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Abstract. Let $\xi \geq 1$ be a countable ordinal. We study the Borel subsets of the plane that can be made $\mathbf{\Pi}^0_\xi$ by refining the Polish topology on the real line. These sets are called potentially $\mathbf{\Pi}^0_\xi$. We give a Hurewicz-like test to recognize potentially $\mathbf{\Pi}^0_\xi$ sets.

1 Preliminaries in dimension one.

Let us recall some results in dimension one before studying Borel subsets of the plane. In descriptive set theory, a standard way to see that a set is complicated is to note that it is more complicated than a well-known example. For instance, we have the following result (see [SR]):

Theorem 1 *(Hurewicz) Let $P_f := \{\alpha \in 2^{\mathbb{N}} / \exists n \in \mathbb{N} \ \forall m \geq n \ \ \alpha(m) = 0\}$, X be a Polish space, and A a Borel subset of X. Then exactly one of the following holds:*
(a) The set A is $\mathbf{\Pi}^0_2(X)$.
(b) There is $u : 2^{\mathbb{N}} \to X$ continuous and one-to-one with $P_f = u^{-1}(A)$.

This result has been generalized to the other Baire classes (see [Lo-SR]). We state this generalization in two parts:

Theorem 2 *(Louveau-Saint Raymond) Let $\xi < \aleph_1$, $A_{1+\xi} \in \mathbf{\Sigma}^0_{1+\xi}(2^{\mathbb{N}})$, X be a Polish space, and A, B disjoint analytic subsets of X. One of the following holds:*
(a) The set A is separable from B by a $\mathbf{\Pi}^0_{1+\xi}(X)$ set.
(b) There is $u : 2^{\mathbb{N}} \to X$ continuous with $A_{1+\xi} \subseteq u^{-1}(A)$ and $2^{\mathbb{N}} \setminus A_{1+\xi} \subseteq u^{-1}(B)$.

If we moreover assume that $A_{1+\xi} \notin \mathbf{\Pi}^0_{1+\xi}$, then this is a dichotomy (in this case, and if $\xi \geq 2$, then we can have u one-to-one).

Theorem 3 *There is a concrete example of $A_{1+\xi} \in \mathbf{\Sigma}^0_{1+\xi}(2^{\mathbb{N}}) \setminus \mathbf{\Pi}^0_{1+\xi}(2^{\mathbb{N}})$, for $\xi < \aleph_1$.*

If we replace P_f (resp., $\mathbf{\Pi}^0_2$) with the set $A_{1+\xi}$ given by Theorem 3 (resp., $\mathbf{\Pi}^0_{1+\xi}$), then we get the generalization of Theorem 1 for $\xi \geq 2$. We state this generalization in two parts for the following reasons:

• Theorem 2 is valid for any $A_{1+\xi} \in \mathbf{\Sigma}^0_{1+\xi}(2^{\mathbb{N}})$, and Theorem 1 is of the form "There is a typical example such that...".

• We will meet again a statement in two parts, in dimension two.

2 Results with the usual notions of reduction.

Let us consider the case of dimension two. The usual notion of comparison for Borel equivalence relations is the Borel reducibility quasi-order (recall that a quasi-order is a reflexive and transitive relation). This means that if X (resp., Y) is a Polish space, and E (resp., F) a Borel equivalence relation on X (resp., Y), then

$$E \leq_B F \;\Leftrightarrow\; \exists u : X \to Y \text{ Borel with } E = (u \times u)^{-1}(F).$$

Note that this makes sense even if E, F are not equivalence relations. We will study a natural invariant for $\leq_B$. Recall the following (see [K]):

Theorem 4 *(Kuratowski) Let X be a Polish space, and (B_n) a sequence of Borel subsets of X. Then there is a finer Polish topology σ on X (and thus having the same Borel sets) making the B_n's clopen.*

In particular, if $u : X \to Y$ is Borel, then there is σ such that $u : [X, \sigma] \to Y$ is continuous. If moreover $E = (u \times u)^{-1}(F)$ and F is in some Baire class Γ, then $E \in \Gamma([X, \sigma]^2)$. This leads to Definition 5, that can be found in [Lo]:

Definition 5 *(Louveau) Let X, Y be Polish spaces, A a Borel subset of $X \times Y$ and Γ a Baire (or Wadge) class. We say that A is $potentially\ in\ \Gamma$ (denoted $A \in pot(\Gamma)$) iff there is a finer Polish topology σ (resp., τ) on X (resp., Y) such that A is in $\Gamma([X, \sigma] \times [Y, \tau])$.*

The previous result shows that this notion makes sense for product topologies. This notion is a natural invariant for $\leq_B$: if F is pot(Γ) and $E \leq_B F$, then E is pot(Γ). Using this notion, A. Louveau showed that the collection of $\mathbf{\Sigma}^0_\xi$ equivalence relations is not cofinal for $\leq_B$, and deduces from this the non existence of a maximum Borel equivalence relation for $\leq_B$ (this non existence result is due to H. Friedman and L. Stanley). A. Louveau has also more recently noticed that one can associate a quasi-order relation $R_A \subseteq (X \times 2)^2$ to $A \subseteq X^2$ as follows:

$$(x, i)\ R_A\ (y, j) \;\Leftrightarrow\; (x, i) = (y, j) \text{ or } [(x, y) \in A \text{ and } (i, j) = (0, 1)].$$

Using this, one can see that, from the point of view of Borel reducibility, the study of Borel quasi-orders is essentially the study of arbitrary Borel subsets of the plane. This strengthens the motivation for studying arbitrary Borel subsets of the plane, from the point of view of potential complexity. We have a result concerning equivalence relations (see [H-K-Lo]):

Theorem 6 *(Harrington-Kechris-Louveau) Let X be a Polish space, E a Borel equivalence relation on X, and $E_0 := \{(\alpha, \beta) \in 2^{\mathbb{N}} \times 2^{\mathbb{N}} / \exists n \in \mathbb{N}\ \forall m \geq n\ \alpha(m) = \beta(m)\}$. Then exactly one of the following holds:*

(a) The relation E is pot$(\mathbf{\Pi}^0_1)$.

(b) $E_0 \leq_B E$ (with u continuous and one-to-one).

We will study other structures than equivalence relations (for example quasi-orders), and even arbitrary Borel subsets of the plane. We need some other notions of comparison. Let X, Y, X', Y' be Polish spaces, and A (resp., A') a Borel subset of $X \times Y$ (resp., $X' \times Y'$). We set

$$A \leq^r_B A' \;\Leftrightarrow\; \exists u : X \to X'\ \exists v : Y \to Y' \text{ Borel with } A = (u \times v)^{-1}(A').$$

We want to extend the previous result to arbitrary Borel subsets of the plane. This works partially (see [L1]):

Theorem 7 *Let $\Delta(2^{\mathbb{N}}) := \{(\alpha,\beta) \in 2^{\mathbb{N}} \times 2^{\mathbb{N}} / \alpha = \beta\}$, $L_0 := \{(\alpha,\beta) \in 2^{\mathbb{N}} \times 2^{\mathbb{N}} / \alpha <_{lex} \beta\}$, X, Y be Polish spaces, and A a pot$(\check{D}_2(\mathbf{\Sigma}^0_1))$ subset of $X \times Y$. Then exactly one of the following holds:*
(a) The set A is pot$(\mathbf{\Pi}^0_1)$.
(b) $\neg\Delta(2^{\mathbb{N}}) \leq^r_B A$ or $L_0 \leq^r_B A$ (with u, v continuous and one-to-one).

Things become more complicated at the level $D_2(\mathbf{\Sigma}^0_1)$ (differences of two open sets; $\check{D}_2(\mathbf{\Sigma}^0_1)$ is the dual Wadge class of unions of a closed set and of an open set).

