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Topological Dynamics of 2D Cellular Automata

Mathieu Sablik^{*1} and Guillaume Theyssier^{**2}

¹ UMPA, (UMR 5669 — CNRS, ENS Lyon), 46, allée d'Italie 69364 Lyon cedex 07
FRANCE

² LAMA, (UMR 5127 — CNRS, Université de Savoie), Campus Scientifique, 73376
Le Bourget-du-lac cedex FRANCE

Abstract. Topological dynamics of cellular automata (CA), inherited from classical dynamical systems theory, has been essentially studied in dimension 1. This paper focus on 2D CA and aims at showing that the situation is different and more complex. The main results are the existence of non sensitive CA without equicontinuous points, the non-recursivity of sensitivity constants and the existence of CA having only non-recursive equicontinuous points. They all show a difference between the 1D and the 2D case. Thanks to these new constructions, we also extend undecidability results concerning topological classification previously obtained in the 1D case.

1 Introduction

Cellular automata were introduced by J. von Neumann as a simple formal model of cellular growth and replication. They consist in a discrete lattice of finite-state machines, called *cells*, which evolve uniformly and synchronously according to a local rule depending only on a finite number of neighbouring cells. A snapshot of the states of the cells at some time of the evolution is called a *configuration*, and a cellular automaton can be view as a global action on the set of configurations.

Despite the apparent simplicity of their definition, cellular automata can have very complex behaviours. One way to try to understand this complexity is to endow the space of configurations with a topology and consider cellular automata as classical dynamical systems. With such a point of view, one can use well-tried tools from dynamical system theory like the notion of sensitivity to initial condition or the notion of equicontinuous point.

This approach has been followed essentially in the case of one dimensional cellular automata. P. Kůrka as shown in [1] that 1D cellular automata are partitioned into two classes:

- \mathcal{E}_q , the set of cellular automata with equicontinuous points,
- \mathcal{S} , the set of sensitive cellular automata.

* mathieu.sablik@umpa.ens-lyon.fr

** guillaume.theyssier@univ-savoie.fr

We stress that this partition result is false in general for classical (continuous) dynamical systems. Thus, it is natural to ask whether this result holds for the model of CA in any dimension, or if it is a “miracle” or an “anomaly” of the one-dimensional case due to the strong constraints on information propagation in this particular setting. One of the main contributions of this paper is to show that this is an anomaly of the 1D case (section 3): there exist a class \mathcal{N} of 2D CA which are neither in \mathcal{E}_q nor in \mathcal{S} .

Each of the sets \mathcal{E}_q and \mathcal{S} has an extremal sub-class: equicontinuous and expansive cellular automata (respectively). This allows to classify cellular automata in four classes according to the degree of sensitivity to initial conditions. The dynamical properties involved in this classification have been intensively studied in the literature for 1D cellular automata (see for instance [1,2,3,4]). Moreover, in [5], the undecidability of this classification is proven, except for the expansivity class whose decidability remains an open problem.

In this paper, we focus on 2D CA and we are particularly interested in differences from the 1D case. As said above, we will prove in section 3 that there is a fundamental difference with respect to the topological dynamics classification, but we will also adopt a computational complexity point of view and show that some properties or parameters which are computable in 1D are non recursive in 2D (proposition 5 and 9 of section 4). To our knowledge, only few dimension-sensitive undecidability results are known for CA ([6,7]). However, we believe that such subtle differences are of great importance in a field where the common belief is that everything interesting is undecidable.

Besides, we establish in section 4 several complexity lower bounds on the classes defined above and extend the undecidability result of [5] to dimension 2. Notably, we show that each of the class \mathcal{E}_q , \mathcal{S} and \mathcal{N} is neither recursively enumerable, nor co-recursively enumerable. This gives new examples of “natural” properties of CA that are harder than the classical problems like reversibility, surjectivity or nilpotency (which are all r.e. or co-r.e.).

