Omega-powers and descriptive set theory
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\( \omega \)-powers and descriptive set theory.

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**Abstract.** We study the sets of the infinite sentences constructible with a dictionary over a finite alphabet, from the viewpoint of descriptive set theory. Among other things, this gives some true co-analytic sets. The case where the dictionary is finite is studied and gives a natural example of a set at the level \( \omega \) of the Wadge hierarchy.

1 Introduction.

We consider the finite alphabet \( n = \{0, \ldots, n - 1\} \), where \( n \geq 2 \) is an integer, and a dictionary over this alphabet, i.e., a subset \( A \) of the set \( n^{<\omega} \) of finite words with letters in \( n \).

**Definition 1** The \( \omega \)-power associated to \( A \) is the set \( A^\omega \) of the infinite sentences constructible with \( A \) by concatenation. So we have
\[
A^\omega := \{a_0a_1 \ldots \in n^\omega / \forall \omega a_i \in A \}.
\]
The \( \omega \)-powers play a crucial role in the characterization of subsets of \( n^\omega \) accepted by finite automata (see Theorem 2.2 in [St1]). We will study these objects from the viewpoint of descriptive set theory. The reader should see [K1] for the classical results of this theory; we will also use the notation of this book. The questions we study are the following:

(1) What are the possible levels of topological complexity for the \( \omega \)-powers? This question was asked by P. Simonnet in [S], and studied in [St2]. O. Finkel (in [F1]) and A. Louveau proved independently that \( \Sigma^1_1 \)-complete \( \omega \)-powers exist. O. Finkel proved in [F2] the existence of a \( \Pi^0_m \)-complete \( \omega \)-power for each integer \( m \geq 1 \).

(2) What is the topological complexity of the set of dictionaries whose associated \( \omega \)-power is of a given level of complexity? This question arises naturally when we look at the characterizations of \( \Pi^0_1 \), \( \Pi^0_2 \) and \( \Sigma^0_1 \) \( \omega \)-powers obtained in [St2] (see Corollary 14 and Lemmas 25, 26).

(3) We will recall that an \( \omega \)-power is an analytic subset of \( n^\omega \). What is the topological complexity of the set of codes for analytic sets which are \( \omega \)-powers? This question was asked by A. Louveau. This question also makes sense for the set of codes for \( \Sigma^0_\xi \) (resp., \( \Pi^0_\xi \)) sets which are \( \omega \)-powers. And also for the set of codes for Borel sets which are \( \omega \)-powers.

As usual with descriptive set theory, the point is not only the computation of topological complexities, but also the hope that these computations will lead to a better understanding of the studied objects. Many sets in this paper won’t be clopen, in particular won’t be recursive. This gives undecidability results.
• We give the answer to Question (2) for the very first levels (\{\emptyset\}, its dual class and \(\Delta_0^1\)). This contains a study of the case where the dictionary is finite. In particular, we show that the set of dictionaries whose associated \(\omega\)-power is generated by a dictionary with two words is a \(\dot{D}_\omega(\Sigma_0^1)\)-complete set. This is a surprising result because this complexity is not clear at all on the definition of the set.

• We give two proofs of the fact that the relation “\(\alpha \in A^\infty\)" is \(\Sigma_1^1\)-complete. One of these proofs is used later to give a partial answer to Question (2). To understand this answer, the reader should see [M] for the basic notions of effective descriptive set theory. Roughly speaking, a set is effectively Borel (resp., effectively Borel in \(A\)) if its construction based on basic clopen sets can be coded with a recursive (resp., recursive in \(A\)) sequence of integers. This answer is the Theorem.

**Theorem.** The following sets are true co-analytic sets:
- \(\{A \in 2^{<\omega} / A^\infty = \emptyset\}\).
- \(\{A \in 2^{<\omega} / A^\infty \in \Sigma_\xi^0 \cap \Delta_1^1(A)\}, \) for \(1 \leq \xi < \omega_1\).
- \(\{A \in 2^{<\omega} / A^\infty \in \Pi_\xi^0 \cap \Delta_1^1(A)\}, \) for \(2 \leq \xi < \omega_1\).

This result also comes from an analysis of Borel \(\omega\)-powers: \(A^\infty\) is Borel if and only if we can choose in a Borel way the decomposition of any sentence of \(A^\infty\) into words of \(A\) (see Lemma 13).

• A natural ordinal rank can be defined on the complement of any \(\omega\)-power, and we study it; its knowledge gives an upper bound of the complexity of the \(\omega\)-power.

• We study the link between Question (1) and the extension ordering on finite sequences of integers.

• Finally, we give some examples of \(\omega\)-powers complete for the classes \(\Delta_0^0, \Sigma_1^0 \oplus \Pi_1^0, D_2(\Sigma_1^0), \dot{D}_2(\Sigma_1^0), \) and \(\dot{D}_2(\Sigma_2^0)\).

## 2 Finitely generated \(\omega\)-powers.

**Notation.** In order to answer to Question (2), we set
\[
\Sigma_0 := \{A \subseteq n^{<\omega} / A^\infty = \emptyset\}, \quad \Pi_0 := \{A \subseteq n^{<\omega} / A^\infty = n^\omega\},
\]
\[
\Delta_1 := \{A \subseteq n^{<\omega} / A^\infty \in \Delta_1^0\},
\]
\[
\Sigma_\xi := \{A \subseteq n^{<\omega} / A^\infty \in \Sigma_\xi^0\}, \quad \Pi_\xi := \{A \subseteq n^{<\omega} / A^\infty \in \Pi_\xi^0\} \quad (\xi \geq 1),
\]
\[
\Delta := \{A \subseteq n^{<\omega} / A^\infty \in \Delta_1^1\}.
\]

• If \(A \subseteq n^{<\omega}\), then we set \(A^- := A \setminus \{\emptyset\}\).

• We define, for \(s \in n^{<\omega}\) and \(\alpha \in n^\omega\), \(\alpha - s := (\alpha(|s|), \alpha(|s| + 1), \ldots)\).

• If \(S \subseteq (n^{<\omega})^{<\omega}\), then we set \(S^* := \{S^* := S(0) \ldots S(|s| - 1) / S \in S\}\).
• We define a recursive map \( \pi : n^\omega \times \omega^\omega \times \omega \rightarrow n^{<\omega} \) by
  \[
  \pi(\alpha, \beta, q) := \begin{cases} 
  (\alpha(0), \ldots, \alpha(\beta[0])) & \text{if } q = 0, \\
  (\alpha(1 + \Sigma_{j<q} \beta[j]), \ldots, \alpha(\Sigma_{j<q} \beta[j])) & \text{otherwise}.
  \end{cases}
  \]
We always have the following equivalence:
  \[
  \alpha \in A^\infty \iff \exists \beta \in \omega^\omega \left[ (\forall m > 0 \beta(m) > 0) \text{ and } (\forall q \in \omega \pi(\alpha, \beta, q) \in A) \right].
  \]

**Proposition 2** (IS) \( A^\infty \in \Sigma^1_1 \) for all \( A \subseteq n^{<\omega} \). If \( A \) is finite, then \( A^\infty \in \Pi^0_1 \).

**Proof.** We define a continuous map \( c : (A^-)^\omega \rightarrow n^\omega \) by the formula \( c((a_i)) := a_0a_1 \ldots \). We have \( A^\infty = c((A^-)^\omega) \), and \( (A^-)^\omega \) is a Polish space (compact if \( A \) is finite).

**Proposition 3** If \( A^\infty \in \Delta^0_1 \), then there exists a finite subset \( B \) of \( A \) such that \( A^\infty = B^\infty \).

**Proof.** Set \( E_k := \{ \alpha \in n^\omega / \alpha[k \in A \text{ and } \alpha \not\prec \alpha[k \in A^\infty] \}. \) It is an open subset of \( n^\omega \) since \( A^\infty \) is open, and \( A^\infty \subseteq \bigcup_{k \geq 0} E_k \). We can find an integer \( p \) such that \( A^\infty \subseteq \bigcup_{0 < k \leq p} E_k \), by compactness of \( A^\infty \). Let \( B := A \cap n^\leq p \). If \( \alpha \in A^\infty \), then we can find an integer \( 0 < k_0 < p \) such that \( \alpha[k_0] \in A \) and \( \alpha \not\prec \alpha[k_0] \in A^\infty \). Thus \( \alpha[k_0] \in B \). Then we do it again with \( \alpha \not\prec \alpha[k_0] \), and so on. Thus we have \( \alpha \in B^\infty = A^\infty \).

**Remark.** This is not true if we only assume that \( A^\infty \) is closed. Indeed, we have the following counter-example, due to O. Finkel:
  \[
  A := \{ s \in 2^{<\omega} / \forall i \leq |s| 2.\text{Card}(\{ j < i / s(j) = 1 \}) \geq i \}.
  \]
We have \( A^\infty = \{ \alpha \in 2^\omega / \forall i \in \omega 2.\text{Card}(\{ j < i / \alpha(j) = 1 \}) \geq i \} \) and if \( B \) is finite and \( B^\infty = A^\infty \), \( B \subseteq A \) and \( 101^20^2 \ldots \notin B^\infty \).

**Theorem 4** (a) \( \Sigma^0_1 = \{ \emptyset, \{ \emptyset \} \} \) is \( \Pi^0_1 \)-complete.
(b) \( \Pi_0 \) is a dense \( \Sigma^0_1 \) subset of \( 2^{<\omega} \). In particular, \( \Pi_0 \) is \( \Sigma^0_1 \)-complete.
(c) \( \Delta_1 \) is a \( K_n \setminus \Pi^0_0 \) subset of \( 2^{<\omega} \). In particular, \( \Delta_1 \) is \( \Sigma^0_2 \)-complete.

**Proof.** (a) Is clear.

(b) If we can find \( m \in \omega \) with \( n^m \subseteq A \), then \( A^\infty = n^\omega \). As \( \{ A \subseteq n^{<\omega} / \exists m \in \omega n^m \subseteq A \} \) is a dense open subset of \( 2^{<\omega} \), the density follows. The formula
  \[
  A \in \Pi_0 \iff \exists m \forall s \in n^m \exists q \leq m s[q] \in A^-
  \]
shows that \( \Pi_0 \) is \( \Sigma^0_1 \), and comes from Proposition 3.

(c) If \( A^\infty \in \Delta^0_1 \), then we can find \( p > 0 \) such that \( A^\infty = (A \cap n^{<\rho})^\infty \), by Proposition 3. So let \( s_1, \ldots, s_k, t_1, \ldots, t_l \in 2^{<\omega} \) be such that \( A^\infty = \bigcup_{1 \leq i \leq k} N_{s_i} = n^\omega \setminus \left( \bigcup_{1 \leq j \leq l} N_{t_j} \right) \). For each \( 1 \leq j \leq l \), and for each sequence \( s \in [(A^-)^{<\omega}]^\ast \setminus \{ \emptyset \} \), \( t_j \notin s \). So we have
  \[
  A^\infty \in \Delta^0_1 \iff \begin{cases} 
  \exists p > 0 \exists k, l \in \omega \exists s_1, \ldots, s_k, t_1, \ldots, t_l \in 2^{<\omega} \bigcup_{1 \leq i \leq k} N_{s_i} = n^\omega \setminus \left( \bigcup_{1 \leq j \leq l} N_{t_j} \right) \text{ and } \forall 1 \leq j \leq l \forall s \in [(A^-)^{<\omega}]^\ast \setminus \{ \emptyset \} t_j \notin s \text{ and } \forall \alpha \in n^\omega \\{ \alpha \notin \bigcup_{1 \leq i \leq k} N_{s_i} \text{ or } \exists \beta \in p^\omega \left[ (\forall m > 0 \beta(m) > 0) \text{ and } (\forall q \in \omega \pi(\alpha, \beta, q) \in A) \right] \}.\end{cases}
  \]
This shows that \( \Delta_1 \) is a \( K_n \) subset of \( 2^{<\omega} \).
To show that it is not $\Pi_2^0$, it is enough to see that its intersection with the closed set

$$\{ A \subseteq n^{<\omega}/A^\infty \neq n^\omega \}$$

is dense and co-dense in this closed set (see (b)), by Baire’s theorem. So let $O$ be a basic clopen subset of $2^{n^{<\omega}}$ meeting this closed set. We may assume that it is of the form

$$\{ A \subseteq n^{<\omega}/\forall i \leq k \ s_i \in A \text{ and } \forall j \leq l \ t_j \notin A \},$$

where $s_0, \ldots, s_k, t_0, \ldots, t_l \in n^{<\omega}$ and $|s_0| > 0$. Let $A := \{ s_i/i \leq k \}$. Then $A \in O$ and $A^\infty$ is in $\Pi_1^0 \setminus \{0, n^\omega\}$. There are two cases.

If $A^\infty \in \Delta^0_1$, then we have to find $B \in O$ with $B^\infty \notin \Delta^0_1$. Let $u_0, \ldots, u_m \in n^{<\omega}$ with $\bigcup_{p \leq m} N_{u_p} = n^\omega \setminus A^\infty$. Let $r \in n \setminus \{ u_0(|u_0| - 1) \}$, $s := u_0 r |u_0| + \max\{ |t_j| \} |j| \leq l$ and $B := A \cup \{ s \}$. Then $B \in O$ and $|s| \in B^\infty$. Let us show that $|s| \in B^\infty$ is not in the interior of $B^\infty$. Otherwise, we could find an integer $q$ such that $N_{|s|} \subseteq B^\infty$. We would have $A^\infty \in \Delta^0_1$. We have $v(u_0(|u_0| - 1))^\infty \in A^\infty$, so $v(u_0(|u_0| - 1))^\infty \in N_{u_0} \cap A^\infty$. But this is absurd. Therefore $B^\infty \notin \Delta^0_1$.

If $A^\infty \notin \Delta^0_1$, then we have to find $B \in O$ such that $B^\infty \in \Delta^0_1 \setminus \{ n^\omega \}$, and $\forall i \leq k \ N_{s_i} = \emptyset$.

So let $s \in n^{<\omega}$ be non constant such that $N_s \cap \bigcup_{i \leq k} N_{s_i} = \emptyset$. We set

$$D := A \cup \bigcup_{r \in n \setminus \{v(0)\}} \{ (r) \} \cup \{ v(0)^{|v|} \},$$

$$B := A \cup \{ s \in n^{<\omega}/|s| > \max_{j \leq l} |t_j| \text{ and } \exists t \in D \ t < s \}. $$

We get $B^\infty = \bigcup_{t \in D} N_t \in \Delta^0_1$ and $N_s \cap B^\infty = \emptyset$.

Now we will study $\mathcal{F} := \{ A \subseteq n^{<\omega}/\exists B \subseteq n^{<\omega} \text{ finite } A^\infty = B^\infty \}$.

