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# $\omega$ -powers and descriptive set theory.

Dominique LECOMTE

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**Abstract.** We study the sets of the infinite sentences constructible with a dictionary over a finite alphabet, from the viewpoint of descriptive set theory. Among other things, this gives some true co-analytic sets. The case where the dictionary is finite is studied and gives a natural example of a set at the level  $\omega$  of the Wadge hierarchy.

## 1 Introduction.

We consider the finite alphabet  $n = \{0, \dots, n-1\}$ , where  $n \geq 2$  is an integer, and a dictionary over this alphabet, i.e., a subset  $A$  of the set  $n^{<\omega}$  of finite words with letters in  $n$ .

**Definition 1** *The  $\omega$ -power associated to  $A$  is the set  $A^\infty$  of the infinite sentences constructible with  $A$  by concatenation. So we have  $A^\infty := \{a_0 a_1 \dots \in n^\omega / \forall i \in \omega \ a_i \in A\}$ .*

The  $\omega$ -powers play a crucial role in the characterization of subsets of  $n^\omega$  accepted by finite automata (see Theorem 2.2 in [St1]). We will study these objects from the viewpoint of descriptive set theory. The reader should see [K1] for the classical results of this theory; we will also use the notation of this book. The questions we study are the following:

- (1) What are the possible levels of topological complexity for the  $\omega$ -powers? This question was asked by P. Simonnet in [S], and studied in [St2]. O. Finkel (in [F1]) and A. Louveau proved independently that  $\Sigma_1^1$ -complete  $\omega$ -powers exist. O. Finkel proved in [F2] the existence of a  $\Pi_m^0$ -complete  $\omega$ -power for each integer  $m \geq 1$ .
- (2) What is the topological complexity of the set of dictionaries whose associated  $\omega$ -power is of a given level of complexity? This question arises naturally when we look at the characterizations of  $\Pi_1^0$ ,  $\Pi_2^0$  and  $\Sigma_1^0$   $\omega$ -powers obtained in [St2] (see Corollary 14 and Lemmas 25, 26).
- (3) We will recall that an  $\omega$ -power is an analytic subset of  $n^\omega$ . What is the topological complexity of the set of codes for analytic sets which are  $\omega$ -powers? This question was asked by A. Louveau. This question also makes sense for the set of codes for  $\Sigma_\xi^0$  (resp.,  $\Pi_\xi^0$ ) sets which are  $\omega$ -powers. And also for the set of codes for Borel sets which are  $\omega$ -powers.

As usual with descriptive set theory, the point is not only the computation of topological complexities, but also the hope that these computations will lead to a better understanding of the studied objects. Many sets in this paper won't be clopen, in particular won't be recursive. This gives undecidability results.

- We give the answer to Question (2) for the very first levels ( $\{\emptyset\}$ , its dual class and  $\Delta_1^0$ ). This contains a study of the case where the dictionary is finite. In particular, we show that the set of dictionaries whose associated  $\omega$ -power is generated by a dictionary with two words is a  $\check{D}_\omega(\Sigma_1^0)$ -complete set. This is a surprising result because this complexity is not clear at all on the definition of the set.

- We give two proofs of the fact that the relation “ $\alpha \in A^\infty$ ” is  $\Sigma_1^1$ -complete. One of these proofs is used later to give a partial answer to Question (2). To understand this answer, the reader should see [M] for the basic notions of effective descriptive set theory. Roughly speaking, a set is effectively Borel (resp., effectively Borel in  $A$ ) if its construction based on basic clopen sets can be coded with a recursive (resp., recursive in  $A$ ) sequence of integers. This answer is the

**Theorem.** *The following sets are true co-analytic sets:*

- $\{A \in 2^{n^{<\omega}}/A^\infty \in \Delta_1^1(A)\}$ .
- $\{A \in 2^{n^{<\omega}}/A^\infty \in \Sigma_\xi^0 \cap \Delta_1^1(A)\}$ , for  $1 \leq \xi < \omega_1$ .
- $\{A \in 2^{n^{<\omega}}/A^\infty \in \Pi_\xi^0 \cap \Delta_1^1(A)\}$ , for  $2 \leq \xi < \omega_1$ .

This result also comes from an analysis of Borel  $\omega$ -powers:  $A^\infty$  is Borel if and only if we can choose in a Borel way the decomposition of any sentence of  $A^\infty$  into words of  $A$  (see Lemma 13). This analysis is also related to Question (3) and to some Borel uniformization result for  $G_\delta$  sets locally with Borel projections. We will specify these relations.

- A natural ordinal rank can be defined on the complement of any  $\omega$ -power, and we study it; its knowledge gives an upper bound of the complexity of the  $\omega$ -power.
- We study the link between Question (1) and the extension ordering on finite sequences of integers.
- Finally, we give some examples of  $\omega$ -powers complete for the classes  $\Delta_1^0$ ,  $\Sigma_1^0 \oplus \Pi_1^0$ ,  $D_2(\Sigma_1^0)$ ,  $\check{D}_2(\Sigma_1^0)$ ,  $\check{D}_3(\Sigma_1^0)$  and  $\check{D}_2(\Sigma_2^0)$ .

## 2 Finitely generated $\omega$ -powers.

**Notation.** In order to answer to Question (2), we set

$$\Sigma_0 := \{A \subseteq n^{<\omega}/A^\infty = \emptyset\}, \quad \Pi_0 := \{A \subseteq n^{<\omega}/A^\infty = n^\omega\},$$

$$\Delta_1 := \{A \subseteq n^{<\omega}/A^\infty \in \Delta_1^0\},$$

$$\Sigma_\xi := \{A \subseteq n^{<\omega}/A^\infty \in \Sigma_\xi^0\}, \quad \Pi_\xi := \{A \subseteq n^{<\omega}/A^\infty \in \Pi_\xi^0\} \quad (\xi \geq 1),$$

$$\Delta := \{A \subseteq n^{<\omega}/A^\infty \in \Delta_1^1\}.$$

- If  $A \subseteq n^{<\omega}$ , then we set  $A^- := A \setminus \{\emptyset\}$ .
- We define, for  $s \in n^{<\omega}$  and  $\alpha \in n^\omega$ ,  $\alpha - s := (\alpha(|s|), \alpha(|s| + 1), \dots)$ .
- If  $\mathcal{S} \subseteq (n^{<\omega})^{<\omega}$ , then we set  $\mathcal{S}^* := \{S^* := S(0) \dots S(|s| - 1)/S \in \mathcal{S}\}$ .

- We define a recursive map  $\pi : n^\omega \times \omega^\omega \times \omega \rightarrow n^{<\omega}$  by

$$\pi(\alpha, \beta, q) := \begin{cases} (\alpha(0), \dots, \alpha(\beta[0])) & \text{if } q = 0, \\ (\alpha(1 + \sum_{j < q} \beta[j]), \dots, \alpha(\sum_{j \leq q} \beta[j])) & \text{otherwise.} \end{cases}$$

We always have the following equivalence:

$$\alpha \in A^\infty \Leftrightarrow \exists \beta \in \omega^\omega [(\forall m > 0 \ \beta(m) > 0) \text{ and } (\forall q \in \omega \ \pi(\alpha, \beta, q) \in A)].$$

**Proposition 2** (*[SJ]*)  $A^\infty \in \Sigma_1^1$  for all  $A \subseteq n^{<\omega}$ . If  $A$  is finite, then  $A^\infty \in \Pi_1^0$ .

**Proof.** We define a continuous map  $c : (A^-)^\omega \rightarrow n^\omega$  by the formula  $c((a_i)) := a_0 a_1 \dots$ . We have  $A^\infty = c[(A^-)^\omega]$ , and  $(A^-)^\omega$  is a Polish space (compact if  $A$  is finite).  $\square$

**Proposition 3** If  $A^\infty \in \Delta_1^0$ , then there exists a finite subset  $B$  of  $A$  such that  $A^\infty = B^\infty$ .

**Proof.** Set  $E_k := \{\alpha \in n^\omega / \alpha[k \in A \text{ and } \alpha - \alpha[k \in A^\infty]\}$ . It is an open subset of  $n^\omega$  since  $A^\infty$  is open, and  $A^\infty \subseteq \bigcup_{k > 0} E_k$ . We can find an integer  $p$  such that  $A^\infty \subseteq \bigcup_{0 < k \leq p} E_k$ , by compactness of  $A^\infty$ . Let  $B := A \cap n^{\leq p}$ . If  $\alpha \in A^\infty$ , then we can find an integer  $0 < k_0 \leq p$  such that  $\alpha[k_0 \in A \text{ and } \alpha - \alpha[k_0 \in A^\infty]$ . Thus  $\alpha[k_0 \in B$ . Then we do it again with  $\alpha - \alpha[k_0$ , and so on. Thus we have  $\alpha \in B^\infty = A^\infty$ .  $\square$

**Remark.** This is not true if we only assume that  $A^\infty$  is closed. Indeed, we have the following counterexample, due to O. Finkel:

$$A := \{s \in 2^{<\omega} / \forall i \leq |s| \ 2 \cdot \text{Card}(\{j < i / s(j) = 1\}) \geq i\}.$$

We have  $A^\infty = \{\alpha \in 2^\omega / \forall i \in \omega \ 2 \cdot \text{Card}(\{j < i / \alpha(j) = 1\}) \geq i\}$  and if  $B$  is finite and  $B^\infty = A^\infty$ ,  $B \subseteq A$  and  $101^2 0^2 \dots \notin B^\infty$ .

**Theorem 4** (a)  $\Sigma_0 = \{\emptyset, \{\emptyset\}\}$  is  $\Pi_1^0$ -complete.

(b)  $\Pi_0$  is a dense  $\Sigma_1^0$  subset of  $2^{n^{<\omega}}$ . In particular,  $\Pi_0$  is  $\Sigma_1^0$ -complete.

(c)  $\Delta_1$  is a  $K_\sigma \setminus \Pi_2^0$  subset of  $2^{n^{<\omega}}$ . In particular,  $\Delta_1$  is  $\Sigma_2^0$ -complete.

**Proof.** (a) Is clear.

(b) If we can find  $m \in \omega$  with  $n^m \subseteq A$ , then  $A^\infty = n^\omega$ . As  $\{A \subseteq n^{<\omega} / \exists m \in \omega \ n^m \subseteq A\}$  is a dense open subset of  $2^{n^{<\omega}}$ , the density follows. The formula

$$A \in \Pi_0 \Leftrightarrow \exists m \forall s \in n^m \exists q \leq m \ s[q \in A^-]$$

shows that  $\Pi_0$  is  $\Sigma_1^0$ , and comes from Proposition 3.

(c) If  $A^\infty \in \Delta_1^0$ , then we can find  $p > 0$  such that  $A^\infty = (A \cap n^{\leq p})^\infty$ , by Proposition 3. So let  $s_1, \dots, s_k, t_1, \dots, t_l \in 2^{<\omega}$  be such that  $A^\infty = \bigcup_{1 \leq i \leq k} N_{s_i} = n^\omega \setminus (\bigcup_{1 \leq j \leq l} N_{t_j})$ . For each  $1 \leq j \leq l$ , and for each sequence  $s \in [(A^-)^{<\omega}]^* \setminus \{\emptyset\}$ ,  $t_j \not\prec s$ . So we have

$$A^\infty \in \Delta_1^0 \Leftrightarrow \begin{cases} \exists p > 0 \ \exists k, l \in \omega \ \exists s_1, \dots, s_k, t_1, \dots, t_l \in 2^{<\omega} \ \bigcup_{1 \leq i \leq k} N_{s_i} = n^\omega \setminus (\bigcup_{1 \leq j \leq l} N_{t_j}) \\ \text{and } \forall 1 \leq j \leq l \ \forall s \in [(A^-)^{<\omega}]^* \setminus \{\emptyset\} \ t_j \not\prec s \text{ and } \forall \alpha \in n^\omega \\ \{\alpha \notin \bigcup_{1 \leq i \leq k} N_{s_i} \text{ or } \exists \beta \in p^\omega [(\forall m > 0 \ \beta(m) > 0) \text{ and } (\forall q \in \omega \ \pi(\alpha, \beta, q) \in A)]\}. \end{cases}$$

This shows that  $\Delta_1$  is a  $K_\sigma$  subset of  $2^{n^{<\omega}}$ .

To show that it is not  $\Pi_2^0$ , it is enough to see that its intersection with the closed set

$$\{A \subseteq n^{<\omega} / A^\infty \neq n^\omega\}$$

is dense and co-dense in this closed set (see (b)), by Baire's theorem. So let  $O$  be a basic clopen subset of  $2^{n^{<\omega}}$  meeting this closed set. We may assume that it is of the form

$$\{A \subseteq n^{<\omega} / \forall i \leq k \ s_i \in A \text{ and } \forall j \leq l \ t_j \notin A\},$$

where  $s_0, \dots, s_k, t_0, \dots, t_l \in n^{<\omega}$  and  $|s_0| > 0$ . Let  $A := \{s_i / i \leq k\}$ . Then  $A \in O$  and  $A^\infty$  is in  $\Pi_1^0 \setminus \{\emptyset, n^\omega\}$ . There are two cases.

If  $A^\infty \in \Delta_1^0$ , then we have to find  $B \in O$  with  $B^\infty \notin \Delta_1^0$ . Let  $u_0, \dots, u_m \in n^{<\omega}$  with  $\bigcup_{p \leq m} N_{u_p} = n^\omega \setminus A^\infty$ . Let  $r \in n \setminus \{u_0(|u_0| - 1)\}$ ,  $s := u_0 r^{|u_0| + \max_{j \leq l} |t_j|}$  and  $B := A \cup \{s\}$ . Then  $B \in O$  and  $s^\infty \in B^\infty$ . Let us show that  $s^\infty$  is not in the interior of  $B^\infty$ . Otherwise, we could find an integer  $q$  such that  $N_{s^q} \subseteq B^\infty$ . We would have  $\alpha := s^q u_0 u_0 (|u_0| - 1) r^\infty \in B^\infty$ . As  $N_{u_0} \cap A^\infty = \emptyset$ , the decomposition of  $\alpha$  into nonempty words of  $B$  would start with  $q$  times  $s$ . If this decomposition could go on, then we would have  $u_0 = (u_0(|u_0| - 1))^{|u_0|}$ . Let  $v \in n^{<\omega}$  be such that  $N_v \subseteq A^\infty$ . We have  $v(u_0(|u_0| - 1))^\infty \in A^\infty$ , so  $(u_0(|u_0| - 1))^\infty \in N_{u_0} \cap A^\infty$ . But this is absurd. Therefore  $B^\infty \notin \Delta_1^0$ .

If  $A^\infty \notin \Delta_1^0$ , then we have to find  $B \in O$  such that  $B^\infty \in \Delta_1^0 \setminus \{n^\omega\}$ . Notice that  $n^\omega \neq \bigcup_{i \leq k} N_{s_i}$ . So let  $v \in n^{<\omega}$  be non constant such that  $N_v \cap \bigcup_{i \leq k} N_{s_i} = \emptyset$ . We set

$$D := A \cup \bigcup_{r \in n \setminus \{v(0)\}} \{(r)\} \cup \{v(0)^{|v|}\},$$

$B := A \cup \{s \in n^{<\omega} / |s| > \max_{j \leq l} |t_j| \text{ and } \exists t \in D \ t \prec s\}$ . We get  $B^\infty = \bigcup_{t \in D} N_t \in \Delta_1^0$  and

$$N_v \cap B^\infty = \emptyset,$$

so  $B^\infty \neq n^\omega$ . □

Now we will study  $\mathcal{F} := \{A \subseteq n^{<\omega} / \exists B \subseteq n^{<\omega} \text{ finite } A^\infty = B^\infty\}$ .

