Omega-powers and descriptive set theory

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Abstract. We study the sets of the infinite sentences constructible with a dictionary over a finite alphabet, from the viewpoint of descriptive set theory. Among other things, this gives some true co-analytic sets. The case where the dictionary is finite is studied and gives a natural example of a set at the level ω of the Wadge hierarchy.

1 Introduction.

We consider the finite alphabet $n = \{0, \dots, n-1\}$, where $n \geq 2$ is an integer, and a dictionary over this alphabet, i.e., a subset A of the set $n^{<\omega}$ of finite words with letters in n.

Definition 1 *The ω-power associated to A is the set A^∞ of the infinite sentences constructible with A by concatenation. So we have $A^\infty := \{a_0 a_1 \dots \in n^\omega / \forall i \in \omega \;\; a_i \in A\}$.*

The ω-powers play a crucial role in the characterization of subsets of n^ω accepted by finite automata (see Theorem 2.2 in [St1]). We will study these objects from the viewpoint of descriptive set theory. The reader should see [K1] for the classical results of this theory; we will also use the notation of this book. The questions we study are the following:

(1) What are the possible levels of topological complexity for the ω-powers? This question was asked by P. Simonnet in [S], and studied in [St2]. O. Finkel (in [F1]) and A. Louveau proved independently that $\mathbf{\Sigma}^1_1$-complete ω-powers exist. O. Finkel proved in [F2] the existence of a $\mathbf{\Pi}^0_m$-complete ω-power for each integer $m \geq 1$.

(2) What is the topological complexity of the set of dictionaries whose associated ω-power is of a given level of complexity? This question arises naturally when we look at the characterizations of $\mathbf{\Pi}^0_1$, $\mathbf{\Pi}^0_2$ and $\mathbf{\Sigma}^0_1$ ω-powers obtained in [St2] (see Corollary 14 and Lemmas 25, 26).

(3) We will recall that an ω-power is an analytic subset of n^ω. What is the topological complexity of the set of codes for analytic sets which are ω-powers? This question was asked by A. Louveau. This question also makes sense for the set of codes for $\mathbf{\Sigma}^0_\xi$ (resp., $\mathbf{\Pi}^0_\xi$) sets which are ω-powers. And also for the set of codes for Borel sets which are ω-powers.

As usual with descriptive set theory, the point is not only the computation of topological complexities, but also the hope that these computations will lead to a better understanding of the studied objects. Many sets in this paper won't be clopen, in particular won't be recursive. This gives undecidability results.

• We give the answer to Question (2) for the very first levels ($\{\emptyset\}$, its dual class and $\mathbf{\Delta}^0_1$). This contains a study of the case where the dictionary is finite. In particular, we show that the set of dictionaries whose associated ω-power is generated by a dictionary with two words is a $\check{D}_\omega(\mathbf{\Sigma}^0_1)$-complete set. This is a surprising result because this complexity is not clear at all on the definition of the set.

• We give two proofs of the fact that the relation "$\alpha \in A^\infty$" is $\mathbf{\Sigma}^1_1$-complete. One of these proofs is used later to give a partial answer to Question (2). To understand this answer, the reader should see [M] for the basic notions of effective descriptive set theory. Roughly speaking, a set is effectively Borel (resp., effectively Borel in A) if its construction based on basic clopen sets can be coded with a recursive (resp., recursive in A) sequence of integers. This answer is the

Theorem. *The following sets are true co-analytic sets:*

- $\{A \in 2^{n^{<\omega}} / A^\infty \in \Delta^1_1(A)\}$.
- $\{A \in 2^{n^{<\omega}} / A^\infty \in \mathbf{\Sigma}^0_\xi \cap \Delta^1_1(A)\}$, *for* $1 \leq \xi < \omega_1$.
- $\{A \in 2^{n^{<\omega}} / A^\infty \in \mathbf{\Pi}^0_\xi \cap \Delta^1_1(A)\}$, *for* $2 \leq \xi < \omega_1$.

This result also comes from an analysis of Borel ω-powers: A^∞ is Borel if and only if we can choose in a Borel way the decomposition of any sentence of A^∞ into words of A (see Lemma 13). This analysis is also related to Question (3) and to some Borel uniformization result for G_δ sets locally with Borel projections. We will specify these relations.

• A natural ordinal rank can be defined on the complement of any ω-power, and we study it; its knowledge gives an upper bound of the complexity of the ω-power.

• We study the link between Question (1) and the extension ordering on finite sequences of integers.

• Finally, we give some examples of ω-powers complete for the classes $\mathbf{\Delta}^0_1$, $\mathbf{\Sigma}^0_1 \oplus \mathbf{\Pi}^0_1$, $D_2(\mathbf{\Sigma}^0_1)$, $\check{D}_2(\mathbf{\Sigma}^0_1)$, $\check{D}_3(\mathbf{\Sigma}^0_1)$ and $\check{D}_2(\mathbf{\Sigma}^0_2)$.

2 Finitely generated ω-powers.

Notation. In order to answer to Question (2), we set

$$\mathbf{\Sigma}_0 := \{A \subseteq n^{<\omega} / A^\infty = \emptyset\},\ \mathbf{\Pi}_0 := \{A \subseteq n^{<\omega} / A^\infty = n^\omega\},$$

$$\mathbf{\Delta}_1 := \{A \subseteq n^{<\omega} / A^\infty \in \mathbf{\Delta}^0_1\},$$

$$\mathbf{\Sigma}_\xi := \{A \subseteq n^{<\omega} / A^\infty \in \mathbf{\Sigma}^0_\xi\},\ \mathbf{\Pi}_\xi := \{A \subseteq n^{<\omega} / A^\infty \in \mathbf{\Pi}^0_\xi\}\ \ (\xi \geq 1),$$

$$\mathbf{\Delta} := \{A \subseteq n^{<\omega} / A^\infty \in \mathbf{\Delta}^1_1\}.$$

• If $A \subseteq n^{<\omega}$, then we set $A^- := A \setminus \{\emptyset\}$.

• We define, for $s \in n^{<\omega}$ and $\alpha \in n^\omega$, $\alpha - s := (\alpha(|s|), \alpha(|s|+1), ...)$.

• If $\mathcal{S} \subseteq (n^{<\omega})^{<\omega}$, then we set $\mathcal{S}^* := \{S^* := S(0) \dots S(|s|-1) / S \in \mathcal{S}\}$.

• We define a recursive map $\pi : n^\omega \times \omega^\omega \times \omega \to n^{<\omega}$ by

$$\pi(\alpha,\beta,q) := \begin{cases} (\alpha(0),\dots,\alpha(\beta[0])) \text{ if } q=0, \\ (\alpha(1+\Sigma_{j<q}\,\beta[j]),...,\alpha(\Sigma_{j\leq q}\,\beta[j])) \text{ otherwise.} \end{cases}$$

We always have the following equivalence:

$$\alpha\in A^\infty \;\Leftrightarrow\; \exists\beta\in\omega^\omega\;\; [(\forall m>0\;\; \beta(m)>0) \text{ and } (\forall q\in\omega\;\; \pi(\alpha,\beta,q)\in A)].$$

Proposition 2 *([S]) $A^\infty \in \mathbf{\Sigma}^1_1$ for all $A \subseteq n^{<\omega}$. If A is finite, then $A^\infty \in \mathbf{\Pi}^0_1$.*

Proof. We define a continuous map $c : (A^-)^\omega \to n^\omega$ by the formula $c((a_i)) := a_0a_1\dots$ We have $A^\infty = c[(A^-)^\omega]$, and $(A^-)^\omega$ is a Polish space (compact if A is finite). □

Proposition 3 *If $A^\infty \in \mathbf{\Delta}^0_1$, then there exists a finite subset B of A such that $A^\infty = B^\infty$.*

Proof. Set $E_k := \{\alpha \in n^\omega / \alpha\lceil k \in A \text{ and } \alpha - \alpha\lceil k \in A^\infty\}$. It is an open subset of n^ω since A^∞ is open, and $A^\infty \subseteq \bigcup_{k>0} E_k$. We can find an integer p such that $A^\infty \subseteq \bigcup_{0<k\leq p} E_k$, by compactness of A^∞. Let $B := A \cap n^{\leq p}$. If $\alpha \in A^\infty$, then we can find an integer $0 < k_0 \leq p$ such that $\alpha\lceil k_0 \in A$ and $\alpha - \alpha\lceil k_0 \in A^\infty$. Thus $\alpha\lceil k_0 \in B$. Then we do it again with $\alpha - \alpha\lceil k_0$, and so on. Thus we have $\alpha \in B^\infty = A^\infty$. □

Remark. This is not true if we only assume that A^∞ is closed. Indeed, we have the following counter-example, due to O. Finkel:

$$A := \{s \in 2^{<\omega} / \forall i \leq |s|\;\; 2.\text{Card}(\{j < i / s(j) = 1\}) \geq i\}.$$

We have $A^\infty = \{\alpha \in 2^\omega / \forall i \in \omega\;\; 2.\text{Card}(\{j < i / \alpha(j) = 1\}) \geq i\}$ and if B is finite and $B^\infty = A^\infty$, $B \subseteq A$ and $101^2 0^2 \dots \notin B^\infty$.

Theorem 4 *(a) $\mathbf{\Sigma}_0 = \{\emptyset, \{\emptyset\}\}$ is $\mathbf{\Pi}^0_1$-complete.*
(b) $\mathbf{\Pi}_0$ is a dense $\mathbf{\Sigma}^0_1$ subset of $2^{n^{<\omega}}$. In particular, $\mathbf{\Pi}_0$ is $\mathbf{\Sigma}^0_1$-complete.
(c) $\mathbf{\Delta}_1$ is a $K_\sigma \setminus \mathbf{\Pi}^0_2$ subset of $2^{n^{<\omega}}$. In particular, $\mathbf{\Delta}_1$ is $\mathbf{\Sigma}^0_2$-complete.

Proof. (a) Is clear.

(b) If we can find $m \in \omega$ with $n^m \subseteq A$, then $A^\infty = n^\omega$. As $\{A \subseteq n^{<\omega} / \exists m \in \omega\; n^m \subseteq A\}$ is a dense open subset of $2^{n^{<\omega}}$, the density follows. The formula

$$A \in \mathbf{\Pi}_0 \;\Leftrightarrow\; \exists m\, \forall s \in n^m\, \exists q \leq m\;\; s\lceil q \in A^-$$

shows that $\mathbf{\Pi}_0$ is Σ^0_1, and comes from Proposition 3.

(c) If $A^\infty \in \mathbf{\Delta}^0_1$, then we can find $p > 0$ such that $A^\infty = (A \cap n^{\leq p})^\infty$, by Proposition 3. So let $s_1, \dots, s_k, t_1, \dots, t_l \in 2^{<\omega}$ be such that $A^\infty = \bigcup_{1\leq i\leq k} N_{s_i} = n^\omega \setminus (\bigcup_{1\leq j\leq l} N_{t_j})$. For each $1 \leq j \leq l$, and for each sequence $s \in [(A^-)^{<\omega}]^* \setminus \{\emptyset\}$, $t_j \not\prec s$. So we have

$$A^\infty\in\mathbf{\Delta}^0_1 \;\Leftrightarrow\; \begin{cases} \exists p>0\;\; \exists k,l\in\omega\;\; \exists s_1,\dots,s_k,t_1,\dots,t_l\in 2^{<\omega}\;\; \bigcup_{1\leq i\leq k} N_{s_i} = n^\omega\setminus(\bigcup_{1\leq j\leq l} N_{t_j}) \\ \text{and}\;\; \forall 1\leq j\leq l\;\; \forall s\in[(A^-)^{<\omega}]^*\setminus\{\emptyset\}\;\; t_j\not\prec s \;\text{ and }\; \forall\alpha\in n^\omega \\ \{\alpha\notin\bigcup_{1\leq i\leq k}N_{s_i} \text{ or } \exists\beta\in p^\omega\; [(\forall m>0\;\; \beta(m)>0) \text{ and } (\forall q\in\omega\;\; \pi(\alpha,\beta,q)\in A)]\}. \end{cases}$$

This shows that $\mathbf{\Delta}_1$ is a K_σ subset of $2^{n^{<\omega}}$.

To show that it is not $\mathbf{\Pi}^0_2$, it is enough to see that its intersection with the closed set

$$\{A \subseteq n^{<\omega} / A^\infty \neq n^\omega\}$$

is dense and co-dense in this closed set (see (b)), by Baire's theorem. So let O be a basic clopen subset of $2^{n^{<\omega}}$ meeting this closed set. We may assume that it is of the form

$$\{A \subseteq n^{<\omega} / \forall i \leq k \;\; s_i \in A \text{ and } \forall j \leq l \;\; t_j \notin A\},$$

where $s_0, \ldots, s_k, t_0, \ldots, t_l \in n^{<\omega}$ and $|s_0| > 0$. Let $A := \{s_i / i \leq k\}$. Then $A \in O$ and A^∞ is in $\mathbf{\Pi}^0_1 \setminus \{\emptyset, n^\omega\}$. There are two cases.

If $A^\infty \in \mathbf{\Delta}^0_1$, then we have to find $B \in O$ with $B^\infty \notin \mathbf{\Delta}^0_1$. Let $u_0, \ldots, u_m \in n^{<\omega}$ with $\bigcup_{p \leq m} N_{u_p} = n^\omega \setminus A^\infty$. Let $r \in n \setminus \{u_0(|u_0|-1)\}$, $s := u_0 r^{|u_0| + \max_{j \leq l} |t_j|}$ and $B := A \cup \{s\}$. Then $B \in O$ and $s^\infty \in B^\infty$. Let us show that s^∞ is not in the interior of B^∞. Otherwise, we could find an integer q such that $N_{s^q} \subseteq B^\infty$. We would have $\alpha := s^q u_0 u_0 (|u_0| - 1) r^\infty \in B^\infty$. As $N_{u_0} \cap A^\infty = \emptyset$, the decomposition of α into nonempty words of B would start with q times s. If this decomposition could go on, then we would have $u_0 = (u_0(|u_0| - 1))^{|u_0|}$. Let $v \in n^{<\omega}$ be such that $N_v \subseteq A^\infty$. We have $v(u_0(|u_0| - 1))^\infty \in A^\infty$, so $(u_0(|u_0| - 1))^\infty \in N_{u_0} \cap A^\infty$. But this is absurd. Therefore $B^\infty \notin \mathbf{\Delta}^0_1$.

If $A^\infty \notin \mathbf{\Delta}^0_1$, then we have to find $B \in O$ such that $B^\infty \in \mathbf{\Delta}^0_1 \setminus \{n^\omega\}$. Notice that $n^\omega \neq \bigcup_{i \leq k} N_{s_i}$. So let $v \in n^{<\omega}$ be non constant such that $N_v \cap \bigcup_{i \leq k} N_{s_i} = \emptyset$. We set

$$D := A \cup \bigcup_{r \in n \setminus \{v(0)\}} \{(r)\} \cup \{v(0)^{|v|}\},$$

$B := A \cup \{s \in n^{<\omega} / |s| > \max_{j \leq l} |t_j| \text{ and } \exists t \in D \; t \prec s\}$. We get $B^\infty = \bigcup_{t \in D} N_t \in \mathbf{\Delta}^0_1$ and

$$N_v \cap B^\infty = \emptyset,$$

so $B^\infty \neq n^\omega$. □

Now we will study $\mathcal{F} := \{A \subseteq n^{<\omega} / \exists B \subseteq n^{<\omega} \text{ finite } A^\infty = B^\infty\}$.