2 Definitions

Let \mathcal{A} be a finite set and $\mathbb{M} = \mathbb{Z}$ (for the one dimensional case) or \mathbb{Z}^2 (for the two dimensional case). We consider $\mathcal{A}^{\mathbb{M}}$, the *configuration space* of \mathbb{M} -indexed sequences in \mathcal{A} . If \mathcal{A} is endowed with the discrete topology, $\mathcal{A}^{\mathbb{M}}$ is compact, perfect and totally disconnected in the product topology. Moreover one can define a metric on $\mathcal{A}^{\mathbb{M}}$ compatible with this topology:

$$\forall x, y \in \mathcal{A}^{\mathbb{M}}, \quad d_C(x, y) = 2^{-\min\{\|i\|_{\infty} : x_i \neq y_i, i \in \mathbb{M}\}}.$$

Let $\mathbb{U} \subset \mathbb{M}$. For $x \in \mathcal{A}^{\mathbb{M}}$, denote $x_{\mathbb{U}} \in \mathcal{A}^{\mathbb{U}}$ the restriction of x to \mathbb{U} . Let $\mathbb{U} \subset \mathbb{M}$ be a finite subset, Σ is a *subshift of finite type of order \mathbb{U}* if there exists $\mathcal{F} \subset \mathcal{A}^{\mathbb{U}}$ such that $x \in \Sigma \iff x_{m+\mathbb{U}} \in \mathcal{F} \quad \forall m \in \mathbb{M}$. In other words, Σ can be viewed as a tiling where the allowed patterns are in \mathcal{F} .

In the sequel, we will consider *tile sets* and ask whether they can tile the plane or not. In our formalism, a tile set is a subshift of finite type: a set of states (the

tiles) given together with a set of allowed patterns (the tiling constraints). We will restrict to 2×1 and 1×2 patterns (dominos) since it is sufficient to have the undecidability results of Berger [8].

A *cellular automaton* (CA) is a pair $(\mathcal{A}^{\mathbb{M}}, F)$ where $F : \mathcal{A}^{\mathbb{M}} \rightarrow \mathcal{A}^{\mathbb{M}}$ is defined by $F(x)_m = \overline{F}((x_{m+u})_{u \in \mathbb{U}})$ for all $x \in \mathcal{A}^{\mathbb{M}}$ and $m \in \mathbb{M}$ where $\mathbb{U} \subset \mathbb{Z}$ is a finite set named *neighbourhood* and $\overline{F} : \mathcal{A}^{\mathbb{U}} \rightarrow \mathcal{A}$ is a *local rule*. The radius of F is $r(F) = \max\{\|u\|_{\infty} : u \in \mathbb{U}\}$. By Hedlund's theorem [9], it is equivalent to say that F is a continuous function which commutes with the shift ($\sigma^m \circ F = F \circ \sigma^m$ for all $m \in \mathbb{M}$).

We recall here general definitions of topological dynamics used all along the article. Let (X, d) be a metric space and $F : X \rightarrow X$ be a continuous function.

- $x \in X$ is an *equicontinuous point* if for all $\varepsilon > 0$, there exists $\delta > 0$, such that for all $y \in X$, if $d(x, y) < \delta$ then $d(F^n(x), F^n(y)) < \varepsilon$ for all $n \in \mathbb{N}$.
- (X, F) is *sensitive* if there exists $\varepsilon > 0$ such that for all $\delta > 0$ and $x \in X$, there exists $y \in X$ and $n \in \mathbb{N}$ such that $d(x, y) < \delta$ and $d(F^n(x), F^n(y)) > \varepsilon$.

3 Non Sensitive and Without Any Equicontinuous Point

In this section, we will construct a 2D CA which has no equicontinuous point and is not sensitive to initial conditions. This is in contrast with dimension 1 where any non-sensitive CA must have equicontinuous points as shown in [1].

The CA (denoted by F in the following) is made of two components:

- an *obstacle component* which is almost static and for which only finite type conditions are checked and corrections are made locally ;
- a *particle component* whose overall behaviour is to move left and to bypass obstacles using a mechanism of separation into two parts.

Formally, F has a Moore's neighbourhood of radius 2 (25 neighbours) and a state set \mathcal{A} with 12 elements : $\mathcal{A} = \{U, D, 0, 1, \downarrow, \uparrow, \leftarrow, \rightarrow, \swarrow, \searrow, \nearrow, \nwarrow\}$ where the subset $\mathcal{A}_F = \{1, \downarrow, \uparrow, \leftarrow, \rightarrow, \swarrow, \searrow, \nearrow, \nwarrow\}$ corresponds to the obstacle component and $\{U, D, 0\}$ to the particle component.