**Proposition 5**  $\mathcal{F}$  is a co-nowhere dense  $\Sigma_2^0$ -hard subset of  $2^{n^{<\omega}}$ .

**Proof.** By Proposition 3, if  $A^\infty = n^\omega$ , then there exists an integer  $p$  such that  $A^\infty = (A \cap n^{\leq p})^\infty$ , so  $\Pi_0 \subseteq \mathcal{F}$  and, by Theorem 4,  $\mathcal{F}$  is co-nowhere dense. We define a continuous map  $\phi : 2^\omega \rightarrow 2^{n^{<\omega}}$  by the formula  $\phi(\gamma) := \{0^k 1 / \gamma(k) = 1\}$ . If  $\gamma \in P_f := \{\alpha \in 2^\omega / \exists p \forall m \geq p \ \alpha(m) = 0\}$ , then  $\phi(\gamma) \in \mathcal{F}$ . If  $\gamma \notin P_f$ , then the concatenation map is an homeomorphism from  $\phi(\gamma)^\omega$  onto  $\phi(\gamma)^\infty$ , thus  $\phi(\gamma)^\infty$  is not  $K_\sigma$ . So  $\phi(\gamma) \notin \mathcal{F}$ , by Proposition 2. Thus the preimage of  $\mathcal{F}$  by  $\phi$  is  $P_f$ , and  $\mathcal{F}$  is  $\Sigma_2^0$ -hard. □

Let  $\mathcal{G}_p := \{A \subseteq n^{<\omega} / \exists s_1, \dots, s_p \in n^{<\omega} \ A^\infty = \{s_1, \dots, s_p\}^\infty\}$ , so that  $\mathcal{F} = \bigcup_p \mathcal{G}_p$ . We have  $\mathcal{G}_0 = \Sigma_0$ , so  $\mathcal{G}_0$  is  $\Pi_1^0 \setminus \Sigma_1^0$ .

**Proposition 6**  $\mathcal{G}_1$  is  $\Pi_1^0 \setminus \Sigma_1^0$ . In particular,  $\mathcal{G}_1$  is  $\Pi_1^0$ -complete.

**Proof.** If  $p \in \omega \setminus \{0\}$ , then  $\{0, 1^p\} \notin \mathcal{G}_1$  since  $B^\infty = \{s^\infty\}$  if  $B = \{s\}$ . Thus  $\{0\}$  is not an interior point of  $\mathcal{G}_1$  since the sequence  $(\{0, 1^p\})_{p>0}$  tends to  $\{0\}$ . So  $\mathcal{G}_1 \notin \Sigma_1^0$ .

- Let  $(A_m) \subseteq \mathcal{G}_1$  tending to  $A \subseteq n^{<\omega}$ . If  $A \subseteq \{\emptyset\}$ , then  $A^\infty = \emptyset = \{\emptyset\}^\infty$ , so  $A \in \mathcal{G}_1$ . If  $A \not\subseteq \{\emptyset\}$ , then let  $t \in A^-$  and  $\alpha_0 := t^\infty$ . There exists an integer  $m_0$  such that  $t \in A_m$  for  $m \geq m_0$ . Thus we may assume that  $t \in A_m$  and  $A_m^\infty \neq \emptyset$ . So let  $s_m \in n^{<\omega} \setminus \{\emptyset\}$  be such that  $A_m^\infty = \{s_m\}^\infty = \{s_m^\infty\}$ . We have  $s_m^\infty = \alpha_0$ . Let  $b := \min\{a \in \omega \setminus \{0\} / (\alpha_0 \upharpoonright a)^\infty = \alpha_0\}$ .

- We will show that  $A_m \subseteq \{(\alpha_0 \upharpoonright b)^q / q \in \omega\}$ . Let  $s \in A_m \setminus \{\emptyset\}$ . As  $s^\infty = \alpha_0$ , we can find an integer  $a > 0$  such that  $s = \alpha_0 \upharpoonright a$ , and  $b \leq a$ . Let  $r < b$  and  $q$  be integers so that  $a = q \cdot b + r$ . We have, if  $r > 0$ ,

$$\begin{aligned} \alpha_0 &= (\alpha_0 \upharpoonright a)^\infty = (\alpha_0 \upharpoonright b)^\infty = (\alpha_0 \upharpoonright q \cdot b)^\infty (\alpha_0 \upharpoonright a - \alpha_0 \upharpoonright q \cdot b)^\infty \alpha_0 \\ &= (\alpha_0 \upharpoonright b)^q (\alpha_0 \upharpoonright a - \alpha_0 \upharpoonright q \cdot b)^\infty \alpha_0 = (\alpha_0 \upharpoonright a - \alpha_0 \upharpoonright q \cdot b)^\infty \alpha_0 = (\alpha_0 \upharpoonright r)^\infty \alpha_0 = (\alpha_0 \upharpoonright r)^\infty. \end{aligned}$$

Thus, by minimality of  $b$ ,  $r = 0$  and we are done.

- Let  $u \in A$ . We can find an integer  $m_u$  such that  $u \in A_m$  for  $m \geq m_u$ . So there exists an integer  $q_u$  such that  $u = (\alpha_0 \upharpoonright b)^{q_u}$ . Therefore  $A^\infty = \{(\alpha_0 \upharpoonright b)^\infty\} = \{\alpha_0 \upharpoonright b\}^\infty$  and  $A \in \mathcal{G}_1$ .  $\square$

**Remark.** Notice that this shows that we can find  $w \in n^{<\omega} \setminus \{\emptyset\}$  such that  $A \subseteq \{w^q / q \in \omega\}$  if  $A \in \mathcal{G}_1$ . Now we study  $\mathcal{G}_2$ . The next lemma is just Corollary 6.2.5 in [Lo].

**Lemma 7** Two finite sequences which commute are powers of the same finite sequence.

**Proof.** Let  $x$  and  $y$  be finite sequences with  $xy = yx$ . Then the subgroup of the free group on  $n$  generators generated by  $x$  and  $y$  is abelian, hence isomorphic to  $\mathbb{Z}$ . One generator of this subgroup must be a finite sequence  $u$  such that  $x$  and  $y$  are both powers of  $u$ .  $\square$

**Lemma 8** Let  $A \in \mathcal{G}_2$ . Then there exists a finite subset  $F$  of  $A$  such that  $A^\infty = F^\infty$ .

**Proof.** We will show more. Let  $A \notin \mathcal{G}_1$  satisfying  $A^\infty = \{s_1, s_2\}^\infty$ , with  $|s_1| \leq |s_2|$ . Then

(a) The decomposition of  $\alpha$  into words of  $\{s_1, s_2\}$  is unique for each  $\alpha \in A^\infty$  (this is a consequence of Corollaries 6.2.5 and 6.2.6 in [Lo]).

(b)  $s_2 s_1 \perp s_1^q s_2$  for each integer  $q > 0$ , and  $s_2 s_1 \wedge s_1^q s_2 = s_1 s_2 \wedge s_2 s_1$ .

(c)  $A \subseteq [\{s_1, s_2\}^{<\omega}]^*$ .

- We prove the first two points. We split into cases.

2.1.  $s_1 \perp s_2$ .

The result is clear.

2.2.  $s_1 \prec_{\neq} s_2 \not\prec s_1^\infty$ .

Here also, the result is clear (cut  $\alpha$  into words of length  $|s_1|$ ).

2.3.  $s_1 \prec_{\neq} s_2 \prec s_1^\infty$ .

We can write  $s_2 = s_1^m s$ , where  $m > 0$  and  $s \prec_{\neq} s_1$ . Thus  $s_2 s_1 = s_1^m s s_1$  and  $s_1^{m+1} s \prec s_1^q s_2$  if  $q > 0$ . But  $s_1^m s s_1 \perp s_1^{m+1} s$  otherwise  $s s_1 = s_1 s$ , and  $s, s_1 s_2$  would be powers of some sequence, which contradicts  $A \notin \mathcal{G}_1$ .

• We prove (c). Let  $t \in A$ , so that  $t s_1^\infty, t s_2 s_1^\infty \in A^\infty$ . These sequences split after  $t(s_1 s_2 \wedge s_2 s_1)$ , and the decomposition of  $t s_1^\infty$  (resp.,  $t s_2 s_1^\infty$ ) into words of  $\{s_1, s_2\}$  starts with  $u s_i$  (resp.,  $u s_{3-i}$ ), where  $u \in [\{s_1, s_2\}^{<\omega}]^*$ . So  $t s_1^\infty$  and  $t s_2 s_1^\infty$  split after  $u(s_1 s_2 \wedge s_2 s_1)$  by (b). But we must have  $t = u$  because of the position of the splitting point.

• We prove Lemma 8. If  $A \in \mathcal{G}_0$ , then  $F := \emptyset$  works. If  $A \in \mathcal{G}_1 \setminus \mathcal{G}_0$ , then let  $w \in n^{<\omega} \setminus \{\emptyset\}$  such that  $A \subseteq \{w^q/q \in \omega\}$ , and  $q > 0$  such that  $w^q \in A$ . Then  $F := \{w^q\}$  works. So we may assume that  $A \notin \mathcal{G}_1$ , and  $A^\infty = \{s_1, s_2\}^\infty$ . As  $A^\infty \subseteq \bigcup_{t \in A^-} \{\alpha \in N_t/s_1 s_2 \wedge s_2 s_1 \prec \alpha - t\}$  is compact, we get a finite subset  $F$  of  $A^-$  such that  $A^\infty \subseteq \bigcup_{t \in F} \{\alpha \in N_t/s_1 s_2 \wedge s_2 s_1 \prec \alpha - t\}$ . We have  $F^\infty \subseteq A^\infty$ . If  $\alpha \in A^\infty$ , then let  $t \in F$  such that  $t \prec \alpha$ . By (c), we have  $t \in [\{s_1, s_2\}^{<\omega}]^*$ . The sequence  $t$  is the beginning of the decomposition of  $\alpha$  into words of  $\{s_1, s_2\}$ . Thus  $\alpha - t \in A^\infty$  and we can go on like this. This shows that  $\alpha \in F^\infty$ .  $\square$

**Remark.** The inclusion of  $A^\infty = \{s_1, s_2\}^\infty$  into  $\{t_1, t_2\}^\infty$  does not imply  $\{s_1, s_2\} \subseteq [\{t_1, t_2\}^{<\omega}]^*$ , even if  $A \notin \mathcal{G}_1$ . Indeed, take  $s_1 := 01, s_2 := t_1 := 0$  and  $t_2 := 10$ . But we have

$$|t_1| + |t_2| \leq |s_1| + |s_2|,$$

which is the case in general:

**Lemma 9** *Let  $A, B \notin \mathcal{G}_1$  satisfying  $A^\infty = \{s_1, s_2\}^\infty \subseteq B^\infty = \{t_1, t_2\}^\infty$ . Then there is  $j \in 2$  such that  $|t_{1+i}| \leq |s_{1+[i+j \bmod 2]}|$  for each  $i \in 2$ . In particular,  $|t_1| + |t_2| \leq |s_1| + |s_2|$ .*

**Proof.** We may assume that  $|s_1| \leq |s_2|$ . Let, for  $i = 1, 2$ ,  $(w_m^i)_m \subseteq \{t_1, t_2\}$  be sequences such that  $s_1^\infty = w_0^1 w_1^1 \dots$  (resp.,  $s_2 s_1^\infty = w_0^2 w_1^2 \dots$ ). By the proof of Lemma 8, there is a minimal integer  $m_0$  satisfying  $w_{m_0}^1 \neq w_{m_0}^2$ . We let  $u := w_0^1 \dots w_{m_0-1}^1$ . The sequences  $s_1^\infty$  and  $s_2 s_1^\infty$  split after  $s_1 s_2 \wedge s_2 s_1 = u(t_1 t_2 \wedge t_2 t_1)$ . Similarly,  $s_1^\infty$  and  $s_1 s_2 s_1^\infty$  split after  $s_1(s_1 s_2 \wedge s_2 s_1) = v(t_1 t_2 \wedge t_2 t_1)$ , where  $v \in [\{t_1, t_2\}^{<\omega}]^* \setminus \{\emptyset\}$ . So we get  $s_1 u = v$ . Similarly, with the sequences  $s_2 s_1^\infty$  and  $s_2^2 s_1^\infty$ , we see that  $s_2 u \in [\{t_1, t_2\}^{<\omega}]^* \setminus \{\emptyset\}$ . So we may assume that  $u \neq \emptyset$  since  $\{s_1, s_2\} \notin \mathcal{G}_1$ . If  $t_1 \not\prec t_2$ , then we may assume that  $\emptyset \neq t_1 \prec_{\neq} t_2$ . So we may assume that we are not in the case  $t_2 \prec t_1^\infty$ . Indeed, otherwise  $t_2 = t_1^m t$ , where  $\emptyset \prec_{\neq} t \prec_{\neq} t_1$  (see the proof of Lemma 8). Moreover,  $t_1$  doesn't finish  $t_2$ , otherwise we would have  $t_1 = t(t_1 - t) = (t_1 - t)t$  and  $t, t_1 - t, t_1, t_2$  would be powers of the same sequence, which contradicts  $\{t_1, t_2\} \notin \mathcal{G}_1$ . As  $s_i u \in [\{t_1, t_2\}^{<\omega}]^*$ , this shows that  $s_i \in [\{t_1, t_2\}^{<\omega}]^*$ . So we are done since  $\{s_1, s_2\} \notin \mathcal{G}_1$  as before.

Assume for example that  $t_2 = w_{m_0}^1$ . Let  $m'$  be maximal with  $t_1^{m'} \prec t_2$ . Notice that

$$ut_1^{m'} \prec s_1s_2 \prec s_1s_2s_1^\infty.$$

We have  $ut_2 \prec s_1s_2s_1^\infty$ , otherwise we would obtain  $ut_1^{m'+1} \prec s_1s_2s_1^\infty \wedge s_2s_1^\infty = s_1s_2 \wedge s_2s_1 \prec s_1^\infty$ , which is absurd. So we get  $|t_2| \leq |s_1|$  since  $|u|+|t_2|+|t_1t_2 \wedge t_2t_1| \leq |s_1|+|s_1s_2 \wedge s_2s_1|$ . Similarly,  $|t_1| \leq |s_2|$  since  $ut_1^{m'+1} \prec s_2^2s_1^\infty$ . The argument is similar if  $t_2 = w_{m_0}^2$  (we get  $|t_i| \leq |s_i|$  in this case for  $i = 1, 2$ ).  $\square$

**Corollary 10**  $\mathcal{G}_2$  is a  $\check{D}_\omega(\Sigma_1^0) \setminus D_\omega(\Sigma_1^0)$  set. In particular,  $\mathcal{G}_2$  is  $\check{D}_\omega(\Sigma_1^0)$ -complete.

**Proof.** We will apply the Hausdorff derivation to  $\mathcal{G} \subseteq 2^{n^{<\omega}}$ . This means that we define a decreasing sequence  $(F_\xi)_{\xi < \omega_1}$  of closed subsets of  $2^{n^{<\omega}}$  as follows:

$$F_\xi := \overline{\left( \bigcap_{\eta < \xi} F_\eta \right)} \cap \mathcal{G} \text{ if } \xi \text{ is even, } \quad \overline{\left( \bigcap_{\eta < \xi} F_\eta \right)} \setminus \mathcal{G} \text{ if } \xi \text{ is odd.}$$

Recall that if  $\xi$  is even, then  $F_\xi = \emptyset$  is equivalent to  $\mathcal{G} \in D_\xi(\Sigma_1^0)$ . Indeed, we set  $U_\xi := \check{F}_\xi$ . We have  $U_{\xi+1} \setminus U_\xi = F_\xi \setminus F_{\xi+1} \subseteq \mathcal{G}$  if  $\xi$  is even and  $U_{\xi+1} \setminus U_\xi \subseteq \check{\mathcal{G}}$  if  $\xi$  is odd. Similarly,  $U_\xi \setminus (\bigcup_{\eta < \xi} U_\eta) \subseteq \check{\mathcal{G}}$  if  $\xi$  is limit. If  $F_\xi = \emptyset$ , then let  $\eta$  be minimal such that  $F_\eta = \emptyset$ . We have  $\mathcal{G} = \bigcup_{\theta \leq \eta, \theta \text{ odd}} U_\theta \setminus (\bigcup_{\rho < \theta} U_\rho)$ . If  $\eta$  is odd, then  $\check{\mathcal{G}} = \bigcup_{\theta < \eta, \theta \text{ even}} U_\theta \setminus (\bigcup_{\rho < \theta} U_\rho) \in D_\eta(\Sigma_1^0)$ , thus  $\mathcal{G} \in \check{D}_\eta(\Sigma_1^0) \subseteq D_\xi(\Sigma_1^0)$ . If  $\eta$  is even, then  $\mathcal{G} = \bigcup_{\theta < \eta, \theta \text{ odd}} U_\theta \setminus (\bigcup_{\rho < \theta} U_\rho) \in D_\eta(\Sigma_1^0)$  and the same conclusion is true. Conversely, if  $\mathcal{G} \in D_\xi(\Sigma_1^0)$ , then let  $(V_\eta)_{\eta < \xi}$  be an increasing sequence of open sets with  $\mathcal{G} = \bigcup_{\eta < \xi, \eta \text{ odd}} V_\eta \setminus (\bigcup_{\theta < \eta} V_\theta)$ . By induction, we check that  $F_\eta \subseteq \check{V}_\eta$  if  $\eta < \xi$ . This clearly implies that  $F_\xi = \emptyset$  because  $\xi$  is even.