Proposition 5 *$\mathcal{F}$ is a co-nowhere dense $\mathbf{\Sigma}^0_2$-hard subset of $2^{n^{<\omega}}$.*

Proof. By Proposition 3, if $A^\infty = n^\omega$, then there exists an integer p such that $A^\infty = (A \cap n^{\leq p})^\infty$, so $\mathbf{\Pi}_0 \subseteq \mathcal{F}$ and, by Theorem 4, $\mathcal{F}$ is co-nowhere dense. We define a continuous map $\phi : 2^\omega \to 2^{n^{<\omega}}$ by the formula $\phi(\gamma) := \{0^k 1 / \gamma(k) = 1\}$. If $\gamma \in P_f := \{\alpha \in 2^\omega / \exists p \; \forall m \geq p \;\; \alpha(m) = 0\}$, then $\phi(\gamma) \in \mathcal{F}$. If $\gamma \notin P_f$, then the concatenation map is an homeomorphism from $\phi(\gamma)^\omega$ onto $\phi(\gamma)^\infty$, thus $\phi(\gamma)^\infty$ is not K_σ. So $\phi(\gamma) \notin \mathcal{F}$, by Proposition 2. Thus the preimage of $\mathcal{F}$ by ϕ is P_f, and $\mathcal{F}$ is $\mathbf{\Sigma}^0_2$-hard. □

Let $\mathcal{G}_p := \{A \subseteq n^{<\omega} / \exists s_1, \ldots, s_p \in n^{<\omega} \; A^\infty = \{s_1, \ldots, s_p\}^\infty\}$, so that $\mathcal{F} = \bigcup_p \mathcal{G}_p$. We have $\mathcal{G}_0 = \mathbf{\Sigma}_0$, so $\mathcal{G}_0$ is $\mathit{\Pi}^0_1 \setminus \mathbf{\Sigma}^0_1$.

Proposition 6 *$\mathcal{G}_1$ is $\mathbf{\Pi}^0_1 \setminus \mathbf{\Sigma}^0_1$. In particular, $\mathcal{G}_1$ is $\mathbf{\Pi}^0_1$-complete.*

Proof. If $p \in \omega \setminus \{0\}$, then $\{0, 1^p\} \notin \mathcal{G}_1$ since $B^\infty = \{s^\infty\}$ if $B = \{s\}$. Thus $\{0\}$ is not an interior point of $\mathcal{G}_1$ since the sequence $(\{0, 1^p\})_{p>0}$ tends to $\{0\}$. So $\mathcal{G}_1 \notin \mathbf{\Sigma}^0_1$.

• Let $(A_m) \subseteq \mathcal{G}_1$ tending to $A \subseteq n^{<\omega}$. If $A \subseteq \{\emptyset\}$, then $A^\infty = \emptyset = \{\emptyset\}^\infty$, so $A \in \mathcal{G}_1$. If $A \not\subseteq \{\emptyset\}$, then let $t \in A^-$ and $\alpha_0 := t^\infty$. There exists an integer m_0 such that $t \in A_m$ for $m \geq m_0$. Thus we may assume that $t \in A_m$ and $A_m^\infty \neq \emptyset$. So let $s_m \in n^{<\omega} \setminus \{\emptyset\}$ be such that $A_m^\infty = \{s_m\}^\infty = \{s_m^\infty\}$. We have $s_m^\infty = \alpha_0$. Let $b := \min\{a \in \omega \setminus \{0\} / (\alpha_0 \lceil a)^\infty = \alpha_0\}$.

• We will show that $A_m \subseteq \{(\alpha_0 \lceil b)^q / q \in \omega\}$. Let $s \in A_m \setminus \{\emptyset\}$. As $s^\infty = \alpha_0$, we can find an integer $a > 0$ such that $s = \alpha_0 \lceil a$, and $b \leq a$. Let $r < b$ and q be integers so that $a = q.b + r$. We have, if $r > 0$,

$$\begin{aligned}\alpha_0 &= (\alpha_0 \lceil a)^\infty = (\alpha_0 \lceil b)^\infty = (\alpha_0 \lceil q.b)(\alpha_0 \lceil a - \alpha_0 \lceil q.b)\alpha_0 \\ &= (\alpha_0 \lceil b)^q (\alpha_0 \lceil a - \alpha_0 \lceil q.b)\alpha_0 = (\alpha_0 \lceil a - \alpha_0 \lceil q.b)\alpha_0 = (\alpha_0 \lceil r)\alpha_0 = (\alpha_0 \lceil r)^\infty.\end{aligned}$$

Thus, by minimality of b, $r = 0$ and we are done.

• Let $u \in A$. We can find an integer m_u such that $u \in A_m$ for $m \geq m_u$. So there exists an integer q_u such that $u = (\alpha_0 \lceil b)^{q_u}$. Therefore $A^\infty = \{(\alpha_0 \lceil b)^\infty\} = \{\alpha_0 \lceil b\}^\infty$ and $A \in \mathcal{G}_1$. □

Remark. Notice that this shows that we can find $w \in n^{<\omega} \setminus \{\emptyset\}$ such that $A \subseteq \{w^q / q \in \omega\}$ if $A \in \mathcal{G}_1$. Now we study $\mathcal{G}_2$. The next lemma is just Corollary 6.2.5 in [Lo].

Lemma 7 *Two finite sequences which commute are powers of the same finite sequence.*

Proof. Let x and y be finite sequences with $xy = yx$. Then the subgroup of the free group on n generators generated by x and y is abelian, hence isomorphic to $\mathbb{Z}$. One generator of this subgroup must be a finite sequence u such that x and y are both powers of u. □

Lemma 8 *Let $A \in \mathcal{G}_2$. Then there exists a finite subset F of A such that $A^\infty = F^\infty$.*

Proof. We will show more. Let $A \notin \mathcal{G}_1$ satisfying $A^\infty = \{s_1, s_2\}^\infty$, with $|s_1| \leq |s_2|$. Then

(a) The decomposition of α into words of $\{s_1, s_2\}$ is unique for each $\alpha \in A^\infty$ (this is a consequence of Corollaries 6.2.5 and 6.2.6 in [Lo]).
(b) $s_2 s_1 \perp s_1{}^q s_2$ for each integer $q > 0$, and $s_2 s_1 \wedge s_1{}^q s_2 = s_1 s_2 \wedge s_2 s_1$.
(c) $A \subseteq [\{s_1, s_2\}^{<\omega}]^*$.

• We prove the first two points. We split into cases.

2.1. $s_1 \perp s_2$.

The result is clear.

2.2. $s_1 \prec_{\neq} s_2 \not\prec s_1^\infty$.

Here also, the result is clear (cut α into words of length $|s_1|$).

2.3. $s_1 \prec_{\neq} s_2 \prec s_1^\infty$.

We can write $s_2 = s_1^m s$, where $m > 0$ and $s \prec_{\neq} s_1$. Thus $s_2 s_1 = s_1^m s s_1$ and $s_1^{m+1} s \prec s_1^q s_2$ if $q > 0$. But $s_1^m s s_1 \perp s_1^{m+1} s$ otherwise $s s_1 = s_1 s$, and s, s_1 s_2 would be powers of some sequence, which contradicts $A \notin \mathcal{G}_1$.

• We prove (c). Let $t \in A$, so that $t s_1^\infty$, $t s_2 s_1^\infty \in A^\infty$. These sequences split after $t(s_1 s_2 \wedge s_2 s_1)$, and the decomposition of $t s_1^\infty$ (resp., $t s_2 s_1^\infty$) into words of $\{s_1, s_2\}$ starts with $u s_i$ (resp., $u s_{3-i}$), where $u \in [\{s_1, s_2\}^{<\omega}]^*$. So $t s_1^\infty$ and $t s_2 s_1^\infty$ split after $u(s_1 s_2 \wedge s_2 s_1)$ by (b). But we must have $t = u$ because of the position of the splitting point.

• We prove Lemma 8. If $A \in \mathcal{G}_0$, then $F := \emptyset$ works. If $A \in \mathcal{G}_1 \setminus \mathcal{G}_0$, then let $w \in n^{<\omega} \setminus \{\emptyset\}$ such that $A \subseteq \{w^q / q \in \omega\}$, and $q > 0$ such that $w^q \in A$. Then $F := \{w^q\}$ works. So we may assume that $A \notin \mathcal{G}_1$, and $A^\infty = \{s_1, s_2\}^\infty$. As $A^\infty \subseteq \bigcup_{t \in A^-} \{\alpha \in N_t / s_1 s_2 \wedge s_2 s_1 \prec \alpha - t\}$ is compact, we get a finite subset F of A^- such that $A^\infty \subseteq \bigcup_{t \in F} \{\alpha \in N_t / s_1 s_2 \wedge s_2 s_1 \prec \alpha - t\}$. We have $F^\infty \subseteq A^\infty$. If $\alpha \in A^\infty$, then let $t \in F$ such that $t \prec \alpha$. By (c), we have $t \in [\{s_1, s_2\}^{<\omega}]^*$. The sequence t is the beginning of the decomposition of α into words of $\{s_1, s_2\}$. Thus $\alpha - t \in A^\infty$ and we can go on like this. This shows that $\alpha \in F^\infty$. □

Remark. The inclusion of $A^\infty = \{s_1, s_2\}^\infty$ into $\{t_1, t_2\}^\infty$ does not imply $\{s_1, s_2\} \subseteq [\{t_1, t_2\}^{<\omega}]^*$, even if $A \notin \mathcal{G}_1$. Indeed, take $s_1 := 01$, $s_2 := t_1 := 0$ and $t_2 := 10$. But we have

$$|t_1| + |t_2| \leq |s_1| + |s_2|,$$

which is the case in general:

Lemma 9 *Let A, $B \notin \mathcal{G}_1$ satisfying $A^\infty = \{s_1, s_2\}^\infty \subseteq B^\infty = \{t_1, t_2\}^\infty$. Then there is $j \in 2$ such that $|t_{1+i}| \leq |s_{1+[i+j \bmod 2]}|$ for each $i \in 2$. In particular, $|t_1| + |t_2| \leq |s_1| + |s_2|$.*

Proof. We may assume that $|s_1| \leq |s_2|$. Let, for $i = 1, 2$, $(w_m^i)_m \subseteq \{t_1, t_2\}$ be sequences such that $s_1^\infty = w_0^1 w_1^1 \ldots$ (resp., $s_2 s_1^\infty = w_0^2 w_1^2 \ldots$). By the proof of Lemma 8, there is a minimal integer m_0 satisfying $w_{m_0}^1 \neq w_{m_0}^2$. We let $u := w_0^1 \ldots w_{m_0-1}^1$. The sequences s_1^∞ and $s_2 s_1^\infty$ split after $s_1 s_2 \wedge s_2 s_1 = u(t_1 t_2 \wedge t_2 t_1)$. Similarly, s_1^∞ and $s_1 s_2 s_1^\infty$ split after $s_1(s_1 s_2 \wedge s_2 s_1) = v(t_1 t_2 \wedge t_2 t_1)$, where $v \in [\{t_1, t_2\}^{<\omega}]^* \setminus \{\emptyset\}$. So we get $s_1 u = v$. Similarly, with the sequences $s_2 s_1^\infty$ and $s_2^2 s_1^\infty$, we see that $s_2 u \in [\{t_1, t_2\}^{<\omega}]^* \setminus \{\emptyset\}$. So we may assume that $u \neq \emptyset$ since $\{s_1, s_2\} \notin \mathcal{G}_1$. If $t_1 \not\perp t_2$, then we may assume that $\emptyset \neq t_1 \prec_{\neq} t_2$. So we may assume that we are not in the case $t_2 \prec t_1^\infty$. Indeed, otherwise $t_2 = t_1^m t$, where $\emptyset \prec_{\neq} t \prec_{\neq} t_1$ (see the proof of Lemma 8). Moreover, t_1 doesn't finish t_2, otherwise we would have $t_1 = t(t_1 - t) = (t_1 - t)t$ and t, $t_1 - t$, t_1, t_2 would be powers of the same sequence, which contradicts $\{t_1, t_2\} \notin \mathcal{G}_1$. As $s_i u \in [\{t_1, t_2\}^{<\omega}]^*$, this shows that $s_i \in [\{t_1, t_2\}^{<\omega}]^*$. So we are done since $\{s_1, s_2\} \notin \mathcal{G}_1$ as before.

Assume for example that $t_2 = w^1_{m_0}$. Let m' be maximal with $t_1^{m'} \prec t_2$. Notice that

$$ut_1^{m'} \prec s_1s_2 \prec s_1s_2s_1^\infty.$$

We have $ut_2 \prec s_1s_2s_1^\infty$, otherwise we would obtain $ut_1^{m'+1} \prec s_1s_2s_1^\infty \wedge s_2s_1^\infty = s_1s_2 \wedge s_2s_1 \prec s_1^\infty$, which is absurd. So we get $|t_2| \leq |s_1|$ since $|u|+|t_2|+|t_1t_2 \wedge t_2t_1| \leq |s_1|+|s_1s_2 \wedge s_2s_1|$. Similarly, $|t_1| \leq |s_2|$ since $ut_1^{m'+1} \prec s_2^2s_1^\infty$. The argument is similar if $t_2 = w^2_{m_0}$ (we get $|t_i| \leq |s_i|$ in this case for $i = 1, 2$). □

Corollary 10 *$\mathcal{G}_2$ is a $\check{D}_\omega(\mathbf{\Sigma}^0_1) \setminus D_\omega(\mathbf{\Sigma}^0_1)$ set. In particular, $\mathcal{G}_2$ is $\check{D}_\omega(\mathbf{\Sigma}^0_1)$-complete.*

Proof. We will apply the Hausdorff derivation to $\mathcal{G} \subseteq 2^{n^{<\omega}}$. This means that we define a decreasing sequence $(F_\xi)_{\xi<\omega_1}$ of closed subsets of $2^{n^{<\omega}}$ as follows:

$$F_\xi := \overline{\left(\bigcap_{\eta<\xi} F_\eta\right) \cap \mathcal{G}} \text{ if } \xi \text{ is even}, \quad \overline{\left(\bigcap_{\eta<\xi} F_\eta\right) \setminus \mathcal{G}} \text{ if } \xi \text{ is odd}.$$

Recall that if ξ is even, then $F_\xi = \emptyset$ is equivalent to $\mathcal{G} \in D_\xi(\mathbf{\Sigma}^0_1)$. Indeed, we set $U_\xi := \check{F}_\xi$. We have $U_{\xi+1} \setminus U_\xi = F_\xi \setminus F_{\xi+1} \subseteq \mathcal{G}$ if ξ is even and $U_{\xi+1} \setminus U_\xi \subseteq \check{\mathcal{G}}$ if ξ is odd. Similarly, $U_\xi \setminus (\bigcup_{\eta<\xi} U_\eta) \subseteq \check{\mathcal{G}}$ if ξ is limit. If $F_\xi = \emptyset$, then let η be minimal such that $F_\eta = \emptyset$. We have $\mathcal{G} = \bigcup_{\theta\leq\eta,\ \theta \text{ odd}} U_\theta \setminus (\bigcup_{\rho<\theta} U_\rho)$. If η is odd, then $\check{\mathcal{G}} = \bigcup_{\theta<\eta,\ \theta \text{ even}} U_\theta \setminus (\bigcup_{\rho<\theta} U_\rho) \in D_\eta(\mathbf{\Sigma}^0_1)$, thus $\mathcal{G} \in \check{D}_\eta(\mathbf{\Sigma}^0_1) \subseteq D_\xi(\mathbf{\Sigma}^0_1)$. If η is even, then $\mathcal{G} = \bigcup_{\theta<\eta,\ \theta \text{ odd}} U_\theta \setminus (\bigcup_{\rho<\theta} U_\rho) \in D_\eta(\mathbf{\Sigma}^0_1)$ and the same conclusion is true. Conversely, if $\mathcal{G} \in D_\xi(\mathbf{\Sigma}^0_1)$, then let $(V_\eta)_{\eta<\xi}$ be an increasing sequence of open sets with $\mathcal{G} = \bigcup_{\eta<\xi,\ \eta \text{ odd}} V_\eta \setminus (\bigcup_{\theta<\eta} V_\theta)$. By induction, we check that $F_\eta \subseteq \check{V}_\eta$ if $\eta < \xi$. This clearly implies that $F_\xi = \emptyset$ because ξ is even.