**Proposition 5** $\mathcal{F}$ is a co-nowhere dense $\Sigma^0_2$-hard subset of $2^{n^{<\omega}}$.

**Proof.** By Proposition 3, if $A^\infty = n^\omega$, then there exists an integer $p$ such that $A^\infty = (A \cap n^{<\omega})^\infty$, so $\Pi_0 \subseteq \mathcal{F}$ and, by Theorem 4, $\mathcal{F}$ is co-nowhere dense. We define a continuous map $\phi : 2^\omega \rightarrow 2^{n^{<\omega}}$ by the formula $\phi(\gamma) := \{ 0^k 1/\gamma(k) = 1 \}$. If $\gamma \in P_f := \{ \alpha \in 2^\omega/\exists p \forall m \geq p \ \alpha(m) = 0 \}$, then $\phi(\gamma) \in \mathcal{F}$. If $\gamma \notin P_f$, then the concatenation map is an homeomorphism from $\phi(\gamma)^\infty$ onto $\phi(\gamma)\infty$, thus $\phi(\gamma)\infty$ is not $K_\sigma$. So $\phi(\gamma) \notin \mathcal{F}$, by Proposition 2. Thus the preimage of $\mathcal{F}$ by $\phi$ is $P_f$, and $\mathcal{F}$ is $\Sigma^0_2$-hard. □

Let $\mathcal{G}_p := \{ A \subseteq n^{<\omega}/\exists s_1, \ldots, s_p \in n^{<\omega} A^\infty = \{ s_1, \ldots, s_p \}^\infty \}$, so that $\mathcal{F} = \bigcup_p \mathcal{G}_p$. We have $\mathcal{G}_0 = \Sigma_0$, so $\mathcal{G}_0$ is $\Pi^0_1 \setminus \Sigma^0_1$. \[4\]
Proposition 6 \( G_1 \) is \( \Pi^0_1 \setminus \Sigma^0_1 \). In particular, \( G_1 \) is \( \Pi^0_1 \)-complete.

**Proof.** If \( p \in \omega \setminus \{0\} \), then \( \{0, 1^p\} \notin G_1 \) since \( B^\infty = \{s^\infty\} \) if \( B = \{s\} \). Thus \( \{0\} \) is not an interior point of \( G_1 \) since the sequence \( \{(0, 1^p)\}_{p>0} \) tends to \( \{0\} \). So \( G_1 \notin \Sigma^0_1 \).

- Let \((A_m) \subseteq G_1\) tending to \( A \subseteq n^{<\omega} \). If \( A \subseteq \{\emptyset\} \), then \( A^\infty = \{\emptyset\}^\infty \), so \( A \in G_1 \). If \( A \not\subseteq \{\emptyset\} \), then let \( t \in A^\infty \) and \( \alpha_0 := t^\infty \). There exists an integer \( m_0 \) such that \( t \in A_m \) for \( m \geq m_0 \). Thus we may assume that \( t \in A_m \) and \( A_m^\infty \neq \emptyset \). So let \( s_m \in n^{<\omega} \setminus \{\emptyset\} \) be such that \( A_m^\infty = \{s_m\}^\infty = \{s_m^\infty\} \). We have \( s_m^\infty = \alpha_0 \). Let \( b := \min\{a \in \omega \setminus \{0\}\} / (\alpha_0[a]^\infty = \alpha_0 \} \).

- We will show that \( A_m = \{(\alpha_0[b]^q/q \in \omega\} \). Let \( s \in A_m \setminus \{\emptyset\} \). As \( s^\infty = \alpha_0 \), we can find an integer \( a > 0 \) such that \( s = \alpha_0[a] \), and \( b \leq a \). Let \( r < b \) and \( q \) be integers so that \( a = qb + r \). We have, if \( r > 0 \),

\[
\begin{align*}
\alpha_0 &= (\alpha_0[a]^\infty = (\alpha_0[b]^\infty = (\alpha_0[q,b](\alpha_0[a] - \alpha_0[q,b]) \alpha_0 \\
&= (\alpha_0[b]^q(\alpha_0[a] - \alpha_0[q,b]) \alpha_0 = (\alpha_0[a] - \alpha_0[q,b]) \alpha_0 = (\alpha_0[r]) \alpha_0 = (\alpha_0[r]^\infty.
\end{align*}
\]

Thus, by minimality of \( b, \ r = 0 \) and we are done.

- Let \( u \in A \). We can find an integer \( m_u \) such that \( u \in A_m \) for \( m \geq m_u \). So there exists an integer \( q_u \) such that \( u = (\alpha_0[b]^q_u) \). Therefore \( A^\infty = \{ (\alpha_0[b]^\infty \} = \{\alpha_0[b]^\infty \} \) and \( A \in G_1 \). \( \square \)

**Remark.** Notice that this shows that we can find \( w \in n^{<\omega} \setminus \{\emptyset\} \) such that \( A \subseteq \{w^q/q \in \omega\} \) if \( A \in G_1 \). Now we study \( G_2 \). The next lemma is just Corollary 6.2.5 in [Lo].

**Lemma 7** Two finite sequences which commute are powers of the same finite sequence.

**Proof.** Let \( x \) and \( y \) be finite sequences with \( xy = yx \). Then the subgroup of the free group on \( n \) generators generated by \( x \) and \( y \) is abelian, hence isomorphic to \( \mathbb{Z} \). One generator of this subgroup must be a finite sequence \( u \) such that \( x \) and \( y \) are both powers of \( u \). \( \square \)

**Lemma 8** Let \( A \in G_2 \). Then there exists a finite subset \( F \) of \( A \) such that \( A^\infty = F^\infty \).

**Proof.** We will show more. Let \( A \notin G_1 \) satisfying \( A^\infty = \{s_1, s_2\}^\infty \), with \( |s_1| \leq |s_2| \). Then

(a) The decomposition of \( \alpha \) into words of \( \{s_1, s_2\} \) is unique for each \( \alpha \in A^\infty \) (this is a consequence of Corollaries 6.2.5 and 6.2.6 in [Lo]).

(b) \( s_2s_1 \perp s_1s_2 \) for each integer \( q > 0 \), and \( s_2s_1 \wedge s_1^qs_2 = s_1s_2 \wedge s_2s_1 \).

(c) \( A \subseteq \{[s_1, s_2]^{<\omega}\}^* \).

- We prove the first two points. We split into cases.

2.1. \( s_1 \perp s_2 \).

The result is clear.
2.2. \( s_1 \not<_{\neq} s_2 \not< s_1^\infty \).

Here also, the result is clear (cut \( \alpha \) into words of length \(|s_1|\)).

2.3. \( s_1 \not<_{\neq} s_2 \not< s_1^\infty \).

We can write \( s_2 = s_1^m s \), where \( m > 0 \) and \( s \not<_{\neq} s_1 \). Thus \( s_2 s_1 = s_1^m s s_1 \) and \( s_1^{m+1} s \not< s_1^q s_2 \) if \( q > 0 \). But \( s_1^m s s_1 \perp s_1^{m+1} s \) otherwise \( s s_1 = s_1 s \), and \( s, s_1, s_2 \) would be powers of some sequence, which contradicts \( A \not\subseteq G_1 \).

• We prove (c). Let \( t \in A \), so that \( ts_1^\infty, ts_2 s_1^\infty \in A^\infty \). These sequences split after \( t(s_1 s_2 \wedge s_2 s_1) \), and the decomposition of \( ts_1^\infty \) (resp., \( ts_2 s_1^\infty \)) into words of \( \{s_1, s_2\} \) starts with \( u s_i \) (resp., \( u s_3 \ldots \)), where \( u \in [\{s_1, s_2\}^\infty]^* \). So \( ts_1^\infty \) and \( ts_2 s_1^\infty \) split after \( u(s_1 s_2 \wedge s_2 s_1) \) by (b). But we must have \( t = u \) because of the position of the splitting point.

• We prove Lemma 8. If \( A \in G_0 \), then \( F := 0 \) works. If \( A \in G_1 \setminus G_0 \), then let \( w \in n^{\omega} \setminus \{0\} \) such that \( A \subseteq \{w^q/q \in \omega\} \), and \( q > 0 \) such that \( w^q \in A \). Then \( F := \{w^q\} \) works. So we may assume that \( A \not\subseteq G_1 \), and \( A^\infty = \{s_1, s_2\}^\infty \). As \( A^\infty \subseteq \bigcup_{\alpha \in A} \{\alpha \in N_i/s_1 s_2 \wedge s_2 s_1 < \alpha - t\} \) is compact, we get a finite subset \( F \) of \( A^\infty \) such that \( A^\infty \subseteq \bigcup_{f \in F} \{\alpha \in N_i/s_1 s_2 \wedge s_2 s_1 < \alpha - t\} \). We have \( F^\infty \subseteq A^\infty \). If \( \alpha \in A^\infty \), then let \( t \in F \) such that \( \alpha \prec t \). By (c), we have \( t \in \{\{s_1, s_2\}^\infty\}^* \). The sequence \( t \) is the beginning of the decomposition of \( \alpha \) into words of \( \{s_1, s_2\} \). Thus \( \alpha - t \in A^\infty \) and we can go on like this. This shows that \( \alpha \in F^\infty \).

\[ \square \]

Remark. The inclusion of \( A^\infty = \{s_1, s_2\}^\infty \) into \( \{t_1, t_2\}^\infty \) does not imply \( \{s_1, s_2\} \subseteq \{t_1, t_2\}^{\omega, *}, \) even if \( A \not\subseteq G_1 \). Indeed, take \( s_1 := 01, s_2 := t_1 := 0 \) and \( t_2 := 10 \). But we have

\[ |t_1| + |t_2| \leq |s_1| + |s_2|, \]

which is the case in general:

Lemma 9 Let \( A, B \notin G_1 \) satisfying \( A^\infty = \{s_1, s_2\}^\infty \subseteq B^\infty = \{t_1, t_2\}^\infty \). Then there is \( j \in 2 \) such that \( |t_1 + i| \leq |s_1 + [i + j \mod 2]| \) for each \( i \in 2 \). In particular, \( |t_1| + |t_2| \leq |s_1| + |s_2| \).

Proof. We may assume that \( |s_1| \leq |s_2| \). Let, for \( i = 1, 2 \), \( (w_i)^m \subseteq \{t_1, t_2\} \) be sequences such that \( s_i^c = w_0^i w_1^i \ldots \) (resp., \( s_2 s_1^\infty = w_0^i w_1^i \ldots \)). By the proof of Lemma 8, there is a minimal integer \( m_0 \) satisfying \( w_0^i \neq w_0^{m_0} \). We let \( u := w_0^1 \ldots w_0^{m_0-1} \). The sequences \( s_1^c \) and \( s_2 s_1^\infty \) split after \( s_1 s_2 \wedge s_2 s_1 = u(t_1 t_2 \wedge t_2 t_1) \). Similarly, \( s_1^c \) and \( s_1 s_2 s_1^\infty \) split after \( s_1(s_1 s_2 \wedge s_2 s_1) = v(t_1 t_2 \wedge t_2 t_1) \), where \( v \in \{t_1, t_2\}^{\omega, *}, \) \( \emptyset \}. \) So we get \( s_1 u = v \). Similarly, with the sequences \( s_2 s_1^\infty \) and \( s_2 s_1^\infty \), we see that \( s_2 u \in \{t_1, t_2\}^{\omega, *}, \emptyset \}. \) So we may assume that \( u \not= 0 \) since \( \{s_1, s_2\} \notin G_1 \). If \( t_1 \not= t_2 \), then we may assume that \( 0 \not= t_1 \not<_{\neq} t_2 \). So we may assume that we are not in the case \( t_2 < t_1 \). Indeed, otherwise \( t_2 = t_1^\infty, \) where \( \emptyset \not<_{\neq} \emptyset \). We have that \( t_1 \not= t_2 \). Otherwise we would have \( t_1 = t(t_1 - t) = (t_1 - t)t \) and \( t, t_1 - t, t_1, t_2 \) would be powers of the same sequence, which contradicts \( \{t_1, t_2\} \notin G_1 \). As \( s_1 u \in \{t_1, t_2\}^{\omega}, \) this shows that \( s_i \in \{t_1, t_2\}^{\omega, *}, \) \( s_i \notin G_1 \) as before.
Assume for example that \( t_2 = w_{m_0}^1 \). Let \( m' \) be maximal with \( t_1^{m'} < t_2 \). Notice that
\[
vt_1^{m'} < s_1 s_2 < s_1 s_2 s_2 s_1^\infty.
\]
We have \( ut_2 < s_1 s_2 s_2^\infty \), otherwise we would obtain \( ut_1^{m' + 1} < s_1 s_2 s_2^\infty \land s_2 s_1^\infty = s_1 s_2 \land s_2 s_1 < s_1^\infty \), which is absurd. So we get \( |t_2| \leq |s_1| \) since \(|u| + |t_2| + |t_1| t_2 t_1| \leq |s_1| + |s_1 s_2 \land s_2 s_1| \). Similarly, \( |t_1| \leq |s_2| \) since \( ut_1^{m' + 1} < s_2 s_1^\infty \). The argument is similar if \( t_2 = w_{m_0}^2 \) (we get \( |t_i| \leq |s_i| \) in this case for \( i = 1, 2 \)).

**Corollary 10** \( G_2 \) is a \( D_\omega(\Sigma_1^0) \setminus D_\omega(\Sigma_0^0) \) set. In particular, \( G_2 \) is \( D_\omega(\Sigma_0^0) \)-complete.

**Proof.** We will apply the Hausdorff derivation to \( G \subseteq 2^{n^<\omega} \). This means that we define a decreasing sequence \((F_\xi)_{\xi<\omega_1}\) of closed subsets of \( 2^{n^<\omega} \) as follows:
\[
F_\xi := \bigcap_{\eta<\xi} F_\eta \quad \text{if} \xi \text{ is even,} \quad \bigcap_{\eta<\xi} F_\eta \setminus G \quad \text{if} \xi \text{ is odd.}
\]
Recall that if \( \xi \) is even, then \( F_\xi = \emptyset \) is equivalent to \( G \subseteq D_\xi(\Sigma_1^0) \). Indeed, we set \( U_\xi := \hat{F}_\xi \). We have \( U_{\xi+1} \setminus U_\xi = F_\xi \setminus F_{\xi+1} \subseteq \hat{G} \) if \( \xi \) is even and \( U_{\xi+1} \setminus U_\xi \subseteq \hat{G} \) if \( \xi \) is odd. Similarly, \( U_\xi \setminus (\bigcup_{\eta<\xi} U_\eta) \subseteq \hat{G} \) if \( \xi \) is limit. If \( F_\xi = \emptyset \), then let \( \eta \) be minimal such that \( F_\eta = \emptyset \). We have \( G = \bigcup_{\theta<\eta, \theta \text{ odd}} U_\theta \setminus \bigcup_{\rho<\theta} U_\rho \). If \( \eta \) is odd, then \( G \subseteq \bigcup_{\theta<\eta, \theta \text{ even}} U_\theta \setminus \bigcup_{\rho<\theta} U_\rho \subseteq D_\eta(\Sigma_1^0) \) and the same conclusion is true. Conversely, if \( G \subseteq D_\xi(\Sigma_1^0) \), then let \( (V_\eta)_{\eta<\xi} \) be an increasing sequence of open sets with \( G \subseteq \bigcup_{\eta<\xi, \eta \text{ odd}} V_\eta \setminus \bigcup_{\eta<\xi} V_\eta \). By induction, we check that \( F_\eta \subseteq V_\eta \) if \( \eta < \xi \).
This clearly implies that \( F_\xi = \emptyset \) because \( \xi \) is even.