• We will show that if  $A \notin \mathcal{G}_1$  satisfies  $A^\infty = \{s_1, s_2\}^\infty$ , then  $A \notin F_M := F_M(\mathcal{G}_2)$ , where  $M$  is the smallest odd integer greater than or equal to  $f(s_1, s_2) := 2^{\sum_{l \leq |s_1|+|s_2|-2} n^{2(|s_1|+|s_2|-l)}}$ .

We argue by contradiction:  $A$  is the limit of  $(A_q)$ , where  $A_q \in F_{M-1} \setminus \mathcal{G}_2$ . Lemma 8 gives a finite subset  $F$  of  $A$ , and we may assume that  $F \subseteq A_q$  for each  $q$ . Thus we have  $A^\infty \subseteq A_q^\infty$ , and the inclusion is strict. Thus we can find  $s^q \in [A_q^{<\omega}]^*$  such that  $N_{s^q} \cap A^\infty = \emptyset$ . Let  $s_0^q, \dots, s_{m_q}^q \in A_q$  be such that  $s^q = s_0^q \dots s_{m_q}^q$ .

Now  $A_q$  is the limit of  $(A_{q,r})_r$ , where  $A_{q,r} \in F_{M-2} \cap \mathcal{G}_2$ , and we may assume that

$$\{s_0^q, \dots, s_{m_q}^q\} \cup F \subseteq A_{q,r}$$

for each  $r$ , and that  $A_{q,r} \notin \mathcal{G}_1$  because  $A_q \notin \mathcal{G}_1 \subseteq \mathcal{G}_2$ . Let  $s_1^{q,r}, s_2^{q,r}$  such that  $A_{q,r}^\infty = \{s_1^{q,r}, s_2^{q,r}\}^\infty$ . By Lemma 9 we have  $|s_1^{q,r}| + |s_2^{q,r}| \leq |s_1| + |s_2|$ . Now we let  $B_0 := A_{0,0}$  and  $s_i^0 := s_i^{0,0}$  for  $i = 1, 2$ . We have  $B_0 \in F_{M-2} \cap \mathcal{G}_2 \setminus \mathcal{G}_1$ ,  $A^\infty \subseteq \neq B_0^\infty = \{s_1^0, s_2^0\}^\infty$ , and

$$|s_1^0| + |s_2^0| \leq |s_1| + |s_2|.$$



Now we iterate this: for each  $0 < k < n^{2(|s_1|+|s_2|)}$ , we get  $B_k \in F_{M-2(k+1)} \cap \mathcal{G}_2 \setminus \mathcal{G}_1$  such that  $B_{k-1}^\infty \subsetneq B_k^\infty = \{s_1^k, s_2^k\}^\infty$  and  $|s_1^k| + |s_2^k| \leq |s_1^{k-1}| + |s_2^{k-1}|$ . We can find  $k_0 < n^{2(|s_1|+|s_2|)}$  such that  $|s_1^{k_0}| + |s_2^{k_0}| < |s_1^{k_0-1}| + |s_2^{k_0-1}|$  (with the convention  $s_i^{-1} := s_i$ ). We set  $C_0 := B_{k_0}$ ,  $t_i^0 := s_i^{k_0}$ . So we have  $C_0 \in F_{M-2(k_0+1)} \cap \mathcal{G}_2 \setminus \mathcal{G}_1$ ,  $C_0^\infty = \{t_1^0, t_2^0\}^\infty$  and  $|t_1^0| + |t_2^0| < |s_1| + |s_2|$ . Now we iterate this: for each  $l \leq |s_1| + |s_2| - 2$ , we get  $t_1^l, t_2^l, k_l < n^{2(|t_1^{l-1}|+|t_2^{l-1}|)}$  and

$$C_l \in F_{M-2\sum_{m \leq l} (k_m+1)} \cap \mathcal{G}_2 \setminus \mathcal{G}_1$$

satisfying  $C_l^\infty = \{t_1^l, t_2^l\}^\infty$  and  $|t_1^l| + |t_2^l| < |t_1^{l-1}| + |t_2^{l-1}|$  (with the convention  $t_i^{-1} := s_i$ ). We have  $|t_1^l| + |t_2^l| \leq |s_1| + |s_2| - 1 - l$ , thus

$$2\sum_{l \leq |s_1|+|s_2|-2} (k_l + 1) \leq 2\sum_{l \leq |s_1|+|s_2|-2} n^{2(|t_1^{l-1}|+|t_2^{l-1}|)} \leq f(s_1, s_2)$$

and this construction is possible. But we have  $|t_1^{|s_1|+|s_2|-2}| + |t_2^{|s_1|+|s_2|-2}| \leq 1$ , thus  $C_{|s_1|+|s_2|-2} \in \mathcal{G}_1$ , which is absurd.

Let  $A \notin \mathcal{G}_2$ . As  $A \notin \mathcal{G}_1$ , we can find  $s, t \in A$  which are not powers of the same sequence. Indeed, let  $s \in A^-$  and  $u$  with minimal length such that  $s$  is a power of  $u$ . Then any  $t \in A \setminus \{u^q/q \in \omega\}$  works, because if  $s$  and  $t$  are powers of  $w$ , then  $w$  has to be a power of  $u$ . Indeed, as  $u \prec w$ ,  $w = u^k v$  with  $v \prec u$ , and  $v$  has to be a power of  $u$  by minimality of  $|u|$  and Lemma 7. Assume that moreover  $A \in F_{2k+2}$ . Now  $A$  is the limit of  $(A_{k,r})_r \subseteq F_{2k+1} \cap \mathcal{G}_2$  for each integer  $k$ , and we may assume that  $s, t \in A_{k,r} \notin \mathcal{G}_1$ . Let  $s_1^{k,r}, s_2^{k,r}$  be such that  $A_{k,r}^\infty = \{s_1^{k,r}, s_2^{k,r}\}^\infty$ . By Lemma 9 we have  $|s_1^{k,r}| + |s_2^{k,r}| \leq |s| + |t|$  and  $f(s_1^{k,r}, s_2^{k,r}) \leq f(s, t)$ . By the preceding point, we must have

$$2k + 1 < f(s, t).$$

Thus  $\bigcap_m F_m \subseteq \mathcal{G}_2$ . Notice that  $F_{m+1}(\check{\mathcal{G}}_2) \subseteq F_m$ , so that  $F_\omega(\check{\mathcal{G}}_2) = \emptyset$  and  $\mathcal{G}_2 \in \check{D}_\omega(\Sigma_1^0)$ .

• Now let us show that  $\{0\} \in F_\omega(\mathcal{G}_2)$  (this will imply  $\mathcal{G}_2 \notin D_\omega(\Sigma_1^0)$ ). It is enough to see that

$$\{0\} \in \bigcap_m F_m.$$

Let  $E(x)$  be the biggest integer less than or equal to  $x$ ,  $p_{k,s} := 2^{k+1-E(|s|/2)}$  and  $k \in \omega$ . We define  $A_\emptyset := \{0\}$  and, for  $s \in (\omega \setminus \{0, 1\})^{\leq 2k+1}$  and  $m > 1$ ,  $A_{sm} := A_s \cup \{(01^{p_{k,s}})^m; (0^2 1^{p_{k,s}})^m\}$  if  $|s|$  is even,  $A_s \cup \{s \in [\{0, 1^{p_{k,s}}\}^{<\omega}]^*/m \leq |s| \leq m + p_{k,s}\}$  if  $|s|$  is odd. Let us show that  $A_s \in \mathcal{G}_2$  (resp.,  $\check{\mathcal{G}}_2$ ) if  $|s|$  is even (resp., odd). First by induction we get  $A_{sm} \subseteq \{0, 1^{p_{k,s}}\}^{<\omega}$ . Therefore  $A_{sm}^\infty = \{0, 1^{p_{k,s}}\}^\infty$  if  $|s|$  is odd, because if  $\alpha$  is in  $\{0, 1^{p_{k,s}}\}^\infty$  and  $t \in [\{0, 1^{p_{k,s}}\}^{<\omega}]^*$  with minimal length  $\geq m$  begins  $\alpha$ , then  $t \in A_{sm}$ . Now if  $|s|$  is even and  $A_{sm}^\infty = \{s_1, s_2\}^\infty$ , then  $0^\infty \in \{s_1, s_2\}^\infty$ , thus for example  $s_1 = 0^{k+1}$ .  $(01^{p_{k,s}})^\infty \in \{s_1, s_2\}^\infty$ , thus  $s_2 \prec (01^{p_{k,s}})^\infty$  and  $|s_2| \geq |(01^{p_{k,s}})^m|$  since  $s_2 0^\infty \in \{s_1, s_2\}^\infty$ . But then  $(0^2 1^{p_{k,s}})^\infty \notin \{s_1, s_2\}^\infty$  since  $m > 1$ . Thus  $A_{sm} \notin \mathcal{G}_2$ .

As  $(A_{sm})_m$  tends to  $A_s$  and  $(A_s)_{|s|=2k+2} \subseteq \mathcal{G}_2$ , we deduce from this that  $A_s$  is in  $F_{2k+1-|s|} \setminus \mathcal{G}_2$  if  $|s| \leq 2k + 1$  is odd, and that  $A_s \in F_{2k+1-|s|} \cap \mathcal{G}_2$  if  $|s| \leq 2k + 1$  is even. Therefore  $\{0\}$  is in  $\bigcap_k F_{2k+1} = \bigcap_m F_m$ .  $\square$

**Remarks.** (1) The end of this proof also shows that  $\mathcal{G}_p \notin D_\omega(\Sigma_1^0)$  if  $p \geq 2$ . Indeed,  $\{0\} \in F_\omega(\mathcal{G}_p)$ . The only thing to change is the definition of  $A_{sm}$  if  $|s|$  is even: we set

$$A_{sm} := A_s \cup \{(0^{j+1}1^{pk,s})^m / j < p\}.$$

(2) If  $\{s_1, s_2\} \notin \mathcal{G}_1$  and  $\{s_1, s_2\}^\infty = \{t_1, t_2\}^\infty$ , then  $\{s_1, s_2\} = \{t_1, t_2\}$ . Indeed,  $\{t_1, t_2\} \notin \mathcal{G}_1$ , thus by Lemma 9 we get  $|s_1| + |s_2| = |t_1| + |t_2|$ . By (c) in the proof of Lemma 8 and the previous fact,  $s_i = t_{\varepsilon_i}^{a_i}$ , where  $a_i > 0$ ,  $\varepsilon_i$ ,  $i \in \{1, 2\}$ . As  $\{s_1, s_2\} \notin \mathcal{G}_1$ ,  $\varepsilon_1 \neq \varepsilon_2$ . Thus  $a_i = 1$ .

**Conjecture 1.** Let  $A \in \mathcal{F}$ . Then there exists a finite subset  $F$  of  $A$  such that  $A^\infty = F^\infty$ .

**Conjecture 2.** Let  $p \geq 1$ ,  $A, B \notin \mathcal{G}_p$  with  $A^\infty = \{s_1, \dots, s_q\}^\infty \subseteq B^\infty = \{t_1, \dots, t_{p+1}\}^\infty$ . Then  $\sum_{1 \leq i \leq p+1} |t_i| \leq \sum_{1 \leq i \leq q} |s_i|$ .

**Conjecture 3.** We have  $\mathcal{G}_{p+1} \setminus \mathcal{G}_p \in D_\omega(\Sigma_1^0)$  for each  $p \geq 1$ . In particular,  $\mathcal{F} \in K_\sigma \setminus \Pi_2^0$ .

Notice that Conjectures 1 and 2 imply Conjecture 3. Indeed,  $\mathcal{F} = \mathcal{G}_1 \cup \bigcup_{p \geq 1} \mathcal{G}_{p+1} \setminus \mathcal{G}_p$ , so  $\mathcal{F} \in K_\sigma$  if  $\mathcal{G}_{p+1} \setminus \mathcal{G}_p \in D_\omega(\Sigma_1^0) \subseteq \Delta_2^0$ , by Proposition 6. By Proposition 5 we have  $\mathcal{F} \notin \Pi_2^0$ . It is enough to see that  $F_\omega := F_\omega(\mathcal{G}_{p+1} \setminus \mathcal{G}_p) = \emptyset$ . We argue as in the proof of Corollary 10. This time,  $f(s_1, \dots, s_q) := 2^{\sum_{l \leq \sum_{1 \leq i \leq q} |s_i| - 2}} n^{q(\sum_{1 \leq i \leq q} |s_i| - l)}$  for  $s_1, \dots, s_q \in n^{<\omega}$ . The fact to notice is that  $A \notin F_M(\mathcal{G}_{p+1} \setminus \mathcal{G}_p)$  if  $A \notin \mathcal{G}_p$  satisfies  $A^\infty = \{s_1, \dots, s_{p+1}\}^\infty$  and  $M$  is the minimal odd integer greater than or equal to  $f(s_1, \dots, s_{p+1})$ . So if  $A \in F_{2k+2} \cap \mathcal{F} \setminus \mathcal{G}_p$ , then Conjecture 1 gives a finite subset  $F := \{s_1, \dots, s_q\}$  of  $A$ . The set  $A$  is the limit of  $(A_{k,r})_r \subseteq F_{2k+1} \cap \mathcal{G}_{p+1} \setminus \mathcal{G}_p$  for each integer  $k$ , and we may assume that  $F \subseteq A_{k,r}$ . Conjecture 2 implies that  $f(s_1^{k,r}, \dots, s_{p+1}^{k,r}) \leq f(s_1, \dots, s_q)$  and  $2k+1 < f(s_1, \dots, s_q)$ . Thus  $\bigcap_m F_m \subseteq \check{\mathcal{F}} \cup \mathcal{G}_p$ . So  $F_\omega \subseteq \overline{(\check{\mathcal{F}} \cup \mathcal{G}_p) \cap \mathcal{G}_{p+1} \setminus \mathcal{G}_p} = \emptyset$ .

### 3 Is $A^\infty$ Borel?

Now we will see that the maximal complexity is possible. We essentially give O. Finkel's example, in a lightly simpler version.

**Proposition 11** Let  $\Gamma := \Sigma_1^1$  or a Baire class. The existence of  $n \in \omega \setminus 2$  and  $A \subseteq n^{<\omega}$  such that  $A^\infty$  is  $\Gamma$ -complete is equivalent to the existence of  $B \subseteq 2^{<\omega}$  such that  $B^\infty$  is  $\Gamma$ -complete.

**Proof.** Let  $p_n := \min\{p \in \omega / n \leq 2^p\} \geq 1$ . We define  $\phi : n \hookrightarrow 2^{p_n} := \{\sigma_0, \dots, \sigma_{2^{p_n}-1}\}$  by the formula  $\phi(m) := \sigma_m$ ,  $\Phi : n^{<\omega} \hookrightarrow 2^{<\omega}$  by the formula  $\Phi(t) := \phi(t(0)) \dots \phi(t(|t|-1))$  and  $f : n^\omega \hookrightarrow 2^\omega$  by the formula  $f(\gamma) := \phi(\gamma(0))\phi(\gamma(1)) \dots$ . Then  $f$  is an homeomorphism from  $n^\omega$  onto its range and reduces  $A^\infty$  to  $B^\infty$ , where  $B := \Phi[A]$ . The inverse function of  $f$  reduces  $B^\infty$  to  $A^\infty$ . So we are done if  $\Gamma$  is stable under intersection with closed sets. Otherwise,  $\Gamma = \Delta_1^0$  or  $\Sigma_1^0$ . If  $A = \{s \in 2^{<\omega} / 0 \prec s \text{ or } 1^2 \prec s\}$ , then  $A^\infty = N_0 \cup N_{1^2}$ , which is  $\Delta_1^0$ -complete. If  $A = \{s \in 2^{<\omega} / 0 \prec s\} \cup \{10^k 1^{l+1} / k, l \in \omega\}$ , then  $A^\infty = 2^\omega \setminus \{10^\infty\}$ , which is  $\Sigma_1^0$ -complete.  $\square$

**Theorem 12** The set  $I := \{(\alpha, A) \in n^\omega \times 2^{n^{<\omega}} / \alpha \in A^\infty\}$  is  $\Sigma_1^1$ -complete. In fact,

(a) (O. Finkel, see [F1]) There exists  $A_0 \subseteq 2^{<\omega}$  such that  $A_0^\infty$  is  $\Sigma_1^1$ -complete.

