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EMBEDDING COPRODUCTS OF PARTITION LATTICES

FRIEDRICH WEHRUNG

Dedicated to Béla Csákány for his 75th birthday

ABSTRACT. We prove that the lattice $\text{Eq } \Omega$ of all equivalence relations on an infinite set Ω contains, as a 0, 1-sublattice, the 0-coproduct of two copies of itself, thus answering a question by G. M. Bergman. Hence, by using methods initiated by de Bruijn and further developed by Bergman, we obtain that $\text{Eq } \Omega$ also contains, as a sublattice, the coproduct of $2^{\text{card } \Omega}$ copies of itself.

1. INTRODUCTION

Whitman's Theorem [10] states that every lattice L can be embedded into the lattice $\text{Eq } \Omega$ of all equivalence relations on some set Ω . The cardinality of Ω may be taken equal to $\text{card } L + \aleph_0$. There is not much room for improvement of the cardinality bound, as for example, $\text{Eq } \Omega$ cannot be embedded into its dual lattice. (We believe the first printed occurrence of this result to be Proposition 6.2 in G. M. Bergman's recent preprint [2], although it may have already been known for some time.) Hence the question of embeddability into $\text{Eq } \Omega$ of lattices of large cardinality (typically, $\text{card}(\text{Eq } \Omega) = 2^{\text{card } \Omega}$) is nontrivial.

In [2], Bergman also extends results of N. G. de Bruijn [3, 4] by proving various embedding results of large powers or copowers of structures such as symmetric groups, endomorphism rings, and monoids of self-maps of an infinite set Ω , into those same structures. The nature of the underlying general argument is categorical. The problem whether the lattice $\text{Eq } \Omega$ contains a coproduct (sometimes called "free product" by universal algebraists) of two, or more, copies of itself, was stated as an open question in a preprint version of that paper. In the present note, we solve this problem in the affirmative.

The idea of our proof is the following. The lattice $\text{Eq } \Omega$ of all equivalence relations on Ω is naturally isomorphic to the ideal lattice $\text{Id } K$ of the lattice K of all *finitely generated* equivalence relations, that is, those equivalence relations containing only finitely many non-diagonal pairs. Denote by $K \amalg^0 K$ the coproduct (amalgamation) of two copies of K above the common ideal 0. As $K \amalg^0 K$ has the same cardinality as Ω , it follows from Jónsson's proof of Whitman's Embedding Theorem that the lattice $\text{Id}(K \amalg^0 K)$ embeds into $\text{Eq } \Omega$. Finally, we prove that the ideal lattice functor preserves the coproduct \amalg^0 and one-one-ness (Theorem 5.2), in such a way that $(\text{Id } K) \amalg^0 (\text{Id } K)$ embeds into $\text{Id}(K \amalg^0 K)$. Then it is easy to extend this result to the usual coproduct $(\text{Id } K) \amalg (\text{Id } K)$.

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We also present an example (Example 5.3) that shows that the result of Theorem 5.2 does not extend to amalgamation above a common (infinite) ideal. That is, for a common ideal A of lattices B and C , the canonical homomorphism from $(\text{Id } B) \amalg_{\text{Id } A} (\text{Id } C)$ to $\text{Id}(B \amalg_A C)$ may not be one-to-one.

2. BASIC CONCEPTS

We refer to [7] for unexplained lattice-theoretical notions. For any subsets Q and X in a poset (i.e., partially ordered set) P , we put

$$Q \downarrow X = \{p \in Q \mid (\exists x \in X)(p \leq x)\} \quad \text{and} \quad Q \uparrow X = \{p \in Q \mid (\exists x \in X)(p \geq x)\}.$$

We also write $Q \downarrow x$, resp. $Q \uparrow x$ in case $X = \{x\}$. A subset Q of P is a *lower subset* of P if $Q = P \downarrow Q$.