- We will show that if \( A \notin G_1 \) satisfies \( A^\infty = \{s_1, s_2\}^\infty \), then \( A \notin F_M \) := \( F_M(G_2) \), where \( M \) is the smallest odd integer greater than or equal to \( f(s_1, s_2) := 2^{\Sigma_{i=0}^{s_1}[s_1]+s_2} + 2^{\Sigma_{i=0}^{s_2}[s_2]} \).

We argue by contradiction: \( A \) is the limit of \((A_q)\), where \( A_q \in F_{M-1} \setminus G_2 \). Lemma 8 gives a finite subset \( F \) of \( A \), and we may assume that \( F \subseteq A_q \) for each \( q \). Thus we have \( A^\infty \subseteq A_\infty \), and the inclusion is strict. Thus we can find \( s^q \in [A^\infty_q]^\ast \) such that \( N_{s^q} \cap A^\infty = \emptyset \). Let \( s^q_0, \ldots, s^q_{m_q} \in A_q \) be such that \( s^q = s^q_0 \ldots s^q_{m_q} \).

Now \( A_q \) is the limit of \((A_{q,r})_r\), where \( A_{q,r} \in F_{M-2} \setminus G_2 \), and we may assume that
\[
\{s^q_0, \ldots, s^q_{m_q}\} \cup F \subseteq A_{q,r}
\]
for each \( r \), and that \( A_{q,r} \notin G_1 \) because \( A_q \notin G_1 \subseteq G_2 \). Let \( s^q_{1,r}, s^q_{2,r} \) such that \( A^\infty_{q,r} = \{s^q_{1,r}, s^q_{2,r}\}^\infty \).

By Lemma 9 we have \( |s^q_{1,r}| + |s^q_{2,r}| \leq |s_1| + |s_2| \). Now we let \( B_0 := A_{0,0} \) and \( s^q_i := s^q_{i,0} \) for \( i = 1, 2 \).
We have \( B_0 \in F_{M-2} \setminus G_2 \setminus G_1 \), \( A^\infty \subseteq B_0^\infty = \{s^q_0, s^q_2\}^\infty \), and
\[
|s^q_1| + |s^q_2| \leq |s_1| + |s_2|.
\]
Now we iterate this: for each $0 < k < n^{2(|s_1|+|s_2|)}$, we get $B_k \in F_{M-2(k+1)} \cap \mathcal{G}_2 \setminus \mathcal{G}_1$ such that $B_k^\infty \subseteq B_k \subseteq \{s_1^k, s_2^k\}^\infty$ and $|s_1^k| + |s_2^k| \leq |s_1^{k-1}| + |s_2^{k-1}|$. We can find $k_0 < n^{2(|s_1|+|s_2|)}$ such that $|s_1^{k_0}| + |s_2^{k_0}| < |s_1^{k_0-1}| + |s_2^{k_0-1}|$ (with the convention $s_i^{-1} := s_{i0}$). We set $C_0 := B_{k_0}, t_1^0 := s_i^{k_0}$. So we have $C_0 \in F_{M-2(k_0+1)} \cap \mathcal{G}_2 \setminus \mathcal{G}_1$, $C_0^\infty = \{t_1^0, t_2^0\}^\infty$ and $|t_1^0| + |t_2^0| < |s_1| + |s_2|$. Now we iterate this: for each $l \leq |s_1| + |s_2| - 2$, we get $t_1^l, t_2^l, k_l < n^{2(|t_1^{l-1}|+|t_2^{l-1}|)}$ and

$$C_l \in F_{M-2\sum_{m \leq l} (k_m+1)} \cap \mathcal{G}_2 \setminus \mathcal{G}_1$$

satisfying $C_l^\infty = \{t_1^l, t_2^l\}^\infty$ and $|t_1^l| + |t_2^l| < |t_1^{l-1}| + |t_2^{l-1}|$ (with the convention $t_i^{-1} := s_i$). We have $|t_1^l| + |t_2^l| \leq |s_1| + |s_2| - 1 - l$, thus

$$2\sum_{l \leq |s_1|+|s_2|-2} (k_l + 1) \leq 2\sum_{l \leq |s_1|+|s_2|-2} n^{2(|t_1^{l-1}|+|t_2^{l-1}|)} \leq f(s_1, s_2)$$

and this construction is possible. But we have $|t_1^{|s_1|+|s_2|-2} + |t_2^{|s_1|+|s_2|-2}| \leq 1$, thus $C_{|s_1|+|s_2|-2} \in \mathcal{G}_1$, which is absurd.

Let $A \notin \mathcal{G}_2$. As $A \notin \mathcal{G}_1$, we can find $s, t \in A$ which are not powers of the same sequence. Indeed, let $s \in A^-$ and $u$ with minimal length such that $s$ is a power of $u$. Then any $t \in A \setminus \{s^v\} \in \omega$ works, because if $s$ and $t$ are powers of $u$, then $w$ has to be a power of $u$. Indeed, as $u < w$, $w = u^kv$ with $v < u$, and $v$ has to be a power of $u$ by minimality of $|u|$ and Lemma 7. Assume that moreover $A \in F_{2k+2}$. Now $A$ is the limit of $(A_{k,r})_r \subseteq F_{2k+1} \cap \mathcal{G}_2$ for each integer $k$, and we may assume that $s, t \in A_{k,r} \notin \mathcal{G}_1$. Let $s_1^{k,r}, s_2^{k,r}$ be such that $A_{k,r}^\infty = \{s_1^{k,r}, s_2^{k,r}\}^\infty$. By Lemma 9 we have $|s_1^{k,r}| + |s_2^{k,r}| \leq |s| + |t|$ and $f(s_1^{k,r}, s_2^{k,r}) \leq f(s, t)$. By the preceding point, we must have

$$2k + 1 < f(s, t).$$

Thus $\bigcap F_m \subseteq \mathcal{G}_2$. Notice that $F_{m+1}(\mathcal{G}_2) \subseteq F_m$, so that $F_{\omega}(\mathcal{G}_2) = \emptyset$ and $\mathcal{G}_2 \subseteq \mathcal{D}_\omega(\Sigma)^0_0$.

\* Now let us show that $\{0\} \in F_{\omega}(\mathcal{G}_2)$ (this will imply $\mathcal{G}_2 \notin D_\omega(\Sigma)^0_0$). It is enough to see that

$$\{0\} \in \bigcap F_m.$$

Let $E(x)$ be the biggest integer less than or equal to $x$, $p_k,s := 2^{k+1-E(|s|^2)}$ and $k \in \omega$. We define $A_0 := \{0\}$ and, for $s \in (\omega \setminus \{0\})^{\leq 2k+1}$ and $m > 1$, $A_{sm} := A_s \cup \{01^{p_k,s}m: (02^{p_k,s})^m \}$ if $|s|$ is even, $A_s \cup \{s \in [0, 1p_k,s)^{\omega}\}^* / m \leq |s| \leq m + p_k,s\}$ if $|s|$ is odd. Let us show that $A_s \in \mathcal{G}_2$ (resp., $\mathcal{G}_2$) if $|s|$ is even (resp., odd). First by induction we get $A_{sm} \subseteq \{0, 1p_k,s\}^{<\omega}$. Therefore $A_{sm}^\infty = \{0, 1p_k,s\}^\infty$ if $|s|$ is odd, because if $\alpha$ is in $\{0, 1p_k,s\}^\infty$ and $t \in [0, 1p_k,s)^{\omega}\}^*$ with minimal length $\geq m$ begins $\alpha$, then $t \in A_{sm}$. Now if $|s|$ is even and $A_{sm}^\infty = \{s_1, s_2\}^\infty$, then $0n \in \{s_1, s_2\}^\infty$, thus for example $s_1 = 0^{k+1}, (01p_k,s)^\infty \in \{s_1, s_2\}^\infty$, thus $s_2 \prec (01p_k,s)^\infty$ and $|s_2| \geq |(01p_k,s)^m|$ since $s_20\in \{s_1, s_2\}^\infty$. But then $(02^{p_k,s})^\infty \notin \{s_1, s_2\}^\infty$ since $m > 1$. Thus $A_{sm} \notin \mathcal{G}_2$.

As $(A_{sm})_m$ tends to $A_s$ and $(A_0)_{|s|=2k+1} \subseteq \mathcal{G}_2$, we deduce from this that $A_s$ is in $F_{2k+1-|s|} \setminus \mathcal{G}_2$ if $|s| \leq 2k + 1$ is odd, and that $A_s \in F_{2k+1-|s|} \cap \mathcal{G}_2$ if $|s| \leq 2k + 1$ is even. Therefore $\{0\}$ is in $\bigcap F_{2k+1} = \bigcap F_m$. \]
**Remarks.** (1) The end of this proof also shows that \( G_p \notin D_\omega(\Sigma^0_1) \) if \( p \geq 2 \). Indeed, \( \{0\} \in F_\omega(G_p) \).

The only thing to change is the definition of \( A_{sm} \), if \( |s| \) is even: we set
\[
A_{sm} := A_s \cup \{(j^{p+1}p_k^p)_m/j < p\}.
\]

(2) If \( \{s_1, s_2\} \notin G_1 \) and \( \{s_1, s_2\}^\infty = \{t_1, t_2\}^\infty \), then \( \{s_1, s_2\} = \{t_1, t_2\} \). Indeed, \( \{t_1, t_2\} \notin G_1 \), thus by Lemma 9 we get \( |s_1| + |s_2| = |t_1| + |t_2| \). By (c) in the proof of Lemma 8 and the previous fact, \( s_i = t_i^{a_i} \), where \( a_i > 0, \epsilon_i, i \in \{1, 2\} \). As \( \{s_1, s_2\} \notin G_1, \epsilon_1 \neq \epsilon_2 \). Thus \( a_i = 1 \).

**Conjecture 1.** Let \( A \in \mathcal{F} \). Then there exists a finite subset \( F \) of \( A \) such that \( A^\infty = F^\infty \).

**Conjecture 2.** Let \( p \geq 1, A, B \notin G_p \) with \( A^\infty = \{s_1, \ldots, s_q\} \subseteq B^\infty = \{t_1, \ldots, t_{p+1}\} \). Then \( \Sigma_1 \leq p+1 \leq \Sigma_1 \leq q \).

**Conjecture 3.** We have \( G_{p+1} \setminus G_p \subset D_\omega(\Sigma^0_1) \) for each \( p \geq 1 \). In particular, \( \mathcal{F} \in \mathcal{K}_\sigma \setminus \Pi^0_2 \).

Notice that Conjectures 1 and 2 imply Conjecture 3. Indeed, \( \mathcal{F} = \mathcal{G}_1 \cup \bigcup_{p \geq 1} G_{p+1} \setminus G_p \), so \( \mathcal{F} \in \mathcal{K}_\sigma \) if \( G_{p+1} \setminus G_p \subset D_\omega(\Sigma^0_1) \subseteq \Delta^0_\lambda \), by Proposition 6. By Proposition 5 we have \( \mathcal{F} \notin \Pi^0_2 \). It is enough to see that \( F_\omega := F_\omega(G_{p+1} \setminus G_p) = 0 \). We argue as in the proof of Corollary 10. This time, \( f(s_1, \ldots, s_q) := 2\sum_{i \leq 1 \leq q} |s_i| \) for \( s_1, \ldots, s_q \in n^{<\omega} \). The fact to notice is that \( A \notin F_\omega(G_{p+1} \setminus G_p) \) if \( A \notin G_p \) satisfies \( A^\infty = \{s_1, \ldots, s_{p+1}\} \) and \( M \) is the minimal odd integer greater than or equal to \( f(s_1, \ldots, s_{p+1}) \). So if \( A \in F_{2k+2} \cap \mathcal{F} \setminus G_p \), then Conjecture 1 gives a finite subset \( F = \{s_1, \ldots, s_q\} \) of \( A \). The set \( A \) is the limit of \( (A_{k,r})_r \subseteq F_{2k+1} \cap G_{p+1} \setminus G_p \) for each integer \( k \), and we may assume that \( F \subseteq A_{k,r} \). Conjecture 2 implies that \( f(s_1, \ldots, s_{p+1}) \leq \inf f(s_1, \ldots, s_q) \) and \( 2k + 1 < \inf f(s_1, \ldots, s_q) \). Thus \( \bigcap_m F_m \subseteq \mathcal{F} \cup G_p \). So \( F_\omega \subseteq (\mathcal{F} \cup G_p) \cap G_{p+1} \setminus G_p = 0 \).

**3 Is \( A^\infty \) Borel?**

Now we will see that the maximal complexity is possible. We essentially give O. Finkel’s example, in a slightly simpler version.

**Proposition 11** Let \( \Gamma := \Sigma^1_1 \) or a Baire class. The existence of \( n \in \omega \setminus 2 \) and \( A \subseteq n^{<\omega} \) such that \( A^\infty = \Gamma \)-complete is equivalent to the existence of \( B \subseteq 2^{<\omega} \) such that \( B^\infty = \Gamma \)-complete.