Theorem 8 *(a) There is a perfect $\leq^r_B$-antichain $(A_\alpha)_{\alpha\in 2^{\mathbb{N}}} \subseteq D_2(\mathbf{\Sigma}^0_1)(2^{\mathbb{N}} \times 2^{\mathbb{N}})$ such that A_α is $\leq^r_B$-minimal among $\mathbf{\Delta}^1_1 \setminus$pot$(\mathbf{\Pi}^0_1)$ sets, for any $\alpha \in 2^{\mathbb{N}}$.*
(b) There is a perfect $\leq_B$-antichain $(R_\alpha)_{\alpha\in 2^{\mathbb{N}}}$ such that R_α is $\leq_B$-minimal among $\mathbf{\Delta}^1_1 \setminus$pot$(\mathbf{\Pi}^0_1)$ sets, for any $\alpha \in 2^{\mathbb{N}}$. Moreover, $(R_\alpha)_{\alpha\in 2^{\mathbb{N}}}$ can be taken to be a subclass of any of the following classes:

- Graphs (i.e., irreflexive and symmetric relations).

- Oriented graphs (i.e., irreflexive and antisymmetric relations).

- Quasi-orders.

- Partial orders (i.e., reflexive, antisymmetric and transitive relations).

In other words, the case of equivalence relations, for which we have a unique (up to bi-reducibility) minimal non potentially closed element with Theorem 6, is very specific. Theorem 8.(b) says, among other things, that the mixture between symmetry and transitivity is very strong.

Example. Let us specify the construction of the antichain in (a). We set, for $C \subseteq 2^{<\mathbb{N}}$,

$$A^C := \{(s0\gamma, s1\gamma) / s \in C, \gamma \in 2^{\mathbb{N}}\}.$$

If $0 \in S \subseteq \mathbb{N}$ is infinite, then set $C_S := \{t \in 2^{<\mathbb{N}} / \text{Card}(t) \in S\}$ (where Card(t) is the number of ones in t). Such an S is of the form $S_\beta := \{\Sigma_{i<j}\ (1+\beta(j)) / j \in \mathbb{N}\}$, where $\beta \in \mathbb{N}^{\mathbb{N}}$.

Theorem 9 *The set A^{C_S} is minimal for $[\mathbf{\Delta}^1_1 \setminus$pot$(\mathbf{\Pi}^0_1), \leq^r_B]$ if*

$$\forall p \in \mathbb{N}\ \ \exists k \in \mathbb{N}\ \ \forall q \in \mathbb{N}\ \ \exists c \in \mathbb{N} \cap [q, q+k]\ \ c + (S \cap [0,p]) = S \cap (c + [0,p]).$$

It remains to define $\beta_\alpha \in 2^{\mathbb{N}} \subseteq \mathbb{N}^{\mathbb{N}}$, for $\alpha \in 2^{\mathbb{N}}$. We inductively define a sequence $(s_{\alpha,n})_n \subseteq 2^{<\mathbb{N}}$ as follows: $s_{\alpha,0} := 0$, $s_{\alpha,1} := 1$, $s_{\alpha,n+2} := s_{\alpha,n}^{\alpha(n)+1} s_{\alpha,n+1}^{\alpha(n+1)+1}$. Note that $s_{\alpha,n} \prec_{\neq} s_{\alpha,n+2}$, so that we can define $\beta_\alpha := \lim_{n\to\infty} s_{\alpha,2n} \in 2^{\mathbb{N}}$. It is suitable: $(A^{C_{S_{\beta_\alpha}}})_{\alpha\in 2^{\mathbb{N}}}$ is a perfect antichain made of minimal sets for $[\mathbf{\Delta}^1_1 \setminus \text{pot}(\mathbf{\Pi}^0_1), \leq^r_B]$.

3 Reduction by homomorphism.

Theorem 8.(a) shows that the classical notions of reduction (on the whole product) don't work, at least at the first level. So we must find another notion of comparison.