Let Σ_F be the subshift of finite type of $\mathcal{A}^{\mathbb{Z}^2}$ defined by the set of allowed patterns constituted by all the 3×3 patterns appearing in the following set of finite configurations:

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where $*$ stand for any state in $\mathcal{A} \setminus \mathcal{A}_F$.

In the sequel, a configuration x is said to be *finite* if the set $\{z : x(z) \neq 0\}$ is finite. Moreover, in such a configuration, we call *obstacle* a maximal 4-connected region of states from \mathcal{A}_F .

The following lemma states that finite configurations from Σ_F consist of rectangle obstacles inside a free $\mathcal{A} \setminus \mathcal{A}_F$ background. Moreover, obstacles are spaced enough to ensure that any position “sees” at most one obstacle in its 3×3 neighbourhood.

Lemma 1. *Let $x \in \Sigma_F$ be a finite configuration. For any $z \in \mathbb{Z}^2$ we have the following:*

- either $x(z) \in \mathcal{A}_F$ and z belongs to a rectangular obstacle;
- or $x(z) \notin \mathcal{A}_F$ and the set of positions $\{z' : x(z') \in \mathcal{A}_F \text{ and } \|z' - z\|_\infty \leq 1\}$ is empty or belongs to the same obstacle.

Proof. Straightforward from the definition of Σ_F . □

The local transition function of F can be sketched as follows:

- states from \mathcal{A}_F are turned into 0’s if finite type conditions defining Σ_F are violated locally and left unchanged in any other case ;
- states U and D behave like a left-moving particle when U is just above D in a background of 0’s, and they separate to bypass obstacles, U going over and D going under, until they meet at the opposite position and recompose a left-moving particle (see figure 1).

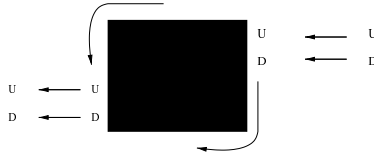


Fig. 1. A particle separating into two parts (U and D) to bypass an obstacle (the black region).

A precise definition of the local transition function of F is the following:

1. if the neighbourhood (5×5 cells) forms a pattern forbidden in Σ_F , then turn into state 0 ;
2. else, apply (if possible) one of the transition rules depending only on the 3×3 neighbourhood detailed in figure 2
3. in any other case, turn into state 0.

The possibility to form arbitrarily large obstacles prevents F from being sensitive to initial conditions.

