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Collisions and their Catenations: Ultimately Periodic Tilings of the Plane

Nicolas Ollinger and Gaétan Richard

Laboratoire d'informatique fondamentale de Marseille (LIF)
Aix-Marseille Université, CNRS,
39 rue Joliot-Curie, 13 013 Marseille, France,
`Gaetan.Richard@lif.univ-mrs.fr`

Abstract. Motivated by the study of cellular automata algorithmics and dynamics, we investigate an extension of ultimately periodic words to two-dimensional infinite words: collisions. A natural composition operation on tilings leads to a catenation operation on collisions. By existence of aperiodic tile sets, ultimately periodic tilings of the plane cannot generate all possible tilings but exhibit some useful properties of their one-dimensional counterparts: ultimately periodic tilings are recursive, very regular, and tiling constraints are easy to preserve by catenation. We show that, for a given catenation scheme of finitely many collisions, the generated set of collisions is semi-linear.

Introduction

The theory of regular languages, sets of one-dimensional sequences of letters sharing some regularities, has been well studied since the sixties. Finite state machines [17], regular languages [13], computing devices with bounded memory, monadic second-order logic [4]: various point of views lead to a same robust notion of regular languages. The concept extends to infinite words and various other one-dimensional structures. Unfortunately, when considering two-dimensional words – partial mappings from the plane \mathbb{Z}^2 to a finite alphabet – such a robust common object fails to emerge: automata on the plane, picture languages, second-order logic, all lead to different notions of regular languages [8]. A first difficulty arises from the definition of a finite word: should it be any partial mapping with a finite support? Should it be rectangles filled with letters? Should it be any mapping with a connected support for some particular connectivity notion? A second difficulty arises from the complexity of two-dimensional patterns: in the simplest case of uniform local constraints, *i.e.* *tilings*, knowing whether a given finite pattern is a factor of a valid tiling (of the whole plane) is already undecidable [1].

In the present paper, we investigate a particular family of recursive tilings of the plane endowed with a catenation operation. Our definition of an ultimately periodic tiling, a *collision*, is inspired by geometrical considerations on

one-dimensional cellular automata space-time diagrams and tilings. It can be thought of as an extension of the notion of ultimately periodic biinfinite words to two-dimensional words. As we will show in this paper, these objects provide a convenient tool to describe synchronization problems in cellular automata algorithms.

One-dimensional cellular automata [12] are dynamical systems whose configurations consist of biinfinite words on a given finite alphabet. The system evolves by applying uniformly and synchronously a locally defined transition rule. The value at each position, or *cell*, of a configuration only depends on the values of the cells on its neighborhood at the previous time step. To discuss the dynamics or to describe algorithmic constructions, it is often convenient to consider space-time diagrams rather than configurations. A space-time diagram is a drawing of a particular orbit of the system: configurations are depicted one on top of the other, from bottom to top, by successively applying the transition rule, as depicted on Fig. 1. This representation permits to draw away the timeline and discuss the structure of emerging two-dimensional patterns. Formally, this is equivalent to consider tilings of half the plane with a special kind of local constraint, oriented by the timeline.



Time goes from bottom to top. Each letter is represented by a different color.

Fig. 1. space-time diagram of a one-dimensional cellular automaton

Let us give first an informal overview of what collisions are and where they come from. An ultimately periodic configuration consists of two infinite periodic words separated by a finite non-periodic word. As transitions of cellular automata are locally defined, the image of an ultimately periodic configuration is an ultimately periodic configuration such that: for each periodic part, the period in the image divides the period in the preimage ; for the non-periodic part, it can only grow by a finite size depending on the local rule. If, by iterating the transition rule of the cellular automaton, the size of the non-periodic part of the configurations remains bounded, then the orbit of the ultimately periodic configuration is, up to a translation, ultimately periodic. When considering this ultimately periodic behavior from the space-time diagram point of view, one can see some kind of *particle*: a localized structure moving with a rational slope in a periodic *background* environment, as depicted on Fig. 2a.

As particles are ultimately periodic configurations, one can construct more complicated configurations by putting particles side by side, ensuring that the non-periodic parts are far enough from each other, and that the periodic parts of two particles put side by side are the same and well aligned. If the non-periodic part of several particles (two or more) becomes near enough in the orbit, complex interactions might occur. If the interaction is localized in both space and time, as depicted on Fig. 2b, this interaction is called a *collision*.

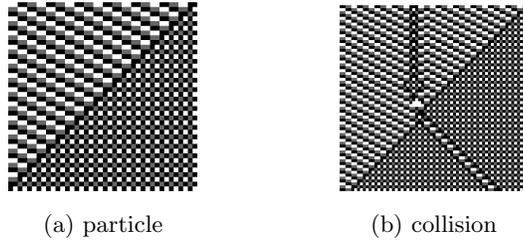


Fig. 2. particles and collisions generated by ultimately periodic configurations

Particles and collisions provide a convenient tool in the study of cellular automata. When constructing two-dimensional cellular automata, like in historical constructions of von Neumann [19] and Codd [5], particles are a convenient way to convey quanta of information from place to place. The most well known example of particle is certainly the glider of the Game of Life used by Conway *et al* to embed computation inside the Game of Life [2] by using particular behavior of glider collisions. When using one-dimensional cellular automata to recognize languages or to compute functions, a classical tool is the notion of signal introduced by Fischer [7] and later developed by Mazoyer and Terrier [15, 16]: signals and their interactions are simple kinds of particles and collisions. Particles appears even in the classification of cellular automata dynamics: in its classification [20], Wolfram identifies what he calls class 4 cellular automata where “(...) *localized structures are produced which on their own are fairly simple, but these structures move around and interact with each other in very complicated ways. (...)*” A first study of particles interaction was proposed by Boccara *and al* [3], latter followed by Crutchfield *and al* [11]: these works focus on particles and bounding the number of possible collisions they can produce. Finally, the proof by Cook of the universality of rule 110 [6] is a typical construction involving a huge number of particles and collisions: once the gadgets and the simulation are described, the main part of the proof consists of proving that particles are well synchronized and that collisions occur exactly as described in the simulation.

When dealing with space-time diagrams consisting of only particles and collisions, a second object is often used: a planar map describing the collisions and their interactions. When identifying particles and collisions in space-time diagrams, in the style of Boccara *and al* [3], one builds the planar map to give a compact description of the diagram, as depicted on Fig. 3. When describing algorithmic computation, in the style of Fischer [7], one describes a family of planar maps as a scheme of the produced space-time diagrams.