We have a positive result with another notion, which is in some sense "half of the Borel reducibility ordering". Let A (resp., A') be an analytic subset of $X\times X$ (resp., $X'\times X'$). We set

$$(X,A)\preceq_B(X',A') \;\Leftrightarrow\; \exists u\colon X\to X' \text{ Borel with } A\subseteq(u\times u)^{-1}(A').$$

This notion essentially makes sense for irreflexive relations (we can take u to be constant if A' is not irreflexive).

Notation. Let $\psi:\mathbb{N}\to 2^{<\mathbb{N}}$ be the natural bijection (i.e., $\psi(0)=\emptyset$, $\psi(1)=0$, $\psi(2)=1$, $\psi(3)=0^2$, $\psi(4)=01$, $\psi(5)=10$, $\psi(6)=1^2$, ...). Note that $|\psi(n)|\leq n$, so that we can define

$$s_n:=\psi(n)0^{n-|\psi(n)|}.$$

The crucial properties of (s_n) are that it is dense (there is n such that $t\prec s_n$, for each $t\in 2^{<\omega}$), and that $|s_n|=n$. We put

$$A_0:=A^{\{s_n/n\in\mathbb{N}\}}=\{(s_n0\gamma,s_n1\gamma)/n\in\mathbb{N},\gamma\in 2^{\mathbb{N}}\}.$$

The symmetric set $s(A_0)$ generated by A_0 is considered in [K-S-T], where the following is essentially proved:

Theorem 10 *(Kechris, Solecki, Todorčević) Let X be a Polish space, and A an analytic subset of $X\times X$. Then exactly one of the following holds:*
(a) $(X,A)\preceq_B(\mathbb{N},\neq)$.
(b) $(2^{\mathbb{N}},A_0)\preceq_B(X,A)$ *(with u continuous).*

In [K-S-T], it is conjectured that we can have u one-to-one in Theorem 10.(b). This is not the case.

4 Reduction on a closed set.

As a consequence of Theorem 10, we have the following:

Theorem 11 *Let X, Y be Polish spaces, and A a Borel subset of $X\times Y$. Then exactly one of the following holds:*
(a) The set A is pot($\mathbf{\Pi}^0_1$).
(b) There are $u:2^{\mathbb{N}}\to X$ and $v:2^{\mathbb{N}}\to Y$ continuous with $A_0=(u\times v)^{-1}(A)\cap\overline{A_0}$.
Moreover, we can neither ensure that u and v are one-to-one, nor remove $\overline{A_0}$.

So we get a minimum non-potentially closed set if we do not ask for a reduction on the whole product. To generalize Theorem 11, the right way to see A_0 seems to be the following. Let T_0 be the tree associated with $\overline{A_0}=A_0\cup\Delta(2^{\mathbb{N}})$:

$$T_0=\{(s,t)\in 2^{<\mathbb{N}}\times 2^{<\mathbb{N}}/s=t \text{ or } \exists n\in\mathbb{N}\,\exists w\in 2^{<\mathbb{N}}\;(s,t)=(s_n0w,s_n1w)\}.$$

The map $\Delta : 2^{\mathbb{N}} \times 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is the symmetric difference: $\Delta(\alpha, \beta)(i) := (\alpha \Delta \beta)(i) = 1$ exactly when $\alpha(i) \neq \beta(i)$, for $i \in \mathbb{N}$. Let $S_1 := \{\gamma \in 2^{\mathbb{N}} / \exists i \in \mathbb{N}\ \gamma(i) = 1\}$ be the typical one-dimensional $\mathbf{\Sigma}^0_1 \setminus \mathbf{\Pi}^0_1$ set. We have

$$A_0 = \{(\alpha, \beta) \in 2^{\mathbb{N}} \times 2^{\mathbb{N}} / (\alpha, \beta) \in [T_0] \text{ and } \alpha \Delta \beta \in S_1\}.$$

This scheme can be generalized. Theorem 11 shows that we cannot have only one minimal non potentially closed set, for the reduction on the whole product. The reduction is possible on a closed set (the closure of A_0). This closure does not explain why we cannot have a reduction on the whole product. This comes from Theorem 8. The orthogonality between the examples appearing in the antichains of its statement comes from different types of cycles. This will give a better explanation than the closure. We will replace the closure with a closed set, that will be seen as the set of branches of some tree on 2×2. This tree will have the acyclicity properties that we need. This leads to the following definition:

Definition 12 *Let R be a relation on a set E.*

• *An R-$path$ is a finite sequence $(e_i)_{i \leq n} \subseteq E$ such that $(e_i, e_{i+1}) \in R$, for $i < n$.*

• *An R-$cycle$ is an R-path $(e_i)_{i \leq n}$ such that $n \geq 3$ and*

$$[0 \leq i \neq j \leq n \text{ and } e_i = e_j] \ \Leftrightarrow\ \{i, j\} = \{0, n\}.$$

• *We say that R is $acyclic$ if there is no R-cycle.*

• *We say that a tree T on 2×2 is $uniformly\ acyclic$ if, for each $p > 0$,*

(a) The relation $T \cap (2^p \times 2^p)$ is irreflexive and antisymmetric.

(b) The symmetric relation $s(T \cap (2^p \times 2^p))$ generated by $T \cap (2^p \times 2^p)$ is acyclic.

The main new results in this paper are the following:

Theorem 13 *(Debs-Lecomte) Let T be a uniformly acyclic tree, $\xi < \aleph_1$, $A_{1+\xi}$ in $\mathbf{\Sigma}^0_{1+\xi}([T])$, X, Y Polish spaces, and A, B disjoint analytic subsets of $X \times Y$. Then one of the following holds:*

(a) The set A is separable from B by a pot$(\mathbf{\Pi}^0_{1+\xi})$ set.

(b) There are $u : 2^{\mathbb{N}} \to X$ and $v : 2^{\mathbb{N}} \to Y$ continuous such that the inclusions $A_{1+\xi} \subseteq (u \times v)^{-1}(A)$ and $[T] \setminus A_{1+\xi} \subseteq (u \times v)^{-1}(B)$ hold.

If we moreover assume that $A_{1+\xi} \notin$ pot$(\mathbf{\Pi}^0_{1+\xi})$, then this is a dichotomy.

This result has initially been shown by D. Lecomte when $1+\xi$ is a successor ordinal. Then G. Debs proved it when $1+\xi$ is a limit ordinal. The proof of Theorem 13 uses the representation Theorem for Borel sets in [D-SR]. Notice that we can deduce Theorem 2 from the proof of Theorem 13. Theorem 13 is the analog of Theorem 2 in dimension two (see [Lo-SR], also Theorem III-2.1 in [D-SR]). The tree T has to be small enough, since there is no possibility to have a reduction on the whole product. But as the same time, T has to be big enough to ensure the existence of complicated sets inside $[T]$:

Theorem 14 *There are concrete examples of:*

(a) A uniformly acyclic tree T.

(b) A set $A_{1+\xi} \in \mathbf{\Sigma}^0_{1+\xi}([T]) \setminus$ pot$(\mathbf{\Pi}^0_{1+\xi})$, for $\xi < \aleph_1$.

This result is the complement of Theorem 13 (which is true with $T := \emptyset$!). Again, the couple Theorems 13-14 is the analog of the couple Theorems 2-3.

5 The examples.

Let us specify the examples of Theorem 14. Let $\varphi=(\varphi_0,\varphi_1):\mathbb{N}\to\mathbb{N}^2$ be the natural bijection. More precisely, we set, for $q\in\mathbb{N}$,

$$M(q):=\max\{m\in\mathbb{N}/\Sigma_{k\leq m}\ k\leq q\}.$$

Then we define $\varphi(q)=(\varphi_0(q),\varphi_1(q)):=(M(q)-q+(\Sigma_{k\leq M(q)}\ k),q-(\Sigma_{k\leq M(q)}\ k))$. One can check that $<i,j>:=\varphi^{-1}(i,j)=(\Sigma_{k\leq i+j}\ k)+j$. More concretely, we get

$$\varphi[\mathbb{N}]=\{(0,0);(1,0);(0,1);(2,0);(1,1);(0,2);\ldots\}.$$

Definition 15 *We say that $E\subseteq\bigcup_{q\in\mathbb{N}}\ 2^q\times 2^q$ is a test if*

(a) $\forall q\in\mathbb{N}\ \ \exists!(s_q,t_q)\in E\cap(2^q\times 2^q)$.