A map $f: K \rightarrow L$ between lattices is *meet-complete* if for each $a \in K$ and each $X \subseteq K$, $a = \bigwedge X$ in K implies that $f(a) = \bigwedge f[X]$ in L . (Observe that we do not require either K or L to be a complete lattice.) When this is required only for nonempty X , we say that f is *nonempty-meet-complete*. *Join-completeness* and *nonempty-join-completeness* of maps are defined dually. We say that f is *complete* (resp., *nonempty-complete*) if it is both meet-complete and join-complete (resp., both nonempty-meet-complete and nonempty-join-complete). We say that f is *lower bounded* if $\{x \in K \mid y \leq f(x)\}$ is either empty or has a least element for each $y \in L$. *Upper bounded* homomorphisms are defined dually. Lower bounded homomorphisms are nonempty-meet-complete and upper bounded homomorphisms are nonempty-join-complete.

An *ideal* of a lattice L is a nonempty lower subset of L closed under finite joins. We denote by $\text{Id } L$ the lattice of all ideals of L . For a lattice homomorphism $f: K \rightarrow L$, the map $\text{Id } f: \text{Id } K \rightarrow \text{Id } L$ defined by

$$(\text{Id } f)(X) = L \downarrow f[X], \quad \text{for each } X \in \text{Id } K,$$

is a nonempty-join-complete lattice embedding. If L is a *0-lattice* (i.e., a lattice with least element), the canonical map $L \rightarrow \text{Id } L$, $x \mapsto L \downarrow x$ is a 0-lattice embedding. The assignment that to every lattice associates its dual lattice L^{op} (i.e., the lattice with the same underlying set as L but reverse ordering) is a category equivalence—and even a category isomorphism—from the category of all lattices to itself, that sends 0-lattices to 1-lattices. For every lattice L , we denote by L° the lattice obtained by adding a new zero element to L .

A lattice L is *upper continuous* if for each $a \in L$ and each upward directed subset $\{x_i \mid i \in I\}$ of L admitting a join, the equality $a \wedge \bigvee_{i \in I} x_i = \bigvee_{i \in I} (a \wedge x_i)$ holds. We shall often use upper continuity in the following form: if I is an upward directed poset and both $(x_i \mid i \in I)$ and $(y_i \mid i \in I)$ are isotone families with respective joins x and y , then the family $(x_i \wedge y_i \mid i \in I)$ has join $x \wedge y$.

Every algebraic lattice is upper continuous, so, for example, $\text{Id } L \cup \{\emptyset\}$ is upper continuous for any lattice L ; hence $\text{Id } L$ is also upper continuous. The lattice $\text{Eq } \Omega$ of all equivalence relations on a set Ω , partially ordered by inclusion, is an algebraic lattice, thus it is upper continuous. Other examples of upper continuous lattices that are not necessarily complete are given in [1]. For example, it follows from [1, Corollary 2.2] that *every finitely presented lattice is upper continuous*.

We denote by $\mathfrak{P}(\Omega)$ the powerset of a set Ω , and by ω the set of all natural numbers.

3. THE FREE LATTICE ON A PARTIAL LATTICE

We recall Dean's description of the free lattice on a partial lattice, see [5] or [6, Section XI.9]. A *partial lattice* is a poset (P, \leq) endowed with partial functions \bigvee and \bigwedge from the nonempty finite subsets of P to P such that if $p = \bigvee X$ (resp., $p = \bigwedge X$), then p is the greatest lower bound (resp., least upper bound) of X in P . An *o-ideal* of P is a lower subset A of P such that $p = \bigvee X$ and $X \subseteq A$ implies that $p \in A$ for each $p \in P$ and each nonempty finite subset X of P . The set $\overline{\text{Id}}P$ of all o-ideals of P , partially ordered by inclusion, is an algebraic lattice. Observe that $\overline{\text{Id}}P = (\text{Id}P) \cup \{\emptyset\}$ in case P is a lattice. O-filters are defined dually; again, the lattice $\overline{\text{Fil}}P$ of all o-filters of P , partially ordered by inclusion, is algebraic. We denote by $\mathcal{J}(A)$ (resp., $\mathcal{F}(A)$) the least o-ideal (resp., o-filter) of P containing a subset A of P .