**Proof.** Let \( p_n := \min \{p \in \omega/n \leq 2^p \} \geq 1 \). We define \( \phi : n \rightarrow 2^{p_n} := \{\sigma_0, \ldots, \sigma_{2^{p_n}-1}\} \) by the formula \( \phi(m) := \sigma_m \), \( \Phi : n^{<\omega} \rightarrow 2^{<\omega} \) by the formula \( \Phi(t) := \phi(t(0)) \cdots \phi(t(|t| - 1)) \) and \( f : n^{<\omega} \rightarrow 2^{<\omega} \) by the formula \( f(\gamma) := \phi(\gamma(0)) \phi(\gamma(1)) \cdots \). Then \( f \) is an homeomorphism from \( n^{<\omega} \) onto its range and reduces \( A^\infty \) to \( B^\infty \), where \( B := \Phi[A] \). The inverse function of \( f \) reduces \( B^\infty \) to \( A^\infty \). So we are done if \( \Gamma \) is stable under intersection with closed sets. Otherwise, \( \Gamma = \Delta^0_1 \) or \( \Sigma^0_1 \). If \( A = \{s \in 2^{<\omega}/0 \prec s \text{ or } 1^2 \prec s\} \), then \( A^\infty = N_0 \cup N_2 \), which is \( \Delta^0_1 \)-complete. If \( A = \{s \in 2^{<\omega}/0 \prec s \} \cup \{10^k1^{l+1}/k, l \in \omega\} \), then \( A^\infty = 2^\omega \setminus \{10^\omega\} \), which is \( \Sigma^0_1 \)-complete. \( \square \)

**Theorem 12** The set \( I := \{(\alpha, A) \in n^{<\omega} \times 2^{<\omega}/\alpha \in A^\infty \} \) is \( \Sigma^1_1 \)-complete. In fact,
(a) (O. Finkel, see [FI]) There exists \( A_0 \subseteq 2^{<\omega} \) such that \( A_0^\infty \) is \( \Sigma^1_1 \)-complete.
(b) There exists \( \alpha_0 \in 2^{<\omega} \) such that \( I_{\alpha_0} \) is \( \Sigma^1_1 \)-complete.
Proof. (a) We set \( L := \{2, 3\} \) and \( T := \{ \tau \subseteq 2^{<\omega} \times L / \forall \langle u, \nu \rangle \in 2^{<\omega} \times L \ [\langle u, \nu \rangle \notin \tau \text{ or } \gamma \text{ subset of } 2 \} \). Then \( \bullet \) The set \( T \) is the set of pruned trees over 2 with labels in \( L \). It is a closed subset of \( 2^{<\omega} \times L \), thus a Polish space. Then we set \( \sigma := \{ \tau \in T / \exists (u, \nu) \in 2^{<\omega} \times L \ [\forall m \geq p \ (u[m, \nu(m)] \in \tau) \} \).

\[ \exists \sigma \in \Sigma^1_1(T) \text{ Let us show that it is complete. We set } T := \{ T \in 2^{<\omega} / T \text{ is a tree} \} \text{ and } I F := \{ T \in T / T \text{ is ill-founded}\} \text{. It is a well-known fact that } T \text{ is a Polish space (it is a closed subset of } 2^{<\omega} \text{), and that } I F \text{ is } \Sigma^1_1 \text{-complete (see [K1])}. \text{ It is enough to find a Borel reduction of } I F \text{ to } \sigma \text{ (see [K2])} . \]

We define \( \psi : \omega^{<\omega} \rightarrow 2^{<\omega} \) by the formula \( \psi(t) := 0|t(0)|0|t(1)|1 \ldots |t(|t|−1)|1 \), and \( \Psi : T \rightarrow T \) by \( \Psi(T) := \{ (u, \nu) \in 2^{<\omega} \times L / \exists t \in T \ u \prec \psi(t) \text{ and } \nu = 3 \text{ if } u = 0, \ 2 + u(|u| − 1) \text{ otherwise} \} \)

The map \( \Psi \) is Baire class one. Let us show that it is a reduction. If \( T \in I F \), then let \( \gamma \in \omega^{\omega} \text{ be such that } \gamma[m] \in T \text{ for each integer } m \). We have \( (\psi(\gamma[m]), 3) \in \Psi(T) \). Let \( w \) be the limit of \( \psi(\gamma[m]) \) and \( \nu(m) := 2 + w(m − 1) \text{ (resp., 3) if } m > 0 \text{ (resp., } m = 0) \). These objects show that \( \Psi(T) \in \sigma \). Conversely, \( T \in I F \) if \( \Psi(T) \in \sigma \).

\( \bullet \) If \( \tau \in T \) and \( m \in \omega \), then we enumerate \( \tau \cap (2^m \times L) := \{ (u_1, v_1), \ldots, (u_{q_m, \tau}, v_{q_m, \tau}) \} \) in the lexicographic ordering. We define \( \varphi : T \rightarrow \omega^{<\omega} \) by the formula

\[ \varphi(\tau) := (u_1^{0,\tau} v_1^{0,\tau} \ldots u_{q_0,\tau}^{0,\tau} v_{q_0,\tau}^{0,\tau})(u_1^{1,\tau} v_1^{1,\tau} \ldots u_{q_1,\tau}^{1,\tau} v_{q_1,\tau}^{1,\tau})4 \ldots \]

The set \( A_0 \) will be made of finite subsequences of sentences in \( \varphi[T] \). We set \( A_0 := \{ u_{q+1}^{m,\tau} v_{q+1}^{m,\tau} \ldots u_{q}^{p,\tau} v_{q}^{p,\tau} / \tau \in T, \ m + 1 < p, \ 0 \leq q \leq q_{m,\tau}, \ 1 \leq r \leq q_{p,\tau}, \ [\text{or } q > 0 \text{ and } \nu_{q,\tau}^{m,\tau} = 3 \text{ and } u_{q,\tau}^{m,\tau} \prec v_{q,\tau}^{p,\tau}) \} \)

(we cut \( u_{q,\tau}^{m,\tau} \prec v_{q,\tau}^{p,\tau} ) \text{ and } v_{q,\tau}^{p,\tau} \text{ with the convention } u_{q,\tau}^{m,\tau} v_{q,\tau}^{p,\tau} \text{ and } v_{q,\tau}^{p,\tau} \text{ at } m > 4 \}. \text{ It is clear that } \varphi \text{ is continuous, and it is enough to see that it reduces } \sigma \text{ to } A_0^\infty . \)

So let us assume that \( \tau \in \sigma \). This means the existence of an infinite branch in the tree with infinitely many 3 labels. We cut \( \varphi(\tau) \) after the first 3 label of the branch corresponding to a sequence of length \( m > 1 \). Then we cut after the first 3 label corresponding to a sequence of length at least \( m + 2 \) of the branch. And so on. This clearly gives a decomposition of \( \varphi(\tau) \) into words in \( A_0 \).

If such a decomposition exists, then the first word is \( u_1^{0,\tau} v_1^{0,\tau} \ldots u_{q_0,\tau}^{0,\tau} v_{q_0,\tau}^{0,\tau} \), and the second is \( u_{q_0,\tau}^{p_0,\tau} v_{q_0,\tau}^{p_0,\tau} \ldots u_{q_1,\tau}^{p_1,\tau} v_{q_1,\tau}^{p_1,\tau} \). So we have \( u_{q_0,\tau}^{p_0,\tau} \prec u_{q_1,\tau}^{p_1,\tau} \). And so on. This gives an infinite branch with infinitely many 3 labels.
• By Proposition 11, we can also have $A_0 \subseteq 2^{<\omega}$.

(b) Let $\alpha_0 := 1010^2 10^3 \ldots (q_i)$ be the sequence of prime numbers: $q_0 := 2$, $q_1 := 3$, $M : \omega^{<\omega} \rightarrow \omega$ defined by $M_s := q_0^{(s)(0)+1} \ldots q_{|s|-1}^{(|s|-1)+1} + 1$, $\phi : \omega^{<\omega} \rightarrow 2^{<\omega} \setminus \{\emptyset\}$ defined by the formulas

$$\phi(\emptyset) := 1010^2 = 1010^{2M_0}$$

and $\phi(sm) := 10^{2M_{\sigma+1}+1} 10^{2M_{\sigma+1}+2} \ldots 10^{2M_m}$, and $\Phi : 2^{\omega^{<\omega}} \rightarrow 2^{\varepsilon^{<\omega}}$ defined by $\Phi(T) := \phi[T]$.

• It is clear that $M_{sm} > M_s$, and that $M$ and $\phi$ are well defined and one-to-one. So $\Phi$ is continuous:

$s \in \phi[T] \iff \exists t \in T$ and $\phi(t) = s$

$s \in \phi[\omega^{<\omega}]$ and $\forall t \in T \phi(t) \neq s$.

If $T \in IF$, then we can find $\beta \in \omega^{\omega}$ such that $\phi(\beta[l]) \in \Phi(T)$ for each integer $l$. Thus

$$\alpha_0 = (1010^{2M_{\beta(0)}} 10^{2M_{\beta(0)+1}} \ldots 10^{2M_{\beta(1)}}) \ldots \in \Phi(T)^{\infty}.$$  

Conversely, if $\alpha_0 \in (\Phi(T))^{\infty}$, then there exist $t_i \in T$ such that $\alpha_0 = \phi(t_0) \phi(t_1) \ldots$ We have $t_0 = \emptyset$, and, if $i > 0$, then $M_{t_i[t_i-1]} = M_{t_i-1}$; from this we deduce that $t_i[t_i] - 1 = t_i - 1$, because $M$ is one-to-one. So let $\beta$ be the limit of the $t_i$’s. We have $\beta[i] = t_i$, thus $\beta \in [T]$ and $T \in IF$. Thus $\Phi|T$ reduces $IF$ to $I_{\alpha_0}$. Therefore this last set is $\Sigma^1_1$-complete. Indeed, it is clear that $I$ is $\Sigma^1_1$:

$$\alpha \in A^{\infty} \iff \exists \beta \in \omega^{\omega} \ [(\forall m > 0 \ (\beta(m) > 0)) \mbox{ and } (\forall q \in \omega \ \pi(\alpha, \beta, q) \in A)].$$

Finally, the map from $T$ into $n^{\omega} \times 2^{n^{<\omega}}$, which associates $(\alpha_0, \Phi(T))$ to $T$ clearly reduces $IF$ to $I$.

So $I$ is $\Sigma^1_1$-complete.

Remark. This proof shows that if $\alpha = s_0 s_1 \ldots$ and $(s_i)$ is an antichain for the extension ordering, then $I_{\alpha}$ is $\Sigma^1_1$-complete (here we have $s_1 = 10^{2i+1} 10^{2i+2}$). To see it, it is enough to notice that $\phi(\emptyset) = s_0$ and $\phi(sm) = s_{M_s} \ldots s_{M_m}$. So $I_{\alpha}$ is $\Sigma^1_1$-complete for a dense set of $\alpha$’s.

We will deduce from this some true co-analytic sets. But we need a lemma, which has its own interest.

Lemma 13  (a) The set $A^{\infty}$ is Borel if and only if there exist a Borel function $f : n^{\omega} \rightarrow \omega^{\omega}$ such that

$$\alpha \in A^{\infty} \iff (\forall m > 0 \ f(\alpha)(m) > 0) \mbox{ and } (\forall q \in \omega \ \pi(\alpha, f(\alpha), q) \in A).$$

(b) Let $\gamma \in \omega^{\omega}$ and $A \subseteq n^{<\omega}$. Then $A^{\infty} \in \Delta^1_1(A, \gamma)$ if and only if, for $\alpha \in n^{\omega}$, we have

$$\alpha \in A^{\infty} \iff \exists \beta \in \Delta^1_1(A, \gamma, \alpha) \ [(\forall m > 0 \ (\beta(m) > 0)) \mbox{ and } (\forall q \in \omega \ \pi(\alpha, \beta, q) \in A)].$$

Proof. The “if” directions in (a) and (b) are clear. We have seen in the proof of Proposition 4 the “if” direction of the equivalences (the existence of an arbitrary $\beta$ is necessary and sufficient). So let us show the “only if” directions.
We define \( f : \omega^\omega \to \omega^\omega \) by the formula \( f(\alpha) := 0^\omega \) if \( \alpha \notin A^\omega \), and, otherwise,

\[
f(\alpha)(0) := \min\{ p \in \omega / \alpha [(p + 1) \in A \} \quad \text{and} \quad \alpha - \alpha[(p + 1) \in A^\omega] \}
\]

\[
f(\alpha)(r + 1) := \min\{ k > 0 / \alpha - \alpha[(1 + \Sigma_{j \leq r} f(\alpha)(j))][k \in A \} \quad \text{and} \quad \alpha - \alpha[(k + 1 + \Sigma_{j \leq r} f(\alpha)(j)) \in A^\omega] \}.
\]

We get \( \pi(\alpha, f(\alpha), 0) = \alpha[f(\alpha)(0) + 1] \in A \) and, if \( q > 0 \),

\[
\pi(\alpha, f(\alpha), q) = (\alpha(1 + \Sigma_{j < q} f(\alpha)[j]), ..., \alpha(\Sigma_{j < q} f(\alpha)[j])) \in A.
\]

As \( f \) is clearly Borel, we are done.

(b) If \( A^\omega \in \Delta_1^1(A, \gamma) \), then so is \( f \) and \( \beta := f(\alpha) \in \Delta_1^1(A, \gamma, \alpha) \) is what we were looking for. \( \square \)

**Remark.** Lemma 13 is a particular case of a more general situation. Actually we have the following uniformization result. It was written after a conversation with G. Debs.

**Proposition 14** Let \( X \) and \( Y \) be Polish spaces, and \( F \in \Pi_0^\infty(X \times Y) \) such that the projection \( \Pi_X[F \cap (X \times Y)] \) is Borel for each \( V \in \Sigma_1^1(Y) \). Then there exists a Borel map \( f : X \to Y \) such that \( (x, f(x)) \in F \) for each \( x \in \Pi_X[F] \).

**Proof.** Let \((Y_n)\) be a basis for the topology of \( Y \) with \( Y_0 := Y, B_0 := \Pi_X[F \cap (X \times Y_n)] \), and \( \tau \) be a finer 0-dimensional Polish topology on \( X \) making the \( B_{n} \)'s clopen (see 13.5 in [K1]). We equip \( X \) with a complete \( \tau \)-compatible metric \( d \). Let \((O_m) \subseteq \Sigma_1^0(X \times Y) \) be decreasing satisfying \( O_0 := X \times Y \) and \( F = \bigcap_m O_m \). We construct a sequence \((Us_s)_{s \in \omega^\omega} \) of clopen subsets of \([B_0, \tau]\) with \( U_0 := B_0 \), and a sequence \((Vs_s)_{s \in \omega^\omega} \) of basic open sets of \( Y \) satisfying

\[
(a) U_s \subseteq \Pi_X[F \cap (Us \times Vs)] \\
(b) \text{diam}_d(U_s), \text{diam}(Vs) \leq \frac{1}{m} \text{ if } s \neq \emptyset \\
(c) U_s = \bigcup_m, \text{disj.} \quad U_{s-m}, \quad Vs_{s-m} \subseteq Vs \\
(d) U_s \times Vs \subseteq O_{|s|}
\]

- Assume that this construction has been achieved. If \( x \notin B_0 \), then we set \( f(x) := y_0 \in Y \) (we may assume that \( F \neq \emptyset \)). Otherwise, we can find a unique sequence \( \gamma \in \omega^\omega \) such that \( x \in V_{\gamma m} \) for each integer \( m \). Thus we can find \( y \in V_{\gamma m} \) such that \( (x, y) \in F \), and \( (V_{\gamma m})_m \) is a decreasing sequence of nonempty closed sets whose diameters tend to 0, which defines a continuous map \( f : [B_0, \tau] \to Y \). If \( x \in B_0 \), then \( (x, f(x)) \in U_{\gamma m} \times V_{\gamma m} \subseteq O_m \), thus \( Gr(f|B_0) \subseteq F \). Notice that \( f : [X, \tau] \to Y \) is continuous, so \( f : X \to Y \) is Borel.