(b) $\forall m,q\in\mathbb{N}\ \ \forall u\in 2^{<\mathbb{N}}\ \ \exists v\in 2^{<\mathbb{N}}\ \ (s_q0uv,t_q1uv)\in E$ *and* $\varphi_0(|t_q1uv|-1)=m$.

(c) $\forall n>0\ \ \exists q<n\ \ \exists w\in 2^{<\mathbb{N}}\ \ s_n=s_q0w$ *and* $t_n=t_q1w$.

We will call T the tree generated by a test $E=\{(s_q,t_q)/q\in\mathbb{N}\}$:

$$T:=\{(s,t)\in 2^{<\mathbb{N}}\times 2^{<\mathbb{N}}/s=t=\emptyset\ \ or\ \ \exists q\in\mathbb{N}\ \ \exists w\in 2^{<\mathbb{N}}\ \ s=s_q0w\ \ and\ \ t=t_q1w\}.$$

One can show the existence of a test, and that T is uniformly acyclic. The uniqueness condition in (a) and condition (c) ensure that T is small enough, and also the acyclicity. The existence condition in (a) and condition (b) ensure that T is big enough. More specifically, if X is a Polish space and σ a finer Polish topology on X, then there is a dense G_δ subset of X on which the two topologies coincide. The first part of condition (b) ensures the possibility to get inside the square of a dense G_δ subset of 2^ω. The examples of Theorem 14.(b) are constructed using the examples in [Lo-SR]. Conditions on the verticals appear, and the second part of condition (b) gives a control on the choice of verticals.

Notation. In [Lo-SR], Lemma 3.3, the map $\rho_0:2^{\mathbb{N}}\to 2^{\mathbb{N}}$ defined as follows is introduced:

$$\rho_0(\varepsilon)(i):=\begin{cases}1\text{ if }\varepsilon(<i,j>)=0,\text{ for each }j\in\mathbb{N},\\ 0\text{ otherwise.}\end{cases}$$

In this paper, $\rho_0^\xi:2^{\mathbb{N}}\to 2^{\mathbb{N}}$ is also defined for $\xi<\aleph_1$ as follows, by induction on ξ (see the proof of Theorem 3.2). We put $\rho_0^0:=\mathrm{Id}_{2^{\mathbb{N}}}$, $\rho_0^{\eta+1}:=\rho_0\circ\rho_0^\eta$. If $\lambda>0$ is limit, then fix $(\xi_k^\lambda)\subseteq\lambda\setminus\{0\}$ such that

$$\Sigma_k\ \xi_k^\lambda=\lambda.$$

For $\varepsilon\in 2^{\mathbb{N}}$ and $k\in\mathbb{N}$, we define $(\varepsilon)^k\in 2^{\mathbb{N}}$ by $(\varepsilon)^k(i):=\varepsilon(i+k)$. We also define $\rho_0^{(k,k+1)}:2^{\mathbb{N}}\to 2^{\mathbb{N}}$ by

$$\rho_0^{(k,k+1)}(\varepsilon)(i):=\begin{cases}\varepsilon(i)\text{ if }i<k,\\ \rho_0^{\xi_k^\lambda}((\varepsilon)^k)(i-k)\text{ if }i\geq k.\end{cases}$$

We set $\rho_0^{(0,k+1)}:=\rho_0^{(k,k+1)}\circ\rho_0^{(k-1,k)}\circ\ldots\circ\rho_0^{(0,1)}$ and $\rho_0^\lambda(\varepsilon)(k):=\rho_0^{(0,k+1)}(\varepsilon)(k)$.