Proposition 1. *F is not sensitive to initial conditions.*

$$\begin{array}{ccc}
\begin{array}{|c|c|c|} \hline * & * & * \\ \hline \mathcal{A}_F & x & * \\ \hline \mathcal{A}_F & \mathcal{A}_F & * \\ \hline \end{array} \mapsto x, & \begin{array}{|c|c|c|} \hline \mathcal{A}_F & \mathcal{A}_F & * \\ \hline \mathcal{A}_F & x & * \\ \hline \mathcal{A}_F & \mathcal{A}_F & * \\ \hline \end{array} \mapsto x, & \begin{array}{|c|c|c|} \hline \mathcal{A}_F & \mathcal{A}_F & \mathcal{A}_F \\ \hline \mathcal{A}_F & x & \mathcal{A}_F \\ \hline \mathcal{A}_F & \mathcal{A}_F & \mathcal{A}_F \\ \hline \end{array} \mapsto x, \\
\begin{array}{|c|c|c|} \hline 0/\mathcal{A}_F & 0 & 0 \\ \hline \mathcal{A}_F & 0 & 0 \\ \hline \mathcal{A}_F & U & 0 \\ \hline \end{array} \mapsto U, & \begin{array}{|c|c|c|} \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline \mathcal{A}_F & U & 0 \\ \hline \end{array} \mapsto U, & \begin{array}{|c|c|c|} \hline 0 & 0 & 0 \\ \hline 0 & 0 & U \\ \hline \mathcal{A}_F & \mathcal{A}_F & D/\mathcal{A}_F \\ \hline \end{array} \mapsto U, \\
\begin{array}{|c|c|c|} \hline 0 & 0 & 0 \\ \hline 0 & 0 & U \\ \hline 0 & 0/\mathcal{A}_F & \mathcal{A}_F \\ \hline \end{array} \mapsto U, & \begin{array}{|c|c|c|} \hline 0 & U & 0/\mathcal{A}_F \\ \hline 0 & 0 & \mathcal{A}_F \\ \hline 0 & 0 & \mathcal{A}_F \\ \hline \end{array} \mapsto U, & \begin{array}{|c|c|c|} \hline 0 & U & \mathcal{A}_F \\ \hline 0 & 0 & \mathcal{A}_F \\ \hline 0 & 0 & D \\ \hline \end{array} \mapsto U, \\
\begin{array}{|c|c|c|} \hline \mathcal{A}_F & D & 0 \\ \hline \mathcal{A}_F & 0 & 0 \\ \hline 0/\mathcal{A}_F & 0 & 0 \\ \hline \end{array} \mapsto D, & \begin{array}{|c|c|c|} \hline \mathcal{A}_F & D & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline \end{array} \mapsto D, & \begin{array}{|c|c|c|} \hline \mathcal{A}_F & \mathcal{A}_F & U/\mathcal{A}_F \\ \hline 0 & 0 & D \\ \hline 0 & 0 & 0 \\ \hline \end{array} \mapsto D, \\
\begin{array}{|c|c|c|} \hline 0 & 0/\mathcal{A}_F & \mathcal{A}_F \\ \hline 0 & 0 & D \\ \hline 0 & 0 & 0 \\ \hline \end{array} \mapsto D, & \begin{array}{|c|c|c|} \hline 0 & 0 & \mathcal{A}_F \\ \hline 0 & 0 & \mathcal{A}_F \\ \hline 0 & D & 0/\mathcal{A}_F \\ \hline \end{array} \mapsto D, & \begin{array}{|c|c|c|} \hline 0 & 0 & U \\ \hline 0 & 0 & \mathcal{A}_F \\ \hline 0 & D & \mathcal{A}_F \\ \hline \end{array} \mapsto D, \\
\begin{array}{|c|c|c|} \hline 0/\mathcal{A}_F & 0/\mathcal{A}_F & U \\ \hline 0/\mathcal{A}_F & 0 & D \\ \hline 0/\mathcal{A}_F & 0/\mathcal{A}_F & 0/\mathcal{A}_F \\ \hline \end{array} \mapsto D, & \begin{array}{|c|c|c|} \hline 0/\mathcal{A}_F & 0/\mathcal{A}_F & 0/\mathcal{A}_F \\ \hline 0/\mathcal{A}_F & 0 & U \\ \hline 0/\mathcal{A}_F & 0/\mathcal{A}_F & D \\ \hline \end{array} \mapsto U
\end{array}$$

Fig. 2. Transition rule of F where x stands for any state in \mathcal{A}_F , ‘*’ means any state in $\mathcal{A} \setminus \mathcal{A}_F$ (2 occurrences of * are independent), and curved arrows mean that the transition is the same for any rotation of the neighbourhood pattern.

Proof. Let $\varepsilon > 0$. Let c_ε be the configuration everywhere equal to 0 except in the square region of side $2\lceil -\log \varepsilon \rceil$ around the centre where there is an obstacle. $\forall y \in \mathcal{A}^{\mathbb{Z}^2}$, if $d(y, c_\varepsilon) \leq \varepsilon/4$ then $\forall t \geq 0$, $d(F^t(c_\varepsilon), F^t(y)) \leq \varepsilon$ since a well-formed obstacle¹ is inalterable for F provided it is surrounded by states in $\mathcal{A} \setminus \mathcal{A}_F$ (cf. 3 first transition rules of case 2 of definition above): this is guaranteed for y by the condition $d(y, c_\varepsilon) \leq \varepsilon/4$. \square

The next lemma shows that Σ_F attracts any finite configuration under the action of F .

Lemma 2. *For any finite configuration x , there exists t_0 such that $\forall t \geq t_0 : F^t(x) \in \Sigma_F$.*

Proof. First, the set $\{z : x(z) \in \mathcal{A}_F\}$ is finite and decreasing under the action of F . Moreover, U and D states can only move left, or move vertically or disappear. Since the total amount of vertical moves for U and D states is bounded by the cardinal of $\{z : x(z) \in \mathcal{A}_F\}$, there is a time t after which no U or D state is a neighbour of a state of \mathcal{A}_F , and each U is above a D in a 0 background (the UD particle is on the left of the finite non-0 region). From this time on, the evolution of cells in a state of \mathcal{A}_F is governed only by the first case of the definition of F . Therefore, after a certain time, finite type conditions defining Σ_F are verified

¹ Precisely, a partial configuration that would form a valid obstacle when completed by 0 everywhere.

everywhere. To conclude, it is easy to check that Σ_F is stable under the action of F . \square

The following lemma establish the key property of the dynamics of F : particles can reach any free position inside a finite field of obstacles from arbitrarily far away from the field.