In the present paper, we formally define particles and collisions, describe how collisions can be catenated, introduce collisions schemes as planar maps and discuss the construction of finite catenations from collisions schemes. All the necessary material is defined in section 1 followed by basic catenation of tilings in section 2. Collisions and their catenations are formally introduced in

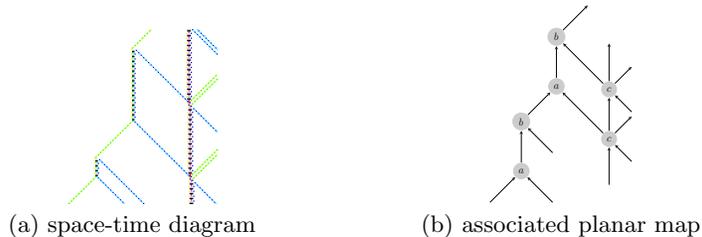


Fig. 3. From space-time diagram to planar map

section 3. Planar maps and the characterization of catenations they encode are treated in section 4.

1 Definitions

In the remaining of this paper, every discussion occurs in the *two-dimensional plane* \mathbb{Z}^2 partially colored with the letters of a given finite alphabet Σ . A *pattern* is a subset of \mathbb{Z}^2 . A *cell* c of a given pattern P is an element $c \in P$. A *vector* is an element of the group $(\mathbb{Z}^2, +)$ of translations in the plane. A *coloring* \mathcal{C} is a partial map from \mathbb{Z}^2 to Σ . The *support* of a coloring \mathcal{C} is denoted by $\text{Sup}(\mathcal{C})$, its restriction to a pattern P is denoted by $\mathcal{C}|_P$.

The *translation* $u \cdot \mathcal{C}$ of a coloring \mathcal{C} by a vector u is the coloring with support $\text{Sup}(\mathcal{C}) + u$ such that, for all $z \in \text{Sup}(\mathcal{C})$, it holds $(u \cdot \mathcal{C})(z + u) = \mathcal{C}(z)$. The *disjoint union* $\mathcal{C} \oplus \mathcal{C}'$ of two colorings \mathcal{C} and \mathcal{C}' is the coloring with support $\text{Sup}(\mathcal{C}) \cup \text{Sup}(\mathcal{C}')$ such that, for all $z \in \text{Sup}(\mathcal{C})$, it holds $\mathcal{C} \oplus \mathcal{C}'(z) = \mathcal{C}(z)$ and for all $z \in \text{Sup}(\mathcal{C}')$, it holds $\mathcal{C} \oplus \mathcal{C}'(z) = \mathcal{C}'(z)$. Colorings and their operations are depicted on Fig. 4.

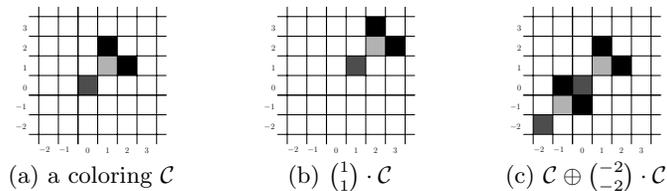


Fig. 4. colorings, translations and disjoint unions

A *tiling constraint* is a pair (V, \mathcal{Y}) where V is a finite pattern and \mathcal{Y} is a subset of Σ^V . A coloring \mathcal{C} *satisfies* a tiling constraint (V, \mathcal{Y}) if for each vector $u \in \mathbb{Z}^2$ such that V subsets $\text{Sup}(u \cdot \mathcal{C})$, it holds $(u \cdot \mathcal{C})|_V \in \mathcal{Y}$. For now on we fix a tiling constraint (V, \mathcal{Y}) . A *tiling* is a coloring with support \mathbb{Z}^2 that satisfies the tiling constraint. The *neighborhood* ∂P of a pattern P is the pattern $\{z + v \mid z \in P, v \in V\}$ of cells constraining P . Two colorings \mathcal{C} and \mathcal{C}' are *undistinguishable* on a pattern P if the equation $\mathcal{C}|_{\partial P} = \mathcal{C}'|_{\partial P}$ holds.

In the following, for geometrical considerations, we will implicitly use variations of discrete forms of the Jordan curve theorem [14]. Two points $\binom{x}{y}, \binom{x'}{y'} \in \mathbb{Z}^2$ are *4-connected* if $\binom{|x-x'|}{|y-y'|} \in \left\{ \binom{1}{0}, \binom{0}{1} \right\}$, *8-connected* if $\binom{|x-x'|}{|y-y'|} \in \left\{ \binom{1}{0}, \binom{0}{1}, \binom{1}{1} \right\}$. A pattern P is *4-connected*, *resp.* *8-connected*, if for each pair of point $z, z' \in P$, there exists a 4-connected, *resp.* 8-connected, path of points of P from z to z' . The discrete Jordan curve theorem states that any non empty 4-connected closed path separates the plane into two 8-connected patterns, the interior and exterior of the path. More generally, a *frontier* is a 4-connected pattern separating the plane into n 8-connected patterns, its *borders*.

2 Catenation of tilings

Let (V, \mathcal{T}) be a tiling constraint and C a set of colorings satisfying this constraint. To generate tilings by catenating colorings in C , the idea is to construct a patchwork of colorings by cutting portions of coloring and glue them together so that tiling constraints are preserved. A simple patchwork of 2 tilings is depicted on Fig. 5.

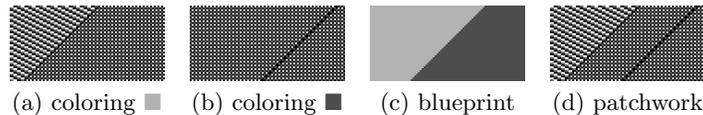


Fig. 5. a patchwork

Definition 1. A patchwork is a tiling \mathcal{T}_ϕ defined for each $z \in \mathbb{Z}^2$ by $\mathcal{T}_\phi(z) = \phi(z)(z)$ where $\phi: \mathbb{Z}^2 \rightarrow C$ is the blueprint of the patchwork such that:

1. $\forall \mathcal{C} \in C, \quad \partial\phi^{-1}(\mathcal{C}) \subseteq \text{Sup}(\mathcal{C});$
2. $\forall z \in \mathbb{Z}^2, \forall v \in V, \quad \phi(z)(z+v) = \phi(z+v)(z+v).$

Patchworks provide a convenient way to combinatorially generate tilings from a set of valid colorings without knowing explicitly the tiling constraint: it is sufficient to know a superset of the tiling neighborhood V and to cut colorings on a big enough boundary containing the same letters.