The set $H_{1+\xi}:=(\rho_0^\xi)^{-1}(\{0^\infty\})$ is also introduced, and the authors show that $H_{1+\xi}$ is $\mathbf{\Pi}^0_{1+\xi}\setminus\mathbf{\Sigma}^0_{1+\xi}$ (see Theorem 3.2).

• The map $S : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is the shift map: $S(\alpha)(i) := \alpha(i+1)$.

• Let T be the tree generated by a test. We put, for $\xi < \aleph_1$,

$$A_{1+\xi} := \{(\alpha,\beta) \in 2^{\mathbb{N}} \times 2^{\mathbb{N}} / (\alpha,\beta) \in [T] \text{ and } S(\alpha \Delta \beta) \notin H_{1+\xi}\}.$$

Then $A_{1+\xi}$ is $\mathbf{\Sigma}^0_{1+\xi}([T]) \setminus \text{pot}(\mathbf{\Pi}^0_{1+\xi})$. We introduce a notation to state the crucial Lemma used to show it:

Notation. We define $p : \mathbb{N}^{<\mathbb{N}} \setminus \{\emptyset\} \to \mathbb{N}$. We actually define $p(s)$ by induction on $|s|$:

$$p(s) := \begin{cases} s(0) \text{ if } |s| = 1, \\ < p(s\lceil(|s|-1)), s(|s|-1) > \text{ otherwise.} \end{cases}$$

Notice that $p|_{\mathbb{N}^n} : \mathbb{N}^n \to \mathbb{N}$ is a bijection for each $n \geq 1$.

Lemma 16 *Let G be a dense G_δ subset of $2^{\mathbb{N}}$. Then there are $\alpha_0 \in G$ and $f : 2^{\mathbb{N}} \to G$ continuous such that, for each $\alpha \in 2^{\mathbb{N}}$,*

(a) $(\alpha_0, f(\alpha)) \in [T]$.

(b) For each $t \in \mathbb{N}^{<\mathbb{N}}$, and each $m \in \mathbb{N}$,

(i) $\alpha(p(tm)) = 1 \ \Rightarrow \ \exists m' \in \mathbb{N} \ \ (\alpha_0 \Delta f(\alpha))(p(tm')+1) = 1$.

(ii) $(\alpha_0 \Delta f(\alpha))(p(tm)+1) = 1 \ \Rightarrow \ \exists m' \in \mathbb{N} \ \ \alpha(p(tm')) = 1$.

6 Complements of the main results.

Now we come to consequences of Theorems 13 and 14. To state them, we need some more notation. We use some tools from effective descriptive set theory (the reader should see [M] for basic notions about it).

Notation. Let X be a recursively presented Polish space. We denote by Δ_X the topology on X generated by $\Delta^1_1(X)$. This topology is Polish (see the proof of Theorem 3.4 in [Lo]). We set

$$\tau_1 := \Delta_X \times \Delta_Y$$

if Y is also a recursively presented Polish space.

• Let $2 \leq \xi < \omega_1^{\text{CK}}$. The topology τ_ξ is generated by $\Sigma^1_1(X \times Y) \cap \mathbf{\Pi}^0_{<\xi}(\tau_1)$. Note that

$$\mathbf{\Sigma}^0_1(\tau_\xi) \subseteq \mathbf{\Sigma}^0_\xi(\tau_1),$$

so that $\mathbf{\Pi}^0_1(\tau_\xi) \subseteq \mathbf{\Pi}^0_\xi(\tau_1)$.

• Recall the existence of Π^1_1 sets $W^X \subseteq \mathbb{N}$, $C^X \subseteq \mathbb{N} \times X$ with $\Delta^1_1(X) = \{C^X_n / n \in W^X\}$,

$$\{(n,x) \in \mathbb{N} \times X / n \in W^X \text{and } x \notin C^X_n\} \in \Pi^1_1(\mathbb{N} \times X)$$

(see [H-K-Lo], Theorem 3.3.1).