The *free lattice* $F_{\mathbf{L}}(P)$ on P is generated, as a lattice, by an isomorphic copy of P , that we shall identify with P . (The subscript \mathbf{L} in $F_{\mathbf{L}}(P)$ stands for the variety of all lattices, as the “free lattice on P ” construction can be carried out in any variety of lattices.) For each $x \in F_{\mathbf{L}}(P)$, the following subsets of P ,

$$\mathcal{J}(x) = P \downarrow x = \{p \in P \mid p \leq x\} \quad \text{and} \quad \mathcal{F}(x) = P \uparrow x = \{p \in P \mid x \leq p\}$$

are, respectively, an o-ideal and an o-filter of P , which can also be evaluated by the following rules:

$$\mathcal{J}(x \vee y) = \mathcal{J}(x) \vee \mathcal{J}(y) \text{ in } \overline{\text{Id}}P, \quad \mathcal{F}(x \vee y) = \mathcal{F}(x) \cap \mathcal{F}(y); \quad (3.1)$$

$$\mathcal{J}(x \wedge y) = \mathcal{J}(x) \cap \mathcal{J}(y), \quad \mathcal{F}(x \wedge y) = \mathcal{F}(x) \vee \mathcal{F}(y) \text{ in } \overline{\text{Fil}}P, \quad (3.2)$$

for all $x, y \in F_{\mathbf{L}}(P)$. The natural partial ordering on $F_{\mathbf{L}}(P)$ satisfies the following “Whitman-type” condition:

$$\begin{aligned} x_0 \wedge x_1 \leq y_0 \vee y_1 &\iff \text{either } (\exists p \in P)(x_0 \wedge x_1 \leq p \leq y_0 \vee y_1) \\ &\text{or there is } i < 2 \text{ such that either } x_i \leq y_0 \vee y_1 \text{ or } x_0 \wedge x_1 \leq y_i, \end{aligned} \quad (3.3)$$

which is also the basis of the inductive definition of that ordering.

4. THE 0-COPRODUCT OF A FAMILY OF LATTICES WITH ZERO

Our development of the 0-coproduct of a family of lattices with zero below bears some similarities with the development of coproducts (called there *free products*) given in [7, Chapter VI]. Nevertheless, as we use the known results about the free lattice on a partial lattice (outlined in Section 3), our presentation becomes significantly shorter.

Let $(L_i \mid i \in I)$ be a family of lattices with zero. Modulo the harmless set-theoretical assumption that $L_i \cap L_j = \{0\}$ for all distinct indices $i, j \in I$, the *coproduct* (often called *free product* by universal algebraists) of $(L_i \mid i \in I)$ can be easily described as $F_{\mathbf{L}}(P)$, where P is the partial lattice whose underlying set is the union $\bigcup_{i \in I} L_i$, whose underlying partial ordering is the one generated by the partial orders on all the L_i s, and whose partial lattice structure consists of all existing joins and meets of nonempty finite subsets in each “component” L_i . We denote this lattice by $L = \coprod_{i \in I}^0 L_i$, the superscript 0 meaning that the coproduct of the L_i s is evaluated in the category of all 0-lattices and 0-preserving homomorphisms, which we shall often emphasize by saying “0-coproduct” instead of just coproduct. We shall also identify each L_i with its canonical copy in L . Of course, the coproduct

of any family of lattices $(L_i \mid i \in I)$ in the variety of all lattices is the sublattice of $\prod_{i \in I}^0 (L_i)^\circ$ generated by the union of the images of the L_i s.

Now we shall analyze further the structure of the 0-coproduct L , in a fashion similar to the development in [7, Chapter VI]. We add a new largest element, denoted by ∞ , to L , and we set $\overline{L}_i = L_i \cup \{\infty\}$ for each $i \in I$. The following lemma is an analogue, for 0-coproducts instead of coproducts, of [7, Theorem VI.1.10].