Topology is a classical tool of symbolic dynamics [10], tilings being exactly the shifts of finite type for two-dimensional words. The set of colorings is endowed with the so called Cantor topology: the product of the discrete topology on $\Sigma \cup \{\perp\}$ where \perp denotes undefined color. This topology is compatible with the following distance on colorings: $d(\mathcal{C}, \mathcal{C}') = 2^{-\min\{|z|, \mathcal{C}(z) \neq \mathcal{C}'(z)\}}$. Let $\mathcal{O}_\mathcal{C}$ be the set of colorings \mathcal{C}' such that $\mathcal{C}'_{|\text{Sup}(\mathcal{C})} = \mathcal{C}_{|\text{Sup}(\mathcal{C})}$. The set of $\mathcal{O}_\mathcal{C}$ for colorings \mathcal{C} with a finite support is a base of clopen sets for the given compact perfect topology.

Proposition 1. *The set of patchworks over C is a compact set.*

Proof. Let \mathcal{T}_i be a sequence of patchworks over C converging to a limit tiling \mathcal{T} . Consider the blueprints ϕ_i of these patchworks. For each cell $z \in \mathbb{Z}^2$, let v_z be the element $(-z \cdot \mathcal{T})|_V$ of \mathcal{Y} . Let $\phi(z)$ be any $\phi_i(z)$ such that $(-z \cdot \phi_i(z))|_V = v_z$ – such a $\phi_i(z)$ always exists by definition of patchworks as \mathcal{T}_i converges to \mathcal{T} . The map ϕ is a blueprint for \mathcal{T} . ■

Proposition 2. *The set of patchworks over C contains the tilings of the closure of C .*

Proof. Let \mathcal{C}_i be a sequence of colorings in C converging to a limit tiling \mathcal{T} . For each \mathcal{C}_i , let P_i be the largest pattern, for inclusion, such that $\mathcal{C}_i|_{P_i} = \mathcal{T}|_{P_i}$. As the sequence \mathcal{C}_i converges to \mathcal{T} , the sequence P_i converges to \mathbb{Z}^2 . Without loss of generality, consider that P_i is an increasing sequence of patterns. For each i let $\delta(i)$ be the smallest j such that $\partial P_i \subseteq P_j$. Consider $P'_n = P_{\delta^n(1)}$, an increasing subsequence of P_i . Construct a blueprint ϕ as follows: for all $z \in \mathbb{Z}^2$, let $\phi(z) = P'_{\min\{n|z \in P'_n\}}$. By construction, this blueprint is valid and its patchwork is \mathcal{T} . ■

Corollary 1. *Let \mathcal{O}_i be a base of open sets of colorings and C be a set of colorings containing at least one element of each \mathcal{O}_i . The set of patchworks over C is the whole set of tilings.* ■

In particular, the set of tiling constraints \mathcal{Y} , viewed as colorings, generates the whole set of tilings. The larger set of colorings with finite support generates the whole set of tilings. But this approach is heterogeneous: we combine colorings to obtain tilings. Can we restrict ourselves to combinations of tilings? More precisely, given a tiling constraint, can we recursively construct a recursive family of tilings T such that the set of patchworks over T is the whole family of tilings?

In the case of one-dimensional tilings, replacing \mathbb{Z}^2 by \mathbb{Z} , it is straightforward that the set of ultimately periodic tilings generates the whole set of tilings: the set of ultimately periodic tilings is a dense set – from any tiling \mathcal{T} and any finite pattern P , one can construct an ultimately periodic tiling \mathcal{T}' such that $\mathcal{T}|_P = \mathcal{T}'|_P$. In the case of two-dimensional tilings, due to the undecidability of the tiling problem [1, 18], there exists no such family.

Proposition 3. *There exists no recursive function that, given a tiling constraint, computes a recursive family of tilings T such that the set of patchworks over T is the whole family of tilings.*

Proof. If such function would exist, it should return an empty family of tilings if and only if the given tiling constraint does not tile the plane. The tiling problem is undecidable. ■

In the next section, we propose an extension of ultimately periodic tilings for two-dimensional tilings with high regularity, focusing on the ease of catenation as there is no hope to capture aperiodic tilings.

3 Ultimately periodic tilings

Biperiodic tilings are among the most regular ones and correspond to the idea of a *background* for cellular automata: a tiling \mathfrak{B} with two non-colinear periodicity vectors u and v such that $\mathfrak{B} = u \cdot \mathfrak{B} = v \cdot \mathfrak{B}$. As backgrounds are objects of dimension 2, if one wants to mix several backgrounds in a same tiling, the interface between two background is of dimension 1. The most regular kind of interface corresponds to the idea of a *particle*: a tiling \mathfrak{P} with two non-colinear vectors, the period u of the particle such that $\mathfrak{P} = u \cdot \mathfrak{P}$ and the period v of its backgrounds such that for all position $z \in \mathbb{Z}^2$, the extracted one-dimensional word $(\mathfrak{P}(z + vi))_{i \in \mathbb{Z}}$ is ultimately periodic. Of course, several particles might meet on the plane, leading to objects of dimension 0 that correspond to the idea of a *collision*. In this paper, an ultimately periodic tiling of the plane is such a collision.

Let $\triangleleft_v(u, u')$ denote the angular portion of the plane, on the right hand side of u , starting in position $v \in \mathbb{Z}^2$ and delimited by the vectors $u, u' \in \mathbb{Z}^2$. Formally, one might geometrically define a collision as follows (and depicted on Fig. 6):

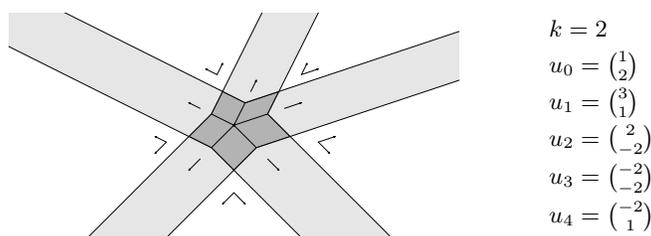


Fig. 6. defining collisions through vectors

Definition 2. A collision is a tiling \mathfrak{C} for which there exists an integer k and a finite cyclic sequence of n vectors $(u_i) \in (\mathbb{Z}^2)^{\mathbb{Z}_n}$ such that, for all $i \in \mathbb{Z}_n$, \mathfrak{C} is u_i -periodic in z , i.e. $\mathfrak{C}(z) = \mathfrak{C}(z + u_i)$, for all positions z inside $\triangleleft_{ku_i}(u_{i-1}, u_{i+1})$.