Lemma 3. *Let $x \in \Sigma_F \cap (\{0\} \cup \mathcal{A}_F)^{\mathbb{Z}^2}$ be a finite configuration. For any z_0 such that $x(z_0) = 0$ there exists a path (z_n) such that:*

1. $\|z_n\|_\infty \rightarrow \infty$
2. $\exists n_0, \forall n \geq n_0$, if x_n is the configuration obtained from x by adding a particle at position z_n (precisely, $x_n(z_n) = U$ and $x_n(z_n + (0, -1)) = D$) then $(F^n(x_n))(z_0) \in \{U, D\}$.

Proof. First, since $x \in \Sigma_F$ and $x(z_0) = 0$, then either $x(z_0 + (0, 1)) = 0$ or $x(z_0 + (0, -1)) = 0$. We will consider only the first case since the proof for the second one is similar. Let (z_n) be the path starting from z_0 defined as follows:

- If $x(z_n + (1, 0)) = 0$ and $x(z_n + (1, -1)) = 0$ then $z_{n+1} = z_n + (1, 0)$.
- Else, position $z_n + (1, 0)$ and/or position $z_n + (1, -1)$ belongs to an obstacle P . Let a , b and c be the positions of the upper-left, upper-right and lower-right outside corners of P and let p be its half perimeter. Then define $z_{n+1}, \dots, z_{n+p+1}$ to be the sequence of positions made of (see figure 3):

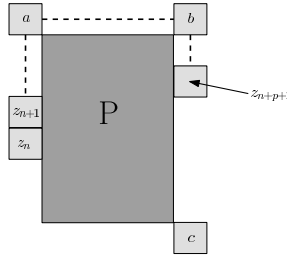


Fig. 3. Definition of the (z_n) path in the presence of obstacles.

- a (possibly empty) vertical segment from z_n to a ,
- the segment $[a; b]$,
- a (possibly empty) vertical segment from b to z_{n+p+1} where z_{n+p+1} is the point on $[b; c]$ such that $z_n a + b z_{n+p+1} = bc$.

We claim that the path (z_n) constructed above has the properties of the lemma. Indeed, one can check that for each case of the inductive construction from z_n to z_m we have:

- $\|z_m\|_\infty > \|z_n\|_\infty$,

– $(F^{m-n}(x_m))(z_n) = U$ and $(F^{m-n}(x_m))(z_n + (0, -1)) = D$ (straightforward from the definition of F). \square

Proposition 2. F has no equicontinuous point.

Proof. Suppose F has an equicontinuous point, precisely a point x which verifies $\forall \varepsilon > 0, \exists \delta : \forall y, d(x, y) \leq \delta \Rightarrow \forall t, d(F^t(x), F^t(y)) \leq \varepsilon$.

Suppose first that there is z_0 such that $x(z_0) = 0$ and let $\varepsilon = 2^{-\|z_0\|_\infty - 1}$. We will show that the hypothesis of x being an equicontinuous point is violated for this particular choice of ε . Consider any $\delta > 0$ and let y be the configuration everywhere equal to 0 except in the centre region of radius $-\log \lceil \delta \rceil$ where it is identical to x . Since y is finite, there exists t_0 such that $y_+ = F^{t_0}(y) \in \Sigma_F$ (by lemma 2). Moreover, the proof of lemma 2 guaranties that for any positive integer t $(F^t(y_+))(z_0) = x(z_0) = 0$. So we can apply lemma 3 on y_+ and position z_0 to get the existence of a path (z_n) allowing particles placed arbitrarily far away from z_0 to reach the position z_0 after a certain time. For any sufficiently large n , we can construct a configuration y' obtained from y by adding a particle at position z_n . By the property of (z_n) , we have the following: $(F^n(y))(z_0) \neq (F^n(y'))(z_0)$ and therefore $d(F^n(y), F^n(y')) > \varepsilon$. Since, if $n > -\log \lceil \delta \rceil$, both y and y' are in the ball of centre x and radius δ , we have the desired contradiction.