(b) There exists  $\alpha_0 \in 2^\omega$  such that  $I_{\alpha_0}$  is  $\Sigma_1^1$ -complete.

**Proof.** (a) We set  $L := \{2, 3\}$  and

$\mathcal{T} := \{ \tau \subseteq 2^{<\omega} \times L / \forall (u, \nu) \in 2^{<\omega} \times L [(u, \nu) \notin \tau] \text{ or}$

$[(\forall v \prec u \exists \mu \in L (v, \mu) \in \tau) \text{ and } ((u, 5 - \nu) \notin \tau) \text{ and } (\exists (\varepsilon, \pi) \in 2 \times L (u\varepsilon, \pi) \in \tau)] \}$ .

The set  $\mathcal{T}$  is the set of pruned trees over 2 with labels in  $L$ . It is a closed subset of  $2^{2^{<\omega} \times L}$ , thus a Polish space. Then we set

$\sigma := \{ \tau \in \mathcal{T} / \exists (\underline{u}, \underline{\nu}) \in 2^\omega \times L^\omega [\forall m (\underline{u}[m], \underline{\nu}(m)) \in \tau] \text{ and } [\forall p \exists m \geq p \underline{\nu}(m) = 3] \}$ .

• Then  $\sigma \in \Sigma_1^1(\mathcal{T})$ . Let us show that it is complete. We set  $\mathcal{T} := \{T \in 2^{\omega^{<\omega}} / T \text{ is a tree}\}$  and  $IF := \{T \in \mathcal{T} / T \text{ is ill-founded}\}$ . It is a well-known fact that  $\mathcal{T}$  is a Polish space (it is a closed subset of  $2^{\omega^{<\omega}}$ ), and that  $IF$  is  $\Sigma_1^1$ -complete (see [K1]). It is enough to find a Borel reduction of  $IF$  to  $\sigma$  (see [K2]).

We define  $\psi : \omega^{<\omega} \hookrightarrow 2^{<\omega}$  by the formula  $\psi(t) := 0^{t(0)}10^{t(1)}1 \dots 0^{t(|t|-1)}1$ , and  $\Psi : \mathcal{T} \rightarrow \mathcal{T}$  by

$\Psi(T) := \{(u, \nu) \in 2^{<\omega} \times L / \exists t \in T u \prec \psi(t) \text{ and } \nu = 3 \text{ if } u = \emptyset, 2 + u(|u| - 1) \text{ otherwise}\}$

$\cup \{(\psi(t)0^{k+1}, 2) / t \in T \text{ and } \forall q \in \omega tq \notin T, k \in \omega\}$ .

The map  $\Psi$  is Baire class one. Let us show that it is a reduction. If  $T \in IF$ , then let  $\gamma \in \omega^\omega$  be such that  $\gamma \upharpoonright m \in T$  for each integer  $m$ . We have  $(\psi(\gamma \upharpoonright m), 3) \in \Psi(T)$ . Let  $\underline{u}$  be the limit of  $\psi(\gamma \upharpoonright m)$  and  $\underline{\nu}(m) := 2 + \underline{u}(m - 1)$  (resp., 3) if  $m > 0$  (resp.,  $m = 0$ ). These objects show that  $\Psi(T) \in \sigma$ . Conversely, we have  $T \in IF$  if  $\Psi(T) \in \sigma$ .

• If  $\tau \in \mathcal{T}$  and  $m \in \omega$ , then we enumerate  $\tau \cap (2^m \times L) := \{(u_1^{m,\tau}, \nu_1^{m,\tau}), \dots, (u_{q_m,\tau}^{m,\tau}, \nu_{q_m,\tau}^{m,\tau})\}$  in the lexicographic ordering. We define  $\varphi : \mathcal{T} \hookrightarrow 5^\omega$  by the formula

$$\varphi(\tau) := (u_1^{0,\tau} \nu_1^{0,\tau} \dots u_{q_0,\tau}^{0,\tau} \nu_{q_0,\tau}^{0,\tau} 4)(u_1^{1,\tau} \nu_1^{1,\tau} \dots u_{q_1,\tau}^{1,\tau} \nu_{q_1,\tau}^{1,\tau} 4) \dots$$

The set  $A_0$  will be made of finite subsequences of sentences in  $\varphi[\mathcal{T}]$ . We set

$$A_0 := \{ u_{q+1}^{m,\tau} \nu_{q+1}^{m,\tau} \dots u_r^{p,\tau} \nu_r^{p,\tau} / \tau \in \mathcal{T}, m+1 < p, 0 \leq q \leq q_{m,\tau}, 1 \leq r \leq q_{p,\tau}, \\ [(m=0 \text{ and } q=0) \text{ or } (q > 0 \text{ and } \nu_q^{m,\tau} = 3 \text{ and } u_q^{m,\tau} \prec u_r^{p,\tau}), \nu_r^{p,\tau} = 3 \}$$

(with the convention  $u_{q_m,\tau+1}^{m,\tau} \nu_{q_m,\tau+1}^{m,\tau} = 4$ ). It is clear that  $\varphi$  is continuous, and it is enough to see that it reduces  $\sigma$  to  $A_0^\infty$ .

So let us assume that  $\tau \in \sigma$ . This means the existence of an infinite branch in the tree with infinitely many 3 labels. We cut  $\varphi(\tau)$  after the first 3 label of the branch corresponding to a sequence of length  $m > 1$ . Then we cut after the first 3 label corresponding to a sequence of length at least  $m + 2$  of the branch. And so on. This clearly gives a decomposition of  $\varphi(\tau)$  into words in  $A_0$ .

If such a decomposition exists, then the first word is  $u_1^{0,\tau} \nu_1^{0,\tau} \dots u_{r_0}^{p_0,\tau} \nu_{r_0}^{p_0,\tau}$ , and the second is  $u_{r_0+1}^{p_0,\tau} \nu_{r_0+1}^{p_0,\tau} \dots u_{r_1}^{p_1,\tau} \nu_{r_1}^{p_1,\tau}$ . So we have  $u_{r_0}^{p_0,\tau} \prec \neq u_{r_1}^{p_1,\tau}$ . And so on. This gives an infinite branch with infinitely many 3 labels.

• By Proposition 11, we can also have  $A_0 \subseteq 2^{<\omega}$ .

(b) Let  $\alpha_0 := 1010^2 10^3 \dots$ ,  $(q_l)$  be the sequence of prime numbers:  $q_0 := 2$ ,  $q_1 := 3$ ,  $M : \omega^{<\omega} \rightarrow \omega$  defined by  $M_s := q_0^{s(0)+1} \dots q_{|s|-1}^{s(|s|-1)+1} + 1$ ,  $\phi : \omega^{<\omega} \rightarrow 2^{<\omega} \setminus \{\emptyset\}$  defined by the formulas

$$\phi(\emptyset) := 1010^2 = 1010^{2M_\emptyset}$$

and  $\phi(sm) := 10^{2M_s+1} 10^{2M_{s+2}} \dots 10^{2M_{sm}}$ , and  $\Phi : 2^{\omega^{<\omega}} \rightarrow 2^{n^{<\omega}}$  defined by  $\Phi(T) := \phi[T]$ .

• It is clear that  $M_{sm} > M_s$ , and that  $M$  and  $\phi$  are well defined and one-to-one. So  $\Phi$  is continuous:

$$\begin{aligned} s \in \phi[T] &\Leftrightarrow \exists t (t \in T \text{ and } \phi(t) = s) \\ &\Leftrightarrow s \in \phi[\omega^{<\omega}] \text{ and } \forall t (t \in T \text{ or } \phi(t) \neq s). \end{aligned}$$

If  $T \in IF$ , then we can find  $\beta \in \omega^\omega$  such that  $\phi(\beta \upharpoonright l) \in \Phi(T)$  for each integer  $l$ . Thus

$$\alpha_0 = (1010^{2M_{\beta \upharpoonright 0}})(10^{2M_{\beta \upharpoonright 0+1}} \dots 10^{2M_{\beta \upharpoonright 1}}) \dots \in (\Phi(T))^\infty.$$

Conversely, if  $\alpha_0 \in (\Phi(T))^\infty$ , then there exist  $t_i \in T$  such that  $\alpha_0 = \phi(t_0)\phi(t_1) \dots$ . We have  $t_0 = \emptyset$ , and, if  $i > 0$ , then  $M_{t_i \upharpoonright |t_i|-1} = M_{t_{i-1}}$ ; from this we deduce that  $t_i \upharpoonright |t_i| - 1 = t_{i-1}$ , because  $M$  is one-to-one. So let  $\beta$  be the limit of the  $t_i$ 's. We have  $\beta \upharpoonright i = t_i$ , thus  $\beta \in [T]$  and  $T \in IF$ . Thus  $\Phi \upharpoonright_T$  reduces  $IF$  to  $I_{\alpha_0}$ . Therefore this last set is  $\Sigma_1^1$ -complete. Indeed, it is clear that  $I$  is  $\Sigma_1^1$ :

$$\alpha \in A^\infty \Leftrightarrow \exists \beta \in \omega^\omega [(\forall m > 0 \beta(m) > 0) \text{ and } (\forall q \in \omega \pi(\alpha, \beta, q) \in A)].$$

Finally, the map from  $\mathcal{T}$  into  $n^\omega \times 2^{n^{<\omega}}$ , which associates  $(\alpha_0, \Phi(T))$  to  $T$  clearly reduces  $IF$  to  $I$ . So  $I$  is  $\Sigma_1^1$ -complete.  $\square$

**Remark.** This proof shows that if  $\alpha = s_0 s_1 \dots$  and  $(s_i)$  is an antichain for the extension ordering, then  $I_\alpha$  is  $\Sigma_1^1$ -complete (here we have  $s_i = 10^{2i+1} 10^{2i+2}$ ). To see it, it is enough to notice that  $\phi(\emptyset) = s_0$  and  $\phi(sm) = s_{M_s} \dots s_{M_{sm}-1}$ . So  $I_\alpha$  is  $\Sigma_1^1$ -complete for a dense set of  $\alpha$ 's.

We will deduce from this some true co-analytic sets. But we need a lemma, which has its own interest.

**Lemma 13** (a) *The set  $A^\infty$  is Borel if and only if there exist a Borel function  $f : n^\omega \rightarrow \omega^\omega$  such that*

$$\alpha \in A^\infty \Leftrightarrow (\forall m > 0 f(\alpha)(m) > 0) \text{ and } (\forall q \in \omega \pi(\alpha, f(\alpha), q) \in A).$$

(b) *Let  $\gamma \in \omega^\omega$  and  $A \subseteq n^{<\omega}$ . Then  $A^\infty \in \Delta_1^1(A, \gamma)$  if and only if, for  $\alpha \in n^\omega$ , we have*

$$\alpha \in A^\infty \Leftrightarrow \exists \beta \in \Delta_1^1(A, \gamma, \alpha) [(\forall m > 0 \beta(m) > 0) \text{ and } (\forall q \in \omega \pi(\alpha, \beta, q) \in A)].$$

**Proof.** The “if” directions in (a) and (b) are clear. We have seen in the proof of Proposition 4 the “if” direction of the equivalences (the existence of an arbitrary  $\beta$  is necessary and sufficient). So let us show the “only if” directions.

(a) We define  $f : \omega^\omega \rightarrow \omega^\omega$  by the formula  $f(\alpha) := 0^\infty$  if  $\alpha \notin A^\infty$ , and, otherwise,

$$f(\alpha)(0) := \min\{p \in \omega / \alpha[(p+1) \in A \text{ and } \alpha - \alpha[(p+1) \in A^\infty]],$$

$$f(\alpha)(r+1) :=$$

$$\min\{k > 0 / [\alpha - \alpha[(1 + \sum_{j \leq r} f(\alpha)(j))][k \in A \text{ and } \alpha - \alpha[(k+1 + \sum_{j \leq r} f(\alpha)(j)) \in A^\infty]].$$

We get  $\pi(\alpha, f(\alpha), 0) = \alpha[f(\alpha)(0) + 1 \in A \text{ and, if } q > 0,$

$$\pi(\alpha, f(\alpha), q) = (\alpha(1 + \sum_{j < q} f(\alpha)(j)), \dots, \alpha(\sum_{j \leq q} f(\alpha)(j))) \in A.$$

As  $f$  is clearly Borel, we are done.

(b) If  $A^\infty \in \Delta_1^1(A, \gamma)$ , then so is  $f$  and  $\beta := f(\alpha) \in \Delta_1^1(A, \gamma, \alpha)$  is what we were looking for.  $\square$

**Remark.** Lemma 13 is a particular case of a more general situation. Actually we have the following uniformization result. It was written after a conversation with G. Debs.

**Proposition 14** *Let  $X$  and  $Y$  be Polish spaces, and  $F \in \Pi_2^0(X \times Y)$  such that the projection  $\Pi_X[F \cap (X \times V)]$  is Borel for each  $V \in \Sigma_1^0(Y)$ . Then there exists a Borel map  $f : X \rightarrow Y$  such that  $(x, f(x)) \in F$  for each  $x \in \Pi_X[F]$ .*

**Proof.** Let  $(Y_n)$  be a basis for the topology of  $Y$  with  $Y_0 := Y$ ,  $B_n := \Pi_X[F \cap (X \times Y_n)]$ , and  $\tau$  be a finer 0-dimensional Polish topology on  $X$  making the  $B_n$ 's clopen (see 13.5 in [K1]). We equip  $X$  with a complete  $\tau$ -compatible metric  $d$ . Let  $(O_m) \subseteq \Sigma_1^0(X \times Y)$  be decreasing satisfying  $O_0 := X \times Y$  and  $F = \bigcap_m O_m$ . We construct a sequence  $(U_s)_{s \in \omega^{<\omega}}$  of clopen subsets of  $[B_0, \tau]$  with  $U_\emptyset := B_0$ , and a sequence  $(V_s)_{s \in \omega^{<\omega}}$  of basic open sets of  $Y$  satisfying

- (a)  $U_s \subseteq \Pi_X[F \cap (U_s \times V_s)]$
- (b)  $\text{diam}_d(U_s), \text{diam}(V_s) \leq \frac{1}{|s|}$  if  $s \neq \emptyset$
- (c)  $U_s = \bigcup_{m, \text{disj.}} U_{s \frown m}, \overline{V_{s \frown n}} \subseteq V_s$
- (d)  $U_s \times V_s \subseteq O_{|s|}$

- Assume that this construction has been achieved. If  $x \notin B_0$ , then we set  $f(x) := y_0 \in Y$  (we may assume that  $F \neq \emptyset$ ). Otherwise, we can find a unique sequence  $\gamma \in \omega^\omega$  such that  $x \in U_{\gamma \upharpoonright m}$  for each integer  $m$ . Thus we can find  $y \in V_{\gamma \upharpoonright m}$  such that  $(x, y) \in F$ , and  $(\overline{V_{\gamma \upharpoonright m}})_m$  is a decreasing sequence of nonempty closed sets whose diameters tend to 0, which defines a continuous map  $f : [B_0, \tau] \rightarrow Y$ . If  $x \in B_0$ , then  $(x, f(x)) \in U_{\gamma \upharpoonright m} \times V_{\gamma \upharpoonright m} \subseteq O_m$ , thus  $\text{Gr}(f|_{B_0}) \subseteq F$ . Notice that  $f : [X, \tau] \rightarrow Y$  is continuous, so  $f : X \rightarrow Y$  is Borel.

- Let us show that the construction is possible. We set  $U_\emptyset := B_0$  and  $V_\emptyset := Y$ . Assume that  $(U_s)_{s \in \omega^{\leq p}}$  and  $(V_s)_{s \in \omega^{\leq p}}$  satisfying conditions (a)-(d) have been constructed, which is the case for  $p = 0$ . Let  $s \in \omega^p$ . If  $(x, y) \in F \cap (U_s \times V_s)$ , then we can find  $U_x \in \Delta_1^0(U_s)$  and a basic open set  $V_y \subseteq Y$  such that  $(x, y) \in U_x \times V_y \subseteq U_s \times \overline{V_y} \subseteq (U_s \times V_s) \cap O_{p+1}$ , and whose diameters are at most  $\frac{1}{p+1}$ . By the Lindelöf property, we can write  $F \cap (U_s \times V_s) \subseteq \bigcup_n U_{x_n} \times V_{y_n}$  and  $F \cap (U_s \times V_s) = \bigcup_n F \cap (U_{x_n} \times V_{y_n})$ .