While characterizing collisions, such a definition does not explicitly identify the different periodic areas of the plane: the finite collision coloring, the uniperiodic particle colorings and the biperiodic background colorings. To manipulate collisions and combine them using the properties of these regular areas, representations are needed that capture these areas. Due to subtleties of discrete geometry, the representation, albeit simple in principle as depicted on Fig. 7, admits more complex definitions. Moreover, we take the tiling constraints into account in the definitions to ensure tilability of any valid combination.

Definition 3. A background representation \mathfrak{B} is a tuple (\mathcal{C}, u, v) where \mathcal{C} is a coloring with finite support, $u, v \in \mathbb{Z}^2$ are non-colinear vector and $\mathfrak{B} = \bigoplus_{i,j \in \mathbb{Z}^2} (iu + jv) \cdot \mathcal{C}$ is the associated tiling.

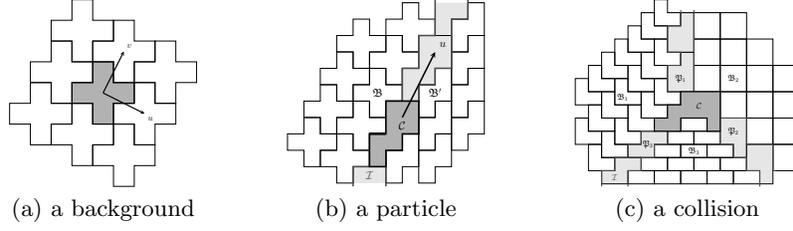


Fig. 7. principle of construction

Definition 4. A particle representation \mathfrak{P} is a tuple $(\mathfrak{B}, \mathcal{C}, u, \mathfrak{B}')$ where \mathfrak{B} and \mathfrak{B}' are backgrounds, \mathcal{C} is a coloring with finite support and $u \in \mathbb{Z}^2$ a vector, satisfying:

1. the support of $\mathcal{I} = \bigoplus_{k \in \mathbb{Z}} ku \cdot \mathcal{C}$ is a frontier with two borders: L , the left one with respect to u , and R , the right one;
2. $\mathfrak{B}|_L \oplus \mathcal{I} \oplus \mathfrak{B}'|_R$ is the tiling \mathfrak{P} associated with \mathfrak{P} ;
3. \mathfrak{P} is undistinguishable from \mathfrak{B} over L and from \mathfrak{B}' over R .

Definition 5. A collision representation \mathfrak{C} is a pair (\mathcal{C}, L) where \mathcal{C} is a finite pattern, L is a finite sequence of n particles $\mathfrak{P}_i = (\mathfrak{B}_i, \mathcal{C}_i, u_i, \mathfrak{B}'_i)$, satisfying:

1. $\forall i \in \mathbb{Z}_n, \mathfrak{B}'_i = \mathfrak{B}_{i+1}$;
2. the support of $\mathcal{I} = \mathcal{C} \oplus \bigoplus_{i \in \mathbb{Z}_n, k \in \mathbb{N}} ku_i \cdot \mathcal{C}_i$ is a frontier with n borders;
3. For all $i \in \mathbb{Z}_n$, the support of $\mathcal{C} \oplus \bigoplus_{k \in \mathbb{N}} (ku_i \cdot \mathcal{C}_i \oplus ku_{i+1} \cdot \mathcal{C}_{i+1})$ is a frontier with two borders: let P_i be the border on the right of \mathfrak{P}_i ;
4. $\mathcal{I} \oplus \bigoplus_i \mathfrak{B}_i|_{P_i}$ is the tiling $\bar{\mathfrak{C}}$ associated with \mathfrak{C} ;
5. For all $i \in \mathbb{Z}_n$, the colorings $\bar{\mathfrak{C}}$ and \mathfrak{B}'_i are undistinguishable over P_i ;
6. For all $i \in \mathbb{Z}_n$, the colorings $\bar{\mathfrak{C}}$ and \mathfrak{P}_i are undistinguishable over $ku_i \cdot \mathcal{C}_i$.

In the following, for any particle $\mathfrak{P} = (\mathfrak{B}, \mathcal{C}, u, \mathfrak{B}')$ inside a collision, the support of the particle is $\bigoplus_{k \in \mathbb{N}} (ku \cdot \mathcal{C})$.

Proposition 4. The tiling associated with a background (resp. particle, collision) representation is a background (resp. particle, collision).

Proof. Let (\mathcal{C}, u, v) be a background representation \mathfrak{B} . As u and v are not collinear, \mathfrak{B} is biperiodic: \mathfrak{B} is a background.

Let $(\mathfrak{B}, \mathcal{C}, u, \mathfrak{B}')$ be a particle representation \mathfrak{P} . As backgrounds are biperiodic, for all background \mathcal{T} and for all vector $u' \in \mathbb{Z}^2$, there is a multiple ku' of u' such that \mathcal{T} is periodic in ku' . Thus, there exists a multiple ku of u such that \mathfrak{P} is periodic in ku and for any vector v not collinear with u , \mathfrak{P} is ultimately periodic on each one-dimensional word extracted using v . Thus, \mathfrak{P} is a particle.

Let $(\mathcal{C}, (\mathfrak{P}_i))$ where $\mathfrak{P}_i = (\mathfrak{B}_i, \mathcal{C}_i, u_i, \mathfrak{B}'_i)$ be a collision representation \mathfrak{C} . Lets construct a collision with vector set (v_i) and constant k as follows. For the vectors (v_i) take the (u_i) sorted in indirect trigonometric order, eventually

adding new vectors to ensure that two consecutive vectors are not collinear. For the constant k , take it large enough so that the support of \mathfrak{C} does not intersect any $\triangleleft_{kv_i}(v_{i-1}, v_{i+1})$ and that for each particle \mathfrak{P}_j , such an intersection occurs only in the case $i = j$: as consecutive (v_j) are non-collinear, this is always possible. In $\bar{\mathfrak{C}}$, each $\triangleleft_{kv_i}(v_{i-1}, v_{i+1})$ is periodic in v_i : $\bar{\mathfrak{C}}$ is a collision. ■

Proposition 5. *The set of background (resp. particle, collision) representations is recursive.*

Proof. Representations being composed of finitely many recursive objects, one only needs to prove that the conditions on these objects are recursively checkable. The tiling constraints conditions are checked on periodic objects: only finitely many tiling constraints need to be considered. The geometrical considerations on frontiers concern sets of periodic patterns in the plane and thus can also be checked by considering finitely many positions in \mathbb{Z}^2 . ■

From now on, we will always consider objects alongside with representations and for the sake of legibility avoid going back and forth from representations to associated tilings.

