepsilon_3}{2} \sum_{n=1}^{m-1} \Delta t \|\delta^1 e_h^n\|_{L^2(\Omega)}^2.$$

Using Theorem 6.2, the fourth and fifth terms are respectively bounded as follows :

$$\begin{aligned} & \left| \sum_{n=1}^{m-1} \Delta t ((u_H^{n+1} - u(t^{n+1})) \cdot \nabla (u(t^{n+1}) - u_h^{n+1}), \delta^1 e_h^n) \right| \\ & \leq \frac{S_4^2}{2\varepsilon_4} (\sup_n \|u(t^{n+1}) - u_H^{n+1}\|_{L^\infty(\Omega)})^2 \sum_{n=1}^{m-1} \Delta t |u(t^{n+1}) - u_h^{n+1}|_{H^1(\Omega)}^2 + \frac{S_4^2 \varepsilon_4}{2} \sum_{n=1}^{m-1} \Delta t \|\delta^1 e_h^n\|_{L^2(\Omega)}^2 \\ & \leq \frac{S_4^2}{2\varepsilon_4} C(H^6 + h^4 + (\Delta t)^4) + \frac{S_4^2 \varepsilon_4}{2} \sum_{n=1}^{m-1} \Delta t \|\delta^1 e_h^n\|_{L^2(\Omega)}^2, \end{aligned}$$

and

$$\left| \sum_{n=1}^{m-1} \Delta t (u(t^{n+1}) \cdot \nabla (u(t^{n+1}) - u_h^{n+1}), \delta^1 e_h^n) \right| \leq \frac{C_{\infty\infty}}{2\varepsilon_5} C(H^6 + h^4 + (\Delta t)^4) + \frac{C_{\infty\infty}\varepsilon_5}{2} \sum_{n=1}^{m-1} \Delta t \|\delta^1 e_h^n\|_{L^2(\Omega)}^2.$$



Thus, after a suitable choice of  $\varepsilon_i, i = 1, \dots, 4$  and by applying the error of the solution computed by one iteration of Euler's scheme established in Proposition 3.3, we obtain

$$\begin{aligned} & \left( \sum_{n=1}^{N-1} \Delta t \|\delta^1 e_h^n\|_{L^2(\Omega)}^2 \right)^{1/2} + \sqrt{\nu} \sup_{1 \leq n \leq N} |e_h^n|_{H^1(\Omega)} + \sqrt{\nu} \sup_{1 \leq n \leq N} |2e_h^n - e_h^{n-1}|_{H^1(\Omega)} \\ & + \sqrt{\nu} \left( \sum_{n=1}^{N-1} |\delta^2 e_h^n|_{H^1(\Omega)}^2 \right)^{1/2} \leq C(h^2 + H^3 + (\Delta t)^2). \end{aligned}$$

□

These two lemmas yield immediately the following theorem.

**Theorem 6.5.** *Under the assumptions of Lemma 6.4, we have :*

$$\left( \sum_{n=1}^{N-1} \Delta t \|p(t^{n+1}) - p_h^{n+1}\|_{L^2(\Omega)}^2 \right)^{1/2} \leq C(h^2 + H^3 + (\Delta t)^2), \quad (6.11)$$

with a constant  $C$  that does not depend on  $h, H$  and  $\Delta t$ .

**Remark 6.6.** *As a consequence,  $h, H$  and  $\Delta t$  satisfy (3.16), then*

$$\left( \sum_{n=1}^{N-1} \Delta t \|p(t^{n+1}) - p_h^{n+1}\|_{L^2(\Omega)}^2 \right)^{1/2} \leq Ch^2. \quad (6.12)$$

This theoretical analysis is confirmed by numerical results cf. [1].

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