If  $x \in U_s$ , then let  $n$  and  $y$  be such that  $(x, y) \in F \cap (U_{x_n} \times V_{y_n})$ . Then

$$x \in O^n := \Pi_X[F \cap (X \times V_{y_n})] \cap U_{x_n} \in \Delta_1^0([B_0, \tau]).$$

Thus  $U_s = \bigcup_n O^n$ . We set  $U_{s \frown n} := O^n \setminus (\bigcup_{p < n} O^p)$  and  $V_{s \frown n} := V_{y_n}$ , and we are done.  $\square$

In our context,  $F = \{(\alpha, \beta) \in n^\omega \times \omega^\omega / (\forall m > 0 \beta(m) > 0) \text{ and } (\forall q \in \omega \pi(\alpha, \beta, q) \in A)\}$ , which is a closed subset of  $X \times Y$ . The projection  $\Pi_X[F \cap (X \times N_s)]$  is Borel if  $A^\infty$  is Borel, since it is  $\{S^* \gamma / S \in (A \cap n^{s(0)+1}) \times \prod_{0 < j < |s|} (A \cap n^{s(j)}) \text{ and } \gamma \in A^\infty\}$ .

**Theorem 15** *The following sets are  $\Pi_1^1 \setminus \Delta_1^1$ :*

(a)  $\Pi := \{(A, \gamma, \theta) \in 2^{n^{<\omega}} \times \omega^\omega \times \omega^\omega / \theta \in \text{WO and } A^\infty \in \Pi_{|\theta|}^0 \cap \Delta_1^1(A, \gamma)\}$ . The same thing is true with  $\Sigma := \{(A, \gamma, \theta) \in 2^{n^{<\omega}} \times \omega^\omega \times \omega^\omega / \theta \in \text{WO and } A^\infty \in \Sigma_{|\theta|}^0 \cap \Delta_1^1(A, \gamma)\}$ .

(b)  $\Sigma_1 := \{A \in 2^{n^{<\omega}} / A^\infty \in \Sigma_1^0 \cap \Delta_1^1(A)\}$ . In fact,  $\Sigma_\xi := \{A \in 2^{n^{<\omega}} / A^\infty \in \Sigma_\xi^0 \cap \Delta_1^1(A)\}$  is  $\Pi_1^1 \setminus \Delta_1^1$  if  $1 \leq \xi < \omega_1$ . Similarly,  $\Pi_\xi := \{A \in 2^{n^{<\omega}} / A^\infty \in \Pi_\xi^0 \cap \Delta_1^1(A)\}$  is  $\Pi_1^1 \setminus \Delta_1^1$  if  $2 \leq \xi < \omega_1$ .

(c)  $\Delta := \{A \in 2^{n^{<\omega}} / A^\infty \in \Delta_1^1(A)\}$ .

**Proof.** Consider the way of coding the Borel sets used in [Lou]. By Lemma 13 we get

$$(A, \gamma, \theta) \in \Pi \Leftrightarrow \begin{cases} \exists p \in \omega \ P(p, A, \gamma, \theta) \text{ and } \forall \alpha \in n^\omega \\ (\alpha \notin A^\infty \text{ or } (p, A, \gamma, \alpha) \in C) \text{ and } [(p, A, \gamma) \in W \text{ and } (p, A, \gamma, \alpha) \notin C] \text{ or} \\ \exists \beta \in \Delta_1^1(A, \gamma, \alpha) \ [(\forall m > 0 \ \beta(m) > 0) \text{ and } (\forall q \in \omega \ \pi(\alpha, \beta, q) \in A)]. \end{cases}$$

This shows that  $\Pi$  is  $\Pi_1^1$ . The same argument works with  $\Sigma$ . From this we can deduce that  $\Sigma_1$  is  $\Pi_1^1$ , if we forget  $\gamma$  and take the section of  $\Sigma$  at  $\theta \in \text{WO} \cap \Delta_1^1$  such that  $|\theta| = 1$ . Similarly,  $\Sigma_\xi$  and  $\Pi_\xi$  are co-analytic if  $\xi \geq 1$ . Forgetting  $\theta$ , we see that the relation “ $A^\infty \in \Delta_1^1(A, \gamma)$ ” is  $\Pi_1^1$ .

• Let us look at the proof of Theorem 12. We will show that if  $\xi \geq 1$  (resp.,  $\xi \geq 2$ ), then  $\Sigma_\xi \setminus I_{\alpha_0}$  (resp.,  $\Pi_\xi \setminus I_{\alpha_0}$ ) is a true co-analytic set. To do this, we will reduce  $WF$  to  $\Sigma_\xi \setminus I_{\alpha_0}$  (resp.,  $\Pi_\xi \setminus I_{\alpha_0}$ ) in a Borel way. We change the definition of  $\Phi$ . We set

$$t \subseteq \alpha_0 \Leftrightarrow \exists k \ t \prec \alpha_0 - \alpha_0[k],$$

$$E := \{(\alpha_0[p]r / p \in \omega \setminus \{2\}, r \in n \setminus \{\alpha_0(p)\})\}, \quad F := \{U^* \not\subseteq \alpha_0 / U \in \phi[T]^{<\omega}\},$$

$$\Phi'(T) := \phi[T] \cup \{s \in n^{<\omega} / \exists t \in E \cup F \ t \prec s\}.$$

This time,  $\Phi'$  is Baire class one, since

$$s \in \Phi'(T) \Leftrightarrow s \in \phi[T] \text{ or } \exists t \in E \ t \prec s \text{ or} \\ \exists U \in (2^{<\omega})^{<\omega} \ (\forall j < |u| \ U(j) \in \phi[T]) \text{ and } U^* \not\subseteq \alpha_0 \text{ and } U^* \prec s.$$

The proof of Theorem 12 remains valid, since if  $\alpha_0 \in (\Phi'(T))^\infty$ , then the decompositions of  $\alpha_0$  into words of  $\Phi'(T)$  are actually decompositions into words of  $\phi[T]$ .

• Let us show that  $(\Phi'(T))^\infty \in \Sigma_1^0 \cap \Delta_1^1(\Phi'(T))$  if  $T \in WF$ . The set  $(\Phi'(T))^\infty$  is

$$\bigcup_{S \in \phi[T]^{<\omega}, l \in n \setminus \{1\}, m \in n \setminus \{0\}} \left[ \left( \bigcup_{s/\exists t \in F \ t \prec s} N_{S^*s} \right) \cup N_{S^*l} \cup N_{S^*1m} \cup (N_{S^*101} \setminus \{S^*\alpha_0\}) \right].$$

If  $\alpha \in n^\omega$ , then  $\alpha$  contains infinitely many  $l \in n \setminus \{1\}$  or finishes with  $1^\infty$ . As  $1^2$  and the sequences beginning with  $l$  are in  $\Phi'(T)$ , the clopen sets are subsets of  $(\Phi'(T))^\infty$  since  $\phi[T]$  and the sequences beginning with  $t \in F$ ,  $l$  or  $1m$  are in  $\Phi'(T)$ . If  $\alpha \in N_{S^*101} \setminus \{S^*\alpha_0\}$ , then let  $p \geq 3$  be maximal such that  $\alpha \upharpoonright (|S^*| + p) = S^*(\alpha_0 \upharpoonright p)$ . We have  $\alpha \in (\Phi'(T))^\infty$  since the sequences beginning with  $(\alpha_0 \upharpoonright p)r$  are in  $\Phi'(T)$ . Thus we get the inclusion into  $(\Phi'(T))^\infty$ .

If  $\alpha \in (\Phi'(T))^\infty$ , then  $\alpha = a_0 a_1 \dots$ , where  $a_i \in \Phi'(T)$ . Either for all  $i$  we have  $a_i \in \phi[T]$ . In this case, there is  $i$  such that  $a_0 \dots a_i \not\subseteq \alpha_0$ , otherwise we could find  $k$  with  $\alpha_0 \upharpoonright k \in (\Phi(T))^\infty$ . But this contradicts the fact that  $T \in WF$ , as in the proof of Theorem 12. So we have  $\alpha \in \bigcup_{\exists t \in F \ t \prec s} N_s$ . Or there exists  $i$  minimal such that  $a_i \notin \phi[T]$ . In this case,

- Either  $\exists t \in E \ t \prec a_i$  and  $\alpha \in \bigcup_{S \in \phi[T]^{<\omega}, l \in n \setminus \{1\}, m \in n \setminus \{0\}} [N_{S^*l} \cup N_{S^*1m} \cup (N_{S^*101} \setminus \{S^*\alpha_0\})]$ ,
- Or  $\exists t \in F \ t \prec a_i$  and  $\alpha \in \bigcup_{S \in \phi[T]^{<\omega}} \bigcup_{s/\exists t \in F \ t \prec s} N_{S^*s}$ .

From this we deduce that  $(\Phi'(T))^\infty$  is  $\Sigma_1^0$ .

Finally, we have

$$\alpha \in (\Phi'(T))^\infty \Leftrightarrow \begin{cases} \exists t \in n^{<\omega} \exists b \in \omega^{<\omega} [(|t| = 1 + \sum_{j < |b|} b(j)) \text{ and } (\forall 0 < m < |b| \ b(m) > 0) \\ \text{and } (\forall q < |b| \ \pi(t0^\infty, b0^\infty, q) \in \Phi'(T))] \text{ and } [\exists l \in n \setminus \{1\} \ tl \prec \alpha \text{ or } t1^2 \prec \alpha]. \end{cases}$$

This shows that  $(\Phi'(T))^\infty$  is  $\Delta_1^1(\Phi'(T))$ .

Therefore,  $\Phi'_{\upharpoonright T}$  reduces  $WF$  to  $\Sigma_\xi \setminus I_{\alpha_0}$  if  $\xi \geq 1$ , and to  $\Pi_\xi \setminus I_{\alpha_0}$  if  $\xi \geq 2$ . So these sets are true co-analytic sets. But  $\Sigma_1 \cap I_{\alpha_0}$  is  $\Pi_1^1$ , by Lemma 13. As  $\Sigma_1 \setminus I_{\alpha_0} = \Sigma_1 \setminus (\Sigma_1 \cap I_{\alpha_0})$ ,  $\Sigma_1$  is not Borel. Thus  $\Sigma$  is not Borel, as before. The argument is similar for  $\Sigma_\xi$ ,  $\Pi_\xi$  ( $\xi \geq 2$ ) and  $\Pi$ . And for  $\Delta$  too.  $\square$

**Question.** Does  $A^\infty \in \Delta_1^1$  imply  $A^\infty \in \Delta_1^1(A)$ ? Probably not. If the answer is positive,  $\Delta$ , and more generally  $\Sigma_\xi$  (for  $\xi \geq 1$ ) and  $\Pi_\xi$  (for  $\xi \geq 2$ ) are true co-analytic sets.

**Remark.** In any case,  $\Delta$  is  $\Sigma_2^1$  because “ $A^\infty \in \Delta_1^1$ ” is equivalent to “ $\exists \gamma \in \omega^\omega \ A^\infty \in \Delta_1^1(A, \gamma)$ ”. This argument shows that  $\Sigma_\xi$  and  $\Pi_\xi$  are  $\Sigma_2^1(\theta)$ , where  $\theta \in WO$  satisfies  $|\theta| = \xi$ . We can say more about  $\Pi_1$ : it is  $\Delta_2^1$ . Indeed, in [St2] we have the following characterization:

$$A^\infty \in \Pi_1^0 \Leftrightarrow \forall \alpha \in n^\omega \ [\forall s \in n^{<\omega} (s \prec \alpha \Rightarrow \exists S \in A^{<\omega} \ s \prec S^*)] \Rightarrow \alpha \in A^\infty.$$

This gives a  $\Pi_2^1$  definition of  $\Pi_1$ . The same fact is true for  $\Sigma_1$ :

**Proposition 16**  $\Sigma_1$  and  $\Pi_1$  are co-nowhere dense  $\Delta_2^1 \setminus D_2(\Sigma_1^0)$  subsets of  $2^{n^{<\omega}}$ . If  $\xi \geq 2$ , then  $\Sigma_\xi$  and  $\Pi_\xi$  are co-nowhere dense  $\Sigma_2^1 \setminus D_2(\Sigma_1^0)$  subsets of  $2^{n^{<\omega}}$ .  $\Delta$  is a co-nowhere dense  $\Sigma_2^1 \setminus D_2(\Sigma_1^0)$  subset of  $2^{n^{<\omega}}$ .

**Proof.** We have seen that  $\Sigma_1$  is  $\Sigma_2^1$ ; it is also  $\Pi_2^1$  because

$$A^\infty \in \Sigma_1^0 \Leftrightarrow \forall \alpha \in n^\omega \quad \alpha \notin A^\infty \text{ or } \exists s \in n^{<\omega} [s \prec \alpha \text{ and } \forall \beta \in n^\omega (s \not\prec \beta \text{ or } \beta \in A^\infty)].$$

By Proposition 4,  $\Pi_0$  is co-nowhere dense, and it is a subset of  $\Sigma_\xi \cap \Pi_\xi \cap \Delta$ . So  $\Sigma_\xi$ ,  $\Pi_\xi$  and  $\Delta$  are co-nowhere dense, and it remains to see that they are not open. It is enough to notice that  $\emptyset$  is not in their interior. Look at the proof of Theorem 12; it shows that for each integer  $m$ , there is a subset  $A_m$  of  $\{s \in 5^{<\omega} / |s| \geq m\}$  such that  $A_m^\infty \notin \Delta_1^1$ . But the argument in the proof of Proposition 11 shows that we can have the same thing in  $n^{<\omega}$  for each  $n \geq 2$ . This gives the result because the sequence  $(A_m)$  tends to  $\emptyset$ .  $\square$

We can say a bit more about  $\Pi_1$  and  $\Sigma_2$ :

**Proposition 17**  $\Pi_1$ ,  $\Pi_1$  and  $\Sigma_2$  are  $\Sigma_2^0$ -hard (so they are not  $\Pi_2^0$ ).

**Proof.** Consider the map  $\phi$  defined in the proof of Proposition 5. By Proposition 2, if  $\gamma \in P_f$ , then  $\phi(\gamma)^\infty$  is  $\Pi_1^0$ . Moreover, as  $\phi(\gamma)$  is an antichain for the extension ordering, the decomposition into words of  $\phi(\gamma)$  is unique. This shows that  $\phi(\gamma)^\infty$  is  $\Delta_1^1$ , because

$$\alpha \in \phi(\gamma)^\infty \Leftrightarrow \exists \beta \in \Delta_1^1(\alpha) [(\forall m > 0 \quad \beta(m) > 0) \text{ and } (\forall q \in \omega \quad \pi(\alpha, \beta, q) \in \phi(\gamma))].$$

So  $\phi(\gamma) \in \Pi_1$  if  $\gamma \in P_f$ . So the preimage of any of the sets in the statement by  $\phi$  is  $P_f$ , and the result follows.  $\square$

## 4 Which sets are $\omega$ -powers?

Now we come to Question (3). Let us specify what we mean by “codes for  $\Gamma$ -sets”, where  $\Gamma$  is a given class, and fix some notation.