4 Finite catenations

A random blueprint of finitely many collisions might produce a tiling which is not a collision, however if the blueprint of the patchwork consists of finitely many 8-connected components, the patchwork is a collision. Using representations of collisions, a more regular family of patchworks can be defined: a catenation induces a patchwork combining collisions by binding pairs of similar particles as depicted on Fig. 3.

Two particles are bindable if they share the same coloring, have opposite vectors and are well aligned so that the colorings superpose well. Formally, two particles $\mathfrak{P} = (\mathfrak{B}, \mathcal{C}, u, \mathfrak{B}')$ and $\tilde{\mathfrak{P}} = (\tilde{\mathfrak{B}}, \tilde{\mathcal{C}}, \tilde{u}, \tilde{\mathfrak{B}}')$ are *bindable*, with n repetitions, if $\tilde{u} = -u$, $\tilde{\mathfrak{B}} = (n-1)u \cdot \mathfrak{B}'$, $\tilde{\mathfrak{B}}' = (n-1)u \cdot \mathfrak{B}$ and $\tilde{\mathcal{C}} = (n-1)u \cdot \mathcal{C}$. The tuple $(\mathfrak{P}, n, \tilde{\mathfrak{P}})$ is a *binding*. The *support* of the binding is the pattern $\bigoplus_{0 \leq i \leq (n-1)} iu \cdot \mathcal{C}$. Intuitively, it corresponds to the support of the particle restricted to its number of repetitions.

Definition 6. *A catenation scheme is a connected planar map whose vertexes are labeled by collisions, and edges (possibly half-infinite) are labeled by particle(s) compatible with adjacent collisions. Moreover, the adjacent particles of a vertex have to correspond to the geometrically ordered set of particles of the associated collision.*

Since a catenation scheme is a planar map, its dual is well defined, and introduces the notion of *face*: a face corresponds to a background area of the tiling. Faces can be either finite or infinite. To transform a catenation scheme into a tiling, one has to tag each finite edge with a binding and ensure that binding informations are consistent.

Definition 7. A catenation is a catenation scheme where all vertices are tagged with a translation vector and all finite edges are tagged by bindings compatible with the particles on the edge and the translated collisions on both sides, so that supports of bindings, finite colorings of translated collisions and translated infinite particles are distinct.

A patchwork is associated to a catenation by constructing a blueprint as follows. For each position $z \in \mathbb{Z}^2$, if z is in the finite coloring of a translated collision, map it to the translated collision, if z is in the support of a finite binding, map it to the particle associated to the binding, if z is in the support of an translated infinite particle, map it to the translated particle, and finally if z is inside a face, map it to the associated background.

Proposition 6. *The patchwork associated with a catenation is a well defined tiling; if the catenation is finite it is a collision.*

Proof. By construction, as all the supports are disjoint, the associated patchwork is a valid embedding of the planar map into the plane. The tiling constraints are valid in the patchwork because of the tiling constraints checking in the definition of representations. If the catenation is finite, the obtained tiling is a collision: its particles are the infinite particles of the catenation, the finite coloring corresponds to the finite part of the plane not filled by infinite particles and their backgrounds. ■

Catenation schemes provide a tool to combine collisions and program complex behaviors in tilings. Catenations provide implementations of such schemes. The main result of this paper states that the set of catenations associated to a catenation scheme is both recursive and very regular.

Theorem 1. *Given a finite catenation scheme, the set of corresponding catenations, defined up to translations, is a recursive semi-linear set.*

Proof. To prove this theorem, we will show how to recursively construct open formulae of Presburger arithmetic: first-order arithmetic over integers with addition and order. These formulae exactly encode the semi-linear set of integers [9]: finite unions of linear sets of integers.

Consider a catenation scheme: we are essentially searching the number of repetitions inside each binding. These numbers will be the unknown of the formula. We split the formula in a conjunction of finitely many subformulae: one for each face. The catenation is valid if and only if each of its face is valid.

For a face, the formula is the conjunction of two formulae: one saying that going around the border of the face returns back to the starting point. This property can be summarised in an equation of the form $\sum_i u_i n_i + \sum_j d_j = 0$ where n_i are the unknown, u_i are the (constant) vectors of particles, and d_j are (constant) translation induced between two consecutive particles by collisions. The second part of the formula expresses the fact that supports of elements are disjoint. For collisions, since the support of finite coloration is finite, it is sufficient to make a (finite) conjunction. For bindings or particles, the number of

points in the support is potentially infinite but consists of only a finite number of equivalence classes modulo the repetition vector. Thus all those point can be described with an universal quantification on the number of repetitions and conjunctions on the classes. Moreover, using order, it is possible to bound the number of repetitions. ■

5 Extensions and open problems

The objects presented here extend to higher dimensions. Of course, the number of objects increases but the approach seems to extend well. The main difficulty to deal with objects in higher dimensions is the embedding of the whole complexity of dimension n in dimension $n + 1$. Another point is to confront the extension with algorithmics of the literature.

In cellular automata algorithmics, constructions might involve infinite catenations. Whereas infinite catenation schemes and associated catenations are well defined, Presburger arithmetic cannot help in the case of infinitely many variables and equations. We need tools to capture constructions involving regular infinite catenation schemes.

References

- [1] R. Berger, The undecidability of the domino problem, *Memoirs American Mathematical Society*, **66**(1966).
- [2] E. R. Berlekamp, J. H. Conway, and R. K. Guy, *Winning ways for your mathematical plays. Vol. 2*, Academic Press Inc. [Harcourt Brace Jovanovich Publishers], London, 1982, games in particular.
- [3] N. Boccara, J. Nasser, and M. Roger, Particlelike structures and their interactions in spatiotemporal patterns generated by one-dimensional deterministic cellular-automaton rules, *Phys. Rev. A*, **44**(1991), no. 2, 866–875.
- [4] J. R. Büchi, On a decision method in restricted second order arithmetic, in *Proceedings of the International Congress on Logic, Methodology, and Philosophy of Science, Berkeley, 1960*, pages 1–11, Stanford University Press, 1962.
- [5] E. F. Codd, *Cellular Automata*, Academic Press, New York, 1968.
- [6] M. Cook, Universality in elementary cellular automata, *Complex Systems*, **15**(2004), 1–40.
- [7] P. C. Fischer, Generation of primes by a one-dimensional real-time iterative array, *Journal of the ACM*, **12**(1965), no. 3, 388–394.
- [8] D. Giammarresi and A. Restivo, Two-dimensional languages, in A. Salomaa and G. Rozenberg, editors, *Handbook of Formal Languages*, volume 3, Beyond Words, pages 215–267, Springer-Verlag, Berlin, 1997.
- [9] S. Ginsburg and E. H. Spanier, Semigroups, presburger formulas, and languages, *Pacific Journal of Mathematics*, **16**(1966), 285–296.
- [10] G. A. Hedlund, Endomorphisms and automorphisms of the shift dynamical system, *Mathematical Systems Theory*, **3**(1969), 320–375.
- [11] W. Hordijk, C. R. Shalizi, and J. P. Crutchfield, Upper bound on the products of particle interactions in cellular automata, *Phys. D*, **154**(2001), no. 3-4, 240–258.