Lemma 4.1. *For each $x \in L$ and each $i \in I$, there are a largest element of L_i below x and a least element of \overline{L}_i above x with respect to the ordering of $L \cup \{\infty\}$. Furthermore, if we denote these elements by $x_{(i)}$ and $x^{(i)}$, respectively, then the following formulas hold:*

$$\begin{aligned} p_{(i)} &= p^{(i)} = p, \text{ if } p \in L_i; \\ p_{(i)} &= 0 \text{ and } p^{(i)} = \infty, \text{ if } p \in P \setminus L_i; \\ (x \vee y)_{(i)} &= x_{(i)} \vee y_{(i)} \text{ and } (x \wedge y)_{(i)} = x_{(i)} \wedge y_{(i)}; \\ (x \vee y)^{(i)} &= x^{(i)} \vee y^{(i)}; \\ (x \wedge y)^{(i)} &= \begin{cases} 0, & \text{if } x^{(j)} \wedge y^{(j)} = 0 \text{ for some } j \in I, \\ x^{(i)} \wedge y^{(i)}, & \text{otherwise,} \end{cases} \end{aligned} \tag{4.1}$$

for each $x, y \in L$ and each $i \in I$.

Proof. For an element x of L , abbreviate by “ $x_{(i)}$ exists” (resp., “ $x^{(i)}$ exists”) the statement that $L_i \downarrow x$ is a principal ideal in L_i (resp., $\overline{L}_i \uparrow x$ is a principal filter in \overline{L}_i), and then denote by $x_{(i)}$ (resp., $x^{(i)}$) the largest element of $L_i \downarrow x$ (resp., the least element of $\overline{L}_i \uparrow x$). Denote by K the set of all $x \in L$ such that both $x_{(i)}$ and $x^{(i)}$ exist for each $i \in I$. It is clear that K contains P and that both $p_{(i)}$ and $p^{(i)}$ are given by the first two formulas of (4.1), for any $p \in P$. Furthermore, it follows immediately from the definition of K that

$$\mathcal{J}(z) = \bigcup_{i \in I} (L_i \downarrow z_{(i)}), \tag{4.2}$$

$$\mathcal{F}(z) = \bigcup_{i \in I} (L_i \uparrow z^{(i)}), \tag{4.3}$$

for each $z \in K$. We shall establish that K is a sublattice of L . So let $x, y \in K$, put $u = x \wedge y$ and $v = x \vee y$. It is straightforward that for each $i \in I$, both $u_{(i)}$ and $v^{(i)}$ exist, and

$$u_{(i)} = x_{(i)} \wedge y_{(i)}, \quad v^{(i)} = x^{(i)} \vee y^{(i)}. \tag{4.4}$$

Now we shall prove that $v_{(i)}$ exists and is equal to $x_{(i)} \vee y_{(i)}$. By the induction hypothesis, (4.2) holds at both x and y . So, as $\mathcal{J}(v) = \mathcal{J}(x) \vee \mathcal{J}(y)$, in order to get the asserted existence and description of the elements $v_{(i)}$, it suffices to prove that

$$\bigcup_{i \in I} (L_i \downarrow x_{(i)}) \vee \bigcup_{i \in I} (L_i \downarrow y_{(i)}) = \bigcup_{i \in I} (L_i \downarrow (x_{(i)} \vee y_{(i)})). \tag{4.5}$$

The containment from left to right is obvious, and each $x_{(i)} \vee y_{(i)}$ is contained in any o-ideal of P containing $\{x_{(i)}, y_{(i)}\}$, so it suffices to prove that the right hand side of (4.5) is an o-ideal of P . As the join operation in P is internal to each L_i , this set is closed under joins. As each L_i is a lower subset of P , this set is also a lower subset of P . This establishes the desired result for the $v_{(i)}$ s.