- For the Borel classes, we will essentially consider the  $2^\omega$ -universal sets used in [K1] (see Theorem 22.3). For  $\xi \geq 1$ ,  $\mathcal{U}^{\xi, \mathcal{A}}$  (resp.  $\mathcal{U}^{\xi, \mathcal{M}}$ ) is  $2^\omega$ -universal for  $\Sigma_\xi^0(n^\omega)$  (resp.  $\Pi_\xi^0(n^\omega)$ ). So we have

$$- \mathcal{U}^{1, \mathcal{A}} = \{(\gamma, \alpha) \in 2^\omega \times n^\omega / \exists p \in \omega \quad \gamma(p) = 0 \text{ and } s_p^n \prec \alpha\}, \text{ where } (s_p^n)_p \text{ enumerates } n^{<\omega}.$$

$$- \mathcal{U}^{\xi, \mathcal{M}} = \neg \mathcal{U}^{\xi, \mathcal{A}}, \text{ for each } \xi \geq 1.$$

$$- \mathcal{U}^{\xi, \mathcal{A}} = \{(\gamma, \alpha) \in 2^\omega \times n^\omega / \exists p \in \omega \quad ((\gamma)_p, \alpha) \in \mathcal{U}^{\eta, \mathcal{M}}\} \text{ if } \xi = \eta + 1.$$

$$- \mathcal{U}^{\xi, \mathcal{A}} = \{(\gamma, \alpha) \in 2^\omega \times n^\omega / \exists p \in \omega \quad ((\gamma)_p, \alpha) \in \mathcal{U}^{\eta_p, \mathcal{M}}\} \text{ if } \xi \text{ is the limit of the strictly increasing sequence of odd ordinals } (\eta_p).$$

- For the class  $\Sigma_1^1$ , we fix some bijection  $p \mapsto ((p)_0, (p)_1)$  between  $\omega$  and  $\omega^2$ . We set

$$(\gamma, \alpha) \in \mathcal{U} \Leftrightarrow \exists \beta \in 2^\omega \quad (\forall m \exists p \geq m \quad \beta(p) = 1) \text{ and } (\forall p \quad [\gamma(p) = 1 \text{ or } s_{(p)_0}^2 \not\prec \beta \text{ or } s_{(p)_1}^n \not\prec \alpha]).$$

It is not hard to see that  $\mathcal{U}$  is  $2^\omega$ -universal for  $\Sigma_1^1(n^\omega)$ , and we use it here because of the compactness of  $2^\omega \times n^\omega$ , rather than the  $\omega^\omega$ -universal set for  $\Sigma_1^1(n^\omega)$  given in [K1] (see Theorem 14.2).



• For the class  $\Delta_1^1$ , it is different because there is no universal set. But we can use the  $\Pi_1^1$  set of codes  $D \subseteq 2^\omega$  for the Borel sets in [K1] (see Theorem 35.5). We may assume that  $D$ ,  $S$  and  $P$  are effective, by [M].

• The sets we are interested in are the following:

$$\begin{aligned}\mathcal{A}_\xi &:= \{\gamma \in 2^\omega / \mathcal{U}_\gamma^{\xi, \mathcal{A}} \text{ is an } \omega\text{-power}\}, & \mathcal{M}_\xi &:= \{\gamma \in 2^\omega / \mathcal{U}_\gamma^{\xi, \mathcal{M}} \text{ is an } \omega\text{-power}\} \\ \mathcal{B} &:= \{d \in D / D_d \text{ is an } \omega\text{-power}\}, \\ \mathcal{A} &:= \{\gamma \in 2^\omega / \mathcal{U}_\gamma \text{ is an } \omega\text{-power}\}.\end{aligned}$$

As we mentioned in the introduction, Lemma 13 is also related to Question (3). A rough answer to this question is  $\Sigma_3^1$ . Indeed, we have, for  $\gamma \in 2^\omega$ ,

$$\gamma \in \mathcal{A} \Leftrightarrow \exists A \in 2^{n^{<\omega}} \forall \alpha \in n^\omega ([(\gamma, \alpha) \notin \mathcal{U} \text{ or } \alpha \in A^\infty] \text{ and } [\alpha \notin A^\infty \text{ or } (\gamma, \alpha) \in \mathcal{U}]).$$

With Lemma 13, we have a better estimation of the complexity of  $\mathcal{B}$ : it is  $\Sigma_2^1$ . Indeed, for  $d \in D$ ,

$$\begin{aligned}D_d \text{ is an } \omega\text{-power} &\Leftrightarrow \exists A \in 2^{n^{<\omega}} \forall \alpha \in n^\omega ([ (d, \alpha) \notin S \text{ or } \exists \beta \in \Delta_1^1(A, d, \alpha) \\ & \quad [(\forall m > 0 \beta(m) > 0) \text{ and } (\forall q \in \omega \pi(\alpha, \beta, q) \in A)] \text{ and } [\alpha \notin A^\infty \text{ or } (d, \alpha) \in P]).\end{aligned}$$

This argument also shows that  $\mathcal{A}_\xi$  and  $\mathcal{M}_\xi$  are  $\Sigma_2^1$ . We can say more about these two sets.

**Proposition 18** *If  $1 \leq \xi < \omega_1$ , then  $\mathcal{A}_\xi$  and  $\mathcal{M}_\xi$  are  $\Sigma_2^1 \setminus D_2(\Sigma_1^0)$  co-meager subsets of  $2^\omega$ . If moreover  $\xi = 1$ , then they are co-nowhere dense.*

**Proof.** We set  $E_1 := \{\gamma \in 2^\omega / \mathcal{U}_\gamma^{1, \mathcal{A}} = n^\omega\}$ ,  $E_{\eta+1} := \{\gamma \in 2^\omega / \forall p (\gamma)_p \in E_\eta\}$  if  $\eta \geq 1$ , and  $E_\xi := \{\gamma \in 2^\omega / \forall p (\gamma)_p \in E_{\eta_p}\}$  (where  $(\eta_p)$  is a strictly increasing sequence of odd ordinals cofinal in the limit ordinal  $\xi$ ). If  $s \in 2^{<\omega}$ , then we set  $\gamma(p) = s(p)$  if  $p < |s|$ , 0 otherwise. Then  $s \prec \gamma$  and  $\mathcal{U}_\gamma^{1, \mathcal{A}} = n^\omega$ , so  $E_1$  is dense. If  $\gamma_0 \in E_1$ , then for all  $\alpha \in n^\omega$  we can find an integer  $p$  such that  $\gamma_0(p) = 0$  and  $s_p^n \prec \alpha$ . By compactness of  $n^\omega$  we can find a finite subset  $F$  of  $\{p \in \omega / \gamma_0(p) = 0\}$  such that for each  $\alpha \in n^\omega$ ,  $s_p^n \prec \alpha$  for some  $p \in F$ . Now  $\{\gamma \in 2^\omega / \forall p \in F \gamma(p) = 0\}$  is an open neighborhood of  $\gamma_0$  and a subset of  $E_1$ . So  $E_1$  is an open subset of  $2^\omega$ . Now the map  $\gamma \mapsto (\gamma)_p$  is continuous and open, so  $E_{\eta+1}$  and  $E_\xi$  are dense  $G_\delta$  subsets of  $2^\omega$ . Then we notice that  $E_\xi$  is a subset of  $\{\gamma \in 2^\omega / \mathcal{U}_\gamma^{\xi, \mathcal{A}} = n^\omega\}$  (resp.,  $\{\gamma \in 2^\omega / \mathcal{U}_\gamma^{1, \mathcal{A}} = \emptyset\}$ ) if  $\xi$  is odd (resp., even). Indeed, this is clear for  $\xi = 1$ . Then we use the formulas  $\mathcal{U}_\gamma^{\eta+1, \mathcal{A}} = \bigcup_p \neg \mathcal{U}_{(\gamma)_p}^{\eta, \mathcal{A}}$  and  $\mathcal{U}_\gamma^{\xi, \mathcal{A}} = \bigcup_p \neg \mathcal{U}_{(\gamma)_p}^{\eta_p, \mathcal{A}}$ , and by induction we are done. As  $\emptyset$  and  $n^\omega$  are  $\omega$ -powers, we get the results about Baire category. Now it remains to see that  $\mathcal{A}_\xi$  and  $\mathcal{M}_\xi$  are not open. But by induction again  $1^\infty \in \mathcal{A}_\xi \cap \mathcal{M}_\xi$ , so it is enough to see that  $1^\infty$  is not in the interior of these sets.

• Let us show that, for  $O \in \Delta_1^0(n^\omega) \setminus \{\emptyset, n^\omega\}$  and for each integer  $m$ , we can find  $\gamma, \gamma' \in 2^\omega$  such that  $\gamma(j) = \gamma'(j) = 1$  for  $j < m$ ,  $\mathcal{U}_\gamma^{\xi, \mathcal{A}} = O$  and  $\mathcal{U}_{\gamma'}^{\xi, \mathcal{M}} = O$ .

For  $\xi = 1$ , write  $O = \bigcup_p N_{s_{q_k}^n}$ , where  $q_k \geq m$ . Let  $\gamma(q) := 0$  if there exists  $k$  such that  $q = q_k$ ,  $\gamma(q) := 1$  otherwise. The same argument applied to  $\check{O}$  gives the complete result for  $\xi = 1$ .

Now we argue by induction. Let  $\gamma_p \in 2^\omega$  be such that  $\gamma_p(q) = 1$  for  $\langle p, q \rangle < m$  and  $\mathcal{U}_{(\gamma_p)}^{n, \mathcal{M}} = O$ . Then define  $\gamma$  by  $\gamma(\langle p, q \rangle) := \gamma_p(q)$ ; we have  $\gamma(j) = 1$  if  $j < m$  and  $\mathcal{U}_\gamma^{n+1, \mathcal{A}} = \bigcup_p \mathcal{U}_{(\gamma_p)}^{n, \mathcal{M}} = O$ . The argument with  $\check{O}$  still works. The argument is similar for limit ordinals.

• Now we apply this fact to  $O := N_{(0)}$ . This gives  $\gamma_p, \gamma'_p \in N_{1^p}$  such that  $\mathcal{U}_{\gamma_p}^{\xi, \mathcal{A}} = N_{(0)}$  and  $\mathcal{U}_{\gamma'_p}^{\xi, \mathcal{M}} = N_{(0)}$ . But  $(\gamma_p), (\gamma'_p)$  tend to  $1^\infty$ ,  $\gamma_p \notin \mathcal{A}_\xi$  and  $\gamma'_p \notin \mathcal{M}_\xi$ .  $\square$

**Corollary 19**  $\mathcal{A}_1$  is  $\check{D}_2(\Sigma_1^0) \setminus D_2(\Sigma_1^0)$ . In particular,  $\mathcal{A}_1$  is  $\check{D}_2(\Sigma_1^0)$ -complete.

**Proof.** By the preceding proof, it is enough to see that  $\mathcal{A}_1 \setminus \{1^\infty\}$  is open. So let  $\gamma_0 \in \mathcal{A}_1 \setminus \{1^\infty\}$ ,  $p_0$  in  $\omega$  with  $\gamma_0(p_0) = 0$ , and  $A_0 \subseteq n^{<\omega}$  with  $\mathcal{U}_{\gamma_0}^{1, \mathcal{A}} = A_0^\infty$ . If  $\alpha \in n^\omega$ , then  $s_{p_0}^n \alpha \in \mathcal{U}_{\gamma_0}^{1, \mathcal{A}}$ , so we can find  $m > 0$  such that  $\alpha - \alpha \upharpoonright m \in A_0^\infty$ ; thus there exists an integer  $p$  such that  $\gamma_0(p) = 0$  and  $s_p^n \prec \alpha - \alpha \upharpoonright m$ . By compactness of  $n^\omega$ , there are finite sets  $F \subseteq \omega \setminus \{0\}$  and  $G \subseteq \{p \in \omega / \gamma_0(p) = 0\}$  such that  $n^\omega = \bigcup_{m \in F, p \in G} \{\alpha \in n^\omega / s_p^n \prec \alpha - \alpha \upharpoonright m\}$ .

We set  $A_\gamma := \{s \in n^{<\omega} / \exists p \gamma(p) = 0 \text{ and } s_p^n \prec s\}$  for  $\gamma \in 2^\omega$ , so that  $A_\gamma^\infty \subseteq \mathcal{U}_\gamma^{1, \mathcal{A}}$ . Assume that  $\gamma(p) = 0$  for each  $p \in G$  and let  $\alpha \in \mathcal{U}_\gamma^{1, \mathcal{A}}$ . Let  $p^0 \in \omega$  be such that  $\gamma(p^0) = 0$  and  $s_{p^0}^n \prec \alpha$ . We can find  $m_0 > 0$  and  $p^1 \in G$  such that  $s_{p^1}^n \prec \alpha - \alpha \upharpoonright (|s_{p^0}^n| + m_0)$ , and  $\alpha \upharpoonright (|s_{p^0}^n| + m_0) \in A_\gamma$ . Then we can find  $m_1 > 0$  and  $p^2 \in G$  such that  $s_{p^2}^n \prec \alpha - \alpha \upharpoonright (|s_{p^0}^n| + m_0 + |s_{p^1}^n| + m_1)$ , and

$$\alpha \upharpoonright (|s_{p^0}^n| + m_0 + |s_{p^1}^n| + m_1) - \alpha \upharpoonright (|s_{p^0}^n| + m_0) \in A_\gamma.$$

And so on. Thus  $\alpha \in A_\gamma^\infty$  and  $\{\gamma \in 2^\omega / \forall p \in G \gamma(p) = 0\}$  is a clopen neighborhood of  $\gamma_0$  and a subset of  $\mathcal{A}_1$ .  $\square$

**Proposition 20**  $\mathcal{A}$  is  $\Sigma_3^1 \setminus D_2(\Sigma_1^0)$  and is co-nowhere dense.

**Proof.** Let  $U := \{\gamma \in 2^\omega / \forall \beta \in 2^\omega \forall \alpha \in n^\omega \exists p [\gamma(p) = 0 \text{ and } s_{(p)_0}^2 \prec \beta \text{ and } s_{(p)_1}^n \prec \alpha]\}$ . By compactness of  $2^\omega \times n^\omega$ ,  $U$  is a dense open subset of  $2^\omega$ . Moreover, if  $\gamma \in U$ , then  $\mathcal{U}_\gamma = \emptyset$ , so  $U \subseteq \mathcal{A}$  and  $\mathcal{A}$  is co-nowhere dense. It remains to see that  $\mathcal{A}$  is not open, as in the proof of Proposition 18. As  $\mathcal{U}_{1^\infty} = n^\omega$ ,  $1^\infty \in \mathcal{A}$ . Let  $p$  be an integer satisfying  $s_{(p)_0}^2 = \emptyset$  and  $s_{(p)_1}^n = 0^q$ . We set  $\gamma_p(m) := 0$  if and only if  $m = p$ , and also  $P_\infty := \{\alpha \in 2^\omega / \forall r \exists m \geq r \alpha(m) = 1\}$ . Then  $(\gamma_p)$  tends to  $1^\infty$  and we have

$$\begin{aligned} \mathcal{U}_{\gamma_p} &= \{\alpha \in n^\omega / \exists \beta \in P_\infty \forall m \ m \neq p \text{ or } s_{(m)_0}^2 \not\prec \beta \text{ or } s_{(m)_1}^n \not\prec \alpha\} \\ &= \{\alpha \in n^\omega / \exists \beta \in P_\infty (\beta, \alpha) \notin 2^\omega \times N_{0^q}\} = \neg N_{0^q}. \end{aligned}$$

So  $\gamma_p \notin \mathcal{A}$ .  $\square$

## 5 Ordinal ranks and $\omega$ -powers.

**Notation.** The fact that the  $\omega$ -powers are  $\Sigma_1^1$  implies the existence of a co-analytic rank on the complement of  $A^\infty$  (see 34.4 in [K1]). We will consider a natural one, defined as follows. We set, for  $\alpha \in n^\omega$ ,  $T_A(\alpha) := \{S \in (A^-)^{<\omega} / S^* \prec \alpha\}$ . This is a tree on  $A^-$ , which is well founded if and only if  $\alpha \notin A^\infty$ .

The rank of this tree is the announced rank  $R_A : \neg A^\infty \rightarrow \omega_1$  (see page 10 in [K1]): we have  $R_A(\alpha) := \rho(T_A(\alpha))$ . Let  $\phi : A^- \rightarrow \omega$  be one-to-one, and  $\tilde{\phi}(S) := (\phi[S(0)], \dots, \phi[S(|s| - 1)])$  for  $S \in (A^-)^{<\omega}$ . This allows us to define the map  $\Phi$  from the set of trees on  $A^-$  into the set of trees on  $\omega$ , which associates  $\{\tilde{\phi}(S)/S \in T\}$  to  $T$ . As  $\tilde{\phi}$  is one-to-one,  $\Phi$  is continuous:

$$t \in \Phi(T) \Leftrightarrow t \in \tilde{\phi}[(A^-)^{<\omega}] \text{ and } \tilde{\phi}^{-1}(t) \in T.$$

Moreover,  $T$  is well-founded if and only if  $\Phi(T)$  is well-founded. Thus, if  $\alpha \notin A^\infty$ , then we have  $\rho(T_A(\alpha)) = \rho(\Phi[T_A(\alpha)])$  because  $\tilde{\phi}$  is strictly monotone (see page 10 in [K1]). Thus  $R_A$  is a co-analytic rank because the function from  $n^\omega$  into the set of trees on  $\omega^{<\omega}$  which associates  $\Phi[T_A(\alpha)]$  to  $\alpha$  is continuous, and because the rank of the well-founded trees on  $\omega$  defines a co-analytic rank (see 34.6 in [K1]). We set

$$R(A) := \sup\{R_A(\alpha)/\alpha \notin A^\infty\}.$$

By the boundedness theorem,  $A^\infty$  is Borel if and only if  $R(A) < \omega_1$  (see 34.5 and 35.23 in [K1]). We can ask the question of the link between the complexity of  $A^\infty$  and the ordinal  $R(A)$  when  $A^\infty$  is Borel.