- [12] J. Kari, Theory of cellular automata: a survey, *Theoretical Computer Science*, **334**(2005), 3–33.
- [13] S. C. Kleene, Representation of events in nerve nets and finite automata, in C. Shannon and J. McCarthy, editors, *Automata Studies*, pages 3–41, Princeton University Press, 1956.
- [14] T. Y. Kong and A. Rosenfeld, Digital topology: introduction and survey, *Comput. Vision Graph. Image Process.*, **48**(1989), no. 3, 357–393.
- [15] J. Mazoyer, Computations on cellular automata: some examples, in *Cellular automata (Saissac, 1996)*, pages 77–118, Kluwer Acad. Publ., Dordrecht, 1999.
- [16] J. Mazoyer and V. Terrier, Signals in one-dimensional cellular automata, *Theoretical Computer Science*, **217**(1999), no. 1, 53–80, cellular automata (Milan, 1996).
- [17] M. Minsky, *Computation: Finite and Infinite Machines*, Prentice Hall, Englewood Cliffs, 1967.
- [18] R. M. Robinson, Undecidability and nonperiodicity for tilings of the plane, *Inventiones Mathematicae*, **12**(1971), 177–209.
- [19] J. von Neumann, *Theory of Self-Reproducing Automata*, University of Illinois Press, Urbana, Ill., 1966.
- [20] S. Wolfram, Universality and complexity in cellular automata, *Physica D. Non-linear Phenomena*, **10**(1984), no. 1-2, 1–35, cellular automata (Los Alamos, N.M., 1983).

A Appendix

In this appendix, we provide the referee with two examples. The first section illustrates how Theorem 1 might be applied on a catenation scheme. The second section is a complete example using the tools of this paper to describe the synchronization details of a cellular automaton construction.

A.1 An example of face resolution

In this section, we give an example of decision procedure for a face in a catenation scheme. Our example uses the simplest alphabet: $\Sigma = \{\square, \blacksquare\}$. The tiling constraint considered is the whole set of elements on $V = \{(0, -1), (0, 0), (0, 1)\}$ (i.e. (V, Σ^V)).

We choose a set of three collisions depicted in Fig. 8c with corresponding particles (Fig. 8b) and backgrounds (Fig. 8a). Relative positions of particles in collisions are indicated with a circle.

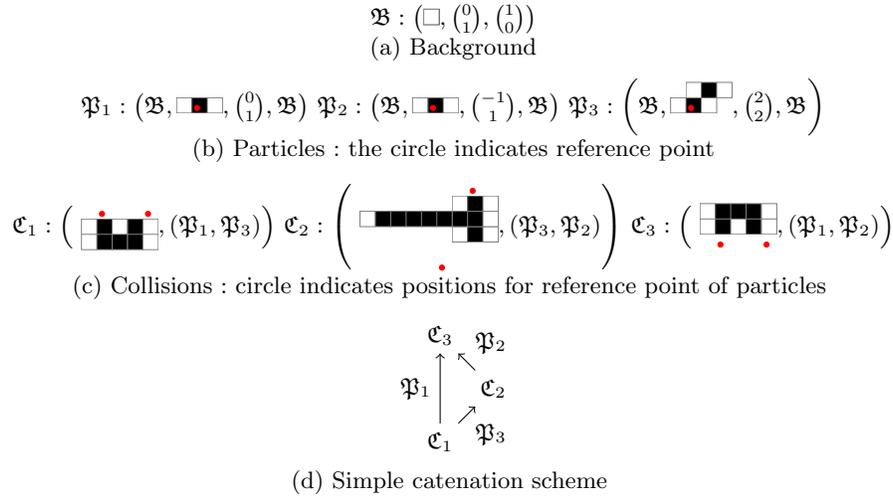


Fig. 8. A simple catenation scheme

Let us now study the catenation scheme given in Fig 8d. This scheme has two faces (one interior and one exterior). Let us restrict ourselves to the interior face and find all possible catenations. To do this, we must introduce the number of repetition for each binding: let us call n_1 (resp. n_2, n_3) the number of repetition of the particle \mathfrak{P}_1 (resp. $\mathfrak{P}_2, \mathfrak{P}_3$).

The formula is divided in two parts. The first part ensure that going around the face take us back to the starting point. This is ensured by adding the translations due to particles in binding and the translation to go from one particle to

an other (due to collision) and saying the result must be null. In our case, the resulting equation is :

$$(n_1 - 1) \binom{0}{1} + \binom{3}{0} + (n_2 - 1) \binom{1}{-1} + \binom{-2}{-5} + (n_3 - 1) \binom{-2}{-2} + \binom{-3}{0} = \binom{0}{0}$$

The second part of the formula concerns non-overlapping of supports. This is obtained by ensuring that all their points are distinct. The form of the formula differs according to the element (binding, particle or collision) whose support is considered. Let us first give an example for support of collisions. To say that collisions $\mathfrak{C}_1 = (\mathcal{C}_1, L_1)$ and $\mathfrak{C}_3 = (\mathcal{C}_3, L_3)$ are disjoint. Since the number of points in each support of non periodic coloring is finite, it is possible to make the conjunction of all pair of points being distinct. This is written:

$$\bigwedge_{z \in \text{Sup}(\mathcal{C}_1)} \bigwedge_{z' \in \text{Sup}(\mathcal{C}_3)} z \neq \left(z' + (n_1 - 1) \binom{0}{1} \right)$$