Suppose now that $\forall z, x(z) \in \mathcal{A}_F$. There must exist some z_0 such that $x(z_0) \neq 1$ (since the uniform configuration everywhere equal to 1 is not an equicontinuous point). It follows from the definition of Σ_F that z_0 belongs to a forbidden pattern for Σ_F . Therefore $(F(x))(z_0) = 0$ and we are brought back to the previous case of this proof. \square

4 Undecidability of Topological Classification Revisited

The undecidability of topological classification has been proven in [5] for 1D CA (except for the expansivity class). As showed above, the situation is different in dimension 2 since sensitivity and presence of equicontinuous points are not complementary classes. Therefore we have to consider a partition into three classes: \mathcal{E}_q , \mathcal{S} and \mathcal{N} .

We will use simulations of Turing machine by tile sets in the classical way (originally suggested by Wang [10]): the tiling represents the space-time diagram of the computation and the transition rule of the Turing machine are converted into tiling constraints. Without loss of generality, we only consider Turing machines working on a semi-infinite tape with a single final state. The i^{th} such machine in a standard enumeration is denoted by \mathcal{M}_i . In the sequel we use the following notations:

1. to each \mathcal{M}_i we associate a tile set T_i whose constraints ensure the simulation of \mathcal{M}_i as mentioned above;
2. when constructing a CA G , we denote by Σ_G the subshift of its admissible obstacles, which plays the same role as Σ_F for F with some differences detailed below.

In [5], a recursive construction is given which produce either a 1D sensitive CA or a 1D CA with equicontinuous points according to whether a Turing machine halts on the empty input. By noticing that a 1D CA is sensitive (resp. has equicontinuous points) in the 1D topology if and only if it is sensitive (resp. has equicontinuous points) in the 2D topology when viewed as a 2D CA, we get the following proposition.

Proposition 3. *There is a recursive function $\Phi_1 : \mathbb{N} \rightarrow AC$ such that $\Phi_1(i) \in \mathcal{E}_q$ if \mathcal{M}_i halts on the empty input and $\Phi_1(i) \in \mathcal{S}$ else.*

However, this is not enough to establish the overall undecidability of the topological classification of 2D CA. The main concern of this section is to complete proposition 3 in order to prove a stronger and more complete undecidability result summarised in the following theorem.

Theorem 1. *Each of the class \mathcal{E}_q , \mathcal{S} and \mathcal{N} is neither r.e. nor co-r.e. Moreover any pair of them is recursively inseparable.*

The proof of this theorem rely on different variants of the construction of the automaton F above. Each time, the construction scheme is the same, and the desired property is obtained by adding various contents inside obstacles and slightly changing the rules of destruction of obstacles according to that content.

The next proposition use a mechanism to bound or not the size of admissible obstacles according to whether a Turing machine halts or not.

Proposition 4. *There is a recursive function $\Phi_2 : \mathbb{N} \rightarrow AC$ such that $\Phi_2(i) \in \mathcal{S}$ if \mathcal{M}_i halts on the empty input and $\Phi_2(i) \in \mathcal{N}$ else.*

Proof (sketch). The CA $\Phi_2(i)$ is constructed from F by replacing the state 1 by the tile set T_i and by defining the obstacle subshift $\Sigma_{\Phi_2(i)}$ through the set of all 3×3 patterns satisfying the following conditions:

- constraints used in the simulation of \mathcal{M}_i by T_i apply;
- when replacing states of T_i by 1, the resulting pattern must be an admissible pattern for Σ_F ;
- the only allowed state as upper-right neighbour of \nearrow is q_0 , the initial state in the simulation of \mathcal{M}_i by T_i ;
- the only allowed state above \uparrow is the blank tape symbol of the simulation of \mathcal{M}_i by T_i when \nearrow is not in the neighbourhood;
- no final state of the simulation of \mathcal{M}_i by T_i is allowed anywhere.

It is clear from these conditions, that any admissible obstacle of $\Sigma_{\Phi_2(i)}$ contains the beginning of the computation of \mathcal{M}_i on the empty tape. Since any field of obstacles allowed in $\Sigma_{\Phi_2(i)}$ is also allowed in Σ_F , the proof of proposition 2 is still valid and $\Phi_2(i)$ is therefore either in class \mathcal{S} or in class \mathcal{N} . Moreover, as shown in the proof of proposition 1, the fact that $\Phi_2(i)$ is sensitive or not depends only on the existence of arbitrarily large admissible obstacle. So, by construction, $\Phi_2(i) \in \mathcal{S}$ if and only if \mathcal{M}_i halts on the empty input. \square

Before going on with the different constructions needed to prove theorem 1, let us stress the dynamical consequence of the construction of proposition 4.