• We will show that if $A \notin \mathcal{G}_1$ satisfies $A^\infty = \{s_1, s_2\}^\infty$, then $A \notin F_M := F_M(\mathcal{G}_2)$, where M is the smallest odd integer greater than or equal to $f(s_1, s_2) := 2\Sigma_{l\leq|s_1|+|s_2|-2}\, n^{2(|s_1|+|s_2|-l)}$.

We argue by contradiction: A is the limit of (A_q), where $A_q \in F_{M-1} \setminus \mathcal{G}_2$. Lemma 8 gives a finite subset F of A, and we may assume that $F \subseteq A_q$ for each q. Thus we have $A^\infty \subseteq A_q^\infty$, and the inclusion is strict. Thus we can find $s^q \in [A_q^{<\omega}]^*$ such that $N_{s^q} \cap A^\infty = \emptyset$. Let $s^q_0, \ldots, s^q_{m_q} \in A_q$ be such that $s^q = s^q_0 \ldots s^q_{m_q}$.

Now A_q is the limit of $(A_{q,r})_r$, where $A_{q,r} \in F_{M-2} \cap \mathcal{G}_2$, and we may assume that

$$\{s^q_0,\ \ldots,\ s^q_{m_q}\} \cup F \subseteq A_{q,r}$$

for each r, and that $A_{q,r} \notin \mathcal{G}_1$ because $A_q \notin \mathcal{G}_1 \subseteq \mathcal{G}_2$. Let $s^{q,r}_1$, $s^{q,r}_2$ such that $A^\infty_{q,r} = \{s^{q,r}_1, s^{q,r}_2\}^\infty$. By Lemma 9 we have $|s^{q,r}_1|+|s^{q,r}_2| \leq |s_1|+|s_2|$. Now we let $B_0 := A_{0,0}$ and $s^0_i := s^{0,0}_i$ for $i = 1,\ 2$. We have $B_0 \in F_{M-2} \cap \mathcal{G}_2 \setminus \mathcal{G}_1$, $A^\infty \subseteq_{\neq} B^\infty_0 = \{s^0_1, s^0_2\}^\infty$, and

$$|s^0_1| + |s^0_2| \leq |s_1| + |s_2|.$$

Now we iterate this: for each $0 < k < n^{2(|s_1|+|s_2|)}$, we get $B_k \in F_{M-2(k+1)} \cap \mathcal{G}_2 \setminus \mathcal{G}_1$ such that $B^\infty_{k-1} \subseteq_{\neq} B^\infty_k = \{s^k_1, s^k_2\}^\infty$ and $|s^k_1| + |s^k_2| \leq |s^{k-1}_1| + |s^{k-1}_2|$. We can find $k_0 < n^{2(|s_1|+|s_2|)}$ such that $|s^{k_0}_1| + |s^{k_0}_2| < |s^{k_0-1}_1| + |s^{k_0-1}_2|$ (with the convention $s^{-1}_i := s_i$). We set $C_0 := B_{k_0}$, $t^0_i := s^{k_0}_i$. So we have $C_0 \in F_{M-2(k_0+1)} \cap \mathcal{G}_2 \setminus \mathcal{G}_1$, $C^\infty_0 = \{t^0_1, t^0_2\}^\infty$ and $|t^0_1| + |t^0_2| < |s_1| + |s_2|$. Now we iterate this: for each $l \leq |s_1| + |s_2| - 2$, we get t^l_1, t^l_2, $k_l < n^{2(|t^{l-1}_1|+|t^{l-1}_2|)}$ and

$$C_l \in F_{M-2\Sigma_{m\leq l}\ (k_m+1)} \cap \mathcal{G}_2 \setminus \mathcal{G}_1$$

satisfying $C^\infty_l = \{t^l_1, t^l_2\}^\infty$ and $|t^l_1| + |t^l_2| < |t^{l-1}_1| + |t^{l-1}_2|$ (with the convention $t^{-1}_i := s_i$). We have $|t^l_1| + |t^l_2| \leq |s_1| + |s_2| - 1 - l$, thus

$$2\Sigma_{l\leq|s_1|+|s_2|-2}\ (k_l + 1) \leq 2\Sigma_{l\leq|s_1|+|s_2|-2}\ n^{2(|t^{l-1}_1|+|t^{l-1}_2|)} \leq f(s_1, s_2)$$

and this construction is possible. But we have $|t^{|s_1|+|s_2|-2}_1| + |t^{|s_1|+|s_2|-2}_2| \leq 1$, thus $C_{|s_1|+|s_2|-2} \in \mathcal{G}_1$, which is absurd.

Let $A \notin \mathcal{G}_2$. As $A \notin \mathcal{G}_1$, we can find $s,\ t \in A$ which are not powers of the same sequence. Indeed, let $s \in A^-$ and u with minimal length such that s is a power of u. Then any $t \in A \setminus \{u^q / q \in \omega\}$ works, because if s and t are powers of w, then w has to be a power of u. Indeed, as $u \prec w$, $w = u^k v$ with $v \prec u$, and v has to be a power of u by minimality of $|u|$ and Lemma 7. Assume that moreover $A \in F_{2k+2}$. Now A is the limit of $(A_{k,r})_r \subseteq F_{2k+1} \cap \mathcal{G}_2$ for each integer k, and we may assume that $s,\ t \in A_{k,r} \notin \mathcal{G}_1$. Let $s^{k,r}_1$, $s^{k,r}_2$ be such that ${A_{k,r}}^\infty = \{s^{k,r}_1, s^{k,r}_2\}^\infty$. By Lemma 9 we have $|s^{k,r}_1| + |s^{k,r}_2| \leq |s| + |t|$ and $f(s^{k,r}_1, s^{k,r}_2) \leq f(s, t)$. By the preceding point, we must have

$$2k + 1 < f(s, t).$$

Thus $\bigcap_m\ F_m \subseteq \mathcal{G}_2$. Notice that $F_{m+1}(\check{\mathcal{G}}_2) \subseteq F_m$, so that $F_\omega(\check{\mathcal{G}}_2) = \emptyset$ and $\mathcal{G}_2 \in \check{D}_\omega(\mathbf{\Sigma}^0_1)$.

• Now let us show that $\{0\} \in F_\omega(\mathcal{G}_2)$ (this will imply $\mathcal{G}_2 \notin D_\omega(\mathbf{\Sigma}^0_1)$). It is enough to see that

$$\{0\} \in \bigcap_m F_m.$$

Let $E(x)$ be the biggest integer less than or equal to x, $p_{k,s} := 2^{k+1-E(|s|/2)}$ and $k \in \omega$. We define $A_\emptyset := \{0\}$ and, for $s \in (\omega \setminus \{0, 1\})^{\leq 2k+1}$ and $m > 1$, $A_{sm} := A_s \cup \{(01^{p_{k,s}})^m; (0^2 1^{p_{k,s}})^m\}$ if $|s|$ is even, $A_s \cup \{s \in [\{0, 1^{p_{k,s}}\}^{<\omega}]^* / m \leq |s| \leq m + p_{k,s}\}$ if $|s|$ is odd. Let us show that $A_s \in \mathcal{G}_2$ (resp., $\check{\mathcal{G}}_2$) if $|s|$ is even (resp., odd). First by induction we get $A_{sm} \subseteq \{0, 1^{p_{k,s}}\}^{<\omega}$. Therefore $A^\infty_{sm} = \{0, 1^{p_{k,s}}\}^\infty$ if $|s|$ is odd, because if α is in $\{0, 1^{p_{k,s}}\}^\infty$ and $t \in [\{0, 1^{p_{k,s}}\}^{<\omega}]^*$ with minimal length $\geq m$ begins α, then $t \in A_{sm}$. Now if $|s|$ is even and $A^\infty_{sm} = \{s_1, s_2\}^\infty$, then $0^\infty \in \{s_1, s_2\}^\infty$, thus for example $s_1 = 0^{k+1}$. $(01^{p_{k,s}})^\infty \in \{s_1, s_2\}^\infty$, thus $s_2 \prec (01^{p_{k,s}})^\infty$ and $|s_2| \geq |(01^{p_{k,s}})^m|$ since $s_2 0^\infty \in \{s_1, s_2\}^\infty$. But then $(0^2 1^{p_{k,s}})^\infty \notin \{s_1, s_2\}^\infty$ since $m > 1$. Thus $A_{sm} \notin \mathcal{G}_2$.

As $(A_{sm})_m$ tends to A_s and $(A_s)_{|s|=2k+2} \subseteq \mathcal{G}_2$, we deduce from this that A_s is in $F_{2k+1-|s|} \setminus \mathcal{G}_2$ if $|s| \leq 2k + 1$ is odd, and that $A_s \in F_{2k+1-|s|} \cap \mathcal{G}_2$ if $|s| \leq 2k + 1$ is even. Therefore $\{0\}$ is in $\bigcap_k\ F_{2k+1} = \bigcap_m\ F_m$. □

Remarks. (1) The end of this proof also shows that $\mathcal{G}_p \notin D_\omega(\mathbf{\Sigma}^0_1)$ if $p \geq 2$. Indeed, $\{0\} \in F_\omega(\mathcal{G}_p)$. The only thing to change is the definition of A_{sm} if $|s|$ is even: we set

$$A_{sm} := A_s \cup \{(0^{j+1}1^{p_{k,s}})^m / j < p\}.$$

(2) If $\{s_1, s_2\} \notin \mathcal{G}_1$ and $\{s_1, s_2\}^\infty = \{t_1, t_2\}^\infty$, then $\{s_1, s_2\} = \{t_1, t_2\}$. Indeed, $\{t_1, t_2\} \notin \mathcal{G}_1$, thus by Lemma 9 we get $|s_1| + |s_2| = |t_1| + |t_2|$. By (c) in the proof of Lemma 8 and the previous fact, $s_i = t_{\varepsilon_i}^{a_i}$, where $a_i > 0$, $\varepsilon_i,\ i \in \{1, 2\}$. As $\{s_1, s_2\} \notin \mathcal{G}_1$, $\varepsilon_1 \neq \varepsilon_2$. Thus $a_i = 1$.

Conjecture 1. Let $A \in \mathcal{F}$. Then there exists a finite subset F of A such that $A^\infty = F^\infty$.

Conjecture 2. Let $p \geq 1$, A, $B \notin \mathcal{G}_p$ with $A^\infty = \{s_1, \ldots, s_q\}^\infty \subseteq B^\infty = \{t_1, \ldots, t_{p+1}\}^\infty$. Then $\Sigma_{1 \leq i \leq p+1} |t_i| \leq \Sigma_{1 \leq i \leq q} |s_i|$.

Conjecture 3. We have $\mathcal{G}_{p+1} \setminus \mathcal{G}_p \in D_\omega(\mathbf{\Sigma}^0_1)$ for each $p \geq 1$. In particular, $\mathcal{F} \in K_\sigma \setminus \mathbf{\Pi}^0_2$.

Notice that Conjectures 1 and 2 imply Conjecture 3. Indeed, $\mathcal{F} = \mathcal{G}_1 \cup \bigcup_{p \geq 1} \mathcal{G}_{p+1} \setminus \mathcal{G}_p$, so $\mathcal{F} \in K_\sigma$ if $\mathcal{G}_{p+1} \setminus \mathcal{G}_p \in D_\omega(\mathbf{\Sigma}^0_1) \subseteq \mathbf{\Delta}^0_2$, by Proposition 6. By Proposition 5 we have $\mathcal{F} \notin \mathbf{\Pi}^0_2$. It is enough to see that $F_\omega := F_\omega(\mathcal{G}_{p+1} \setminus \mathcal{G}_p) = \emptyset$. We argue as in the proof of Corollary 10. This time, $f(s_1, \ldots, s_q) := 2\Sigma_{l \leq \Sigma_{1 \leq i \leq q} |s_i| - 2}\ n^{q(\Sigma_{1 \leq i \leq q} |s_i| - l)}$ for $s_1,\ \ldots,\ s_q \in n^{<\omega}$. The fact to notice is that $A \notin F_M(\mathcal{G}_{p+1} \setminus \mathcal{G}_p)$ if $A \notin \mathcal{G}_p$ satisfies $A^\infty = \{s_1, \ldots, s_{p+1}\}^\infty$ and M is the minimal odd integer greater than or equal to $f(s_1, \ldots, s_{p+1})$. So if $A \in F_{2k+2} \cap \mathcal{F} \setminus \mathcal{G}_p$, then Conjecture 1 gives a finite subset $F := \{s_1, \ldots, s_q\}$ of A. The set A is the limit of $(A_{k,r})_r \subseteq F_{2k+1} \cap \mathcal{G}_{p+1} \setminus \mathcal{G}_p$ for each integer k, and we may assume that $F \subseteq A_{k,r}$. Conjecture 2 implies that $f(s_1^{k,r}, \ldots, s_{p+1}^{k,r}) \leq f(s_1, \ldots, s_q)$ and $2k + 1 < f(s_1, \ldots, s_q)$. Thus $\bigcap_m F_m \subseteq \check{\mathcal{F}} \cup \mathcal{G}_p$. So $F_\omega \subseteq \overline{(\check{\mathcal{F}} \cup \mathcal{G}_p) \cap \mathcal{G}_{p+1} \setminus \mathcal{G}_p} = \emptyset$.

3 Is A^∞ Borel?

Now we will see that the maximal complexity is possible. We essentially give O. Finkel's example, in a lightly simpler version.

Proposition 11 *Let $\Gamma := \mathbf{\Sigma}^1_1$ or a Baire class. The existence of $n \in \omega \setminus 2$ and $A \subseteq n^{<\omega}$ such that A^∞ is Γ-complete is equivalent to the existence of $B \subseteq 2^{<\omega}$ such that B^∞ is Γ-complete.*

Proof. Let $p_n := \min\{p \in \omega / n \leq 2^p\} \geq 1$. We define $\phi : n \hookrightarrow 2^{p_n} := \{\sigma_0, \ldots, \sigma_{2^{p_n}-1}\}$ by the formula $\phi(m) := \sigma_m$, $\Phi : n^{<\omega} \hookrightarrow 2^{<\omega}$ by the formula $\Phi(t) := \phi(t(0)) \ldots \phi(t(|t| - 1))$ and $f : n^\omega \hookrightarrow 2^\omega$ by the formula $f(\gamma) := \phi(\gamma(0))\phi(\gamma(1)) \ldots$ Then f is an homeomorphism from n^ω onto its range and reduces A^∞ to B^∞, where $B := \Phi[A]$. The inverse function of f reduces B^∞ to A^∞. So we are done if Γ is stable under intersection with closed sets. Otherwise, $\Gamma = \mathbf{\Delta}^0_1$ or $\mathbf{\Sigma}^0_1$. If $A = \{s \in 2^{<\omega} / 0 \prec s \text{ or } 1^2 \prec s\}$, then $A^\infty = N_0 \cup N_{1^2}$, which is $\mathbf{\Delta}^0_1$-complete. If $A = \{s \in 2^{<\omega} / 0 \prec s\} \cup \{10^k1^{l+1} / k, l \in \omega\}$, then $A^\infty = 2^\omega \setminus \{10^\infty\}$, which is $\mathbf{\Sigma}^0_1$-complete. $\square$

Theorem 12 *The set $I := \{(\alpha, A) \in n^\omega \times 2^{n^{<\omega}} / \alpha \in A^\infty\}$ is $\mathbf{\Sigma}^1_1$-complete. In fact,*
(a) (O. Finkel, see [F1]) There exists $A_0 \subseteq 2^{<\omega}$ such that A_0^∞ is $\mathbf{\Sigma}^1_1$-complete.
(b) There exists $\alpha_0 \in 2^\omega$ such that I_{α_0} is $\mathbf{\Sigma}^1_1$-complete.