The $(n_1 - 1) \binom{0}{1}$ corresponds to the translation induced between \mathfrak{C}_1 and \mathfrak{C}_3 by the binding. One other possible case involves binding. In this case, the size of the support is no longer finite. However, the points are at regular coordinates (due to the repetition axe). Let us take the example of supports of $\mathfrak{P}_1 = (\mathfrak{B}, \mathcal{C}, \binom{0}{1}, \mathfrak{B}')$ and $\mathfrak{C}_2 = (\mathcal{C}_2, L_2)$. Non overlap of those supports can be expressed with the formula :

$$\forall i, \bigwedge_{z \in \text{Sup}(\mathcal{C})} \bigwedge_{z' \in \text{Sup}(\mathcal{C}_2)} 0 \leq i < (n_1 - 1) \Rightarrow \left(z + i \binom{0}{1} \neq z' + (n_3 - 1) \binom{2}{2} \right)$$

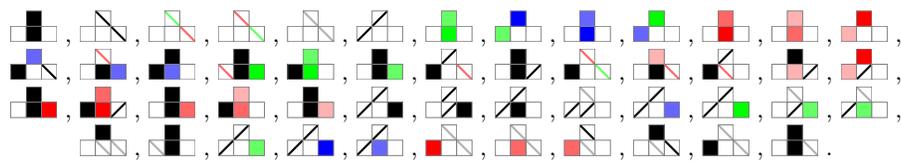
The same method also applies for particles that are not part of bindings.

Making the conjunction of all those formulae gives a unique formula on Presburger arithmetic whose solution correspond to catenation. In this example, we can find that the first part gives solutions on the form: $n_1 = 4k + 7, n_2 = 2k + 2, n_3 = k, k \in \mathbb{N}$. Moreover, the second part adds the condition $k > 2$ (checking the support of \mathfrak{P}_1 and \mathfrak{C}_2).

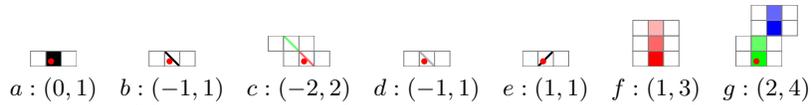
A.2 A complete example of synchronisation

This section is dedicated to give a simple but complete example using the catenation of collisions to solve synchronisation problems in cellular automata. For this, we construct a cellular automaton with 15 states which is able to “calculate” the Syracuse sequence.

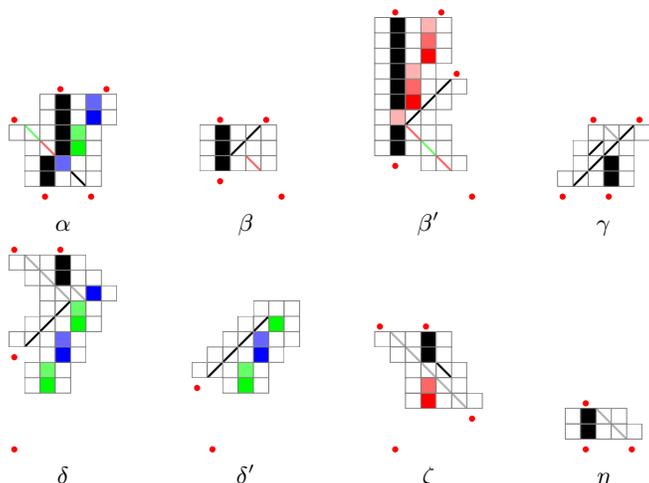
This automaton is based on particles and collisions. First the Syracuse sequence is a integer sequence satisfying the successor of n if calculated with the function f such that $f(n) = n/2$ if n is even, and $f(n) = 3n + 1$ if n is odd. The automaton neighbourhood implies a support of the tiling constraint $V = \{(-1, 0), (0, 0), (1, 0), (0, 1)\}$. Since the automaton is deterministic, for every triplet l, c, r there is exactly one element z such that $(l, c, r, z) \in \mathcal{Y}$ (z is the



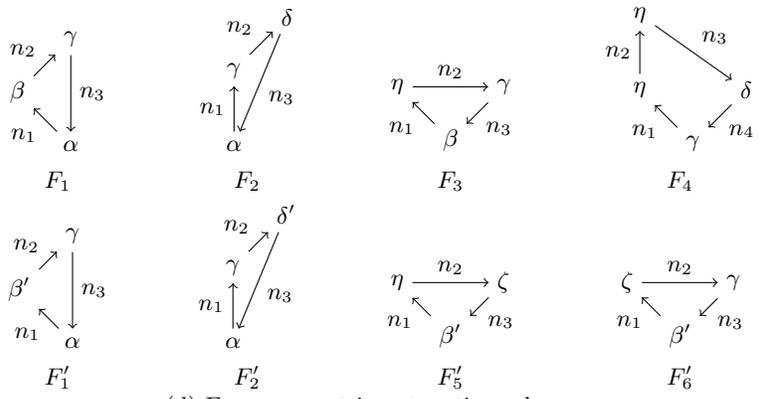
(a) Set of constraints: all non defined behavior maps on \square



(b) Set of particles (background is omitted since uniq)



(c) Set of Collisions : circle indicates relative position of particles (names of particles are omitted)



(d) Faces present in catenation schemes

Fig. 9. Elements of the automaton

element in position $(1, 0)$). The set of constraint (i.e. the local transition rule) is defined in Fig. 9(a).

As for the previous example, we take a set of collisions depicted in Fig.9c along with included particles (Fig.9b). Backgrounds are not depicted since they consist only the uniformly white one (i.e $(\square, \binom{0}{1}, \binom{1}{0})$). The desired behaviors are depicted in Fig. 10. Faces present in those behaviors are depicted in Fig 9d. Let us study those faces. As for the previous example, conditions on faces can be divided into two parts: in our example, we focus only on the first equation. Due to the form of patterns in collisions and particles, the second part does not bring any supplementary conditions. The equation for faces are explicit below.