**Proposition 21** *If  $\xi < \omega_1$ ,  $r \in \omega$  and  $R(A) = \omega \cdot \xi + r$ , then  $A^\infty \in \Sigma_{2, \xi+1}^0$ .*

**Proof.** The reader should see [L] for operations on ordinals.

• If  $0 < \lambda < \omega_1$  is a limit ordinal, then let  $(\lambda_q)$  be a strictly increasing co-final sequence in  $\lambda$ , with  $\lambda_q = \omega \cdot \theta + q$  if  $\lambda = \omega \cdot (\theta + 1)$ , and  $\lambda_q = \omega \cdot \xi_q$  if  $\lambda = \omega \cdot \xi$ , where  $(\xi_q)$  is a strictly increasing co-final sequence in the limit ordinal  $\xi$  otherwise. By induction, we define

$$\begin{aligned} E_0 &:= \{\alpha \in n^\omega / \forall s \in A^- \ s \not\prec \alpha\}, \\ E_{\theta+1} &:= \{\alpha \in n^\omega / \forall s \in A^- \ s \not\prec \alpha \text{ or } \alpha - s \in E_\theta\}, \\ E_\lambda &:= \{\alpha \in n^\omega / \forall s \in A^- \ s \not\prec \alpha \text{ or } \exists q \in \omega \ \alpha - s \in E_{\lambda_q}\}. \end{aligned}$$

• Let us show that  $E_{\omega \cdot \xi + r} \in \Pi_{2, \xi+1}^0$ . We may assume that  $\xi \neq 0$  and that  $r = 0$ . If  $\xi = \theta + 1$ , then  $E_{\lambda_q} \in \Pi_{2, \theta+1}^0$  by induction hypothesis, thus  $E_{\omega \cdot \xi + r} \in \Pi_{2, \theta+3}^0 = \Pi_{2, \xi+1}^0$ . Otherwise,  $E_{\lambda_q} \in \Pi_{2, \xi_q+1}^0$  by induction hypothesis, thus  $E_{\omega \cdot \xi + r} \in \Pi_{\xi+1}^0 = \Pi_{2, \xi+1}^0$ .

• Let us show that if  $\alpha \in A^\infty$ , then  $\alpha \notin E_{\omega \cdot \xi + r}$ . If  $\xi = r = 0$ , it is clear. If  $r = m + 1$  and  $s \in A^-$  satisfies  $s \prec \alpha$  and  $\alpha - s \in A^\infty$ , then we have  $\alpha - s \notin E_{\omega \cdot \xi + m}$  by induction hypothesis, thus  $\alpha \notin E_{\omega \cdot \xi + r}$ . If  $r = 0$  and  $s \in A^-$  satisfies  $s \prec \alpha$  and  $\alpha - s \in A^\infty$ , then we have  $\alpha - s \notin E_{\lambda_q}$  for each integer  $q$ , by induction hypothesis, thus  $\alpha \notin E_{\omega \cdot \xi + r}$ .

• Let  $s \in A^-$  such that  $s \prec \alpha \notin A^\infty$ . We have

$$\begin{aligned} \rho(T_A(\alpha - s)) &= \sup\{\rho_{T_A(\alpha-s)}(t) + 1 / t \in T_A(\alpha - s)\} \\ &\leq \sup\{\rho_{T_A(\alpha)}((s)t) + 1 / (s)t \in T_A(\alpha)\} \\ &\leq \rho_{T_A(\alpha)}((s)) + 1 \\ &\leq \rho_{T_A(\alpha)}(\emptyset) < \rho(T_A(\alpha)). \end{aligned}$$

The first inequality comes from the fact that the map from  $T_A(\alpha - s)$  into  $T_A(\alpha)$ , which associates  $(s)t$  to  $t$  is strictly monotone (see page 10 in [K1]). We have

$$\rho(T_A(\alpha)) \geq [\sup\{\rho(T_A(\alpha - s)) / s \in A^-, s \prec \alpha\}] + 1.$$

Let us show that we actually have equality. We have

$$\rho(T_A(\alpha)) = \rho_{T_A(\alpha)}(\emptyset) + 1 = \sup\{\rho_{T_A(\alpha)}((s)) + 1 / s \in A^-, s \prec \alpha\} + 1.$$

Therefore, it is enough to notice that if  $s \in A^-$  and  $s \prec \alpha$ , then  $\rho_{T_A(\alpha)}((s)) \leq \rho_{T_A(\alpha-s)}(\emptyset)$ . But this comes from the fact that the map from  $\{S \in T_A(\alpha) / S(0) = s\}$  into  $T_A(\alpha - s)$ , which associates  $S - (s)$  to  $S$ , preserves the extension ordering (see page 352 in [K1]).

• Let us show that, if  $\alpha \notin A^\infty$ , then “ $\rho(T_A(\alpha)) \leq \omega \cdot \xi + r + 1$ ” is equivalent to “ $\alpha \in E_{\omega \cdot \xi + r}$ ”. We do it by induction on  $\omega \cdot \xi + r$ . If  $\xi = r = 0$ , then it is clear. If  $r = m + 1$ , then “ $\rho(T_A(\alpha)) \leq \omega \cdot \xi + r + 1$ ” is equivalent to “ $\forall s \in A^-, s \not\prec \alpha$  or  $\rho(T_A(\alpha - s)) \leq \omega \cdot \xi + m + 1$ ”, by the preceding point. This is equivalent to “ $\forall s \in A^-, s \not\prec \alpha$  or  $\alpha - s \in E_{\omega \cdot \xi + m}$ ”, which is equivalent to “ $\alpha \in E_{\omega \cdot \xi + r}$ ”. If  $r = 0$ , then “ $\rho(T_A(\alpha)) \leq \omega \cdot \xi + r + 1$ ” is equivalent to “ $\forall s \in A^-, s \not\prec \alpha$  or there exists an integer  $q$  such that  $\rho(T_A(\alpha - s)) \leq \lambda_q + 1$ ”. This is equivalent to “ $\forall s \in A^-, s \not\prec \alpha$  or there exists an integer  $q$  such that  $\alpha - s \in E_{\lambda_q}$ ”, which is equivalent to “ $\alpha \in E_{\omega \cdot \xi + r}$ ”.

• If  $\alpha \notin A^\infty$ , then  $\rho(T_A(\alpha)) \leq \omega \cdot \xi + r + 1$ . By the preceding point,  $\alpha \in E_{\omega \cdot \xi + r}$ . Thus we have  $A^\infty = \neg E_{\omega \cdot \xi + r} \in \Sigma_{2 \cdot \xi + 1}^0$ .  $\square$

We can find an upper bound for the rank  $R$ , for some Borel classes:

**Proposition 22** (a)  $A^\infty = n^\omega$  if and only if  $R(A) = 0$ .

(b) If  $A^\infty = \emptyset$ , then  $R(A) = 1$ .

(c) If  $A^\infty \in \Delta_1^0$ , then  $R(A) < \omega$ , and there exists  $A_p \subseteq 2^{<\omega}$  such that  $A_p^\infty \in \Delta_1^0$  and  $R(A_p) = p$  for each integer  $p$ .

(d) If  $A^\infty \in \Pi_1^0$ , then  $R(A) \leq \omega$ , and  $(A^\infty \notin \Sigma_1^0) \Leftrightarrow R(A) = \omega$ .

**Proof.** (a) If  $\alpha \notin A^\infty$ , then  $\emptyset \in T_A(\alpha)$  and  $\rho(T_A(\alpha)) \geq \rho_{T_A(\alpha)}(\emptyset) + 1 \geq 1$ .

(b) We have  $T_A(\alpha) = \{\emptyset\}$  for each  $\alpha$ , and  $\rho(T_A(\alpha)) = \rho_{T_A(\alpha)}(\emptyset) + 1 = 1$ .

(c) By compactness, there exists  $s_1, \dots, s_p \in n^{<\omega}$  such that  $A^\infty = \bigcup_{1 \leq m \leq p} N_{s_m} \in \Delta_1^0$ . If  $\alpha \notin A^\infty$ , then we have  $N_{\alpha \upharpoonright \max_{1 \leq m \leq p} |s_m|} \subseteq \neg A^\infty$ , thus  $\rho(T_A(\alpha)) \leq \max_{1 \leq m \leq p} |s_m| + 1 < \omega$ . So we get the first point. To see the second one, we set  $A_0 := 2^{<\omega}$ . If  $p > 0$ , then we set

$$A_p := \{0^2\} \cup \bigcup_{q \leq p} \{s \in 2^{<\omega} / 0^{2q} 1 \prec s\} \cup \{s \in 2^{<\omega} / 0^{2p+1} \prec s\}.$$

Then  $A_p^\infty = \bigcup_{q \leq p} N_{0^{2q} 1} \cup N_{0^{2p+1}} \in \Delta_1^0$ . If  $\alpha_p := 0^{2p-1} 1^\infty$ , then  $\rho(T_{A_p}(\alpha_p)) = p$ . If  $\alpha \notin A_p^\infty$ , then  $\rho(T_{A_p}(\alpha)) \leq p$ .

(d) If  $A^\infty \in \mathbf{\Pi}_1^0$  and  $\alpha \notin A^\infty$ , then let  $s \in n^{<\omega}$  with  $\alpha \in N_s \subseteq \neg A^\infty$ . Then  $\rho(T_A(\alpha)) \leq |s| + 1$ . Thus  $R(A) \leq \omega$ . If  $A^\infty \notin \mathbf{\Sigma}_1^0$ , then we have  $R(A) \geq \omega$ , by Proposition 21. Thus  $R(A) = \omega$ . Conversely, we apply (c).  $\square$

**Remark.** Notice that it is not true that if the Wadge class  $\langle A^\infty \rangle$ , having  $A^\infty$  as a complete set, is a subclass of  $\langle B^\infty \rangle$ , then  $R(A) \leq R(B)$ . Indeed, for  $A$  we take the example  $A_2$  in (c), and for  $B$  we take the example for  $\mathbf{\Sigma}_1^0$  that we met in the proof of Proposition 11. If we exchange the roles of  $A$  and  $B$ , then we see that the converse is also false. This example  $A$  for  $\mathbf{\Sigma}_1^0$  shows that Proposition 21 is optimal for  $\xi = 0$  since  $R(A) = 1$  and  $A^\infty \in \mathbf{\Sigma}_1^0 \setminus \mathbf{\Pi}_1^0$ . We can say more: it is not true that if  $A^\infty = B^\infty$ , then  $R(A) \leq R(B)$ . We use again (c): we take  $A := A_2$  and  $B := A \setminus \{0^2\}$ . We have  $A^\infty = B^\infty = A_2^\infty$ ,  $R(A) = 2$  and  $R(B) = 1$ .

**Proposition 23** *For each  $\xi < \omega_1$ , there exists  $A_\xi \subseteq 2^{<\omega}$  with  $A_\xi^\infty \in \mathbf{\Sigma}_1^0$  and  $R(A_\xi) \geq \xi$ .*

**Proof.** We use the notation in the proof of Theorem 15. Let  $T \in \mathcal{T}$ , and  $\varphi : T \rightarrow T_{\Phi'(T)}(\alpha_0)$  defined by the formula  $\varphi(s) := (\phi(s[0]), \dots, \phi(s[|s| - 1]))$ . Then  $\varphi$  is strictly monotone. If  $T \in WF$ , then  $\alpha_0 \notin (\Phi'(T))^\infty$  and  $T_{\Phi'(T)}(\alpha_0) \in WF$ . In this case,  $\rho(T) \leq \rho(T_{\Phi'(T)}(\alpha_0)) = R_{\Phi'(T)}(\alpha_0)$  (see page 10 in [K1]). Let  $T_\xi \in WF$  be a tree with rank at least  $\xi$  (see 34.5 and 34.6 in [K1]). We set  $A_\xi := \Phi'(T_\xi)$ . It is clear that  $A_\xi$  is what we were looking for.  $\square$

**Remark.** Let  $\psi : 2^{n^{<\omega}} \rightarrow \{\text{Trees on } n^{<\omega}\}$  defined by  $\psi(A) := T_A(\alpha_0)$ , and  $r : \neg I_{\alpha_0} \rightarrow \omega_1$  defined by  $r(A) := \rho(T_A(\alpha_0))$ . Then  $\psi$  is continuous, thus  $r$  is a  $\mathbf{\Pi}_1^1$ -rank on

$$\psi^{-1}(\{\text{Well-founded trees on } n^{<\omega}\}) = \neg I_{\alpha_0}.$$

By the boundedness theorem, the rank  $r$  and  $R$  are not bounded on  $\neg I_{\alpha_0}$ . Proposition 23 specifies this result. It shows that  $R$  is not bounded on  $\mathbf{\Sigma}_1 \setminus I_{\alpha_0}$ .

## 6 The extension ordering.

**Proposition 24** *We equip  $A$  with the extension ordering.*

(a) *If  $A \subseteq n^{<\omega}$  is an antichain, then  $A^\infty$  is in  $\{\emptyset\} \cup \{n^\omega\} \cup [\mathbf{\Pi}_1^0 \setminus \mathbf{\Sigma}_1^0] \cup [II_2^0(A) \setminus \mathbf{\Sigma}_2^0]$ , and any of these cases is possible.*

(b) *If  $A \subseteq n^{<\omega}$  has finite antichains, then  $A^\infty \in \mathbf{\Pi}_2^0$  (and is not  $\mathbf{\Sigma}_2^0$  in general).*

**Proof.** Let  $G := \{\alpha \in n^\omega / \forall r \exists m \exists p \geq r \ \alpha \upharpoonright m \in [(A^-)^p]^*\}$ . Then  $G \in II_2^0(A)$  and contains  $A^\infty$ . Conversely, if  $\alpha \in G$ , then we have  $T_A(\alpha) \cap (A^-)^p \neq \emptyset$  for each integer  $p$ , thus  $T_A(\alpha)$  is infinite.

(a) If  $A$  is an antichain, then each sequence in  $T_A(\alpha)$  has at most one extension in this tree adding one to the length. Thus  $T_A(\alpha)$  is finite splitting. This implies that  $T_A(\alpha)$  has an infinite branch if  $\alpha \in G$ , by König's lemma. Therefore  $A^\infty = G \in II_2^0(A)$ .

- If we take  $A := \emptyset$ , then  $A$  is an antichain and  $A^\infty = \emptyset$ .
- If we take  $A := \{(0), \dots, (n-1)\}$ , then  $A$  is an antichain and  $A^\infty = n^\omega$ .
- If  $A^\infty \notin \{\emptyset, n^\omega\}$ , then  $A^\infty \notin \Sigma_1^0$ . Indeed, let  $\alpha_0 \notin A^\infty$  and  $s_0 \in A^-$ . By uniqueness of the decomposition into words of  $A^-$ , the sequence  $(s_0^n \alpha_0)_n \subseteq n^\omega \setminus A^\infty$  tends to  $s_0^\infty \in A^\infty$ .
- If we take  $A := \{(0)\}$ , then  $A$  is an antichain and  $A^\infty = \{0^\infty\} \in \Pi_1^0 \setminus \Sigma_1^0$ .
- If  $A$  is finite, then  $A^\infty$  is  $\Pi_1^0 \setminus \Sigma_1^0$  or is in  $\{\emptyset, n^\omega\}$ , by the facts above and Proposition 2.
- If  $A$  is infinite, then  $A^\infty \notin \Sigma_2^0$  because the map  $c$  in the proof of Proposition 2 is a homeomorphism and  $(A^-)^\omega$  is not  $K_\sigma$ .
- If  $A := \{0^k 1/k \in \omega\}$ , then  $A$  is an antichain and  $A^\infty = P_\infty$ , which is  $\Pi_2^0 \setminus \Sigma_2^0$ .

(b) The intersection of  $P_\infty$  with  $N_1$  can be made with the chain  $\{10^k/k \in \omega\}$ . So let us assume that  $A$  has finite antichains.