Proof. (a) We set $L := \{2,3\}$ and

$$\mathcal{T} := \{\, \tau \subseteq 2^{<\omega} \times L / \forall (u,\nu) \in 2^{<\omega} \times L \ \ [(u,\nu) \notin \tau] \text{ or}$$

$$[(\forall v \prec u \ \exists \mu \in L \ (v,\mu) \in \tau) \text{ and } ((u, 5-\nu) \notin \tau) \text{ and } (\exists (\varepsilon,\pi) \in 2 \times L \ (u\varepsilon,\pi) \in \tau)] \,\}.$$

The set $\mathcal{T}$ is the set of pruned trees over 2 with labels in L. It is a closed subset of $2^{2^{<\omega} \times L}$, thus a Polish space. Then we set

$$\sigma := \{\tau \in \mathcal{T} / \exists (\underline{u}, \underline{\nu}) \in 2^{\omega} \times L^{\omega} \ [\forall m \ (\underline{u}\lceil m, \underline{\nu}(m)) \in \tau] \text{ and } [\forall p \ \exists m \geq p \ \ \underline{\nu}(m) = 3]\}.$$

• Then $\sigma \in \mathbf{\Sigma}^1_1(\mathcal{T})$. Let us show that it is complete. We set $\mathcal{T} := \{T \in 2^{\omega^{<\omega}} / T \text{ is a tree}\}$ and $IF := \{T \in \mathcal{T} / T \text{ is ill-founded}\}$. It is a well-known fact that $\mathcal{T}$ is a Polish space (it is a closed subset of $2^{\omega^{<\omega}}$), and that IF is $\mathbf{\Sigma}^1_1$-complete (see [K1]). It is enough to find a Borel reduction of IF to σ (see [K2]).

We define $\psi : \omega^{<\omega} \hookrightarrow 2^{<\omega}$ by the formula $\psi(t) := 0^{t(0)}10^{t(1)}1 \dots 0^{t(|t|-1)}1$, and $\Psi : \mathcal{T} \to \mathcal{T}$ by

$$\Psi(T) := \{(u,\nu) \in 2^{<\omega} \times L / \exists t \in T \ u \prec \psi(t) \text{ and } \nu = 3 \text{ if } u = \emptyset, \ 2 + u(|u|-1) \text{ otherwise}\}$$

$$\cup \{(\psi(t)0^{k+1}, 2) / t \in T \text{ and } \forall q \in \omega \ tq \notin T, k \in \omega\}.$$

The map Ψ is Baire class one. Let us show that it is a reduction. If $T \in IF$, then let $\gamma \in \omega^{\omega}$ be such that $\gamma\lceil m \in T$ for each integer m. We have $(\psi(\gamma\lceil m), 3) \in \Psi(T)$. Let $\underline{u}$ be the limit of $\psi(\gamma\lceil m)$ and $\underline{\nu}(m) := 2 + \underline{u}(m-1)$ (resp., 3) if $m > 0$ (resp., $m = 0$). These objects show that $\Psi(T) \in \sigma$. Conversely, we have $T \in IF$ if $\Psi(T) \in \sigma$.

• If $\tau \in \mathcal{T}$ and $m \in \omega$, then we enumerate $\tau \cap (2^m \times L) := \{(u_1^{m,\tau}, \nu_1^{m,\tau}), \dots, (u_{q_{m,\tau}}^{m,\tau}, \nu_{q_{m,\tau}}^{m,\tau})\}$ in the lexicographic ordering. We define $\varphi : \mathcal{T} \hookrightarrow 5^{\omega}$ by the formula

$$\varphi(\tau) := (u_1^{0,\tau}\nu_1^{0,\tau} \dots u_{q_{0,\tau}}^{0,\tau}\nu_{q_{0,\tau}}^{0,\tau}4)(u_1^{1,\tau}\nu_1^{1,\tau} \dots u_{q_{1,\tau}}^{1,\tau}\nu_{q_{1,\tau}}^{1,\tau}4)\dots$$

The set A_0 will be made of finite subsequences of sentences in $\varphi[\mathcal{T}]$. We set

$$A_0 := \{\, u_{q+1}^{m,\tau}\nu_{q+1}^{m,\tau} \dots u_r^{p,\tau}\nu_r^{p,\tau} / \tau \in \mathcal{T}, \ m+1 < p, \ 0 \leq q \leq q_{m,\tau}, \ 1 \leq r \leq q_{p,\tau},$$
$$[(m = 0 \text{ and } q = 0) \text{ or } (q > 0 \text{ and } \nu_q^{m,\tau} = 3 \text{ and } u_q^{m,\tau} \prec u_r^{p,\tau})], \ \nu_r^{p,\tau} = 3 \,\}$$

(with the convention $u_{q_{m,\tau}+1}^{m,\tau}\nu_{q_{m,\tau}+1}^{m,\tau} = 4$). It is clear that φ is continuous, and it is enough to see that it reduces σ to A_0^{∞}.

So let us assume that $\tau \in \sigma$. This means the existence of an infinite branch in the tree with infinitely many 3 labels. We cut $\varphi(\tau)$ after the first 3 label of the branch corresponding to a sequence of length $m > 1$. Then we cut after the first 3 label corresponding to a sequence of length at least $m+2$ of the branch. And so on. This clearly gives a decomposition of $\varphi(\tau)$ into words in A_0.

If such a decomposition exists, then the first word is $u_1^{0,\tau}\nu_1^{0,\tau} \dots u_{r_0}^{p_0,\tau}\nu_{r_0}^{p_0,\tau}$, and the second is $u_{r_0+1}^{p_0,\tau}\nu_{r_0+1}^{p_0,\tau} \dots u_{r_1}^{p_1,\tau}\nu_{r_1}^{p_1,\tau}$. So we have $u_{r_0}^{p_0,\tau} \prec_{\neq} u_{r_1}^{p_1,\tau}$. And so on. This gives an infinite branch with infinitely many 3 labels.

• By Proposition 11, we can also have $A_0 \subseteq 2^{<\omega}$.

(b) Let $\alpha_0 := 1010^2 10^3 \ldots$, (q_l) be the sequence of prime numbers: $q_0 := 2$, $q_1 := 3$, $M : \omega^{<\omega} \to \omega$ defined by $M_s := q_0^{s(0)+1} \ldots q_{|s|-1}^{s(|s|-1)+1} + 1$, $\phi : \omega^{<\omega} \to 2^{<\omega} \setminus \{\emptyset\}$ defined by the formulas

$$\phi(\emptyset) := 1010^2 = 1010^{2M_\emptyset}$$

and $\phi(sm) := 10^{2M_s+1} 10^{2M_s+2} \ldots 10^{2M_{sm}}$, and $\Phi : 2^{\omega^{<\omega}} \to 2^{n^{<\omega}}$ defined by $\Phi(T) := \phi[T]$.

• It is clear that $M_{sm} > M_s$, and that M and ϕ are well defined and one-to-one. So Φ is continuous:

$$\begin{aligned} s \in \phi[T] &\Leftrightarrow \exists t \ \ (t \in T \text{ and } \phi(t) = s) \\ &\Leftrightarrow s \in \phi[\omega^{<\omega}] \text{ and } \forall t \ \ (t \in T \text{ or } \phi(t) \neq s). \end{aligned}$$

If $T \in IF$, then we can find $\beta \in \omega^\omega$ such that $\phi(\beta\lceil l) \in \Phi(T)$ for each integer l. Thus

$$\alpha_0 = (1010^{2M_{\beta\lceil 0}})(10^{2M_{\beta\lceil 0}+1} \ldots 10^{2M_{\beta\lceil 1}}) \ldots \in (\Phi(T))^\infty.$$

Conversely, if $\alpha_0 \in (\Phi(T))^\infty$, then there exist $t_i \in T$ such that $\alpha_0 = \phi(t_0)\phi(t_1)\ldots$ We have $t_0 = \emptyset$, and, if $i > 0$, then $M_{t_i\lceil |t_i|-1} = M_{t_{i-1}}$; from this we deduce that $t_i\lceil |t_i| - 1 = t_{i-1}$, because M is one-to-one. So let β be the limit of the t_i's. We have $\beta\lceil i = t_i$, thus $\beta \in [T]$ and $T \in IF$. Thus $\Phi_{\lceil \mathcal{T}}$ reduces IF to I_{α_0}. Therefore this last set is $\mathbf{\Sigma}^1_1$-complete. Indeed, it is clear that I is Σ^1_1:

$$\alpha \in A^\infty \Leftrightarrow \ \exists \beta \in \omega^\omega \ \ [(\forall m > 0 \ \ \beta(m) > 0) \text{ and } (\forall q \in \omega \ \ \pi(\alpha, \beta, q) \in A)].$$

Finally, the map from $\mathcal{T}$ into $n^\omega \times 2^{n^{<\omega}}$, which associates $(\alpha_0, \Phi(T))$ to T clearly reduces IF to I. So I is $\mathbf{\Sigma}^1_1$-complete. □

Remark. This proof shows that if $\alpha = s_0 s_1 \ldots$ and (s_i) is an antichain for the extension ordering, then I_α is $\mathbf{\Sigma}^1_1$-complete (here we have $s_i = 10^{2i+1}10^{2i+2}$). To see it, it is enough to notice that $\phi(\emptyset) = s_0$ and $\phi(sm) = s_{M_s} \ldots s_{M_{sm}-1}$. So I_α is $\mathbf{\Sigma}^1_1$-complete for a dense set of α's.

We will deduce from this some true co-analytic sets. But we need a lemma, which has its own interest.

Lemma 13 *(a) The set A^∞ is Borel if and only if there exist a Borel function $f : n^\omega \to \omega^\omega$ such that*

$$\alpha \in A^\infty \ \Leftrightarrow \ (\forall m > 0 \ \ f(\alpha)(m) > 0) \text{ and } (\forall q \in \omega \ \ \pi(\alpha, f(\alpha), q) \in A).$$

(b) Let $\gamma \in \omega^\omega$ and $A \subseteq n^{<\omega}$. Then $A^\infty \in \Delta^1_1(A, \gamma)$ if and only if, for $\alpha \in n^\omega$, we have

$$\alpha \in A^\infty \Leftrightarrow \ \exists \beta \in \Delta^1_1(A, \gamma, \alpha) \ \ [(\forall m > 0 \ \ \beta(m) > 0) \text{ and } (\forall q \in \omega \ \ \pi(\alpha, \beta, q) \in A)].$$

Proof. The "if" directions in (a) and (b) are clear. We have seen in the proof of Proposition 4 the "if" direction of the equivalences (the existence of an arbitrary β is necessary and sufficient). So let us show the "only if" directions.

(a) We define $f : n^\omega \to \omega^\omega$ by the formula $f(\alpha) := 0^\infty$ if $\alpha \notin A^\infty$, and, otherwise,

$$f(\alpha)(0) := \min\{p \in \omega / \alpha\lceil(p+1) \in A \text{ and } \alpha - \alpha\lceil(p+1) \in A^\infty\},$$

$f(\alpha)(r+1) :=$

$$\min\{k > 0 / [\alpha - \alpha\lceil(1 + \Sigma_{j\leq r}\, f(\alpha)(j))]\lceil k \in A \text{ and } \alpha - \alpha\lceil(k+1+\Sigma_{j\leq r}\, f(\alpha)(j)) \in A^\infty\}.$$

We get $\pi(\alpha, f(\alpha), 0) = \alpha\lceil f(\alpha)(0) + 1 \in A$ and, if $q > 0$,

$$\pi(\alpha, f(\alpha), q) = \ (\alpha(1 + \Sigma_{j<q}\, f(\alpha)[j]), ..., \alpha(\Sigma_{j\leq q}\, f(\alpha)[j])) \ \in A.$$

As f is clearly Borel, we are done.

(b) If $A^\infty \in \Delta^1_1(A, \gamma)$, then so is f and $\beta := f(\alpha) \in \Delta^1_1(A, \gamma, \alpha)$ is what we were looking for. □

Remark. Lemma 13 is a particular case of a more general situation. Actually we have the following uniformization result. It was written after a conversation with G. Debs.

Proposition 14 *Let X and Y be Polish spaces, and $F \in \mathbf{\Pi}^0_2(X \times Y)$ such that the projection $\Pi_X[F \cap (X \times V)]$ is Borel for each $V \in \mathbf{\Sigma}^0_1(Y)$. Then there exists a Borel map $f : X \to Y$ such that $(x, f(x)) \in F$ for each $x \in \Pi_X[F]$.*

Proof. Let (Y_n) be a basis for the topology of Y with $Y_0 := Y$, $B_n := \Pi_X[F \cap (X \times Y_n)]$, and τ be a finer 0-dimensional Polish topology on X making the B_n's clopen (see 13.5 in [K1]). We equip X with a complete τ-compatible metric d. Let $(O_m) \subseteq \mathbf{\Sigma}^0_1(X \times Y)$ be decreasing satisfying $O_0 := X \times Y$ and $F = \bigcap_m O_m$. We construct a sequence $(U_s)_{s\in\omega^{<\omega}}$ of clopen subsets of $[B_0, \tau]$ with $U_\emptyset := B_0$, and a sequence $(V_s)_{s\in\omega^{<\omega}}$ of basic open sets of Y satisfying

$$\begin{array}{l}(a)\ U_s \subseteq \Pi_X[F \cap (U_s \times V_s)]\\ (b)\ \mathrm{diam}_d(U_s),\ \mathrm{diam}(V_s) \leq \frac{1}{|s|} \text{ if } s \neq \emptyset\\ (c)\ U_s = \bigcup_{m,\text{disj.}}\ U_{s^\frown m},\ \overline{V_{s^\frown n}} \subseteq V_s\\ (d)\ U_s \times V_s \subseteq O_{|s|}\end{array}$$

• Assume that this construction has been achieved. If $x \notin B_0$, then we set $f(x) := y_0 \in Y$ (we may assume that $F \neq \emptyset$). Otherwise, we can find a unique sequence $\gamma \in \omega^\omega$ such that $x \in U_{\gamma\lceil m}$ for each integer m. Thus we can find $y \in V_{\gamma\lceil m}$ such that $(x, y) \in F$, and $(\overline{V_{\gamma\lceil m}})_m$ is a decreasing sequence of nonempty closed sets whose diameters tend to 0, which defines a continuous map $f : [B_0, \tau] \to Y$. If $x \in B_0$, then $(x, f(x)) \in U_{\gamma\lceil m} \times V_{\gamma\lceil m} \subseteq O_m$, thus $Gr(f_{|B_0}) \subseteq F$. Notice that $f : [X, \tau] \to Y$ is continuous, so $f : X \to Y$ is Borel.

• Let us show that the construction is possible. We set $U_\emptyset := B_0$ and $V_\emptyset := Y$. Assume that $(U_s)_{s\in\omega^{\leq p}}$ and $(V_s)_{s\in\omega^{\leq p}}$ satisfying conditions (a)-(d) have been constructed, which is the case for $p = 0$. Let $s \in \omega^p$. If $(x, y) \in F \cap (U_s \times V_s)$, then we can find $U_x \in \mathbf{\Delta}^0_1(U_s)$ and a basic open set $V_y \subseteq Y$ such that $(x, y) \in U_x \times V_y \subseteq U_x \times \overline{V_y} \subseteq (U_s \times V_s) \cap O_{p+1}$, and whose diameters are at most $\frac{1}{p+1}$. By the Lindelöf property, we can write $F \cap (U_s \times V_s) \subseteq \bigcup_n U_{x_n} \times V_{y_n}$ and $F \cap (U_s \times V_s) = \bigcup_n F \cap (U_{x_n} \times V_{y_n})$.