It is well-known that for any 1D sensitive CA of radius r , 2^{-2r} is always the maximal admissible sensitivity constant (see for instance [1]). Thanks to the construction above it is easy to construct CA with tiny sensitivity constants as shown by the following proposition.

Proposition 5. *The (maximal admissible) sensitivity constant of sensitive 2D CA cannot be recursively (lower-)bounded in the number of states and the neighbourhood size.*

Proof. It is straightforward to check that for each CA $\Phi_2(i)$ where \mathcal{M}_i halts after n steps on the empty input, the maximal admissible obstacle is of height $O(n)$ and of width at least $O(\log(n))$. The proposition follows since the sensitivity constant of any CA $\Phi_2(i) \in \mathcal{S}$ is precisely $2^{-l/2+1}$ where l is the minimum between the largest height and the largest width of admissible obstacles. \square

Back to the path towards theorem 1, the following proposition uses the same ideas as proposition 4 but exchanging the role of halting and non-halting computations.

Proposition 6. *There is a recursive function $\Phi_3 : \mathbb{N} \rightarrow AC$ such that $\Phi_3(i) \in \mathcal{N}$ if \mathcal{M}_i halts on the empty input and $\Phi_3(i) \in \mathcal{S}$ else.*

Proof (sketch). Following the proof of proposition 4, $\Sigma_{\Phi_3(i)}$ is the same as $\Sigma_{\Phi_2(i)}$ with the following differences:

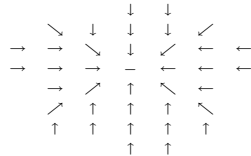
- the final state q_f in the simulation of \mathcal{M}_i by T_i is allowed;
- the only states allowed at the right and above an occurrence of q_f are q_f , \downarrow and \leftarrow ;
- the only state allowed as a lower-left neighbour of \swarrow is q_f .

It follows directly that an admissible obstacle must contain the final state. Moreover, if \mathcal{M}_i halts on the empty input, admissible obstacles can be arbitrarily large. The proposition follows by a straightforward adaptation of the proofs of propositions 1 and 2. \square

The properties of the CA F and the other constructions above rely on the fact that obstacles able to stop or deviate particles cannot be fit together to form larger obstacles. Thus, F and other CA have no equicontinuous point. In the following, we will use a new kind of obstacles: they are protected from particles by a boundary as the classical obstacles of F , but they are made only of successive boundaries like onion skins. With this new construction it is not difficult to build an equicontinuous point provided there are arbitrarily large valid obstacles. The next proposition use this idea to reduce existence of equicontinuous point to a tiling problem.

Proposition 7. *There is a recursive function Φ_4 which associate to any tile set T a CA $\Phi_4(T)$ which is in class \mathcal{E}_q if T tiles the plane and in class \mathcal{N} else.*

Proof (sketch). Given a tile set T , the AC $\Phi_4(T)$ is identical to F , except that it has a second kind of obstacles, called T -obstacles. T -obstacles are square patterns of states from the set $E = T \times X$ with $X = \{\downarrow, \uparrow, \leftarrow, \rightarrow, \swarrow, \searrow, \nwarrow, \nearrow, -\}$ and where the T component is a valid tiling and the X component is made from the set of 2×2 patterns appearing in the following finite configuration:



The X component is used to give everywhere in T -obstacles a local notion of *inside* and *outside* as depicted by figure 4: roughly speaking, arrows point to the inside region. Other constraints concerning T -obstacles are checked locally:

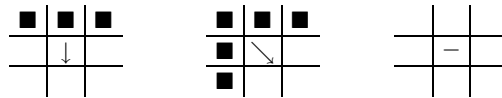


Fig. 4. Inside (white) and outside (black) positions for states of X . Two arrows in X which are equal up to a 90 degrees rotation (e.g. \downarrow and \leftarrow) have similar inside/outside positions up to the same rotation.

- any pair of obstacles must be at least 2 cells away from each other;
- their shape must be a square and this is ensured by requiring that any cell of a T -obstacle can have $\{0, U, D\}$ neighbours only in its outside region;
- the behaviour of particles with T -obstacles is the same as with classical obstacles (they can not cross them and states U and D separate to bypass them).