- Let us show that  $A$  is the union of a finite set and of a finite union of infinite subsets of sets of the form  $A_{\alpha_m} := \{s \in n^{<\omega} / s \prec \alpha_m\}$ . Let us enumerate  $A := \{s_r/r \in \omega\}$ . We construct a sequence  $(A_m)$ , finite or not, of subsets of  $A$ . We do it by induction on  $r$ , to decide in which set  $A_m$  the sequence  $s_r$  is. First,  $s_0 \in A_0$ . Assume that  $s_0, \dots, s_r$  have been put into  $A_0, \dots, A_{p_r}$ , with  $p_r \leq r$  and  $A_m \cap \{s_0, \dots, s_r\} \neq \emptyset$  if  $m \leq p_r$ . We choose  $m \leq p_r$  minimal such that  $s_{r+1}$  is compatible with all the sequences in  $A_m \cap \{s_0, \dots, s_r\}$ , we put  $s_{r+1}$  into  $A_m$  and we set  $p_{r+1} := p_r$  if possible. Otherwise, we put  $s_{r+1}$  into  $A_{p_{r+1}}$  and we set  $p_{r+1} := p_r + 1$ .

Let us show that there are only finitely many infinite  $A_m$ 's. If  $A_m$  is infinite, then there exists a unique sequence  $\alpha_m \in n^\omega$  such that  $A_m \subseteq A_{\alpha_m}$ . Let us argue by contradiction: there exists an infinite sequence  $(m_q)_q$  such that  $A_{m_q}$  is infinite. Let  $t_0$  be the common beginning of the  $\alpha_{m_q}$ 's. There exists  $\varepsilon_0 \in n$  such that  $N_{t_0\varepsilon_0} \cap \{\alpha_{m_q}/q \in \omega\}$  is infinite. We choose a sequence  $u_0$  in  $A$  extending  $t_0\mu_0$ , where  $\mu_0 \neq \varepsilon_0$ . Then we do it again: let  $t_0\varepsilon_0t_1$  be the common beginning of the elements of  $N_{t_0\varepsilon_0} \cap \{\alpha_{m_q}/q \in \omega\}$ . There exists  $\varepsilon_1 \in n$  such that  $N_{t_0\varepsilon_0t_1\varepsilon_1} \cap \{\alpha_{m_q}/q \in \omega\}$  is infinite. We choose a sequence  $u_1$  in  $A$  extending  $t_0\varepsilon_0t_1\mu_1$ , where  $\mu_1 \neq \varepsilon_1$ . The sequence  $(u_l)$  is an infinite antichain in  $A$ . But this is absurd. Now let us choose the longest sequence in each nonempty finite  $A_m$ ; this gives an antichain in  $A$  and the result.

- Now let  $\alpha \in G$ . There are two cases. Either for each  $m$  and for each integer  $k$ ,  $\alpha \upharpoonright k \notin [A^{<\omega}]^*$  or  $\alpha \upharpoonright k \neq \alpha_m$ . In this case,  $T_A(\alpha)$  is finite splitting. As  $T_A(\alpha)$  is infinite,  $T_A(\alpha)$  has an infinite branch witnessing that  $\alpha \in A^\infty$ , by König's lemma. Otherwise,  $\alpha \in \bigcup_{s \in [A^{<\omega}]^*, m} \{s\alpha_m\}$ , which is countable. Thus  $G \setminus A^\infty \in \Sigma_2^0$  and  $A^\infty = G \setminus (G \setminus A^\infty) \in \Pi_2^0$ .  $\square$

## 7 Examples.

- We have seen examples of subsets  $A$  of  $2^{<\omega}$  such that  $A^\infty$  is complete for the classes  $\{\emptyset\}$ ,  $\{n^\omega\}$ ,  $\Delta_1^0$ ,  $\Sigma_1^0$ ,  $\Pi_1^0$ ,  $\Pi_2^0$  and  $\Sigma_1^1$ . We will give some more examples, for some classes of Borel sets. Notice that to show that a set in such a non self-dual class is complete, it is enough to show that it is true (see 21.E, 22.10 and 22.26 in [K1]).

• For the class  $\Sigma_1^0 \oplus \Pi_1^0 := \{(U \cap O) \cup (F \setminus O) / U \in \Sigma_1^0, O \in \Delta_1^0, F \in \Pi_1^0\}$ , we can take  $A := \{s \in 2^{<\omega} / 0^2 1 \prec s \text{ or } s = 0^2 \text{ or } \exists p \in \omega \ 10^p 1 \prec s\}$ , since  $A^\infty = \{0^\infty\} \cup \bigcup_q N_{0^{2q+2}1} \cup N_1 \setminus \{10^\infty\}$ .

• For the class  $\check{D}_2(\Sigma_1^0) := \{U \cup F / U \in \Sigma_1^0, F \in \Pi_1^0\}$ , we can take Example 9 in [St2]:  $A := \{s \in 2^{<\omega} / 0 \prec s \text{ or } \exists q \in \omega \ (101)^q 1^3 \prec s \text{ or } s = 10^2\}$ . We have

$$A^\infty = \bigcup_{p \in \omega} [N_{(10^2)^{p0}} \cup (\bigcup_{q \in \omega} N_{(10^2)^p (101)^q 1^3})] \cup \{(10^2)^\infty\},$$

which is a  $\neg D_2(\Sigma_1^0)$  set. Towards a contradiction, assume that  $A^\infty$  is  $D_2(\Sigma_1^0)$ :

$$A^\infty = U_1 \cap F = U \cup F_2,$$

where the  $U$ 's are open and the  $F$ 's are closed. Let  $O$  be a clopen set separating  $\neg U_1$  from  $F_2$  (see 22.C in [K1]). Then  $A^\infty = (U \cap O) \cup (F \setminus O)$  would be in  $\Sigma_1^0 \oplus \Pi_1^0$ . If  $(10^2)^\infty \in O$ , then we would have  $N_{(10^2)^{p_0}} \subseteq O$  for some integer  $p_0$ . But the sequence  $((10^2)^p (1^2 0)^\infty)_{p \geq p_0} \subseteq O \setminus U$  and tends to  $(10^2)^\infty$ , which is absurd. If  $(10^2)^\infty \notin O$ , then we would have  $N_{(10^2)^{q_0}} \subseteq \neg O$  for some integer  $q_0$ . But the sequence  $((10^2)^{q_0} (101)^q 1^\infty)_{q \geq q_0} \subseteq F \setminus O$  and tends to  $(10^2)^{q_0} (101)^\infty$ , which is absurd.

• For the class  $D_2(\Sigma_1^0)$ , we can take  $A := [A_1^{<\omega}]^* \setminus [A_0^{<\omega}]^*$ , where  $A_0 := \{010, 01^2\}$  and

$$A_1 := \{010, 01^2, 0^2, 0^3, 10^2, 1^2 0, 10^3, 1^2 0^2\}.$$

We have  $A^\infty = A_1^\infty \setminus A_0^\infty$ . Indeed, as  $A \subseteq [A_1^{<\omega}]^*$ , we have  $A^\infty \subseteq A_1^\infty$ . If  $\alpha \in A_0^\infty$ , then its decomposition into words of  $A_1$  is unique and made of words in  $A_0$ . Thus  $\alpha \notin A^\infty$  and

$$A^\infty \subseteq A_1^\infty \setminus A_0^\infty.$$

Conversely, if  $\alpha = a_0 a_1 \dots \in A_1^\infty \setminus A_0^\infty$ , with  $a_i \in A_1^-$ , then there are two cases. Either there are infinitely many indexes  $i$  (say  $i_0, i_1, \dots$ ) such that  $a_i \notin A_0$ . In this case, the words  $a_0 \dots a_{i_0}$ ,  $a_{i_0+1} \dots a_{i_1}, \dots$ , are in  $A$  and  $\alpha \in A^\infty$ . Or there exists a maximal index  $i$  such that  $a_i \notin A_0$ . In this case,  $a_0 \dots a_i 0, 10^2, 1^2 0 \in A$ , thus  $\alpha \in A^\infty = A_1^\infty \setminus A_0^\infty$ . Proposition 2 shows that  $A \in D_2(\Sigma_1^0)$ . If  $A^\infty = U \cup F$ , with  $U \in \Sigma_1^0$  and  $F \in \Pi_1^0$ , then we have  $U = \emptyset$  because  $A_1^\infty$  is nowhere dense (every sequence in  $A_1$  contains 0, thus the sequences in  $A_1^\infty$  have infinitely many 0's). Thus  $A^\infty$  would be closed. But this contradicts the fact that  $((01^2)^n 0^\infty)_n \subseteq A^\infty$  and tends to  $(01^2)^\infty \notin A^\infty$ . Thus  $A^\infty$  is a true  $D_2(\Sigma_1^0)$  set.

• For the class  $\check{D}_3(\Sigma_1^0)$ , we can take  $A := ([A_2^{<\omega}]^* \setminus [A_1^{<\omega}]^*) \cup [A_0^{<\omega}]^*$ , where  $A_0 := \{0^2\}$ ,  $A_1 := \{0^2, 01\}$ , and  $A_2 := \{0^2, 01, 10, 10^2\}$ . We have  $A^\infty = (A_2^\infty \setminus A_1^\infty) \cup A_0^\infty$ . Indeed, as  $A \subseteq [A_2^{<\omega}]^*$ , we have  $A^\infty \subseteq A_2^\infty$ . If  $\alpha \in A_1^\infty$ , then its decomposition into words of  $A_2^-$  is unique and made of words in  $A_1$ . If moreover  $\alpha \notin A_0^\infty$ , then it is clear that  $\alpha \notin A^\infty$  and

$$A^\infty \subseteq (A_2^\infty \setminus A_1^\infty) \cup A_0^\infty.$$

Conversely, it is clear that  $A_0^\infty \subseteq A^\infty$ . If  $\alpha = a_0 a_1 \dots \in A_2^\infty \setminus A_1^\infty$ , then the argument above still works. We have to check that  $s := a_0 \dots a_{i_0} \notin [A_1^{<\omega}]^*$ . It is clear if  $a_{i_0} = 10$ . Otherwise,  $a_{i_0} = 10^2$  and we argue by contradiction.

The length of  $s$  is even and the decomposition of  $s$  into words of  $A_1$  is unique. It finishes with  $0^2$ , and the even coordinates of the sequence  $s$  are 0. Therefore,  $a_{i_0-1} = 0^2$  or  $10$ ; we have the same thing with  $a_{i_0-2}, a_{i_0-3}, \dots$ . Because of the parity, some 0 remains at the beginning. But this is absurd. Now we have to check that  $a_0 \dots a_i 0 \notin [A_1^{<\omega}]^*$ . It is clear if  $a_i = 10^2$ . Otherwise,  $a_i = 10$  and the argument above works.

Finally, we have to check that if  $\gamma \in A_1^\infty$ , then  $\gamma - (0) \in A^\infty$ . There is a sequence  $p_0, p_1, \dots$ , finite or not, such that  $\gamma = (0^{2p_0})(01)(0^{2p_1})(01)\dots 0^\infty$ . Therefore

$$\gamma - (0) = (0^{2p_0}10)(0^{2p_1}10)\dots(0^2)^\infty \in A^\infty.$$

If we set  $U_i := \neg A_{2-i}^\infty$ , then we see that  $A^\infty \in \check{D}_3(\Sigma_1^0)$ . If  $\alpha$  finishes with  $1^\infty$ , then  $\alpha \notin A_2^\infty$ ; thus  $A_2^\infty$  is nowhere dense, just like  $A^\infty$ . Thus if  $A^\infty = (U_2 \setminus U_1) \cup U_0$  with  $U_i$  open, then  $U_0 = \emptyset$ . By uniqueness of the decomposition of a sentence in  $A_i^\infty$  into words of  $A_{i+1}$ , we see that  $A_i^\infty$  is nowhere dense in  $A_{i+1}^\infty$ . So let  $x_\emptyset \in A_0^\infty$ ,  $(x_n) \subseteq A_1^\infty \setminus A_0^\infty$  converging to  $x_\emptyset$ , and  $(x_{n,m})_m \subseteq A_2^\infty \setminus A_1^\infty$  converging to  $x_n$ . Then  $x_{n,m} \in U_1$ , which is absurd. Thus  $A^\infty \notin D_3(\Sigma_1^0)$ .

• For the class  $\check{D}_2(\Sigma_2^0)$ , we can take  $A := \{s \in 2^{<\omega} / 1^2 \prec s \text{ or } s = (0)\}$ . We can write

$$A^\infty = (\{0^\infty\} \cup \bigcup_p N_{0^p 1^2}) \cap (P_f \cup \{\alpha \in 2^\omega / \forall n \exists m \geq n \alpha(m) = \alpha(m+1) = 1\}).$$

Then  $A^\infty \notin D_2(\Sigma_2^0)$ , otherwise  $A^\infty \cap N_{1^2} \in D_2(\Sigma_2^0)$  and would be a comeager subset of  $N_{1^2}$ . We could find  $s \in 2^{<\omega}$  with even length such that  $A^\infty \cap N_{1^2 s} \in \mathbf{\Pi}_2^0$ . We define a continuous function  $f : 2^\omega \rightarrow 2^\omega$  by formulas  $f(\alpha)(2n) := \alpha(n)$  if  $n > \frac{|s|+1}{2}$ ,  $(1^2 s)(2n)$  otherwise, and  $f(\alpha)(2n+1) := 0$  if  $n > \frac{|s|}{2}$ ,  $(1^2 s)(2n+1)$  otherwise. It reduces  $P_f$  to  $A^\infty \cap N_{1^2 s}$ , which is absurd.



**Summary of the complexity results in this paper:**

	Baire category	complexity   $\xi = 1$	$\xi = 2$	$\xi \geq 3$
$\Sigma_0$	nowhere dense	$\Pi_1^0 \setminus \Sigma_1^0$		
$\Pi_0$	co-nowhere dense	$\Sigma_1^0 \setminus \Pi_1^0$		
$\Delta_1$	co-nowhere dense	$K_\sigma \setminus \Pi_2^0$		
$\Sigma_\xi$	co-nowhere dense	$\Pi_1^1 \setminus \Delta_1^1$		$\Pi_1^1 \setminus \Delta_1^1$
$\Pi_\xi$	co-nowhere dense	$\Pi_1^1 \setminus \Pi_2^0$	$\Pi_1^1 \setminus \Delta_1^1$	$\Pi_1^1 \setminus \Delta_1^1$
$\Delta$	co-nowhere dense	$\Pi_1^1 \setminus \Delta_1^1$		
$\Sigma_\xi$	co-nowhere dense	$\Delta_2^1 \setminus D_2(\Sigma_1^0)$	$\Sigma_2^1 \setminus \Pi_2^0$	$\Sigma_2^1 \setminus D_2(\Sigma_1^0)$
$\Pi_\xi$	co-nowhere dense	$\Delta_2^1 \setminus \Pi_2^0$	$\Sigma_2^1 \setminus D_2(\Sigma_1^0)$	$\Sigma_2^1 \setminus D_2(\Sigma_1^0)$
$\Delta$	co-nowhere dense	$\Sigma_2^1 \setminus D_2(\Sigma_1^0)$		
$\mathcal{G}_\xi (\xi \in \omega)$		$\Pi_1^0 \setminus \Sigma_1^0$ nowhere dense	$\check{D}_\omega(\Sigma_1^0) \setminus D_\omega(\Sigma_1^0)$	$\Pi_1^1 \setminus D_\omega(\Sigma_1^0)$
$\mathcal{F}$		$\Pi_1^1 \setminus \Pi_2^0$		
$\mathcal{A}_\xi$	co-meager	$\check{D}_2(\Sigma_1^0) \setminus D_2(\Sigma_1^0)$ co-nowhere dense	$\Sigma_2^1 \setminus D_2(\Sigma_1^0)$	$\Sigma_2^1 \setminus D_2(\Sigma_1^0)$
$\mathcal{M}_\xi$	co-meager	$\Sigma_2^1 \setminus D_2(\Sigma_1^0)$ co-nowhere dense	$\Sigma_2^1 \setminus D_2(\Sigma_1^0)$	$\Sigma_2^1 \setminus D_2(\Sigma_1^0)$
$\mathcal{B}$		$\Sigma_2^1$		
$\mathcal{A}$	co-nowhere dense	$\Sigma_3^1 \setminus D_2(\Sigma_1^0)$		

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