If $x \in U_s$, then let n and y be such that $(x,y) \in F \cap (U_{x_n} \times V_{y_n})$. Then

$$x \in O^n := \mathit{\Pi}_X[F \cap (X \times V_{y_n})] \cap U_{x_n} \in \mathbf{\Delta}^0_1([B_0,\tau]).$$

Thus $U_s = \bigcup_n O^n$. We set $U_{s^\frown n} := O^n \setminus (\bigcup_{p<n} O^p)$ and $V_{s^\frown n} := V_{y_n}$, and we are done. □

In our context, $F = \{(\alpha,\beta) \in n^\omega \times \omega^\omega / (\forall m > 0 \;\; \beta(m) > 0)$ and $(\forall q \in \omega \;\; \pi(\alpha,\beta,q) \in A)\}$, which is a closed subset of $X \times Y$. The projection $\Pi_X[F \cap (X \times N_s)]$ is Borel if A^∞ is Borel, since it is $\{S^*\gamma / S \in (A \cap n^{s(0)+1}) \times \Pi_{0<j<|s|} (A \cap n^{s(j)})$ and $\gamma \in A^\infty\}$.

Theorem 15 *The following sets are $\mathit{\Pi}^1_1 \setminus \mathbf{\Delta}^1_1$:*
(a) $\mathit{\Pi} := \{(A,\gamma,\theta) \in 2^{n^{<\omega}} \times \omega^\omega \times \omega^\omega / \theta \in$ WO and $A^\infty \in \mathbf{\Pi}^0_{|\theta|} \cap \mathit{\Delta}^1_1(A,\gamma)\}$. The same thing is true with $\mathit{\Sigma} := \{(A,\gamma,\theta) \in 2^{n^{<\omega}} \times \omega^\omega \times \omega^\omega / \theta \in$ WO and $A^\infty \in \mathbf{\Sigma}^0_{|\theta|} \cap \mathit{\Delta}^1_1(A,\gamma)\}$.
(b) $\mathit{\Sigma}_1 := \{A \in 2^{n^{<\omega}} / A^\infty \in \mathbf{\Sigma}^0_1 \cap \mathit{\Delta}^1_1(A)\}$. In fact, $\mathit{\Sigma}_\xi := \{A \in 2^{n^{<\omega}} / A^\infty \in \mathbf{\Sigma}^0_\xi \cap \mathit{\Delta}^1_1(A)\}$ is $\mathbf{\Pi}^1_1 \setminus \mathbf{\Delta}^1_1$ if $1 \leq \xi < \omega_1$. Similarly, $\mathit{\Pi}_\xi := \{A \in 2^{n^{<\omega}} / A^\infty \in \mathbf{\Pi}^0_\xi \cap \mathit{\Delta}^1_1(A)\}$ is $\mathbf{\Pi}^1_1 \setminus \mathbf{\Delta}^1_1$ if $2 \leq \xi < \omega_1$.
(c) $\mathit{\Delta} := \{A \in 2^{n^{<\omega}} / A^\infty \in \mathit{\Delta}^1_1(A)\}$.

Proof. Consider the way of coding the Borel sets used in [Lou]. By Lemma 13 we get

$$(A,\gamma,\theta) \in \mathit{\Pi} \Leftrightarrow \begin{cases} \exists p \in \omega \;\; P(p,A,\gamma,\theta) \text{ and } \forall \alpha \in n^\omega \\ (\alpha \notin A^\infty \text{ or } (p,A,\gamma,\alpha) \in C) \text{ and } ([(p,A,\gamma) \in W \text{ and } (p,A,\gamma,\alpha) \notin C] \text{ or} \\ \exists \beta \in \mathit{\Delta}^1_1(A,\gamma,\alpha) \;\; [(\forall m > 0 \;\; \beta(m) > 0) \text{ and } (\forall q \in \omega \;\; \pi(\alpha,\beta,q) \in A)]). \end{cases}$$

This shows that $\mathit{\Pi}$ is $\mathit{\Pi}^1_1$. The same argument works with $\mathit{\Sigma}$. From this we can deduce that $\mathit{\Sigma}_1$ is $\mathit{\Pi}^1_1$, if we forget γ and take the section of $\mathit{\Sigma}$ at $\theta \in \mathrm{WO} \cap \mathit{\Delta}^1_1$ such that $|\theta| = 1$. Similarly, $\mathit{\Sigma}_\xi$ and $\mathit{\Pi}_\xi$ are co-analytic if $\xi \geq 1$. Forgetting θ, we see that the relation "$A^\infty \in \mathit{\Delta}^1_1(A,\gamma)$" is $\mathit{\Pi}^1_1$.

• Let us look at the proof of Theorem 12. We will show that if $\xi \geq 1$ (resp., $\xi \geq 2$), then $\mathit{\Sigma}_\xi \setminus I_{\alpha_0}$ (resp., $\mathit{\Pi}_\xi \setminus I_{\alpha_0}$) is a true co-analytic set. To do this, we will reduce WF to $\mathit{\Sigma}_\xi \setminus I_{\alpha_0}$ (resp., $\mathit{\Pi}_\xi \setminus I_{\alpha_0}$) in a Borel way. We change the definition of Φ. We set

$$t \subseteq \alpha_0 \;\Leftrightarrow\; \exists k \;\; t \prec \alpha_0 - \alpha_0 \lceil k,$$

$$E := \{(\alpha_0 \lceil p) r / p \in \omega \setminus \{2\}, r \in n \setminus \{\alpha_0(p)\}\}, \qquad F := \{U^* \not\subseteq \alpha_0 / U \in \phi[T]^{<\omega}\},$$
$$\Phi'(T) := \phi[T] \cup \{s \in n^{<\omega} / \exists t \in E \cup F \;\; t \prec s\}.$$

This time, Φ' is Baire class one, since

$$\begin{aligned} s \in \Phi'(T) \;\Leftrightarrow\; & s \in \phi[T] \text{ or } \exists t \in E \;\; t \prec s \text{ or} \\ & \exists U \in (2^{<\omega})^{<\omega} \;\; (\forall j < |u| \; U(j) \in \phi[T]) \text{ and } U^* \not\subseteq \alpha_0 \text{ and } U^* \prec s. \end{aligned}$$

The proof of Theorem 12 remains valid, since if $\alpha_0 \in (\Phi'(T))^\infty$, then the decompositions of α_0 into words of $\Phi'(T)$ are actually decompositions into words of $\phi[T]$.

• Let us show that $(\Phi'(T))^\infty \in \mathbf{\Sigma}^0_1 \cap \Delta^1_1(\Phi'(T))$ if $T \in WF$. The set $(\Phi'(T))^\infty$ is

$$\bigcup_{S\in\phi[T]^{<\omega},l\in n\setminus\{1\},m\in n\setminus\{0\}} [(\bigcup_{s/\exists t\in F\ t\prec s} N_{S^*s}) \cup N_{S^*l} \cup N_{S^*1m} \cup (N_{S^*101} \setminus \{S^*\alpha_0\})].$$

If $\alpha \in n^\omega$, then α contains infinitely many $l \in n \setminus \{1\}$ or finishes with 1^∞. As 1^2 and the sequences beginning with l are in $\Phi'(T)$, the clopen sets are subsets of $(\Phi'(T))^\infty$ since $\phi[T]$ and the sequences beginning with $t \in F$, l or $1m$ are in $\Phi'(T)$. If $\alpha \in N_{S^*101} \setminus \{S^*\alpha_0\}$, then let $p \geq 3$ be maximal such that $\alpha\lceil(|S^*|+p) = S^*(\alpha_0\lceil p)$. We have $\alpha \in (\Phi'(T))^\infty$ since the sequences beginning with $(\alpha_0\lceil p)r$ are in $\Phi'(T)$. Thus we get the inclusion into $(\Phi'(T))^\infty$.

If $\alpha \in (\Phi'(T))^\infty$, then $\alpha = a_0a_1\ldots$, where $a_i \in \Phi'(T)$. Either for all i we have $a_i \in \phi[T]$. In this case, there is i such that $a_0\ldots a_i \not\sqsubseteq \alpha_0$, otherwise we could find k with $\alpha_0 - \alpha_0\lceil k \in (\Phi(T))^\infty$. But this contradicts the fact that $T \in WF$, as in the proof of Theorem 12. So we have $\alpha \in \bigcup_{\exists t\in F\ t\prec s} N_s$. Or there exists i minimal such that $a_i \notin \phi[T]$. In this case,

- Either $\exists t \in E\ t \prec a_i$ and $\alpha \in \bigcup_{S\in\phi[T]^{<\omega},l\in n\setminus\{1\},m\in n\setminus\{0\}} [N_{S^*l} \cup N_{S^*1m} \cup (N_{S^*101} \setminus \{S^*\alpha_0\})]$,

- Or $\exists t \in F\ t \prec a_i$ and $\alpha \in \bigcup_{S\in\phi[T]^{<\omega}} \bigcup_{s/\exists t\in F\ t\prec s} N_{S^*s}$.

From this we deduce that $(\Phi'(T))^\infty$ is $\mathbf{\Sigma}^0_1$.

Finally, we have

$$\alpha\in(\Phi'(T))^\infty \Leftrightarrow \begin{cases} \exists t\in n^{<\omega}\ \exists b\in\omega^{<\omega}\ [(|t|=1+\Sigma_{j<|b|}\ b(j)) \text{ and } (\forall 0<m<|b|\ b(m)>0) \\ \text{and } (\forall q<|b|\ \pi(t0^\infty, b0^\infty, q)\in\Phi'(T))] \text{ and } [\exists l\in n\setminus\{1\}\ tl\prec\alpha \text{ or } t1^2\prec\alpha]. \end{cases}$$

This shows that $(\Phi'(T))^\infty$ is $\Delta^1_1(\Phi'(T))$.

Therefore, $\Phi'_{\lceil\mathcal{T}}$ reduces WF to $\Sigma_\xi \setminus I_{\alpha_0}$ if $\xi \geq 1$, and to $\Pi_\xi \setminus I_{\alpha_0}$ if $\xi \geq 2$. So these sets are true co-analytic sets. But $\Sigma_1 \cap I_{\alpha_0}$ is Π^1_1, by Lemma 13. As $\Sigma_1 \setminus I_{\alpha_0} = \Sigma_1 \setminus (\Sigma_1 \cap I_{\alpha_0})$, Σ_1 is not Borel. Thus Σ is not Borel, as before. The argument is similar for Σ_ξ, Π_ξ ($\xi \geq 2$) and Π. And for Δ too. □

Question. Does $A^\infty \in \mathbf{\Delta}^1_1$ imply $A^\infty \in \Delta^1_1(A)$? Probably not. If the answer is positive, $\mathbf{\Delta}$, and more generally $\mathbf{\Sigma}_\xi$ (for $\xi \geq 1$) and $\mathbf{\Pi}_\xi$ (for $\xi \geq 2$) are true co-analytic sets.

Remark. In any case, $\mathbf{\Delta}$ is Σ^1_2 because "$A^\infty \in \mathbf{\Delta}^1_1$" is equivalent to "$\exists\gamma\in\omega^\omega\ A^\infty \in \Delta^1_1(A,\gamma)$". This argument shows that $\mathbf{\Sigma}_\xi$ and $\mathbf{\Pi}_\xi$ are $\Sigma^1_2(\theta)$, where $\theta \in WO$ satisfies $|\theta| = \xi$. We can say more about $\mathbf{\Pi}_1$: it is Δ^1_2. Indeed, in [St2] we have the following characterization:

$$A^\infty\in\mathbf{\Pi}^0_1 \Leftrightarrow \forall\alpha\in n^\omega\ [\forall s\in n^{<\omega}\ (s\prec\alpha \Rightarrow \exists S\in A^{<\omega}\ s\prec S^*)] \Rightarrow \alpha\in A^\infty.$$

This gives a Π^1_2 definition of $\mathbf{\Pi}_1$. The same fact is true for $\mathbf{\Sigma}_1$:

Proposition 16 *$\mathbf{\Sigma}_1$ and $\mathbf{\Pi}_1$ are co-nowhere dense $\Delta^1_2 \setminus D_2(\mathbf{\Sigma}^0_1)$ subsets of $2^{n^{<\omega}}$. If $\xi \geq 2$, then $\mathbf{\Sigma}_\xi$ and $\mathbf{\Pi}_\xi$ are co-nowhere dense $\mathbf{\Sigma}^1_2 \setminus D_2(\mathbf{\Sigma}^0_1)$ subsets of $2^{n^{<\omega}}$. $\mathbf{\Delta}$ is a co-nowhere dense $\Sigma^1_2 \setminus D_2(\mathbf{\Sigma}^0_1)$ subset of $2^{n^{<\omega}}$.*

Proof. We have seen that $\mathbf{\Sigma}_1$ is Σ_2^1; it is also Π_2^1 because

$$A^\infty \in \mathbf{\Sigma}_1^0 \ \Leftrightarrow\ \forall \alpha \in n^\omega \ \ \alpha \notin A^\infty \text{ or } \exists s \in n^{<\omega}\ [s \prec \alpha \text{ and } \forall \beta \in n^\omega\ (s \not\prec \beta \text{ or } \beta \in A^\infty)].$$

By Proposition 4, $\mathbf{\Pi}_0$ is co-nowhere dense, and it is a subset of $\mathbf{\Sigma}_\xi \cap \mathbf{\Pi}_\xi \cap \mathbf{\Delta}$. So $\mathbf{\Sigma}_\xi$, $\mathbf{\Pi}_\xi$ and $\mathbf{\Delta}$ are co-nowhere dense, and it remains to see that they are not open. It is enough to notice that $\emptyset$ is not in their interior. Look at the proof of Theorem 12; it shows that for each integer m, there is a subset A_m of $\{s \in 5^{<\omega} / |s| \geq m\}$ such that $A_m^\infty \notin \mathbf{\Delta}_1^1$. But the argument in the proof of Proposition 11 shows that we can have the same thing in $n^{<\omega}$ for each $n \geq 2$. This gives the result because the sequence (A_m) tends to $\emptyset$. □

We can say a bit more about $\mathbf{\Pi}_1$ and $\mathbf{\Sigma}_2$:

Proposition 17 *$\mathbf{\Pi}_1$, Π_1 and $\mathbf{\Sigma}_2$ are $\mathbf{\Sigma}_2^0$-hard (so they are not $\mathbf{\Pi}_2^0$).*

Proof. Consider the map ϕ defined in the proof of Proposition 5. By Proposition 2, if $\gamma \in P_f$, then $\phi(\gamma)^\infty$ is $\mathbf{\Pi}_1^0$. Moreover, as $\phi(\gamma)$ is an antichain for the extension ordering, the decomposition into words of $\phi(\gamma)$ is unique. This shows that $\phi(\gamma)^\infty$ is Δ_1^1, because

$$\alpha \in \phi(\gamma)^\infty \ \Leftrightarrow\ \exists \beta \in \Delta_1^1(\alpha)\ [(\forall m > 0\ \ \beta(m) > 0) \text{ and } (\forall q \in \omega\ \ \pi(\alpha, \beta, q) \in \phi(\gamma))].$$

So $\phi(\gamma) \in \Pi_1$ if $\gamma \in P_f$. So the preimage of any of the sets in the statement by ϕ is P_f, and the result follows. □

4 Which sets are ω-powers?

Now we come to Question (3). Let us specify what we mean by "codes for Γ-sets", where Γ is a given class, and fix some notation.

• For the Borel classes, we will essentially consider the 2^ω-universal sets used in [K1] (see Theorem 22.3). For $\xi \geq 1$, $\mathcal{U}^{\xi,\mathcal{A}}$ (resp. $\mathcal{U}^{\xi,\mathcal{M}}$) is 2^ω-universal for $\mathbf{\Sigma}_\xi^0(n^\omega)$ (resp. $\mathbf{\Pi}_\xi^0(n^\omega)$). So we have

- $\mathcal{U}^{1,\mathcal{A}} = \{(\gamma, \alpha) \in 2^\omega \times n^\omega / \exists p \in \omega\ \gamma(p) = 0 \text{ and } s_p^n \prec \alpha\}$, where $(s_p^n)_p$ enumerates $n^{<\omega}$.
- $\mathcal{U}^{\xi,\mathcal{M}} = \neg\, \mathcal{U}^{\xi,\mathcal{A}}$, for each $\xi \geq 1$.
- $\mathcal{U}^{\xi,\mathcal{A}} = \{(\gamma, \alpha) \in 2^\omega \times n^\omega / \exists p \in \omega\ ((\gamma)_p, \alpha) \in \mathcal{U}^{\eta,\mathcal{M}}\}$ if $\xi = \eta + 1$.
- $\mathcal{U}^{\xi,\mathcal{A}} = \{(\gamma, \alpha) \in 2^\omega \times n^\omega / \exists p \in \omega\ ((\gamma)_p, \alpha) \in \mathcal{U}^{\eta_p,\mathcal{M}}\}$ if ξ is the limit of the strictly increasing sequence of odd ordinals (η_p).