It remains to prove that $u^{(i)}$ exists and is equal to z_i , where $z_i = x^{(i)} \wedge y^{(i)}$ if $x^{(j)} \wedge y^{(j)} \neq 0$ for all j , and $z_i = 0$ otherwise. By the induction hypothesis, (4.3) holds at both x and y . So, as $\mathcal{F}(u) = \mathcal{F}(x) \vee \mathcal{F}(y)$, in order to get the asserted existence and description of the elements $u^{(i)}$, it suffices to prove that

$$\bigcup_{i \in I} (L_i \uparrow x^{(i)}) \vee \bigcup_{i \in I} (L_i \uparrow y^{(i)}) = \bigcup_{i \in I} (L_i \uparrow z_i). \quad (4.6)$$

The containment from left to right is obvious. If an o-filter U of P contains $\{x^{(i)}, y^{(i)}\}$ for all $i \in I$, then it also contains all elements $x^{(i)} \wedge y^{(i)}$; in particular, it is equal to P in case $x^{(i)} \wedge y^{(i)} = 0$ for some i . In any case, $z_i \in U$ for all $i \in I$. So it suffices to prove that the right hand side of (4.6) is an o-filter of P . This is trivial in case $z_i = 0$ for some i , so suppose that $z_i \neq 0$ for all i . As the meet operation in P is internal to each L_i , the right hand side of (4.6) is closed under meets. As each $L_i \setminus \{0\}$ is an upper subset of P , this set is also an upper subset of P . This establishes the desired result for the $u^{(i)}$ s. \square

Lemma 4.2. *Let K_i be a 0-sublattice of a lattice L_i , for each $i \in I$. Then the canonical 0-lattice homomorphism $f: \prod_{i \in I}^0 K_i \rightarrow \prod_{i \in I}^0 L_i$ is an embedding.*

Proof. By the amalgamation property for lattices [7, Section V.4], the i -th coprojection from K_i to K is an embedding, for each $i \in I$. Put $L'_i = K \amalg_{K_i} L_i$ for each $i \in I$. Comparing the universal properties, it is immediate that the 0-coproduct L of $(L_i \mid i \in I)$ is also the coproduct of $(L'_i \mid i \in I)$ over K . Again by using the amalgamation property for lattices, all canonical maps from the L'_i s to L are embeddings. So, in particular, the canonical map from their common sublattice K to L is an embedding. \square

We shall call the adjoint maps $\alpha_i: x \mapsto x_{(i)}$ and $\beta_i: x \mapsto x^{(i)}$ the canonical lower, resp. upper adjoint of L onto L_i , resp. \overline{L}_i . Observe that these maps may not be defined in the case of amalgamation of two lattices over a common sublattice, as Example 5.3 will show. (In that example, there is no largest element of B below $b_0 \vee c_0$.)

The following result is an immediate consequence of well-known general properties of adjoint maps.

Corollary 4.3. *The canonical embedding from L_i into L is both lower bounded and upper bounded, for each $i \in I$. In particular, it is a nonempty-complete lattice homomorphism. Furthermore, the lower adjoint α_i is meet-complete while the upper adjoint β_i is nonempty-join-complete.*

In the following lemma, we shall represent the elements of $L = \prod_{i \in I}^0 L_i$ in the form $\mathbf{p}(\vec{a})$, where \mathbf{p} is a lattice term with variables from $I \times \omega$ and the ‘‘vector’’ $\vec{a} = (a_{i,n} \mid (i,n) \in I \times \omega)$ is an element of the cartesian product $\Pi = \prod_{(i,n) \in I \times \omega} L_i$. Define a *support* of \mathbf{p} as a subset J of I such that \mathbf{p} involves only variables from $J \times \omega$. Obviously, \mathbf{p} has a finite support. It is straightforward from (4.1) that $\mathbf{p}(\vec{a})_{(i)} = 0$ and either $\mathbf{p}(\vec{a}) = 0$ or $\mathbf{p}(\vec{a})^{(i)} = \infty$, for each i outside a support of \mathbf{p} .