- Let us show that the construction is possible. We set \( U_0 := B_0 \) and \( V_0 := Y \). Assume that \((Us_s)_{s \in \omega^\omega} \leq p \) and \((Vs_s)_{s \in \omega^\omega} \leq p \) satisfying conditions (a)-(d) have been constructed, which is the case for \( p = 0 \). Let \( s \in \omega^p \). If \( (x, y) \in F \cap (Us \times Vs) \), then we can find \( U_x \in \Delta_0^\infty(Us_s) \) and a basic open set \( V_y \subseteq Y \) such that \( (x, y) \in U_x \times V_y \subseteq U_x \times V_y \subseteq (Us \times Vs) \cap O_{p+1} \), and whose diameters are at most \( \frac{1}{p^{1}} \). By the Lindelöf property, we can write \( F \cap (Us \times Vs) \subseteq \bigcup_n U_{x_n} \times V_{y_n} \) and \( F \cap (Us \times Vs) = \bigcup_n F \cap (Us_n \times V_{y_n}) \).
If $x \in U_s$, then let $n$ and $y$ be such that $(x, y) \in F \cap (U_{x_n} \times V_{y_n})$. Then

$$x \in O^n := \Pi_x [F \cap (X \times V_{y_n})] \cap U_{x_n} \in \Delta_1^0 ([B_0, \tau]).$$

Thus $U_s = \bigcup_n O^n$. We set $U_{s-n} := O^n \setminus (\bigcup_{p<n} O^p)$ and $V_{s-n} := V_{y_n}$, and we are done. \hfill $\square$

In our context, $F = \{(\alpha, \beta) \in n^\omega \times \omega^\omega/(\forall m > 0 \beta(m) > 0) \text{ and } (\forall q \in \omega \pi(\alpha, \beta, q) \in A)\}$, which is a closed subset of $X \times Y$. The projection $\Pi_x[F \cap (X \times N_s)]$ is Borel if $A^\infty$ is Borel, since it is $\{S^\gamma / S \in (A \cap n^{(0)+1}) \times \Pi_{0<j<|s|} (A \cap n^{(j)})\}$ and $\gamma \in A^\infty$.

**Theorem 15** The following sets are $\Pi_1^1 \setminus \Delta_1^1$:

(a) $\Pi := \{(A, \gamma, \theta) \in 2^{n^\omega} \times \omega^\omega \times \omega^\omega / \theta \in \text{WO and } A^\infty \in \Pi_0^0 \cap \Delta_1^1 (A, \gamma)\}$. The same thing is true with $\Sigma := \{(A, \gamma, \theta) \in 2^{n^\omega} \times \omega^\omega \times \omega^\omega / \theta \in \text{WO and } A^\infty \in \Sigma_0^0 \cap \Delta_1^1 (A, \gamma)\}$.

(b) $\Sigma_\xi := \{A \in 2^{n^\omega} / A^\infty \in \Sigma_\xi^0 \cap \Delta_1^1 (A)\}$. In fact, $\Sigma_\xi := \{A \in 2^{n^\omega} / A^\infty \in \Sigma_\xi^0 \cap \Delta_1^1 (A)\}$ is $\Pi_1^1 \setminus \Delta_1^1$ if $1 \leq \xi < \omega_1$. Similarly, $\Pi_\xi := \{A \in 2^{n^\omega} / A^\infty \in \Pi_\xi^0 \cap \Delta_1^1 (A)\}$ is $\Pi_1^1 \setminus \Delta_1^1$ if $2 \leq \xi < \omega_1$.

(c) $\Delta := \{A \in 2^{n^\omega} / A^\infty \in \Delta_1^1 (A)\}$.

**Proof.** Consider the way of coding the Borel sets used in [Lou]. By Lemma 13 we get

$$(A, \gamma, \theta) \in \Pi \iff \exists \alpha \in n^\omega \quad P(p, A, \gamma, \theta) \quad \text{and } \forall \alpha \in n^\omega \quad (\alpha \notin A^\infty \text{ or } (p, A, \gamma, \alpha) \notin C) \quad \text{and} \quad \exists \beta \in \Delta_1^1 (A, \gamma, \alpha) \quad \text{or} \quad \exists \beta \in \Delta_1^1 (A, \gamma, \alpha) \quad \text{or} \quad \exists \beta \in \Delta_1^1 (A, \gamma, \alpha) \quad \text{or} \quad \exists \beta \in \Delta_1^1 (A, \gamma, \alpha) \text{ and } \exists \beta (m > 0, \beta(m) > 0) \text{ and } \forall q \in \omega \pi(\alpha, \beta, q) \in A)\}.$$

This shows that $\Pi$ is $\Pi_1^1$. The same argument works with $\Sigma$. From this we can deduce that $\Sigma_\xi$ is $\Pi_1^1$, if we forget $\gamma$ and take the section of $\Sigma$ at $\theta \in \text{WO} \cap \Delta_1^1$ such that $|\theta| = 1$. Similarly, $\Sigma_\xi$ and $\Pi_\xi$ are co-analytic if $\xi \geq 1$. Forgetting $\theta$, we see that the relation “$A^\infty \in \Delta_1^1 (A, \gamma)$” is $\Pi_1^1$.

- Let us look at the proof of Theorem 12. We will show that if $\xi \geq 1$ (resp., $\xi \geq 2$), then $\Sigma_\xi \setminus I_{a_0}$ (resp., $\Pi_\xi \setminus I_{a_0}$) is a true co-analytic set. To do this, we will reduce $WF$ to $\Sigma_\xi \setminus I_{a_0}$ (resp., $\Pi_\xi \setminus I_{a_0}$) in a Borel way. We change the definition of $\Phi$. We set

$$t \leq a_0 \iff \exists k \quad t < a_0 - a_0[k],$$

$$E := \{(a_0[p] / r \in \omega \setminus \{2\}, r \in n \setminus \{a_0[p]\}\}, \quad F := \{U^* \not\subseteq a_0 / U \in \phi[T]^{<\omega}\},$$

$$\Phi'(T) := \phi[T] \cup \{s \in n^{<\omega} / \exists t \in E \cup F \quad t < s\}.$$ 

This time, $\Phi'$ is Baire class one, since

$$s \in \Phi'(T) \iff s \in \phi[T] \quad \text{or} \quad \exists t \in E \quad t < s \quad \text{or} \quad \exists U \in (2^{<\omega})^{<\omega} \quad (\forall j < |u| \ U(j) \in \phi[T] \) and \quad U^* \not\subseteq a_0 \text{ and } U^* < s.$$ 

The proof of Theorem 12 remains valid, since if $a_0 \in (\Phi'(T))^{<\omega}$, then the decompositions of $a_0$ into words of $\Phi'(T)$ are actually decompositions into words of $\phi[T]$. 

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Let us show that \((\Phi'(T))^\infty \in \Sigma^0_1 \cap \Delta^1_1(\Phi'(T))\) if \(T \in WF\). The set \((\Phi'(T))^\infty\) is

\[
\bigcup_{S \in \phi(T)^{<\omega}, J \in n \setminus \{1\}, m \in n \setminus \{0\}} \left[ \bigcup_{s/\exists t \in F \ t \prec s} N_{S^* \cup N_{S^*+1 \cup (N_{S^*+101} \setminus \{S^* a_0\})}} \right].
\]

If \(\alpha \in n^\omega\), then \(\alpha\) contains infinitely many \(l \in n \setminus \{1\}\) or finishes with \(1^\infty\). As \(1^2\) and the sequences beginning with \(l\) are in \(\Phi'(T)\), the clopen sets are subsets of \((\Phi'(T))^\infty\) since \(\phi[T]\) and the sequences beginning with \(t \in F\), \(l\) or \(lm\) are in \(\Phi'(T)\). If \(\alpha \in (N_{S^*+1} \setminus \{S^* a_0\})\), then \(p \geq 3\) be maximal such that \(\alpha'[\{S^*\} + p] = S^* (a_0 [p])\). We have \(\alpha \in (\Phi'(T))^\infty\) since the sequences beginning with \((a_0 [p])\) are in \(\Phi'(T)\). Thus we get the inclusion into \((\Phi'(T))^\infty\).

If \(\alpha \in (\Phi'(T))^\infty\), then \(\alpha = a_0 a_1 \ldots\), where \(a_i \in \Phi'(T)\). Either for all \(i\) we have \(a_i \in \phi[T]\). In this case, there is \(i\) such that \(a_0 \ldots a_{i-1} \not\subseteq a_0\), otherwise we could find \(k\) with \(a_k = a_0 - a_0[k] \in \Phi'(T))^\infty\). But this contradicts the fact that \(T \in WF\), as in the proof of Theorem 12. So we have \(\alpha \in \bigcup_{\exists t \in F \ t \prec s} N_s\). Or there exists \(i\) minimal such that \(a_i \not\in \phi[T]\). In this case,

- Either \(\exists t \in E t \prec a_i\) and \(\alpha \in \bigcup_{S \in \phi[T]^{<\omega}, J \in n \setminus \{1\}, m \in n \setminus \{0\}} [N_{S^*+1} \cup N_{S^*+1 \cup (N_{S^*+101} \setminus \{S^* a_0\})}],\)
- Or \(\exists t \in F t \prec a_i\) and \(\alpha \in \bigcup_{S \in \phi[T]^{<\omega} \cup \cup_{s/\exists t \in F \ t \prec s} N_s \cup \bigcup_{S \in \phi[T]^{<\omega}} \bigcup_{s/\exists t \in F \ t \prec s} N_s^* s\).

From this we deduce that \((\Phi'(T))^\infty\) is \(\Sigma^0_1\).

Finally, we have

\[
\alpha \in (\Phi'(T))^\infty \iff \left\{ \begin{array}{l}
\exists t \in n^{<\omega} \exists b \in \omega^{<\omega} [(|t| = 1 + \Sigma_{j < |b|} b(j)) \land (\forall 0 < m < |b| b(m) > 0) \\
\land (\forall q < |b| \exists \alpha \in \Phi'(T)) [\exists 3l \in n \setminus \{1\} tl \prec \alpha \lor tl^2 \prec \alpha].
\end{array} \right.
\]

This shows that \((\Phi'(T))^\infty\) is \(\Delta^1_1(\Phi'(T))\).

Therefore, \(\Phi'(T)\) reduces \(WF\) to \(\Sigma^0_1 \setminus \Pi^0_0\) if \(\xi \geq 1\), and to \(\Pi^0_1 \setminus \Pi^0_0\) if \(\xi \geq 2\). So these sets are true co-analytic sets. But \(\Sigma^0_1 \setminus \Pi^0_0 = \Pi^0_1\), by Lemma 13. As \(\Sigma^0_1 \setminus \Pi^0_0 = \Sigma^1_1 \setminus \Pi^0_0\), \(\Sigma^1_1\) is not Borel. Thus \(\Sigma^0_1\) is not Borel, as before. The argument is similar for \(\Sigma^1_1\), \(\Pi^0_1\) (\(\xi \geq 2\)) and \(\Pi^0_2\). And for \(\Delta^0_2\) sure.

**Question.** Does \(A^\infty \in \Delta^1_1\) imply \(A^\infty \in \Delta^1_1(A)\)? Probably not. If the answer is positive, \(\Delta^0_1\) and more generally \(\Sigma^0_1\) (for \(\xi \geq 1\)) and \(\Pi^0_1\) (for \(\xi \geq 2\)) are true co-analytic sets.

**Remark.** In any case, \(\Delta^0_1\) is \(\Sigma^0_1\) because "\(A^\infty \in \Delta^0_1\)" is equivalent to "\(\exists \gamma \in \omega^\omega A^\infty \in \Delta^0_1(A, \gamma)\)". This argument shows that \(\Sigma^0_1\) and \(\Pi^0_1\) are \(\Delta^0_1\) (\(\theta\)), where \(\theta \in WO\) satisfies \(|\theta| = \xi\). We can say more about \(\Pi_1\): it is \(\Delta^0_1\). Indeed, in [St2] we have the following characterization:

\[
A^\infty \in \Pi^0_1 \iff \forall \alpha \in n^\omega \exists \gamma \in n^{<\omega} (s \prec \alpha \Rightarrow \exists S \in A^{<\omega} S \prec S^\gamma) \Rightarrow \alpha \in A^\infty.
\]

This gives a \(\Pi^0_1\) definition of \(\Pi_1\). The same fact is true for \(\Sigma^0_1\).

**Proposition 16** \(\Sigma^0_1\) and \(\Pi^0_1\) are co-nowhere dense \(\Delta^0_2 \setminus D_2(\Sigma^0_1)\) subsets of \(2^{n^{<\omega}}\). If \(\xi \geq 2\), then \(\Sigma^0_1\) and \(\Pi^0_1\) are co-nowhere dense \(\Sigma^0_2 \setminus D_2(\Sigma^0_1)\) subsets of \(2^{n^{<\omega}}\). \(\Delta^0_2\) is a co-nowhere dense \(\Sigma^0_2 \setminus D_2(\Sigma^0_1)\) subset of \(2^{n^{<\omega}}\).
Proof. We have seen that $\Sigma_1$ is $\Sigma^0_2$; it is also $\Pi^1_2$ because

$$A^\infty \in \Sigma^0_1 \iff \forall \alpha \in n^\omega \alpha \notin A^\infty \text{ or } \exists s \in n^{<\omega} [s \prec \alpha \text{ and } \forall \beta \in n^\omega \ (s \neq \beta \text{ or } \beta \in A^\infty)].$$

By Proposition 4, $\Pi_0$ is co-nowhere dense, and it is a subset of $\Sigma_\xi \cap \Pi_\xi \cap \Delta$. So $\Sigma_\xi$, $\Pi_\xi$ and $\Delta$ are co-nowhere dense, and it remains to see that they are not open. It is enough to notice that $\emptyset$ is not in their interior. Look at the proof of Theorem 12; it shows that for each integer $m$, there is a subset $A_m$ of $\{s \in 5^{<\omega} \mid |s| \geq m\}$ such that $A_m^\infty \notin \Delta^1_1$. But the argument in the proof of Proposition 11 shows that we can have the same thing in $n^{<\omega}$ for each $n \geq 2$. This gives the result because the sequence $(A_m)$ tends to $\emptyset$. \hfill \Box

We can say a bit more about $\Pi_1$ and $\Sigma_2$:

Proposition 17 $\Pi_1$, $\Pi_2$ and $\Sigma_2$ are $\Sigma^0_2$-hard (so they are not $\Pi^0_2$).