• Set $\mathrm{pot}(\mathbf{\Pi}^0_0) := \mathbf{\Delta}^1_1(X)\times\mathbf{\Delta}^1_1(Y)$ and, for $\xi<\omega_1^{\mathrm{CK}}$,

$$W_\xi^{X\times Y} := \{p\in W^{X\times Y}/C_p^{X\times Y}\in \mathrm{pot}(\mathbf{\Pi}^0_\xi)\}.$$

We also set $W_{<\xi}^{X\times Y} := \bigcup_{\eta<\xi} W_\eta^{X\times Y}$.

Theorem 17 *(Debs-Lecomte-Louveau) Let T given by Theorem 14, $\xi<\omega_1^{CK}$, $A_{1+\xi}$ given by Theorem 14, and X, Y be recursively presented Polish spaces.*

(1) Let A, B be disjoint Σ^1_1 subsets of $X\times Y$. The following are equivalent:

(a) The set A cannot be separated from B by a pot$(\mathbf{\Pi}^0_{1+\xi})$ set.

(b) The set A cannot be separated from B by a $\Delta^1_1\cap$ pot$(\mathbf{\Pi}^0_{1+\xi})$ set.

(c) The set A cannot be separated from B by a $\mathbf{\Pi}^0_{1+\xi}(\tau_1)$ set.

(d) $\overline{A}^{\tau_{1+\xi}}\cap B\neq\emptyset$.

(e) There are $u:2^{\mathbb{N}}\to X$ and $v:2^{\mathbb{N}}\to Y$ continuous such that the inclusions $A_{1+\xi}\subseteq(u\times v)^{-1}(A)$ and $[T]\setminus A_{1+\xi}\subseteq(u\times v)^{-1}(B)$ hold.

(2) The sets $W_{1+\xi}^{X\times Y}$ and $W_{<1+\xi}^{X\times Y}$ are Π^1_1.

The equivalence between (a), (b) and (c), and also (2), are proved in [Lo]. We can assume this equivalence and (2), then prove Theorems 13, 14, and then prove Theorem 17. We can also prove directly Theorem 17 by induction on ξ. An immediate consequence of this is the following, proved in [Lo]:

Corollary 18 *(Louveau) Let $\xi<\omega_1^{\mathrm{CK}}$, X, Y be recursively presented Polish spaces, and A a Δ^1_1 subset of $X\times Y$. The following are equivalent:*

(a) The set A is pot$(\mathbf{\Pi}^0_{1+\xi})$.

(b) The set A is $\mathbf{\Pi}^0_{1+\xi}(\tau_1)$.

We also have the following consequence of Theorems 13 and 14:

Corollary 19 *(Debs-Lecomte) Let $\xi<\aleph_1$. There is a Borel subset $A_{1+\xi}$ of $2^{\mathbb{N}}\times 2^{\mathbb{N}}$ such that for any Polish spaces X, Y, and for any disjoint analytic subsets A, B of $X\times Y$, exactly one of the following holds:*

(a) The set A is separable from B by a pot$(\mathbf{\Pi}^0_{1+\xi})$ set.

(b) There are $u:2^{\mathbb{N}}\to X$ and $v:2^{\mathbb{N}}\to Y$ continuous such that the inclusions $A_{1+\xi}\subseteq(u\times v)^{-1}(A)$ and $\overline{A_{1+\xi}}\setminus A_{1+\xi}\subseteq(u\times v)^{-1}(B)$ hold.

Moreover we can neither ensure that u and v are one-to-one if $\xi\leq 1$, nor replace $\overline{A_{1+\xi}}\setminus A_{1+\xi}$ with $(2^{\mathbb{N}}\times 2^{\mathbb{N}})\setminus A_{1+\xi}$.

The one-to-one complement is due to D. Lecomte (see Theorem 11 when $\xi=0$, and Theorem 15 in [L2] when $\xi=1$). The latter complement has initially been shown by D. Lecomte when $\xi\leq 1$ (see for example Theorem 11). Then G. Debs found a simpler proof, which moreover works in the general case.

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