Lemma 4.4. *Let Λ be an upward directed poset, let $(\vec{a}^\lambda \mid \lambda \in \Lambda)$ be an isotone family of elements of Π with supremum \vec{a} in Π , and let \mathbf{p} be a lattice term. If all the lattices L_i are upper continuous, then $\mathbf{p}(\vec{a}) = \bigvee_{\lambda \in \Lambda} \mathbf{p}(\vec{a}^\lambda)$ in L .*

Again, Example 5.3 will show that Lemma 4.4 fails to extend to the amalgam of two lattices over a common ideal.

Proof. As $\mathbf{p}(\vec{a})$ is clearly an upper bound for all elements $\mathbf{p}(\vec{a}^\lambda)$, it suffices to prove that for each lattice term \mathbf{q} on $I \times \omega$ and each $\vec{b} \in \Pi$ such that $\mathbf{p}(\vec{a}^\lambda) \leq \mathbf{q}(\vec{b})$ for all $\lambda \in \Lambda$, the inequality $\mathbf{p}(\vec{a}) \leq \mathbf{q}(\vec{b})$ holds. We argue by induction on the sums of the lengths of \mathbf{p} and \mathbf{q} . The case where \mathbf{p} is a projection follows immediately from the second sentence of Corollary 4.3. The case where either \mathbf{p} is a join or \mathbf{q} is a meet is straightforward.

Now suppose that $\mathbf{p} = \mathbf{p}_0 \wedge \mathbf{p}_1$ and $\mathbf{q} = \mathbf{q}_0 \vee \mathbf{q}_1$. We shall make repeated uses of the following easily established principle, which uses only the assumption that Λ is upward directed:

For every positive integer n and every $X_0, \dots, X_{n-1} \subseteq \Lambda$, if $\bigcup_{i < n} X_i$ is cofinal in Λ , then one of the X_i s is cofinal in Λ .

Now we use (3.3). If there exists a cofinal subset Λ' of Λ such that

$$(\forall \lambda \in \Lambda')(\exists i < 2)(\text{either } \mathbf{p}_i(\vec{a}^\lambda) \leq \mathbf{q}(\vec{b}) \text{ or } \mathbf{p}(\vec{a}^\lambda) \leq \mathbf{q}_i(\vec{b})),$$

then there are $i < 2$ and a smaller cofinal subset Λ'' of Λ' such that

$$\begin{aligned} \text{either } & (\forall \lambda \in \Lambda'')(\mathbf{p}_i(\vec{a}^\lambda) \leq \mathbf{q}(\vec{b})) \\ \text{or } & (\forall \lambda \in \Lambda'')(\mathbf{p}(\vec{a}^\lambda) \leq \mathbf{q}_i(\vec{b})). \end{aligned}$$

In the first case, it follows from the induction hypothesis that $\mathbf{p}_i(\vec{a}) \leq \mathbf{q}(\vec{b})$. In the second case, it follows from the induction hypothesis that $\mathbf{p}(\vec{a}) \leq \mathbf{q}_i(\vec{b})$. In both cases, $\mathbf{p}(\vec{a}) \leq \mathbf{q}(\vec{b})$. It remains to consider the case where there exists a cofinal subset Λ' of Λ such that

$$(\forall \lambda \in \Lambda')(\exists c_\lambda \in P)(\mathbf{p}(\vec{a}^\lambda) \leq c_\lambda \leq \mathbf{q}(\vec{b})).$$

It follows from the induction hypothesis that

$$\mathbf{p}_\ell(\vec{a}) = \bigvee_{\lambda \in \Lambda'} \mathbf{p}_\ell(\vec{a}^\lambda), \quad \text{for all } \ell < 2. \quad (4.7)$$

Fix a common finite support J of $\mathbf{p}_0, \mathbf{p}_1, \mathbf{q}_0, \mathbf{q}_1$. Each c_λ belongs to L_i , for some i in the given support J . By using the finiteness of J and by extracting a further cofinal subset of Λ' , we may assume that all those i are equal to the same index $j \in J$. Hence we have reduced the problem to the case where

$$(\forall \lambda \in \Lambda')(\mathbf{p}(\vec{a}^\lambda) \leq c_\lambda \leq \mathbf{q}(\vec{b})), \quad \text{where } c_\lambda = \mathbf{p}(\vec{a}^\lambda)^{(j)} \in L_j. \quad (4.8)$$