Proof. Consider the map $\phi$ defined in the proof of Proposition 5. By Proposition 2, if $\gamma \in P_f$, then $\phi(\gamma)^\infty$ is $\Pi^1_1$. Moreover, as $\phi(\gamma)$ is an antichain for the extension ordering, the decomposition into words of $\phi(\gamma)$ is unique. This shows that $\phi(\gamma)^\infty$ is $\Delta^1_1$, because

$$\alpha \in \phi(\gamma)^\infty \iff \exists \beta \in \Delta^1_1(\alpha) \left[\forall m > 0 \ (\beta(m) > 0) \text{ and } (\forall q \in \omega \ (\pi(\alpha, \beta, q) \in \phi(\gamma)))\right].$$

So $\phi(\gamma) \in \Pi_1$ if $\gamma \in P_f$. So the preimage of any of the sets in the statement by $\phi$ is $P_f$, and the result follows. \hfill \Box

4 Which sets are $\omega$-powers?

Now we come to Question (3). Let us specify what we mean by “codes for $\Gamma$-sets”, where $\Gamma$ is a given class, and fix some notation.

For the Borel classes, we will essentially consider the $2^\omega$-universal sets used in [K1] (see Theorem 22.3). For $\xi \geq 1$, $U^{\xi, A}$ (resp. $U^{\xi, M}$) is $2^\omega$-universal for $\Sigma^\xi_\chi(n^\omega)$ (resp. $\Pi^\xi_\chi(n^\omega)$). So we have

- $U^{1, A} = \{(\gamma, \alpha) \in 2^\omega \times n^\omega \mid \exists p \in \omega \ (\gamma(p) = 0) \text{ and } s^n_p \prec \alpha\}$, where $(s^n_p)_p$ enumerates $n^{<\omega}$.
- $U^{\xi, A} = \sim U^{\xi, A}$, for each $\xi \geq 1$.
- $U^{\xi, A} = \{(\gamma, \alpha) \in 2^\omega \times n^\omega \mid \exists p \in \omega \ (\gamma(p) \in U^{\eta, M})\}$ if $\xi = \eta + 1$.
- $U^{\xi, A} = \{(\gamma, \alpha) \in 2^\omega \times n^\omega \mid \exists p \in \omega \ (\gamma(p) \in U^{\eta, M})\}$ if $\xi$ is the limit of the strictly increasing sequence of odd ordinals $(\eta_p)$.

For the class $\Sigma^1_1$, we fix some bijection $p \mapsto ((p)_0, (p)_1)$ between $\omega$ and $\omega^2$. We set

$$(\gamma, \alpha) \in U \iff \exists \beta \in 2^\omega \ (\forall m \exists p \geq m \ (\beta(p) = 1) \text{ and } (\forall p \ [\gamma(p) = 1 \text{ or } s^2_{(p)_0} \neq \beta \text{ or } s^n_{(p)_1} \neq \alpha])\).$$

It is not hard to see that $U$ is $2^\omega$-universal for $\Sigma^1_1(n^\omega)$, and we use it here because of the compactness of $2^\omega \times n^\omega$, rather than the $\omega^\omega$-universal set for $\Sigma^1_1(n^\omega)$ given in [K1] (see Theorem 14.2).
• The sets we are interested in are the following:

\[ \mathcal{A}_\xi := \{ \gamma \in 2^\omega / \mathcal{U}_\gamma^{\xi, \mathbb{A}} \text{ is an } \omega \text{-power} \}, \quad \mathcal{M}_\xi := \{ \gamma \in 2^\omega / \mathcal{U}_\gamma^{\xi, \mathbb{M}} \text{ is an } \omega \text{-power} \} \]

\[ B := \{ d \in D / D_d \text{ is an } \omega \text{-power} \}, \]

\[ A := \{ \gamma \in 2^\omega / \mathcal{U}_\gamma \text{ is an } \omega \text{-power} \}. \]

As mentioned in the introduction, Lemma 13 is also related to Question (3). A rough answer to this question is \( \Sigma^1_1 \). Indeed, we have, for \( \gamma \in 2^\omega \),

\[ \gamma \in A \iff \exists A \in 2^{n_\omega} \forall \alpha \in n^\omega \left( \left( (\gamma, \alpha) \notin U \text{ or } \alpha \in A^\infty \right) \text{ and } [\alpha \notin A^\infty \text{ or } (\gamma, \alpha) \in U] \right). \]

With Lemma 13, we have a better estimation of the complexity of \( B \): it is \( \mathcal{S}^1_2 \). Indeed, for \( d \in D \),

\[ D_d \text{ is an } \omega \text{-power } \iff \exists A \in 2^{n_\omega} \forall \alpha \in n^\omega \left( [(d, \alpha) \notin S \text{ or } \exists \beta \in \Delta^1_1(A, d, \alpha) \nonumber \right) \]

\[ \nonumber \left( \forall m > 0 \beta(m) > 0 \right) \text{ and } (\forall q \in \omega \pi(\alpha, \beta, q) \in A)) \right) \) \text{ and } [\alpha \notin A^\infty \text{ or } (d, \alpha) \in P]. \]

This argument also shows that \( \mathcal{A}_\xi \) and \( \mathcal{M}_\xi \) are \( \Sigma^1_2 \). We can say more about these two sets.

**Proposition 18** If \( 1 \leq \xi < \omega_1 \), then \( \mathcal{A}_\xi \) and \( \mathcal{M}_\xi \) are \( \Sigma^1_2 \setminus D_2(\Sigma^0_1) \) co-meager subsets of \( 2^\omega \). If moreover \( \xi = 1 \), then they are co-nowhere dense.

**Proof.** We set \( E_1 := \{ \gamma \in 2^\omega / \mathcal{U}^1_\gamma = n^\omega \} \), \( E_{\eta+1} := \{ \gamma \in 2^\omega / \forall p (\gamma)_p \in E_\eta \} \) if \( \eta \geq 1 \), and \( E_\xi := \{ \gamma \in 2^\omega / \forall p (\gamma)_p \in E_\eta \} \) (where \( (\eta_p) \) is a strictly increasing sequence of odd ordinals cofinal in the limit ordinal \( \xi \)). If \( s \in 2^{<\omega} \), then we set \( \gamma(p) = s(p) \) if \( p < |s|, 0 \) otherwise. Then \( s \prec \gamma \) and \( \mathcal{U}^1_\gamma = n^\omega \), so \( E_1 \) is dense. If \( \gamma_0 \in E_1 \), then for all \( \alpha \in n^\omega \) we can find an integer \( p \) such that \( \gamma_0(p) = 0 \) and \( s_\alpha^p < \alpha \). By compactness of \( n^\omega \) we can find a finite subset \( F \) of \( \{ p \in \omega / \gamma_0(p) = 0 \} \) such that for each \( \alpha \in n^\omega, s_\alpha^p < \alpha \) for some \( p \in F \). Now \( \{ \gamma \in 2^\omega / \forall p \in F (\gamma(p) = 0) \} \) is an open neighborhood of \( \gamma_0 \) and a subset of \( E_1 \). So \( E_1 \) is an open subset of \( 2^\omega \). Now the map \( \gamma \mapsto (\gamma)_p \) is continuous and open, so \( E_{\eta+1} \) and \( E_\xi \) are dense \( G_\delta \) subsets of \( 2^\omega \). Then we notice that \( E_\xi \) is a subset of \( \{ \gamma \in 2^\omega / \mathcal{U}^{\xi, \mathbb{A}} = n^\omega \} \) (resp., \( \{ \gamma \in 2^\omega / \mathcal{U}^{\xi, \mathbb{M}} = \emptyset \} \)) if \( \xi \) is odd (resp., even). Indeed, this is clear for \( \xi = 1 \). Then we use the formulas \( \mathcal{U}^{\xi+1, \mathbb{A}} = \bigcup_p \mathcal{U}^{\xi, \mathbb{A}} - \mathcal{U}^{\xi, \mathbb{A}} \) and \( \mathcal{U}^{\xi, \mathbb{M}} = \bigcup_p \mathcal{U}^{\xi, \mathbb{M}} - \mathcal{U}^{\xi, \mathbb{M}} \), and by induction we are done. As \( \emptyset \) and \( n^\omega \) are \( \omega \)-powers, we get the results about Baire category. Now it remains to see that \( \mathcal{A}_\xi \) and \( \mathcal{M}_\xi \) are not open. But by induction again \( 1^\omega \in \mathcal{A}_\xi \cap \mathcal{M}_\xi \), so it is enough to see that \( 1^\omega \) is not in the interior of these sets.

• Let us show that, for \( O \in \Delta^1_1(n^\omega) \setminus \{ 0, n^\omega \} \) and for each integer \( m \), we can find \( \gamma, \gamma' \in 2^\omega \) such that \( \gamma(j) = \gamma'(j) = 1 \) for \( j < m, \mathcal{U}^{\xi, \mathbb{A}} = O \) and \( \mathcal{U}^{\xi, \mathbb{M}} = O \).

For \( \xi = 1 \), write \( O = \bigcup_p N_{q_p} \), where \( q_p \geq m \). Let \( \gamma(q) := 0 \) if there exists \( k \) such that \( q = q_k \), \( \gamma(q) := 1 \) otherwise. The same argument applied to \( O \) gives the complete result for \( \xi = 1 \).
Now we argue by induction. Let $\gamma_p \in 2^\omega$ be such that $\gamma_p(q) = 1$ for $<p,q><m$ and $\mathcal{U}_p^{n,A} = O$. Then define $\gamma$ by $\gamma(<p,q>) := \gamma_p(q)$; we have $\gamma(j) = 1$ if $j < m$ and $\mathcal{U}_l^{n+1,A} = \bigcup_p \mathcal{U}_p^{n,A} = O$. The argument with $\hat{O}$ still works. The argument is similar for limit ordinals.

- Now we apply this fact to $O := N_0$. This gives $\gamma_p, \gamma'_p \in N_1$ such that $\mathcal{U}_{\gamma_p}^{A} = N_0$ and $\mathcal{U}_{\gamma'_p}^{A} = N_0$. But $(\gamma_p), (\gamma'_p)$ tend to $1^\omega$, $\gamma_p \notin A_\xi$ and $\gamma'_p \notin A_\xi$. \hfill \Box

**Corollary 19** $A_1$ is $\bar{D}_2(\Sigma_0^1) \setminus D_2(\Sigma_1^0)$. In particular, $A_1$ is $\bar{D}_2(\Sigma_0^1)$-complete.

**Proof.** By the preceding proof, it is enough to see that $A_1 \setminus \{1^\omega\}$ is open. So let $\gamma_0 \in A_1 \setminus \{1^\omega\}$, $p_0$ in $\omega$ with $\gamma_0(p_0) = 0$, and $A_0 \subseteq n^\omega$ with $\mathcal{U}_0 = 0$. If $\alpha \in n^\omega$, then $\gamma_0(\alpha) = 0$ and $s_p^n < \alpha - \alpha[n \in A_0]$. Thus there exists an integer $p$ such that $\gamma_0(p) = 0$ and $s_p^n \prec \alpha$. By compactness of $n^\omega$, there are finite sets $F \subseteq \omega \setminus \{0\}$ and $G \subseteq \{p \in \omega / \gamma_0(p) = 0\}$ such that $n^\omega = \bigcup_{m \in F, p \in G} \{\alpha \in n^\omega / s_p^n \prec \alpha \}$. We set $A_0 := \{s \in n^\omega / \exists \gamma(p) = 0 \text{ and } s_p^n \prec s\}$ for $\gamma \in 2^\omega$, so that $A_0 \subseteq \gamma_0^{\omega,1}$. Assume that $\gamma(p) = 0$ for each $p \in G$ and let $\alpha \in \gamma_0^{\omega,1}$. Let $p^0 \in \omega$ be such that $\gamma(p^0) = 0$ and $s_{p^0}^n < \alpha$. We can find $m_0 > 0$ and $p^1 \in G$ such that $s_{p^0}^n < \alpha - \alpha[|s_{p^0}^n + m_0|]$, and $\gamma([|s_{p^0}^n + m_0|] + r) = 0$, and $G \subseteq \{p \in \omega / \gamma(p) = 0\}$ such that $s_p^n \prec \alpha - \alpha[|s_p^n + m_0| + |s_p^n| + m_1]$, and $\gamma([|s_p^n + m_0| + |s_p^n| + m_1]) = 0$. And so on. Thus $\alpha \in A_0^{\omega,1}$ and $\{\gamma \in 2^\omega / \exists p \in G \gamma(p) = 0\}$ is a clopen neighborhood of $\gamma_0$ and a subset of $A_1$. \hfill \Box

**Proposition 20** $A$ is $\Sigma_3^1 \setminus D_2(\Sigma_1^0)$ and is co-nowhere dense.

**Proof.** Let $U := \{\gamma \in 2^\omega / \exists \beta \in 2^\omega \forall \alpha \in n^\omega \exists p[\gamma(p) = 0 \text{ and } s_{p^0}^2 \prec \beta \text{ and } s_{p^1}^n \prec \alpha]\}$. By compactness of $2^\omega \times n^\omega$, $U$ is a dense open subset of $2^\omega$. Moreover, if $\gamma \in U$, then $\mathcal{U}_\gamma = \emptyset$, so $U \subseteq A$ and $A$ is co-nowhere dense. It remains to see that $A$ is not open, as in the proof of Proposition 18. As $\mathcal{U}_\omega = n^\omega$, $1^\omega \in A$. Let $p$ be an integer satisfying $s_{p^0}^2 = 0$ and $s_{p^1}^n = 0$. We set $\gamma_p(m) := 0$ if and only if $m = p$, and also $P_\alpha := \{\alpha \in 2^\omega / \exists m \geq r \alpha(m) = 1\}$. Then $(\gamma_p)$ tends to $1^\omega$ and we have $\mathcal{U}_p = \{\alpha \in n^\omega / \exists \beta \in P_\alpha \forall m \neq p \text{ or } s_{(p)^m_0}^2 \neq \beta \text{ or } s_{(p)^m_1}^n \neq \alpha\} = \{\alpha \in n^\omega / \exists \beta \in P_\alpha \beta \in 2^\omega \times n0\}$ = $\neg N_{0\gamma}$. So $\gamma_p \notin A$. \hfill \Box

### 5 Ordinal ranks and $\omega$-powers.

**Notation.** The fact that the $\omega$-powers are $\Sigma_3^1$ implies the existence of a co-analytic rank on the complement of $A^\omega$ (see 34.4 in [K1]). We will consider a natural one, defined as follows. We set, for $\alpha \in n^\omega$, $T_\alpha(\alpha) := \{S \in (A^-)^{<\omega} / S^\alpha < \alpha\}$. This is a tree on $A^-$, which is well founded if and only if $\alpha \notin A^\omega$. 