• For the class $\mathbf{\Sigma}_1^1$, we fix some bijection $p \mapsto ((p)_0, (p)_1)$ between ω and ω^2. We set

$$(\gamma, \alpha) \in \mathcal{U} \ \Leftrightarrow\ \exists \beta \in 2^\omega\ (\forall m\ \exists p \geq m\ \ \beta(p) = 1) \text{ and } (\forall p\ \ [\gamma(p) = 1 \text{ or } s^2_{(p)_0} \not\prec \beta \text{ or } s^n_{(p)_1} \not\prec \alpha]).$$

It is not hard to see that $\mathcal{U}$ is 2^ω-universal for $\mathbf{\Sigma}_1^1(n^\omega)$, and we use it here because of the compactness of $2^\omega \times n^\omega$, rather than the ω^ω-universal set for $\mathbf{\Sigma}_1^1(n^\omega)$ given in [K1] (see Theorem 14.2).

• For the class $\mathbf{\Delta}^1_1$, it is different because there is no universal set. But we can use the $\mathit{\Pi}^1_1$ set of codes $D \subseteq 2^\omega$ for the Borel sets in [K1] (see Theorem 35.5). We may assume that D, S and P are effective, by [M].

• The sets we are interested in are the following:

$$\mathcal{A}_\xi := \{\gamma \in 2^\omega / \mathcal{U}^{\xi,\mathcal{A}}_\gamma \text{ is an } \omega\text{-power}\}, \quad \mathcal{M}_\xi := \{\gamma \in 2^\omega / \mathcal{U}^{\xi,\mathcal{M}}_\gamma \text{ is an } \omega\text{-power}\}$$

$$\mathcal{B} := \{d \in D / D_d \text{ is an } \omega\text{-power}\},$$

$$\mathcal{A} := \{\gamma \in 2^\omega / \mathcal{U}_\gamma \text{ is an } \omega\text{-power}\}.$$

As we mentionned in the introduction, Lemma 13 is also related to Question (3). A rough answer to this question is Σ^1_3. Indeed, we have, for $\gamma \in 2^\omega$,

$$\gamma \in \mathcal{A} \;\Leftrightarrow\; \exists A \in 2^{n^{<\omega}} \; \forall \alpha \in n^\omega \; ([(\gamma, \alpha) \notin \mathcal{U} \text{ or } \alpha \in A^\infty] \text{ and } [\alpha \notin A^\infty \text{ or } (\gamma, \alpha) \in \mathcal{U}]).$$

With Lemma 13, we have a better estimation of the complexity of $\mathcal{B}$: it is Σ^1_2. Indeed, for $d \in D$,

$$D_d \text{ is an } \omega\text{-power} \;\Leftrightarrow\; \exists A \in 2^{n^{<\omega}} \; \forall \alpha \in n^\omega \; ([(d, \alpha) \notin S \;\text{ or }\; \exists \beta \in \Delta^1_1(A, d, \alpha)$$

$$[(\forall m > 0 \;\; \beta(m) > 0) \text{ and } (\forall q \in \omega \;\; \pi(\alpha, \beta, q) \in A)]] \text{ and } [\alpha \notin A^\infty \text{ or } (d, \alpha) \in P]).$$

This argument also shows that $\mathcal{A}_\xi$ and $\mathcal{M}_\xi$ are $\mathbf{\Sigma}^1_2$. We can say more about these two sets.

Proposition 18 *If $1 \leq \xi < \omega_1$, then $\mathcal{A}_\xi$ and $\mathcal{M}_\xi$ are $\mathbf{\Sigma}^1_2 \setminus D_2(\mathbf{\Sigma}^0_1)$ co-meager subsets of 2^ω. If moreover $\xi = 1$, then they are co-nowhere dense.*

Proof. We set $E_1 := \{\gamma \in 2^\omega / \mathcal{U}^{1,\mathcal{A}}_\gamma = n^\omega\}$, $E_{\eta+1} := \{\gamma \in 2^\omega / \forall p \; (\gamma)_p \in E_\eta\}$ if $\eta \geq 1$, and $E_\xi := \{\gamma \in 2^\omega / \forall p \; (\gamma)_p \in E_{\eta_p}\}$ (where (η_p) is a strictly increasing sequence of odd ordinals cofinal in the limit ordinal ξ). If $s \in 2^{<\omega}$, then we set $\gamma(p) = s(p)$ if $p < |s|$, 0 otherwise. Then $s \prec \gamma$ and $\mathcal{U}^{1,\mathcal{A}}_\gamma = n^\omega$, so E_1 is dense. If $\gamma_0 \in E_1$, then for all $\alpha \in n^\omega$ we can find an integer p such that $\gamma_0(p) = 0$ and $s^n_p \prec \alpha$. By compactness of n^ω we can find a finite subset F of $\{p \in \omega / \gamma_0(p) = 0\}$ such that for each $\alpha \in n^\omega$, $s^n_p \prec \alpha$ for some $p \in F$. Now $\{\gamma \in 2^\omega / \forall p \in F \; \gamma(p) = 0\}$ is an open neighborhood of γ_0 and a subset of E_1. So E_1 is an open subset of 2^ω. Now the map $\gamma \mapsto (\gamma)_p$ is continuous and open, so $E_{\eta+1}$ and E_ξ are dense G_δ subsets of 2^ω. Then we notice that E_ξ is a subset of $\{\gamma \in 2^\omega / \mathcal{U}^{\xi,\mathcal{A}}_\gamma = n^\omega\}$ (resp., $\{\gamma \in 2^\omega / \mathcal{U}^{1,\mathcal{A}}_\gamma = \emptyset\}$) if ξ is odd (resp., even). Indeed, this is clear for $\xi = 1$. Then we use the formulas $\mathcal{U}^{\eta+1,\mathcal{A}}_\gamma = \bigcup_p \neg \mathcal{U}^{\eta,\mathcal{A}}_{(\gamma)_p}$ and $\mathcal{U}^{\xi,\mathcal{A}}_\gamma = \bigcup_p \neg \mathcal{U}^{\eta_p,\mathcal{A}}_{(\gamma)_p}$, and by induction we are done. As $\emptyset$ and n^ω are ω-powers, we get the results about Baire category. Now it remains to see that $\mathcal{A}_\xi$ and $\mathcal{M}_\xi$ are not open. But by induction again $1^\infty \in \mathcal{A}_\xi \cap \mathcal{M}_\xi$, so it is enough to see that 1^∞ is not in the interior of these sets.

• Let us show that, for $O \in \mathbf{\Delta}^0_1(n^\omega) \setminus \{\emptyset, n^\omega\}$ and for each integer m, we can find γ, $\gamma' \in 2^\omega$ such that $\gamma(j) = \gamma'(j) = 1$ for $j < m$, $\mathcal{U}^{\xi,\mathcal{A}}_\gamma = O$ and $\mathcal{U}^{\xi,\mathcal{M}}_{\gamma'} = O$.

For $\xi = 1$, write $O = \bigcup_p N_{s^n_{q_k}}$, where $q_k \geq m$. Let $\gamma(q) := 0$ if there exists k such that $q = q_k$, $\gamma(q) := 1$ otherwise. The same argument applied to $\check{O}$ gives the complete result for $\xi = 1$.

Now we argue by induction. Let $\gamma_p \in 2^\omega$ be such that $\gamma_p(q) = 1$ for $<p, q>< m$ and $\mathcal{U}^{\eta,\mathcal{M}}_{(\gamma)_p} = O$. Then define γ by $\gamma(<p, q>) := \gamma_p(q)$; we have $\gamma(j) = 1$ if $j < m$ and $\mathcal{U}^{\eta+1,\mathcal{A}}_{\gamma} = \bigcup_p \mathcal{U}^{\eta,\mathcal{M}}_{(\gamma)_p} = O$. The argument with $\check{O}$ still works. The argument is similar for limit ordinals.

• Now we apply this fact to $O := N_{(0)}$. This gives γ_p, $\gamma'_p \in N_{1^p}$ such that $\mathcal{U}^{\xi,\mathcal{A}}_{\gamma_p} = N_{(0)}$ and $\mathcal{U}^{\xi,\mathcal{M}}_{\gamma'_p} = N_{(0)}$. But (γ_p), (γ'_p) tend to 1^∞, $\gamma_p \notin \mathcal{A}_\xi$ and $\gamma'_p \notin \mathcal{M}_\xi$. □

Corollary 19 *$\mathcal{A}_1$ is $\check{D}_2(\mathbf{\Sigma}^0_1) \setminus D_2(\mathbf{\Sigma}^0_1)$. In particular, $\mathcal{A}_1$ is $\check{D}_2(\mathbf{\Sigma}^0_1)$-complete.*

Proof. By the preceding proof, it is enough to see that $\mathcal{A}_1 \setminus \{1^\infty\}$ is open. So let $\gamma_0 \in \mathcal{A}_1 \setminus \{1^\infty\}$, p_0 in ω with $\gamma_0(p_0) = 0$, and $A_0 \subseteq n^{<\omega}$ with $\mathcal{U}^{1,\mathcal{A}}_{\gamma_0} = A^\infty_0$. If $\alpha \in n^\omega$, then $s^n_{p_0}\alpha \in \mathcal{U}^{1,\mathcal{A}}_{\gamma_0}$, so we can find $m > 0$ such that $\alpha - \alpha\lceil m \in A^\infty_0$; thus there exists an integer p such that $\gamma_0(p) = 0$ and $s^n_p \prec \alpha - \alpha\lceil m$. By compactness of n^ω, there are finite sets $F \subseteq \omega \setminus \{0\}$ and $G \subseteq \{p \in \omega / \gamma_0(p) = 0\}$ such that $n^\omega = \bigcup_{m\in F, p\in G} \{\alpha \in n^\omega / s^n_p \prec \alpha - \alpha\lceil m\}$.

We set $A_\gamma := \{s \in n^{<\omega} / \exists p\, \gamma(p) = 0 \text{ and } s^n_p \prec s\}$ for $\gamma \in 2^\omega$, so that $A^\infty_\gamma \subseteq \mathcal{U}^{1,\mathcal{A}}_\gamma$. Assume that $\gamma(p) = 0$ for each $p \in G$ and let $\alpha \in \mathcal{U}^{1,\mathcal{A}}_\gamma$. Let $p^0 \in \omega$ be such that $\gamma(p^0) = 0$ and $s^n_{p^0} \prec \alpha$. We can find $m_0 > 0$ and $p^1 \in G$ such that $s^n_{p^1} \prec \alpha - \alpha\lceil(|s^n_{p^0}| + m_0)$, and $\alpha\lceil(|s^n_{p^0}| + m_0) \in A_\gamma$. Then we can find $m_1 > 0$ and $p^2 \in G$ such that $s^n_{p^2} \prec \alpha - \alpha\lceil(|s^n_{p^0}| + m_0 + |s^n_{p^1}| + m_1)$, and

$$\alpha\lceil(|s^n_{p^0}| + m_0 + |s^n_{p^1}| + m_1) - \alpha\lceil(|s^n_{p^0}| + m_0) \in A_\gamma.$$

And so on. Thus $\alpha \in A^\infty_\gamma$ and $\{\gamma \in 2^\omega / \forall p \in G\, \gamma(p) = 0\}$ is a clopen neighborhood of γ_0 and a subset of $\mathcal{A}_1$. □

Proposition 20 *$\mathcal{A}$ is $\Sigma^1_3 \setminus D_2(\mathbf{\Sigma}^0_1)$ and is co-nowhere dense.*

Proof. Let $U := \{\gamma \in 2^\omega / \forall \beta \in 2^\omega\ \forall \alpha \in n^\omega\ \exists p\ \ [\gamma(p) = 0 \text{ and } s^2_{(p)_0} \prec \beta \text{ and } s^n_{(p)_1} \prec \alpha]\}$. By compactness of $2^\omega \times n^\omega$, U is a dense open subset of 2^ω. Moreover, if $\gamma \in U$, then $\mathcal{U}_\gamma = \emptyset$, so $U \subseteq \mathcal{A}$ and $\mathcal{A}$ is co-nowhere dense. It remains to see that $\mathcal{A}$ is not open, as in the proof of Proposition 18. As $\mathcal{U}_{1^\infty} = n^\omega$, $1^\infty \in \mathcal{A}$. Let p be an integer satisfying $s^2_{(p)_0} = \emptyset$ and $s^n_{(p)_1} = 0^q$. We set $\gamma_p(m) := 0$ if and only if $m = p$, and also $P_\infty := \{\alpha \in 2^\omega / \forall r\ \exists m \geq r\ \alpha(m) = 1\}$. Then (γ_p) tends to 1^∞ and we have

$$\begin{aligned}\mathcal{U}_{\gamma_p} &= \{\alpha \in n^\omega / \exists \beta \in P_\infty\ \forall m\ \ m \neq p \text{ or } s^2_{(m)_0} \not\prec \beta \text{ or } s^n_{(m)_1} \not\prec \alpha\} \\ &= \{\alpha \in n^\omega / \exists \beta \in P_\infty\ (\beta, \alpha) \notin 2^\omega \times N_{0^q}\} = \neg\, N_{0^q}.\end{aligned}$$

So $\gamma_p \notin \mathcal{A}$. □

5 Ordinal ranks and ω-powers.

Notation. The fact that the ω-powers are $\mathbf{\Sigma}^1_1$ implies the existence of a co-analytic rank on the complement of A^∞ (see 34.4 in [K1]). We will consider a natural one, defined as follows. We set, for $\alpha \in n^\omega$, $T_A(\alpha) := \{S \in (A^-)^{<\omega} / S^* \prec \alpha\}$. This is a tree on A^-, which is well founded if and only if $\alpha \notin A^\infty$.

The rank of this tree is the announced rank $R_A : \neg\, A^\infty \to \omega_1$ (see page 10 in [K1]): we have $R_A(\alpha) := \rho(T_A(\alpha))$. Let $\phi : A^- \to \omega$ be one-to-one, and $\tilde{\phi}(S) := (\phi[S(0)], \ldots, \phi[S(|s|-1)])$ for $S \in (A^-)^{<\omega}$. This allows us to define the map Φ from the set of trees on A^- into the set of trees on ω, which associates $\{\tilde{\phi}(S)/S \in T\}$ to T. As $\tilde{\phi}$ is one-to-one, Φ is continuous:

$$t \in \Phi(T) \Leftrightarrow t \in \tilde{\phi}[(A^-)^{<\omega}] \text{ and } \tilde{\phi}^{-1}(t) \in T.$$

Moreover, T is well-founded if and only if $\Phi(T)$ is well-founded. Thus, if $\alpha \notin A^\infty$, then we have $\rho(T_A(\alpha)) = \rho(\Phi[T_A(\alpha)])$ because $\tilde{\phi}$ is strictly monotone (see page 10 in [K1]). Thus R_A is a co-analytic rank because the function from n^ω into the set of trees on $\omega^{<\omega}$ which associates $\Phi[T_A(\alpha)]$ to α is continuous, and because the rank of the well-founded trees on ω defines a co-analytic rank (see 34.6 in [K1]). We set

$$R(A) := \sup\{R_A(\alpha)/\alpha \notin A^\infty\}.$$

By the boundedness theorem, A^∞ is Borel if and only if $R(A) < \omega_1$ (see 34.5 and 35.23 in [K1]). We can ask the question of the link between the complexity of A^∞ and the ordinal $R(A)$ when A^∞ is Borel.