$$\begin{aligned}
- F_1 &: (n_1 - 1) \binom{-2}{2} + \binom{-1}{5} + (n_2 - 1) \binom{1}{1} + \binom{3}{0} + (n_3 - 1) \binom{0}{-1} + \binom{-3}{-2} = \binom{0}{0} \\
- F_2 &: (n_1 - 1) \binom{0}{1} + \binom{2}{5} + (n_2 - 1) \binom{1}{1} + \binom{0}{-6} + (n_3 - 1) \binom{-2}{-4} + \binom{-3}{0} = \binom{0}{0} \\
- F_3 &: (n_1 - 1) \binom{0}{1} + \binom{3}{0} + (n_2 - 1) \binom{1}{-1} + \binom{-2}{-5} + (n_3 - 1) \binom{-1}{-1} + \binom{-3}{0} = \binom{0}{0} \\
- F_4 &: (n_1 - 1) \binom{-1}{1} + \binom{-3}{0} + (n_2 - 1) \binom{0}{1} + \binom{-3}{3} + (n_3 - 1) \binom{1}{-1} + \binom{3}{0} + (n_4 - 1) \binom{-1}{-1} + \binom{0}{-7} = \binom{0}{0} \\
- F'_1 &: (n_1 - 1) \binom{-2}{2} + \binom{-1}{8} + (n_2 - 1) \binom{1}{1} + \binom{3}{0} + (n_3 - 1) \binom{0}{-1} + \binom{-3}{-2} = \binom{0}{0} \\
- F'_2 &: (n_1 - 1) \binom{0}{1} + \binom{2}{5} + (n_2 - 1) \binom{1}{1} + \binom{1}{-4} + (n_3 - 1) \binom{-2}{-4} + \binom{-3}{0} = \binom{0}{0} \\
- F'_5 &: (n_1 - 1) \binom{0}{1} + \binom{3}{0} + (n_2 - 1) \binom{1}{-1} + \binom{2}{-8} + (n_3 - 1) \binom{-1}{-3} + \binom{-3}{0} = \binom{0}{0} \\
- F'_6 &: (n_1 - 1) \binom{1}{3} + \binom{4}{4} + (n_2 - 1) \binom{1}{-1} + \binom{-2}{-5} + (n_3 - 1) \binom{-1}{-1} + \binom{-1}{4} = \binom{0}{0}
\end{aligned}$$

From these equations, we can extract a set of solutions for each face.

$$\begin{aligned}
- F_1 &: n_1 = x, n_2 = 2x, n_3 = 4x + 1 \\
- F_2 &: n_1 = 2x + 1, n_2 = 2x + 2, n_3 = x + 1 \\
- F_3 &: n_1 = 2x + 6, n_2 = x + 2, n_3 = x \\
- F_4 &: n_1 = y, n_2 = 2x + 6, n_3 = y + x + 2, n_4 = x \\
- F'_1 &: n_1 = x, n_2 = 2x, n_3 = 4x + 4 \\
- F'_2 &: n_1 = 2x, n_2 = 2x + 1, n_3 = x + 1 \\
- F'_5 &: n_1 = 4x + 11, n_2 = x, n_3 = x + 2 \\
- F'_6 &: n_1 = x, n_2 = x, n_3 = 2x
\end{aligned}$$

It is possible to group those solutions together in order to construct a solution for the whole scheme. In our case, it is possible to find solutions for both schemes as depicted in Fig. 10. To conclude one can make the link between the number of repetitions and the spacing between the two consecutive vertical particles in the scheme. For the odd (resp. even) case, the input of x repetitions correspond to a spacing of $2x + 3$ (resp. $2x + 4$) and the output of $6x + 4$ (resp. $x - 2$) correspond to a spacing of $6x + 10 (= 3(2x + 3) + 1)$ (resp. $x + 2 (= (2x + 4)/2)$). Thus, those scheme correspond to space-time diagram with the wanted behavior (some examples can be seen in Fig.11. As the automaton is deterministic, this behavior is the only possible.

In this example, some non-trivial problems of synchronisation (influencing for example the occurrence of collision β/β' or δ/δ') are easily dealt with when combining solutions of faces.

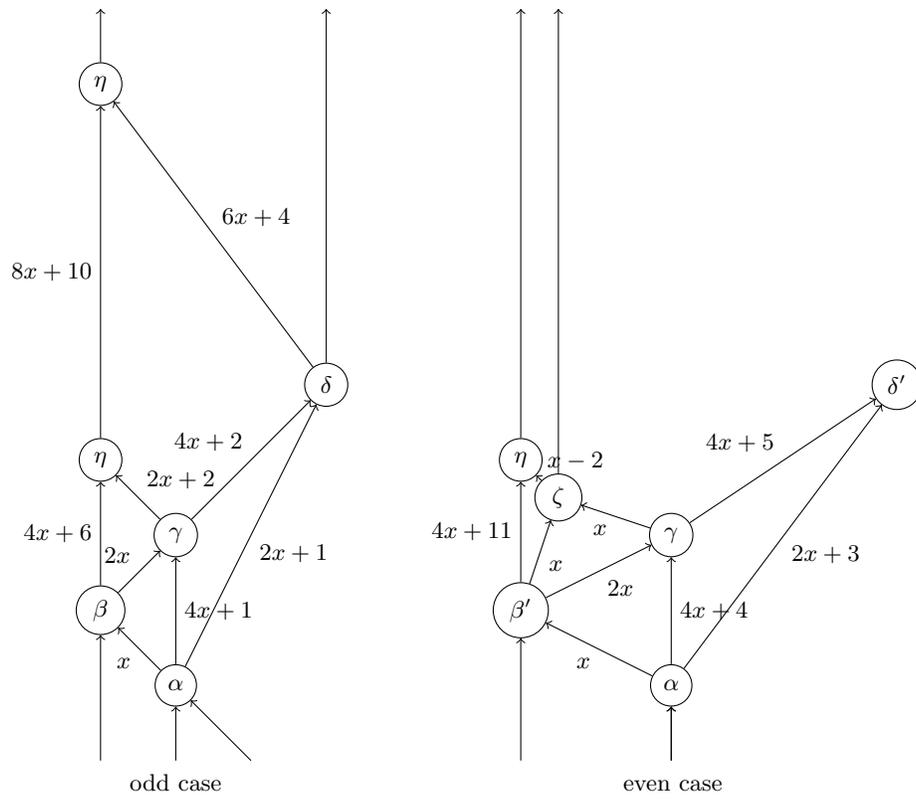


Fig. 10. Symbolic behavior

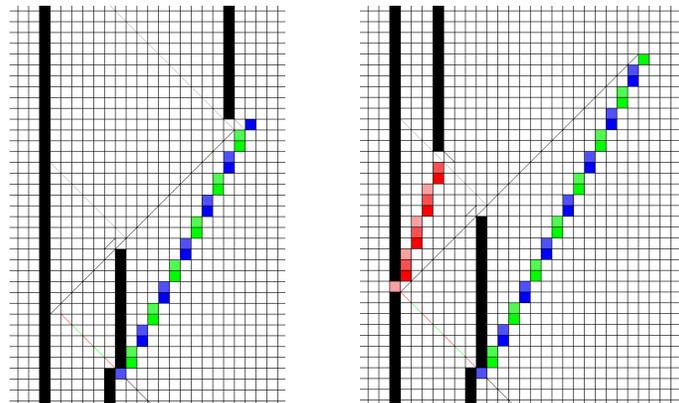


Fig. 11. Examples of space-time diagram