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The rank of this tree is the announced rank $R_A : A^\omega \to \omega_1$ (see page 10 in [K1]): we have $R_A(\alpha) := \rho(T_A(\alpha))$. Let $\phi : A^- \to \omega$ be one-to-one, and $\hat{\phi}(S) := (\phi(S[0]), \ldots, \phi(S[|S| - 1]))$ for $S \in (A^-)^{<\omega}$. This allows us to define the map $\Phi$ from the set of trees on $A^-$ into the set of trees on $\omega$, which associates $\{\hat{\phi}(S)/S \in T\}$ to $T$. As $\phi$ is one-to-one, $\Phi$ is continuous:

$$t \in \Phi(T) \iff t \in \hat{\phi}((A^-)^{<\omega}) \text{ and } \hat{\phi}^{-1}(t) \in T.$$  

Moreover, $T$ is well-founded if and only if $\Phi(T)$ is well-founded. Thus, if $\alpha \notin A^\omega$, then we have $\rho(T_A(\alpha)) = \rho(\Phi(T_A(\alpha)))$ because $\phi$ is strictly monotone (see page 10 in [K1]). Thus $R_A$ is a co-analytic rank because the function from $n^\omega$ into the set of trees on $\omega^{<\omega}$ which associates $\Phi(T_A(\alpha))$ to $\alpha$ is continuous, and because the rank of the well-founded trees on $\omega$ defines a co-analytic rank (see 34.6 in [K1]). We set

$$R(\alpha) := \sup \{R_A(\alpha)/\alpha \notin A^\omega\}.$$  

By the boundedness theorem, $A^\omega$ is Borel if and only if $R(A) < \omega_1$ (see 34.5 and 35.23 in [K1]). We can ask the question of the link between the complexity of $A^\omega$ and the ordinal $R(A)$ when $A^\omega$ is Borel.

**Proposition 21** If $\xi < \omega_1$, $r \in \omega$ and $R(A) = \omega.\xi + r$, then $A^\omega \in \Sigma^r_{2,\xi+1}$.

**Proof.** The reader should see [L] for operations on ordinals.

- If $0 < \lambda < \omega_1$ is a limit ordinal, then let $(\lambda_q)$ be a strictly increasing co-final sequence in $\lambda$, with $\lambda_0 = \omega.\theta + q$ if $\lambda = \omega.(\theta + 1)$, and $\lambda_q = \omega.\xi q$ if $\lambda = \omega.\xi$, where $(\xi_q)$ is a strictly increasing co-final sequence in the limit ordinal $\xi$ otherwise. By induction, we define

$$E_0 := \{\alpha \in n^\omega/\forall s \in A^- s \not\prec \alpha\},$$

$$E_{\theta+1} := \{\alpha \in n^\omega/\forall s \in A^- s \not\prec \alpha \text{ or } \alpha - s \in E_\theta\},$$

$$E_\lambda := \{\alpha \in n^\omega/\forall s \in A^- s \not\prec \alpha \text{ or } \exists q \in \omega \alpha - s \in E_{\lambda_q}\}.$$  

- Let us show that $E_{\omega.\xi+r} \in \Pi^0_{2,\xi+1}$: We may assume that $\xi \neq 0$ and that $r = 0$. If $\xi = \theta + 1$, then $E_{\lambda_q} \in \Pi^0_{2,\theta+1}$ by induction hypothesis, thus $E_{\omega.\xi+r} \in \Pi^0_{2,\theta+3} = \Pi^0_{2,\xi+1}$. Otherwise, $E_{\lambda_q} \in \Pi^0_{2,\xi_q+1}$ by induction hypothesis, thus $E_{\omega.\xi+r} \in \Pi^0_{2,\xi+1}$.

- Let us show that if $\alpha \in A^\omega$, then $\alpha \notin E_{\omega.\xi+r}$. If $\xi = r = 0$, it is clear. If $r = m + 1$ and $s \in A^-$ satisfies $s \prec \alpha$ and $\alpha - s \in A^\omega$, then we have $\alpha - s \notin E_{\omega.\xi+n}$ by induction hypothesis, thus $\alpha \notin E_{\omega.\xi+r}$. If $r = 0$ and $s \in A^-$ satisfies $s \prec \alpha$ and $\alpha - s \in A^\omega$, then we have $\alpha - s \notin E_{\lambda_q}$ for each integer $q$, by induction hypothesis, thus $\alpha \notin E_{\omega.\xi+r}$.

- Let $s \in A^-$ such that $s \prec \alpha \notin A^\omega$. We have

$$\rho(T_A(\alpha - s)) = \sup \{\rho_{T_A(\alpha - s)}(t) \mid t \in T_A(\alpha - s)\}$$

$$\leq \sup \{\rho_{T_A(s)}(s) \mid (s) t \in T_A(\alpha)\}$$

$$\leq \rho_{T_A(\alpha)}(s) + 1$$

$$\leq \rho_{T_A(\alpha)}(0) < \rho(T_A(\alpha)).$$
The first inequality comes from the fact that the map from \( T_A(\alpha - s) \) into \( T_A(\alpha) \), which associates \((s)t \) to \( t \) is strictly monotone (see page 10 in [K1]). We have

\[ \rho(T_A(\alpha)) \geq \left[ \sup\{ \rho(T_A(\alpha - s)) \mid s \in A^- \}, \ s \prec \alpha \right] + 1. \]

Let us show that we actually have equality. We have

\[ \rho(T_A(\alpha)) = \rho_{T_A(\alpha)}(\emptyset) + 1 = \sup\{ \rho_{T_A(\alpha)}(\{s\}) + 1 \mid s \in A^-, s \prec \alpha \} + 1. \]

Therefore, it is enough to notice that if \( s \in A^- \) and \( s \prec \alpha \), then \( \rho_{T_A(\alpha)}(\{s\}) \leq \rho_{T_A(\alpha-s)}(\emptyset) \). But this comes from the fact that the map from \( \{S \in T_A(\alpha) \mid S(0) = s \} \) into \( T_A(\alpha - s) \), which associates \( S - (s) \) to \( S \), preserves the extension ordering (see page 352 in [K1]).

- Let us show that, if \( \alpha \notin A^\infty \), then \( \rho(T_A(\alpha)) \leq \omega, \xi + r + 1 \) is equivalent to \( \alpha \in E_{\omega, \xi + r} \). We do it by induction on \( \omega, \xi \). If \( \xi = r = 0 \), it is clear. If \( r = m + 1 \), then \( \rho(T_A(\alpha)) \leq \omega, \xi + r + 1 \) is equivalent to \( \forall s \in A^-, s \notin \alpha \) or \( \rho(T_A(\alpha-s)) \leq \omega, \xi + m + 1 \), by the preceding point. This is equivalent to \( \forall s \in A^-, s \notin \alpha \) or \( \alpha - s \in E_{\omega, \xi + m} \), which is equivalent to \( \alpha \in E_{\omega, \xi + r} \). If \( r = 0 \), then \( \rho(T_A(\alpha)) \leq \omega, \xi + r + 1 \) is equivalent to \( \forall s \in A^-, s \notin \alpha \) or there exists an integer \( q \) such that \( \rho(T_A(\alpha-s)) \leq \lambda_q + 1 \). This is equivalent to \( \forall s \in A^-, s \notin \alpha \) or there exists an integer \( q \) such that \( \alpha - s \in E_{\lambda_q} \), which is equivalent to \( \alpha \in E_{\omega, \xi + r} \).

- If \( \alpha \notin A^\infty \), then \( \rho(T_A(\alpha)) \leq \omega, \xi + r + 1 \). By the preceding point, \( \alpha \in E_{\omega, \xi + r} \). Thus we have \( A^\infty = \neg E_{\omega, \xi + r} \in \Sigma_2^{\omega, \xi+1}. \)

We can find an upper bound for the rank \( R \), for some Borel classes:

**Proposition 22**  (a) \( A^\infty = n^\omega \) if and only if \( R(A) = 0 \).

(b) If \( A^\infty = \emptyset \), then \( R(A) = 1 \).

(c) If \( A^\infty \in \Delta_1^0 \), then \( R(A) < \omega \), and there exists \( A_p \subseteq 2^{\omega} \) such that \( A^\infty_p \in \Delta_1^0 \) and \( R(A_p) = p \) for each integer \( p \).

(d) If \( A^\infty \in \Pi_1^1 \), then \( R(A) \leq \omega \), and \( A^\infty \notin \Sigma_1^0 \) or \( R(A) = \omega \).

**Proof.** (a) If \( \alpha \notin A^\infty \), then \( \emptyset \in T_A(\alpha) \) and \( \rho(T_A(\alpha)) \geq \rho_{T_A(\alpha)}(\emptyset) + 1 \geq 1 \).

(b) We have \( T_A(\alpha) = \{\emptyset\} \) for each \( \alpha \), and \( \rho(T_A(\alpha)) = \rho_{T_A(\alpha)}(\emptyset) + 1 = 1 \).

(c) By compactness, there exists \( s_1, \ldots, s_p \in n^{\omega} \) such that \( A^\infty = \bigcup_{1 \leq m \leq p} N_{s_m} \in \Delta_1^0 \). If \( \alpha \notin A^\infty \), then we have \( N_{\alpha} \max_{1 \leq m \leq p, |s_m|} \subseteq \neg A^\infty \), thus \( \rho(T_A(\alpha)) \leq \max_{1 \leq m \leq p, |s_m|} + 1 < \omega \). So we get the first point. To see the second one, we set \( A_0 := 2^{\omega} \). If \( p > 0 \), then we set

\[ A_p := \{0^2\} \cup \bigcup_{q \leq p} \{s \in 2^{\omega} / 0^{2q+1} < s\} \cup \{s \in 2^{\omega} / 0^{2p+1} < s\}. \]

Then \( A^\infty_p = \bigcup_{q \leq p} N_{0^2q+1} \cup N_{0^2q+1} \in \Delta_1^0 \). If \( \alpha_p := 0^{2p+1} \infty \), then \( \rho(T_A(\alpha_p)) = p \). If \( \alpha \notin A^\infty_p \), then \( \rho(T_A(\alpha)) \leq p \).
(d) If $A^\omega \in \Pi_1^0$ and $\alpha \not\in A^\omega$, then let $s \in n^{<\omega}$ with $\alpha \in N_s \subseteq \neg A^\omega$. Then $\rho(T_\alpha(\alpha)) \leq |s| + 1$. Thus $R(A) \leq \omega$. If $A^\omega \notin \Sigma_1^0$, then we have $R(A) \geq \omega$, by Proposition 21. Thus $R(A) = \omega$. Conversely, we apply (c). □

Remark. Notice that it is not true that if the Wadge class $< A^\omega >$, having $A^\omega$ as a complete set, is a subclass of $< B^\omega >$, then $R(A) \leq R(B)$. Indeed, for $A$ we take the example $A_2$ in (c), and for $B$ we take the example for $\Sigma_0^1$ that we met in the proof of Proposition 11. If we exchange the roles of $A$ and $B$, then we see that the converse is also false. This example $A$ for $\Sigma_0^1$ shows that Proposition 21 is optimal for $\xi = 0$ since $R(A) = 1$ and $A^\omega \in \Sigma_0^1 \setminus \Pi_1^0$. We can say more: it is not true that if $A^\omega = B^\omega$, then $R(A) \leq R(B)$. We use again (c): we take $A := A_2$ and $B := A \setminus \{0^2\}$. We have $A^\omega = B^\omega = A_2^\omega$, $R(A) = 2$ and $R(B) = 1$.

Proposition 23 For each $\xi < \omega_1$, there exists $A_\xi \subseteq 2^{<\omega}$ with $A_\xi^\omega \in \Sigma_0^1$ and $R(A_\xi) \geq \xi$.

Proof. We use the notation in the proof of Theorem 15. Let $T \in \tau$, and $\varphi : T \rightarrow T_{\varphi(T)}(\alpha_0)$ defined by the formula $\varphi(s) := (\varphi(s[0]), \ldots, \varphi(s[|s| - 1]))$. Then $\varphi$ is strictly monotone. If $T \in WF$, then $\alpha_0 \not\in (\Phi'(T))^\omega$ and $T_{\Phi'(T)}(\alpha_0) \in WF$. In this case, $\rho(T) \leq \rho(T_{\Phi'(T)}(\alpha_0)) = R_{\Phi'(T)}(\alpha_0)$ (see page 10 in [K1]). Let $T_\xi \in WF$ be a tree with rank at least $\xi$ (see 34.5 and 34.6 in [K1]). We set $A_\xi := \Phi'(T_\xi)$. It is clear that $A_\xi$ is what we were looking for. □

Remark. Let $\psi : 2^{n^{<\omega}} \rightarrow \{\text{Trees on } n^{<\omega}\}$ defined by $\psi(A) := T_\alpha(\alpha_0)$, and $r : \neg I_{\alpha_0} \rightarrow \omega_1$ defined by $r(A) := \rho(T_\alpha(\alpha_0))$. Then $\psi$ is continuous, thus $r$ is a $\Pi_1^1$-rank on $\psi^{-1}(\{\text{Well-founded trees on } n^{<\omega}\}) = \neg I_{\alpha_0}$.

By the boundedness theorem, the rank $r$ and $R$ are not bounded on $\neg I_{\alpha_0}$. Proposition 23 specifies this result. It shows that $R$ is not bounded on $\Sigma_1 \setminus I_{\alpha_0}$.

6 The extension ordering.

Proposition 24 We equip $A$ with the extension ordering.

(a) If $A \subseteq n^{<\omega}$ is an antichain, then $A^\omega$ is in $\{\emptyset\} \cup \{n^\omega\} \cup [\Pi_1^0 \setminus \Sigma_1^0] \cup [\Pi_2^0(A) \setminus \Sigma_2^0]$, and any of these cases is possible.

(b) If $A \subseteq n^{<\omega}$ has finite antichains, then $A^\omega \in \Pi_2^0$ (and is not $\Sigma_2^0$ in general).

Proof. Let $G := \{\alpha \in n^\omega / \forall r \exists m \geq r \; \alpha[m \in [(A)^p]^+\}$. Then $G \in \Pi_2^0(A)$ and contains $A^\omega$. Conversely, if $\alpha \in G$, then we have $T_\alpha(\alpha) \cap (A)^p \neq \emptyset$ for each integer $p$, thus $T_\alpha(\alpha)$ is infinite.

(a) If $A$ is an antichain, then each sequence in $T_\alpha(\alpha)$ has at most one extension in this tree adding one to the length. Thus $T_\alpha(\alpha)$ is finite splitting. This implies that $T_\alpha(\alpha)$ has an infinite branch if $\alpha \in G$, by König’s lemma. Therefore $A^\omega = G \in \Pi_2^0(A)$. 