Proposition 21 *If $\xi < \omega_1$, $r \in \omega$ and $R(A) = \omega.\xi + r$, then $A^\infty \in \mathbf{\Sigma}^0_{2.\xi+1}$.*

Proof. The reader should see [L] for operations on ordinals.

• If $0 < \lambda < \omega_1$ is a limit ordinal, then let (λ_q) be a strictly increasing co-final sequence in λ, with $\lambda_q = \omega.\theta + q$ if $\lambda = \omega.(\theta+1)$, and $\lambda_q = \omega.\xi_q$ if $\lambda = \omega.\xi$, where (ξ_q) is a strictly increasing co-final sequence in the limit ordinal ξ otherwise. By induction, we define

$$\begin{aligned} E_0 &:= \{\alpha \in n^\omega / \forall s \in A^-\ s \not\prec \alpha\}, \\ E_{\theta+1} &:= \{\alpha \in n^\omega / \forall s \in A^-\ s \not\prec \alpha \text{ or } \alpha - s \in E_\theta\}, \\ E_\lambda &:= \{\alpha \in n^\omega / \forall s \in A^-\ s \not\prec \alpha \text{ or } \exists q \in \omega\ \alpha - s \in E_{\lambda_q}\}. \end{aligned}$$

• Let us show that $E_{\omega.\xi+r} \in \mathbf{\Pi}^0_{2.\xi+1}$. We may assume that $\xi \neq 0$ and that $r = 0$. If $\xi = \theta + 1$, then $E_{\lambda_q} \in \mathbf{\Pi}^0_{2.\theta+1}$ by induction hypothesis, thus $E_{\omega.\xi+r} \in \mathbf{\Pi}^0_{2.\theta+3} = \mathbf{\Pi}^0_{2.\xi+1}$. Otherwise, $E_{\lambda_q} \in \mathbf{\Pi}^0_{2.\xi_q+1}$ by induction hypothesis, thus $E_{\omega.\xi+r} \in \mathbf{\Pi}^0_{\xi+1} = \mathbf{\Pi}^0_{2.\xi+1}$.

• Let us show that if $\alpha \in A^\infty$, then $\alpha \notin E_{\omega.\xi+r}$. If $\xi = r = 0$, it is clear. If $r = m+1$ and $s \in A^-$ satisfies $s \prec \alpha$ and $\alpha - s \in A^\infty$, then we have $\alpha - s \notin E_{\omega.\xi+m}$ by induction hypothesis, thus $\alpha \notin E_{\omega.\xi+r}$. If $r = 0$ and $s \in A^-$ satisfies $s \prec \alpha$ and $\alpha - s \in A^\infty$, then we have $\alpha - s \notin E_{\lambda_q}$ for each integer q, by induction hypothesis, thus $\alpha \notin E_{\omega.\xi+r}$.

• Let $s \in A^-$ such that $s \prec \alpha \notin A^\infty$. We have

$$\begin{aligned} \rho(T_A(\alpha - s)) &= \sup\{\rho_{T_A(\alpha-s)}(t) + 1 \ / \ t \in T_A(\alpha - s)\} \\ &\leq \sup\{\rho_{T_A(\alpha)}((s)t) + 1 \ / \ (s)t \in T_A(\alpha)\} \\ &\leq \rho_{T_A(\alpha)}((s)) + 1 \\ &\leq \rho_{T_A(\alpha)}(\emptyset) < \rho(T_A(\alpha)). \end{aligned}$$

The first inequality comes from the fact that the map from $T_A(\alpha - s)$ into $T_A(\alpha)$, which associates $(s)t$ to t is strictly monotone (see page 10 in [K1]). We have

$$\rho(T_A(\alpha)) \geq [\sup\{\rho(T_A(\alpha - s)) \ / \ s \in A^-, \ s \prec \alpha\}] + 1.$$

Let us show that we actually have equality. We have

$$\rho(T_A(\alpha)) = \rho_{T_A(\alpha)}(\emptyset) + 1 = \sup\{\rho_{T_A(\alpha)}((s)) + 1 \ / \ s \in A^-, \ s \prec \alpha\} + 1.$$

Therefore, it is enough to notice that if $s \in A^-$ and $s \prec \alpha$, then $\rho_{T_A(\alpha)}((s)) \leq \rho_{T_A(\alpha - s)}(\emptyset)$. But this comes from the fact that the map from $\{S \in T_A(\alpha) \ / \ S(0) = s\}$ into $T_A(\alpha - s)$, which associates $S - (s)$ to S, preserves the extension ordering (see page 352 in [K1]).

• Let us show that, if $\alpha \notin A^\infty$, then "$\rho(T_A(\alpha)) \leq \omega.\xi + r + 1$" is equivalent to "$\alpha \in E_{\omega.\xi + r}$". We do it by induction on $\omega.\xi + r$. If $\xi = r = 0$, then it is clear. If $r = m + 1$, then "$\rho(T_A(\alpha)) \leq \omega.\xi + r + 1$" is equivalent to "$\forall s \in A^-$, $s \not\prec \alpha$ or $\rho(T_A(\alpha - s)) \leq \omega.\xi + m + 1$", by the preceding point. This is equivalent to "$\forall s \in A^-$, $s \not\prec \alpha$ or $\alpha - s \in E_{\omega.\xi + m}$", which is equivalent to "$\alpha \in E_{\omega.\xi + r}$". If $r = 0$, then "$\rho(T_A(\alpha)) \leq \omega.\xi + r + 1$" is equivalent to "$\forall s \in A^-$, $s \not\prec \alpha$ or there exists an integer q such that $\rho(T_A(\alpha - s)) \leq \lambda_q + 1$". This is equivalent to "$\forall s \in A^-$, $s \not\prec \alpha$ or there exists an integer q such that $\alpha - s \in E_{\lambda_q}$", which is equivalent to "$\alpha \in E_{\omega.\xi + r}$".

• If $\alpha \notin A^\infty$, then $\rho(T_A(\alpha)) \leq \omega.\xi + r + 1$. By the preceding point, $\alpha \in E_{\omega.\xi + r}$. Thus we have $A^\infty = \neg\, E_{\omega.\xi + r} \in \mathbf{\Sigma}^0_{2.\xi + 1}$. □

We can find an upper bound for the rank R, for some Borel classes:

Proposition 22 *(a) $A^\infty = n^\omega$ if and only if $R(A) = 0$.*
(b) If $A^\infty = \emptyset$, then $R(A) = 1$.
(c) If $A^\infty \in \mathbf{\Delta}^0_1$, then $R(A) < \omega$, and there exists $A_p \subseteq 2^{<\omega}$ such that $A_p^\infty \in \mathbf{\Delta}^0_1$ and $R(A_p) = p$ for each integer p.
(d) If $A^\infty \in \mathbf{\Pi}^0_1$, then $R(A) \leq \omega$, and $(A^\infty \notin \mathbf{\Sigma}^0_1 \Leftrightarrow R(A) = \omega)$.

Proof. (a) If $\alpha \notin A^\infty$, then $\emptyset \in T_A(\alpha)$ and $\rho(T_A(\alpha)) \geq \rho_{T_A(\alpha)}(\emptyset) + 1 \geq 1$.

(b) We have $T_A(\alpha) = \{\emptyset\}$ for each α, and $\rho(T_A(\alpha)) = \rho_{T_A(\alpha)}(\emptyset) + 1 = 1$.

(c) By compactness, there exists $s_1, \ldots, s_p \in n^{<\omega}$ such that $A^\infty = \bigcup_{1 \leq m \leq p} N_{s_m} \in \mathbf{\Delta}^0_1$. If $\alpha \notin A^\infty$, then we have $N_{\alpha \restriction \max_{1 \leq m \leq p} |s_m|} \subseteq \neg\, A^\infty$, thus $\rho(T_A(\alpha)) \leq \max_{1 \leq m \leq p} |s_m| + 1 < \omega$. So we get the first point. To see the second one, we set $A_0 := 2^{<\omega}$. If $p > 0$, then we set

$$A_p := \{0^2\} \cup \bigcup_{q \leq p} \{s \in 2^{<\omega} / 0^{2q}1 \prec s\} \cup \{s \in 2^{<\omega} / 0^{2p+1} \prec s\}.$$

Then $A_p^\infty = \bigcup_{q \leq p} N_{0^{2q}1} \cup N_{0^{2p+1}} \in \mathbf{\Delta}^0_1$. If $\alpha_p := 0^{2p-1}1^\infty$, then $\rho(T_{A_p}(\alpha_p)) = p$. If $\alpha \notin A_p^\infty$, then $\rho(T_{A_p}(\alpha)) \leq p$.

(d) If $A^\infty \in \mathbf{\Pi}^0_1$ and $\alpha \notin A^\infty$, then let $s \in n^{<\omega}$ with $\alpha \in N_s \subseteq \neg\, A^\infty$. Then $\rho(T_A(\alpha)) \leq |s|+1$. Thus $R(A) \leq \omega$. If $A^\infty \notin \mathbf{\Sigma}^0_1$, then we have $R(A) \geq \omega$, by Proposition 21. Thus $R(A) = \omega$. Conversely, we apply (c). □

Remark. Notice that it is not true that if the Wadge class $< A^\infty >$, having A^∞ as a complete set, is a subclass of $< B^\infty >$, then $R(A) \leq R(B)$. Indeed, for A we take the example A_2 in (c), and for B we take the example for $\mathbf{\Sigma}^0_1$ that we met in the proof of Proposition 11. If we exchange the roles of A and B, then we see that the converse is also false. This example A for $\mathbf{\Sigma}^0_1$ shows that Proposition 21 is optimal for $\xi = 0$ since $R(A) = 1$ and $A^\infty \in \mathbf{\Sigma}^0_1 \setminus \mathbf{\Pi}^0_1$. We can say more: it is not true that if $A^\infty = B^\infty$, then $R(A) \leq R(B)$. We use again (c): we take $A := A_2$ and $B := A \setminus \{0^2\}$. We have $A^\infty = B^\infty = A_2^\infty$, $R(A) = 2$ and $R(B) = 1$.

Proposition 23 *For each $\xi < \omega_1$, there exists $A_\xi \subseteq 2^{<\omega}$ with $A_\xi^\infty \in \mathbf{\Sigma}^0_1$ and $R(A_\xi) \geq \xi$.*

Proof. We use the notation in the proof of Theorem 15. Let $T \in \mathcal{T}$, and $\varphi : T \to T_{\Phi'(T)}(\alpha_0)$ defined by the formula $\varphi(s) := (\phi(s\lceil 0), \ldots, \phi(s\lceil |s|-1))$. Then φ is strictly monotone. If $T \in WF$, then $\alpha_0 \notin (\Phi'(T))^\infty$ and $T_{\Phi'(T)}(\alpha_0) \in WF$. In this case, $\rho(T) \leq \rho(T_{\Phi'(T)}(\alpha_0)) = R_{\Phi'(T)}(\alpha_0)$ (see page 10 in [K1]). Let $T_\xi \in WF$ be a tree with rank at least ξ (see 34.5 and 34.6 in [K1]). We set $A_\xi := \Phi'(T_\xi)$. It is clear that A_ξ is what we were looking for. □

Remark. Let $\psi : 2^{n^{<\omega}} \to \{\text{Trees on } n^{<\omega}\}$ defined by $\psi(A) := T_A(\alpha_0)$, and $r : \neg\, I_{\alpha_0} \to \omega_1$ defined by $r(A) := \rho(T_A(\alpha_0))$. Then ψ is continuous, thus r is a $\mathbf{\Pi}^1_1$-rank on

$$\psi^{-1}(\{\text{Well-founded trees on } n^{<\omega}\}) = \neg\, I_{\alpha_0}.$$

By the boundedness theorem, the rank r and R are not bounded on $\neg\, I_{\alpha_0}$. Proposition 23 specifies this result. It shows that R is not bounded on $\Sigma_1 \setminus I_{\alpha_0}$.

6 The extension ordering.

Proposition 24 *We equip A with the extension ordering.*

(a) If $A \subseteq n^{<\omega}$ is an antichain, then A^∞ is in $\{\emptyset\} \cup \{n^\omega\} \cup [\mathbf{\Pi}^0_1 \setminus \mathbf{\Sigma}^0_1] \cup [\mathit{\Pi}^0_2(A) \setminus \mathbf{\Sigma}^0_2]$, and any of these cases is possible.

(b) If $A \subseteq n^{<\omega}$ has finite antichains, then $A^\infty \in \mathbf{\Pi}^0_2$ (and is not $\mathbf{\Sigma}^0_2$ in general).

Proof. Let $G := \{\alpha \in n^\omega \;/\; \forall r\, \exists m\, \exists p \geq r \;\; \alpha\lceil m \in [(A^-)^p]^*\}$. Then $G \in \mathit{\Pi}^0_2(A)$ and contains A^∞. Conversely, if $\alpha \in G$, then we have $T_A(\alpha) \cap (A^-)^p \neq \emptyset$ for each integer p, thus $T_A(\alpha)$ is infinite.

(a) If A is an antichain, then each sequence in $T_A(\alpha)$ has at most one extension in this tree adding one to the length. Thus $T_A(\alpha)$ is finite splitting. This implies that $T_A(\alpha)$ has an infinite branch if $\alpha \in G$, by König's lemma. Therefore $A^\infty = G \in \mathit{\Pi}^0_2(A)$.

- If we take $A := \emptyset$, then A is an antichain and $A^\infty = \emptyset$.

- If we take $A := \{(0), \dots, (n-1)\}$, then A is an antichain and $A^\infty = n^\omega$.

- If $A^\infty \notin \{\emptyset, n^\omega\}$, then $A^\infty \notin \mathbf{\Sigma}^0_1$. Indeed, let $\alpha_0 \notin A^\infty$ and $s_0 \in A^-$. By uniqueness of the decomposition into words of A^-, the sequence $(s_0^n \alpha_0)_n \subseteq n^\omega \setminus A^\infty$ tends to $s_0^\infty \in A^\infty$.

- If we take $A := \{(0)\}$, then A is an antichain and $A^\infty = \{0^\infty\} \in \mathbf{\Pi}^0_1 \setminus \mathbf{\Sigma}^0_1$.

- If A is finite, then A^∞ is $\mathbf{\Pi}^0_1 \setminus \mathbf{\Sigma}^0_1$ or is in $\{\emptyset, n^\omega\}$, by the facts above and Proposition 2.

- If A is infinite, then $A^\infty \notin \mathbf{\Sigma}^0_2$ because the map c in the proof of Proposition 2 is an homeomorphism and $(A^-)^\omega$ is not K_σ.

- If $A := \{0^k 1 / k \in \omega\}$, then A is an antichain and $A^\infty = P_\infty$, which is $\mathit{\Pi}^0_2 \setminus \mathbf{\Sigma}^0_2$.

(b) The intersection of P_∞ with N_1 can be made with the chain $\{10^k / k \in \omega\}$. So let us assume that A has finite antichains.

• Let us show that A is the union of a finite set and of a finite union of infinite subsets of sets of the form $A_{\alpha_m} := \{s \in n^{<\omega} / s \prec \alpha_m\}$. Let us enumerate $A := \{s_r / r \in \omega\}$. We construct a sequence (A_m), finite or not, of subsets of A. We do it by induction on r, to decide in which set A_m the sequence s_r is. First, $s_0 \in A_0$. Assume that $s_0, \dots, s_r$ have been put into $A_0, \dots, A_{p_r}$, with $p_r \leq r$ and $A_m \cap \{s_0, \dots, s_r\} \neq \emptyset$ if $m \leq p_r$. We choose $m \leq p_r$ minimal such that s_{r+1} is compatible with all the sequences in $A_m \cap \{s_0, \dots, s_r\}$, we put s_{r+1} into A_m and we set $p_{r+1} := p_r$ if possible. Otherwise, we put s_{r+1} into A_{p_r+1} and we set $p_{r+1} := p_r + 1$.