If $\mathbf{p}_0(\vec{a})^{(i)} \wedge \mathbf{p}_1(\vec{a})^{(i)} = 0$ for some $i \in I$, then $\mathbf{p}(\vec{a}) = 0 \leq \mathbf{q}(\vec{b})$ and we are done. Now suppose that $\mathbf{p}_0(\vec{a})^{(i)} \wedge \mathbf{p}_1(\vec{a})^{(i)} \neq 0$ for all $i \in I$. By using (4.7), the finiteness of J , and the upper continuity of L_i , we obtain that there exists a cofinal subset Λ'' of Λ' such that

$$(\forall \lambda \in \Lambda'')(\forall i \in J)(\mathbf{p}_0(\vec{a}^\lambda)^{(i)} \wedge \mathbf{p}_1(\vec{a}^\lambda)^{(i)} \neq 0).$$

In particular, both $\mathbf{p}_0(\vec{a}^\lambda)$ and $\mathbf{p}_1(\vec{a}^\lambda)$ are nonzero for each $\lambda \in \Lambda''$. As J is a common support of \mathbf{p}_0 and \mathbf{p}_1 , the equality $\mathbf{p}_0(\vec{a}^\lambda)^{(i)} \wedge \mathbf{p}_1(\vec{a}^\lambda)^{(i)} = \infty$ holds for all $\lambda \in \Lambda''$ and all $i \in I \setminus J$, hence

$$(\forall \lambda \in \Lambda'')(\forall i \in I)(\mathbf{p}_0(\vec{a}^\lambda)^{(i)} \wedge \mathbf{p}_1(\vec{a}^\lambda)^{(i)} \neq 0).$$

Thus it follows from (4.1) that $c_\lambda = \mathbf{p}(\vec{a}^\lambda)^{(j)} = \mathbf{p}_0(\vec{a}^\lambda)^{(j)} \wedge \mathbf{p}_1(\vec{a}^\lambda)^{(j)}$ for each $\lambda \in \Lambda''$. Hence, by the upper continuity of L_j (and thus of \bar{L}_j), (4.7), and the previously observed fact that the upper adjoint β_j is nonempty-join-complete, $\{c_\lambda \mid \lambda \in \Lambda''\}$ has a join in L_j , which is equal to $\mathbf{p}_0(\vec{a})^{(j)} \wedge \mathbf{p}_1(\vec{a})^{(j)} = \mathbf{p}(\vec{a})^{(j)}$. Therefore, it follows from (4.8) that $\mathbf{p}(\vec{a}) \leq \mathbf{p}(\vec{a})^{(j)} \leq \mathbf{q}(\vec{b})$. \square

5. IDEAL LATTICES AND 0-COPRODUCTS

In this section we fix again a family $(L_i \mid i \in I)$ of lattices with zero, pairwise intersecting in $\{0\}$, and we form $L = \coprod_{i \in I}^0 L_i$. We denote by $\varepsilon_i: \text{Id } L_i \hookrightarrow \text{Id } L$ the 0-lattice homomorphism induced by the canonical embedding $L_i \hookrightarrow L$, for each $i \in I$. By the universal property of the coproduct, there exists a unique 0-lattice homomorphism $\varepsilon: \coprod_{i \in I}^0 \text{Id } L_i \rightarrow \text{Id } L$ such that $\varepsilon_i = \varepsilon \upharpoonright_{\text{Id } L_i}$ for each $i \in I$. Observe that in case I is finite, the lattice $\coprod_{i \in I}^0 \text{Id } L_i$ has $\bigvee_{i \in I} L_i$ as a largest element, and this element is sent by ε to L (because every element of L lies below some join of elements of the L_i s). Hence, *if the index set I is finite, then the map ε preserves the unit as well.*