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- If we take \( A := \emptyset \), then \( A \) is an antichain and \( A^\infty = \emptyset \).
- If we take \( A := \{(0), \ldots, (n - 1)\} \), then \( A \) is an antichain and \( A^\infty = n^\omega \).
- If \( A^\infty \notin \{0, n^\omega\} \), then \( A^\infty \notin \Sigma_1^1 \). Indeed, let \( \alpha_0 \notin A^\infty \) and \( s_0 \in A \). By uniqueness of the decomposition into words of \( A^- \), the sequence \((s_0^{\alpha_0})_n \subseteq n^\omega \setminus A^\infty \) tends to \( s^\infty_0 \in A^\infty \).
- If we take \( A := \{(0)\} \), then \( A \) is an antichain and \( A^\infty = \{0^\infty\} \in \Pi_0^0 \setminus \Sigma_1^0 \).
- If \( A \) is finite, then \( A^\infty \) is \( \Pi_1^0 \setminus \Sigma_0^0 \) or is in \( \{0, n^\omega\} \), by the facts above and Proposition 2.
- If \( A \) is infinite, then \( A^\infty \notin \Sigma_2^0 \) because the map \( c \) in the proof of Proposition 2 is a homeomorphism and \( (A^-)^\omega \) is not \( K_\sigma \).
- If \( A := \{0^k 1/k \in \omega\} \), then \( A \) is an antichain and \( A^\infty = P_\infty \), which is \( \Pi_2^0 \setminus \Sigma_0^0 \).

(b) The intersection of \( P_\infty \) with \( N_1 \) can be made with the chain \( \{10^k/k \in \omega\} \). So let us assume that \( A \) has finite antichains.

Let us show that \( A \) is the union of a finite set and of a finite union of infinite subsets of the form \( A_{\alpha_m} := \{ s \in n^{<\omega} / s < \alpha_m \} \). Let us enumerate \( A := \{s_r/r \in \omega\} \). We construct a sequence \( (A_m) \), finite or not, of subsets of \( A \). We do it by induction on \( r \), to decide in which set \( A_m \) the sequence \( s_r \) is. First, \( s_0 \in A_0 \). Assume that \( s_0, \ldots, s_r \) have been put into \( A_0, \ldots, A_{p_r} \), with \( p_r \leq r \) and \( A_m \cap \{s_0, \ldots, s_r\} \neq \emptyset \) if \( m \leq p_r \). We choose \( m \leq p_r \) minimal such that \( s_{r+1} \) is compatible with all the sequences in \( A_m \cap \{s_0, \ldots, s_r\} \), we put \( s_{r+1} \) into \( A_m \) and we set \( p_{r+1} := p_r \) if possible. Otherwise, we put \( s_{r+1} \) into \( A_{p_r+1} \) and we set \( p_{r+1} := p_r + 1 \).

Let us show that there are only finitely many infinite \( A_m \)'s. If \( A_m \) is infinite, then there exists a unique sequence \( \alpha_m \in n^\omega \) such that \( A_m \subseteq A_{\alpha_m} \). Let us argue by contradiction: there exists an infinite sequence \( (m_q) \) such that \( A_{m_q} \) is infinite. Let \( t_0 \) be the common beginning of the \( \alpha_{m_q} \)'s. There exists \( \varepsilon_0 \in n \) such that \( N_{t_0 \varepsilon_0} \cap \{\alpha_{m_q} / q \in \omega\} \) is infinite. We choose a sequence \( u_0 \) in \( A \) extending \( t_0 \varepsilon_0 \), where \( \mu_0 \neq \varepsilon_0 \). Then we do it again: let \( t_0 \varepsilon_0 \mu_1 \) be the common beginning of the elements of \( N_{t_0 \varepsilon_0 \mu_1} \cap \{\alpha_{m_q} / q \in \omega\} \). There exists \( \varepsilon_1 \in n \) such that \( N_{t_0 \varepsilon_0 \mu_1 \varepsilon_1} \cap \{\alpha_{m_q} / q \in \omega\} \) is infinite. We choose a sequence \( u_1 \) in \( A \) extending \( t_0 \varepsilon_0 \mu_1 \varepsilon_1 \), where \( \mu_1 \neq \varepsilon_1 \). The sequence \( \langle u_i \rangle \) is an infinite antichain in \( A \). But this is absurd. Now let us choose the longest sequence in each nonempty finite \( A_m \); this gives an antichain in \( A \) and the result.

Now let \( \alpha \in G \). There are two cases. Either for each \( m \) and for each integer \( k \), \( \alpha[k] \notin [A^{<\omega}]^k \) or \( \alpha - \alpha[k] \neq \alpha_m \). In this case, \( T_A(\alpha) \) is finite splitting. As \( T_A(\alpha) \) is infinite, \( T_A(\alpha) \) has an infinite branch witnessing that \( \alpha \in A^\infty \), by König’s lemma. Otherwise, \( \alpha \in \bigcup_{s \in [A^{<\omega}]^*, m} \{s \alpha_m\} \), which is countable. Thus \( G \setminus A^\infty \in \Sigma_2^0 \) and \( A^\infty = G \setminus (G \setminus A^\infty) \in \Pi_2^0 \).

7 Examples.

- We have seen examples of subsets \( A \) of \( 2^{<\omega} \) such that \( A^\infty \) is complete for the classes \( \{0\} \), \( \{n^\omega\} \), \( \Delta_1^0 \), \( \Sigma_1^1 \), \( \Pi_1^0 \), \( \Pi_0^1 \) and \( \Sigma_1^1 \). We will give some more examples, for some classes of Borel sets. Notice that to show that a set is such a non self-dual class is complete, it is enough to show that it is true (see 21.E, 22.10 and 22.26 in [K1]).
• For the class $\Sigma_1^0 \oplus \Pi_1^0 := \{(U \cap O) \cup (F \setminus O) / U \in \Sigma_1^0, O \in \Delta_1^0, F \in \Pi_1^0\}$, we can take $A := \{s \in 2^{<\omega}/0^1 \prec s \text{ or } s = 0^2 \text{ or } \exists p \in \omega \ 10^1 \prec s\}$, since $A^\infty = \{0^{\infty}\} \cup \bigcup_{\eta} N_{(0^2)+\eta} \setminus \{10^\infty\}$.

• For the class $\hat{D}_2(\Sigma_1^0) := \{U \cup F / U \in \Sigma_1^0, F \in \Pi_1^0\}$, we can take Example 9 in [St2]: $A := \{s \in 2^{<\omega}/0 < s \text{ or } \exists q \in \omega \ (101)^q 1^3 \prec s \text{ or } s = 10^2\}$. We have

$$A^\infty = \bigcup_{p \in \omega} [N_{(10^2)p} \cup \bigcup_{\eta \in \omega} N_{(10^2)p(101)^\eta}] \cup \{(10^2)^\infty\},$$

which is a $\neg D_2(\Sigma_1^0)$ set. Towards a contradiction, assume that $A^\infty$ is $D_2(\Sigma_1^0)$:

$$A^\infty = U_1 \cap F = U \cup F_2,$$

where the $U$’s are open and the $F$’s are closed. Let $O$ be a clopen set separating $\neg U_1$ from $F_2$ (see 22.C in [K1]). Then $A^\infty = (U \cap O) \cup (F \setminus O)$ would be in $\Sigma_1^0 \oplus \Pi_1^0$. If $(10^2)^\infty \in O$, then we would have $N_{(10^2)p_0} \subseteq O$ for some integer $p_0$. But the sequence $((10^2)^p(1^2)^{p_0}) p \geq p_0 \subseteq O \setminus U$ and tends to $(10^2)^\infty$, which is absurd. If $(10^2)^\infty \notin O$, then we would have $N_{(10^2)p_0} \subseteq \neg O$ for some integer $q_0$. But the sequence $((10^2)^p(101)^{q_0}) q \geq q_0 \subseteq F \setminus O$ and tends to $(10^2)^{q_0}(101)^\infty$, which is absurd.

• For the class $D_2(\Sigma_1^0)$, we can take $A := [A_1^{<\omega}]^* \setminus [A_0^{<\omega}]^*$, where $A_0 := \{010, 01^2\}$ and $A_1 := \{010, 01^2, 0^2, 0^3, 1^2 0^3, 1^2 0^2\}$. We have $A^\infty = A_1^\infty \setminus A_0^\infty$. Indeed, as $A \subseteq [A_1^{<\omega}]^*$, we have $A^\infty \subseteq A_1^\infty$. If $\alpha \in A_0^\infty$, then its decomposition into words of $A_1$ is unique and made of words in $A_0$. Thus $\alpha \notin A^\infty$ and $A^\infty \subseteq A_1^\infty \setminus A_0^\infty$.

Conversely, if $\alpha = a_0 a_1 \ldots \in A_1^\infty \setminus A_0^\infty$, with $a_i \in A_1^\infty$, then there are two cases. Either there are infinitely many indexes $i$ (say $i_0, i_1, \ldots$) such that $a_i \notin A_0$. In this case, the words $a_0 \ldots a_{i_0}, a_{i_0+1} \ldots a_{i_1}, \ldots$ are in $\alpha$ and $\alpha \in A^\infty$. Or there exists a maximal index $i$ such that $a_i \notin A_0$. In this case, $a_0 \ldots a_{i_0}, 1^2 0 \in A$, thus $\alpha \in A^\infty = A_1^\infty \setminus A_0^\infty$. We have $A^\infty = U \cup F$, with $U \in \Sigma_1^0$ and $F \in \Pi_1^0$, then we have $U = \emptyset$ because $A^\infty$ is nowhere dense (every sequence in $A_1$ contains 0, thus the sequences in $A_1^\infty$ have infinitely many 0’s). Thus $A^\infty$ would be closed. But this contradicts the fact that $(10^2)^\infty \notin A^\infty$ and tends to $(10^2)^\infty \notin A^\infty$. Thus $A^\infty$ is a true $D_2(\Sigma_1^0)$ set.

• For the class $\hat{D}_3(\Sigma_1^0)$, we can take $A := ([A_2^{<\omega}]^* \setminus [A_1^{<\omega}]^*) \cup [A_0^{<\omega}]^*$, where $A_0 := \{0^2\}$, $A_1 := \{0^2, 0^1\}$, and $A_2 := \{0^2, 0^1, 1^2, 10, 1^2 0^2\}$. We have $A^\infty = (A_2^\infty \setminus A_1^\infty) \cup A_0^\infty$. Indeed, as $A \subseteq [A_2^{<\omega}]^*$, we have $A^\infty \subseteq A_2^\infty$. If $\alpha \in A_2^\infty$, then its decomposition into words of $A_2^\infty$ is unique and made of words in $A_1$. If moreover $\alpha \notin A_0^\infty$, then it is clear that $\alpha \notin A^\infty$ and $A^\infty \subseteq (A_2^\infty \setminus A_1^\infty) \cup A_0^\infty$.

Conversely, it is clear that $A_1^\infty \subseteq A^\infty$. If $\alpha = a_0 a_1 \ldots \in A_2^\infty \setminus A_1^\infty$, then the argument above still works. We have to check that $s := a_0 \ldots a_{i_0} \notin [A_1^{<\omega}]^*$. It is clear if $a_{i_0} = 10$. Otherwise, $a_{i_0} = 10^2$ and we argue by contradiction.
The length of \( s \) is even and the decomposition of \( s \) into words of \( A_1 \) is unique. It finishes with \( 0^2 \), and the even coordinates of the sequence \( s \) are 0. Therefore, \( a_{i_0 - 2}, a_{i_0 - 3}, \ldots \) Because of the parity, some 0 remains at the beginning. But this is absurd. Now we have to check that \( a_0 \ldots a_i 0 \notin \left[ A_1^{<\omega} \right]^* \). It is clear if \( a_i = 10^2 \). Otherwise, \( a_i = 10 \) and the argument above works.

Finally, we have to check that if \( \gamma \in A_1^{<\omega} \), then \( \gamma - (0) \in A^{<\omega} \). There is a sequence \( p_0, p_1, \ldots \), finite or not, such that \( \gamma = (0^{2p_0})(01)(0^{2p_1})(01)\ldots 0^{<\omega} \). Therefore

\[
\gamma - (0) = (0^{2p_0}10)(0^{2p_1}10)\ldots (0^2)^{<\omega} \in A^{<\omega}.
\]

If we set \( U_i := \neg A_2^{<\omega} \), then we see that \( A^{<\omega} \in D_3(\Sigma_1^0) \). If \( \alpha \) finishes with \( 1^{<\omega} \), then \( \alpha \notin A_2^{<\omega} \); thus \( A_2^{<\omega} \) is nowhere dense, just like \( A^{<\omega} \). Thus if \( A^{<\omega} = (U_2 \setminus U_1) \cup U_0 \) with \( U_1 \) open, then \( U_0 = \emptyset \). By uniqueness of the decomposition of a sentence in \( A_1^{<\omega} \) into words of \( A_{i+1} \), we see that \( A_1^{<\omega} \) is nowhere dense in \( A_1^{<\omega} \). So let \( x_0 \in A_1^{<\omega} \), \( (x_n) \subseteq A_1^{<\omega} \setminus A_0^{<\omega} \) converging to \( x_0 \), and \( (x_{n,m}) \subseteq A_2^{<\omega} \setminus A_1^{<\omega} \) converging to \( x_n \). Then \( x_{n,m} \in U_1 \), which is absurd. Thus \( A^{<\omega} \notin D_3(\Sigma_1^0) \).

- For the class \( D_2(\Sigma_0^0) \), we can take \( A := \{ s \in 2^{<\omega} / 1^2 < s \) or \( s = (0) \} \). We can write

\[
A^{<\omega} = (\{0^\omega\} \cup \bigcup_{P_f} N_{\{p_{12}\}}) \cap (P_f \cup \{ \alpha \in 2^{<\omega} / \forall n \exists m \alpha(m) = \alpha(m+1) = 1 \}).
\]

Then \( A^{<\omega} \notin D_2(\Sigma_0^0) \), otherwise \( A^{<\omega} \cap N_{12} \in D_2(\Sigma_0^0) \) and would be a comeager subset of \( N_{12} \). We could find \( s \in 2^{<\omega} \) with even length such that \( A^{<\omega} \cap N_{12,s} \in \Pi_3^0 \). We define a continuous function \( f : 2^{<\omega} \to 2^{<\omega} \) by formulas \( f(\alpha)(2n) := \alpha(n) \) if \( n > \frac{|\alpha|+1}{2} \), \( (2^s)(2n) \) otherwise, and \( f(\alpha)(2n+1) := 0 \) if \( n > \frac{|\alpha|}{2} \), \( (2^s)(2n+1) \) otherwise. It reduces \( P_f \) to \( A^{<\omega} \cap N_{12,s} \), which is absurd.
Summary of the complexity results in this paper:

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8 References.


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