Let us show that there are only finitely many infinite A_m's. If A_m is infinite, then there exists a unique sequence $\alpha_m \in n^\omega$ such that $A_m \subseteq A_{\alpha_m}$. Let us argue by contradiction: there exists an infinite sequence $(m_q)_q$ such that A_{m_q} is infinite. Let t_0 be the common beginning of the α_{m_q}'s. There exists $\varepsilon_0 \in n$ such that $N_{t_0\varepsilon_0} \cap \{\alpha_{m_q} / q \in \omega\}$ is infinite. We choose a sequence u_0 in A extending $t_0\mu_0$, where $\mu_0 \neq \varepsilon_0$. Then we do it again: let $t_0\varepsilon_0 t_1$ be the common beginning of the elements of $N_{t_0\varepsilon_0} \cap \{\alpha_{m_q} / q \in \omega\}$. There exists $\varepsilon_1 \in n$ such that $N_{t_0\varepsilon_0 t_1 \varepsilon_1} \cap \{\alpha_{m_q} / q \in \omega\}$ is infinite. We choose a sequence u_1 in A extending $t_0\varepsilon_0 t_1 \mu_1$, where $\mu_1 \neq \varepsilon_1$. The sequence (u_l) is an infinite antichain in A. But this is absurd. Now let us choose the longest sequence in each nonempty finite A_m; this gives an antichain in A and the result.

• Now let $\alpha \in G$. There are two cases. Either for each m and for each integer k, $\alpha\lceil k \notin [A^{<\omega}]^*$ or $\alpha - \alpha\lceil k \neq \alpha_m$. In this case, $T_A(\alpha)$ is finite splitting. As $T_A(\alpha)$ is infinite, $T_A(\alpha)$ has an infinite branch witnessing that $\alpha \in A^\infty$, by König's lemma. Otherwise, $\alpha \in \bigcup_{s \in [A^{<\omega}]^*, m} \{s\alpha_m\}$, which is countable. Thus $G \setminus A^\infty \in \mathbf{\Sigma}^0_2$ and $A^\infty = G \setminus (G \setminus A^\infty) \in \mathbf{\Pi}^0_2$. □

7 Examples.

• We have seen examples of subsets A of $2^{<\omega}$ such that A^∞ is complete for the classes $\{\emptyset\}$, $\{n^\omega\}$, $\mathbf{\Delta}^0_1$, $\mathbf{\Sigma}^0_1$, $\mathbf{\Pi}^0_1$, $\mathbf{\Pi}^0_2$ and $\mathbf{\Sigma}^1_1$. We will give some more examples, for some classes of Borel sets. Notice that to show that a set in such a non self-dual class is complete, it is enough to show that it is true (see 21.E, 22.10 and 22.26 in [K1]).

• For the class $\mathbf{\Sigma}^0_1 \oplus \mathbf{\Pi}^0_1 := \{(U \cap O) \cup (F \setminus O) \ / \ U \in \mathbf{\Sigma}^0_1,\ O \in \mathbf{\Delta}^0_1,\ F \in \mathbf{\Pi}^0_1\}$, we can take $A := \{s \in 2^{<\omega} / 0^2 1 \prec s$ or $s = 0^2$ or $\exists p \in \omega\ 10^p 1 \prec s\}$, since $A^\infty = \{0^\infty\} \cup \bigcup_q N_{0^{2q+2}1} \cup N_1 \setminus \{10^\infty\}$.

• For the class $\check{D}_2(\mathbf{\Sigma}^0_1) := \{U \cup F \ / \ U \in \mathbf{\Sigma}^0_1,\ F \in \mathbf{\Pi}^0_1\}$, we can take Example 9 in [St2]: $A := \{s \in 2^{<\omega} \ / \ 0 \prec s$ or $\exists\ q \in \omega\ \ (101)^q 1^3 \prec s$ or $s = 10^2\}$. We have

$$A^\infty = \bigcup_{p \in \omega} [N_{(10^2)^p 0} \cup (\bigcup_{q \in \omega} N_{(10^2)^p (101)^q 1^3})] \cup \{(10^2)^\infty\},$$

which is a $\neg\ D_2(\mathbf{\Sigma}^0_1)$ set. Towards a contradiction, assume that A^∞ is $D_2(\mathbf{\Sigma}^0_1)$:

$$A^\infty = U_1 \cap F = U \cup F_2,$$

where the U's are open and the F's are closed. Let O be a clopen set separating $\neg\ U_1$ from F_2 (see 22.C in [K1]). Then $A^\infty = (U \cap O) \cup (F \setminus O)$ would be in $\mathbf{\Sigma}^0_1 \oplus \mathbf{\Pi}^0_1$. If $(10^2)^\infty \in O$, then we would have $N_{(10^2)^{p_0}} \subseteq O$ for some integer p_0. But the sequence $\left((10^2)^p (1^2 0)^\infty\right)_{p \geq p_0} \subseteq O \setminus U$ and tends to $(10^2)^\infty$, which is absurd. If $(10^2)^\infty \notin O$, then we would have $N_{(10^2)^{q_0}} \subseteq \neg\ O$ for some integer q_0. But the sequence $\left((10^2)^{q_0} (101)^q 1^\infty\right)_{q \geq q_0} \subseteq F \setminus O$ and tends to $(10^2)^{q_0} (101)^\infty$, which is absurd.

• For the class $D_2(\mathbf{\Sigma}^0_1)$, we can take $A := [A_1^{<\omega}]^* \setminus [A_0^{<\omega}]^*$, where $A_0 := \{010, 01^2\}$ and

$$A_1 := \{010, 01^2, 0^2, 0^3, 10^2, 1^2 0, 10^3, 1^2 0^2\}.$$

We have $A^\infty = A_1^\infty \setminus A_0^\infty$. Indeed, as $A \subseteq [A_1^{<\omega}]^*$, we have $A^\infty \subseteq A_1^\infty$. If $\alpha \in A_0^\infty$, then its decomposition into words of A_1 is unique and made of words in A_0. Thus $\alpha \notin A^\infty$ and

$$A^\infty \subseteq A_1^\infty \setminus A_0^\infty.$$

Conversely, if $\alpha = a_0 a_1 \ldots \in A_1^\infty \setminus A_0^\infty$, with $a_i \in A_1^-$, then there are two cases. Either there are infinitely many indexes i (say $i_0, i_1, \ldots$) such that $a_i \notin A_0$. In this case, the words $a_0 \ldots a_{i_0}$, $a_{i_0+1} \ldots a_{i_1}, \ldots$, are in A and $\alpha \in A^\infty$. Or there exists a maximal index i such that $a_i \notin A_0$. In this case, $a_0 \ldots a_i 0,\ 10^2,\ 1^2 0 \in A$, thus $\alpha \in A^\infty = A_1^\infty \setminus A_0^\infty$. Proposition 2 shows that $A \in D_2(\mathbf{\Sigma}^0_1)$. If $A^\infty = U \cup F$, with $U \in \mathbf{\Sigma}^0_1$ and $F \in \mathbf{\Pi}^0_1$, then we have $U = \emptyset$ because A_1^∞ is nowhere dense (every sequence in A_1 contains 0, thus the sequences in A_1^∞ have infinitely many 0's). Thus A^∞ would be closed. But this contradicts the fact that $\left((01^2)^n 0^\infty\right)_n \subseteq A^\infty$ and tends to $(01^2)^\infty \notin A^\infty$. Thus A^∞ is a true $D_2(\mathbf{\Sigma}^0_1)$ set.

• For the class $\check{D}_3(\mathbf{\Sigma}^0_1)$, we can take $A := ([A_2^{<\omega}]^* \setminus [A_1^{<\omega}]^*) \cup [A_0^{<\omega}]^*$, where $A_0 := \{0^2\}$, $A_1 := \{0^2, 01\}$, and $A_2 := \{0^2, 01, 10, 10^2\}$. We have $A^\infty = (A_2^\infty \setminus A_1^\infty) \cup A_0^\infty$. Indeed, as $A \subseteq [A_2^{<\omega}]^*$, we have $A^\infty \subseteq A_2^\infty$. If $\alpha \in A_1^\infty$, then its decomposition into words of A_2^- is unique and made of words in A_1. If moreover $\alpha \notin A_0^\infty$, then it is clear that $\alpha \notin A^\infty$ and

$$A^\infty \subseteq (A_2^\infty \setminus A_1^\infty) \cup A_0^\infty.$$

Conversely, it is clear that $A_0^\infty \subseteq A^\infty$. If $\alpha = a_0 a_1 \ldots \in A_2^\infty \setminus A_1^\infty$, then the argument above still works. We have to check that $s := a_0 \ldots a_{i_0} \notin [A_1^{<\omega}]^*$. It is clear if $a_{i_0} = 10$. Otherwise, $a_{i_0} = 10^2$ and we argue by contradiction.

The length of s is even and the decomposition of s into words of A_1 is unique. It finishes with 0^2, and the even coordinates of the sequence s are 0. Therefore, $a_{i_0-1}=0^2$ or 10; we have the same thing with a_{i_0-2}, a_{i_0-3}, $\ldots$ Because of the parity, some 0 remains at the beginning. But this is absurd. Now we have to check that $a_0 \ldots a_i 0 \notin [A_1^{<\omega}]^*$. It is clear if $a_i = 10^2$. Otherwise, $a_i = 10$ and the argument above works.

Finally, we have to check that if $\gamma \in A_1^\infty$, then $\gamma - (0) \in A^\infty$. There is a sequence $p_0,\ p_1, \ldots$, finite or not, such that $\gamma = (0^{2p_0})(01)(0^{2p_1})(01)\ldots 0^\infty$. Therefore

$$\gamma - (0) = (0^{2p_0}10)(0^{2p_1}10)\ldots(0^2)^\infty \in A^\infty.$$

If we set $U_i := \neg\, A_{2-i}^\infty$, then we see that $A^\infty \in \check{D}_3(\mathbf{\Sigma}_1^0)$. If α finishes with 1^∞, then $\alpha \notin A_2^\infty$; thus A_2^∞ is nowhere dense, just like A^∞. Thus if $A^\infty = (U_2 \setminus U_1) \cup U_0$ with U_i open, then $U_0 = \emptyset$. By uniqueness of the decomposition of a sentence in A_i^∞ into words of A_{i+1}, we see that A_i^∞ is nowhere dense in A_{i+1}^∞. So let $x_\emptyset \in A_0^\infty$, $(x_n) \subseteq A_1^\infty \setminus A_0^\infty$ converging to $x_\emptyset$, and $(x_{n,m})_m \subseteq A_2^\infty \setminus A_1^\infty$ converging to x_n. Then $x_{n,m} \in U_1$, which is absurd. Thus $A^\infty \notin D_3(\mathbf{\Sigma}_1^0)$.

• For the class $\check{D}_2(\mathbf{\Sigma}_2^0)$, we can take $A := \{s \in 2^{<\omega} \ /\ 1^2 \prec s \text{ or } s = (0)\}$. We can write

$$A^\infty = (\{0^\infty\} \cup \bigcup_p N_{0^p 1^2}) \cap (P_f \cup \{\alpha \in 2^\omega / \forall n\ \exists m \geq n\ \alpha(m) = \alpha(m+1) = 1\}).$$

Then $A^\infty \notin D_2(\mathbf{\Sigma}_2^0)$, otherwise $A^\infty \cap N_{1^2} \in D_2(\mathbf{\Sigma}_2^0)$ and would be a comeager subset of N_{1^2}. We could find $s \in 2^{<\omega}$ with even length such that $A^\infty \cap N_{1^2 s} \in \mathbf{\Pi}_2^0$. We define a continuous function $f : 2^\omega \to 2^\omega$ by formulas $f(\alpha)(2n) := \alpha(n)$ if $n > \frac{|s|+1}{2}$, $(1^2 s)(2n)$ otherwise, and $f(\alpha)(2n+1) := 0$ if $n > \frac{|s|}{2}$, $(1^2 s)(2n+1)$ otherwise. It reduces P_f to $A^\infty \cap N_{1^2 s}$, which is absurd.

Summary of the complexity results in this paper:

	Baire category	complexity | $\xi = 1$	$\xi = 2$	$\xi \geq 3$
$\mathbf{\Sigma}_0$	nowhere dense	$\mathit{\Pi}_1^0 \setminus \mathbf{\Sigma}_1^0$		
$\mathbf{\Pi}_0$	co-nowhere dense	$\mathit{\Sigma}_1^0 \setminus \mathbf{\Pi}_1^0$		
$\mathit{\Delta}_1$	co-nowhere dense	$K_\sigma \setminus \mathbf{\Pi}_2^0$		
$\mathit{\Sigma}_\xi$	co-nowhere dense	$\mathit{\Pi}_1^1 \setminus \mathbf{\Delta}_1^1$		$\mathbf{\Pi}_1^1 \setminus \mathbf{\Delta}_1^1$
$\mathit{\Pi}_\xi$	co-nowhere dense	$\mathit{\Pi}_1^1 \setminus \mathbf{\Pi}_2^0$	$\mathit{\Pi}_1^1 \setminus \mathbf{\Delta}_1^1$	$\mathbf{\Pi}_1^1 \setminus \mathbf{\Delta}_1^1$
$\mathit{\Delta}$	co-nowhere dense	$\mathit{\Pi}_1^1 \setminus \mathbf{\Delta}_1^1$		
$\mathbf{\Sigma}_\xi$	co-nowhere dense	$\mathit{\Delta}_2^1 \setminus D_2(\mathbf{\Sigma}_1^0)$	$\mathit{\Sigma}_2^1 \setminus \mathbf{\Pi}_2^0$	$\mathbf{\Sigma}_2^1 \setminus D_2(\mathbf{\Sigma}_1^0)$
$\mathbf{\Pi}_\xi$	co-nowhere dense	$\mathit{\Delta}_2^1 \setminus \mathbf{\Pi}_2^0$	$\mathit{\Sigma}_2^1 \setminus D_2(\mathbf{\Sigma}_1^0)$	$\mathbf{\Sigma}_2^1 \setminus D_2(\mathbf{\Sigma}_1^0)$
$\mathbf{\Delta}$	co-nowhere dense	$\mathit{\Sigma}_2^1 \setminus D_2(\mathbf{\Sigma}_1^0)$		
$\mathcal{G}_\xi$ $(\xi \in \omega)$		$\mathbf{\Pi}_1^0 \setminus \mathbf{\Sigma}_1^0$ nowhere dense	$\check{D}_\omega(\mathbf{\Sigma}_1^0) \setminus D_\omega(\mathbf{\Sigma}_1^0)$	$\mathit{\Pi}_1^1 \setminus D_\omega(\mathbf{\Sigma}_1^0)$
$\mathcal{F}$		$\mathit{\Pi}_1^1 \setminus \mathbf{\Pi}_2^0$		
$\mathcal{A}_\xi$	co-meager	$\check{D}_2(\mathbf{\Sigma}_1^0) \setminus D_2(\mathbf{\Sigma}_1^0)$ co-nowhere dense	$\mathit{\Sigma}_2^1 \setminus D_2(\mathbf{\Sigma}_1^0)$	$\mathbf{\Sigma}_2^1 \setminus D_2(\mathbf{\Sigma}_1^0)$
$\mathcal{M}_\xi$	co-meager	$\mathit{\Sigma}_2^1 \setminus D_2(\mathbf{\Sigma}_1^0)$ co-nowhere dense	$\mathit{\Sigma}_2^1 \setminus D_2(\mathbf{\Sigma}_1^0)$	$\mathbf{\Sigma}_2^1 \setminus D_2(\mathbf{\Sigma}_1^0)$
$\mathcal{B}$		$\mathit{\Sigma}_2^1$		
$\mathcal{A}$	co-nowhere dense	$\mathit{\Sigma}_3^1 \setminus D_2(\mathbf{\Sigma}_1^0)$		

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