The overall dynamics of $\Phi_4(T)$ is similar to that of F with the following exception, which is the key point of the construction: destruction of non-valid T -obstacles is done progressively to preserve as much as possible valid zones inside non-valid obstacles. More precisely any cell in a state from E remains unchanged unless one of the following conditions is verified in which case it turns into state 0:

- if there is an error in the inside neighbourhood;
- if there is a position in the outside neighbourhood such that the pattern formed by that position together with the cell itself is forbidden;
- if there is a state from E in the 5×5 neighbourhood which is not connected to the cell by states from E .

One can check that proposition 1 is still true. Moreover proposition 2 is true if and only if there is a bound on the size of valid T -obstacle. Indeed, lemma 2 and 3 are

always true and the only point which can be eventually false with the CA $\Phi_4(T)$ is the last case in the proof of proposition 2. Precisely, if a valid configuration x such that $\forall z, x(z) \in E$ can be constructed, then it is an equicontinuous point. Otherwise, any x such that $\forall z, x(z) \in E$ contains an error somewhere and the proof scheme of proposition 2 can be applied to $\Phi_4(T)$. The proposition follows since such a valid x can be constructed if and only if T can tile the plane. \square

We have now established all the useful constructions for the main theorem.

Proof (of theorem 1). The fact that any pair of sets among \mathcal{E}_q , \mathcal{S} and \mathcal{N} is recursively inseparable follows directly from propositions 3, 4 and 7 (by Berger's theorem [8]). The fact that \mathcal{E}_q is neither r.e. nor co-r.e. follows from propositions 7 (by Berger's theorem) and 3 respectively, it follows from propositions 6 and 4 for \mathcal{S} , and from propositions 4 and 6 for \mathcal{N} . \square

Remark 1. All the decision problems above admit as input CA of arbitrary radius and state set. However, as in [5], one could find more natural to fix the state set and therefore the topological space in which all properties are expressed. Without entering into full details, one can be quickly convinced that all our constructions and therefore all our undecidability results can be adapted to work with a fixed state set (say $\{0, 1\}$). The key point is that our constructions use a specific subshift as a support to exhibit the desired dynamics and a single "error" state to turn into in case of a violation of subshift conditions. Thus, one can use state 0 as the "error" state and encode every other states used in the construction by a specific pattern of $\{0, 1\}$. Of course, this can be done only with a sufficiently large radius.

To finish this section, we will establish another difference between 1D and 2D concerning the complexity of equicontinuous points. Let us first recall by the following proposition (proof in [1]) that equicontinuous point in 1D CA can be generated by finite words often called "blocking" words.

Proposition 8. *For any 1D CA F with equicontinuous points, there is a word u such that ${}^\infty u {}^\infty$ is an equicontinuous point for F .*

The situation is more complex in 2D, where it is not always possible to find a recursive equicontinuous point as shown by the following proposition.

Proposition 9. *There exists a 2D CA having equicontinuous points, but only nonrecursive ones.*

Proof (sketch). In the construction of proposition 7, CA in class \mathcal{E}_q are constructed from tile sets in such a way that any equicontinuous point must have a component containing a valid tiling. The proposition follows since there exists a tile set T which tiles the plane but produces only non-recursive tilings [11]. \square

5 Open problems

It is well-known that equicontinuous CA are exactly ultimately periodic CA (if they are also bijective, they are periodic). The proof techniques developed by Kari in [6] allows to prove the following theorem.

Theorem 2. *There is no recursive lower-bound on the pre-period and period of 2D equicontinuous CA.*

An interesting open question in the continuation of this paper is to determine whether periods of 1D equicontinuous CA (bijective or not) can be recursively bounded or not. The only known result in 1D is that pre-periods are not recursively bounded (this is essentially the nilpotency problem).

Concerning classes \mathcal{E}_q and \mathcal{S} this time, it is interesting to notice that they are definable in elementary arithmetic. This is due to proposition 8 which shows that the existential quantification over configurations can be replaced by a quantification over finite words in the definition of \mathcal{E}_q . Proposition 9 shows that there is no hope to do the same with 2D CA. This raises the natural question of whether classes \mathcal{E}_q , \mathcal{S} and \mathcal{N} are definable in elementary arithmetic. If they are in the arithmetical hierarchy, it would also be interesting to know at which level.

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