Lemma 5.1. *Let \mathbf{p} be a lattice term on $I \times \omega$ and let $\vec{X} = (X_{i,n} \mid (i,n) \in I \times \omega)$ be an element of $\prod_{(i,n) \in I \times \omega} \text{Id } L_i$. We put $\vec{\varepsilon}\vec{X} = (\varepsilon_i(X_{i,n}) \mid (i,n) \in I \times \omega) \in (\text{Id } L)^{I \times \omega}$. Then the following equality holds.*

$$\mathbf{p}(\vec{\varepsilon}\vec{X}) = L \downarrow \{ \mathbf{p}(\vec{x}) \mid \vec{x} \vec{\varepsilon} \vec{X} \},$$

where “ $\vec{x} \vec{\varepsilon} \vec{X}$ ” stands for $(\forall (i,n) \in I \times \omega)(x_{i,n} \in X_{i,n})$.

Proof. We argue by induction on the length of the term \mathbf{p} . If \mathbf{p} is a projection, then the result follows immediately from the definition of the maps ε_i . If \mathbf{p} is either a join or a meet, then the result follows immediately from the expressions for the join and the meet in the ideal lattice of L , in a fashion similar to the end of the proof of [7, Lemma I.4.8]. \square

Theorem 5.2. *The canonical map $\varepsilon: \coprod_{i \in I}^0 \text{Id } L_i \rightarrow \text{Id}(\coprod_{i \in I}^0 L_i)$ is a 0-lattice embedding.*

Proof. We put again $L = \coprod_{i \in I}^0 L_i$. Let \mathbf{p}, \mathbf{q} be lattice terms in $I \times \omega$ and let $\vec{X} \in \prod_{(i,n) \in I \times \omega} \text{Id } L_i$ such that $\mathbf{p}(\vec{\varepsilon}\vec{X}) \leq \mathbf{q}(\vec{\varepsilon}\vec{X})$ in $\text{Id } L$. We must prove that $\mathbf{p}(\vec{X}) \leq \mathbf{q}(\vec{X})$ in $\coprod_{i \in I}^0 \text{Id } L_i$. For each $\vec{x} \vec{\varepsilon} \vec{X}$, the inequalities $L \downarrow \mathbf{p}(\vec{x}) \leq \mathbf{p}(\vec{\varepsilon}\vec{X}) \leq \mathbf{q}(\vec{\varepsilon}\vec{X})$ hold in $\text{Id } L$, thus, by Lemma 5.1, there exists $\vec{y} \vec{\varepsilon} \vec{X}$ such that $L \downarrow \mathbf{p}(\vec{x}) \leq L \downarrow \mathbf{q}(\vec{y})$ in $\text{Id } L$, that is, $\mathbf{p}(\vec{x}) \leq \mathbf{q}(\vec{y})$ in L . Therefore, by applying the canonical map from $L = \coprod_{i \in I}^0 L_i$ to $\coprod_{i \in I}^0 \text{Id } L_i$ and putting $\vec{L} \downarrow \vec{x} = (L_i \downarrow x_{i,n} \mid (i,n) \in I \times \omega)$, we obtain

$$\mathbf{p}(\vec{L} \downarrow \vec{x}) \leq \mathbf{q}(\vec{L} \downarrow \vec{y}) \leq \mathbf{q}(\vec{X}) \quad \text{in } \coprod_{i \in I}^0 \text{Id } L_i. \quad (5.1)$$

As \vec{X} is equal to the directed join $\bigvee_{\vec{x} \vec{\varepsilon} \vec{X}} (\vec{L} \downarrow \vec{x})$ in $\prod_{(i,n) \in I \times \omega} \text{Id } L_i$ and each $\text{Id } L_i$ is upper continuous, it follows from Lemma 4.4 that

$$\mathbf{p}(\vec{X}) = \bigvee \left(\mathbf{p}(\vec{L} \downarrow \vec{x}) \mid \vec{x} \vec{\varepsilon} \vec{X} \right) \quad \text{in } \coprod_{i \in I}^0 \text{Id } L_i.$$

Therefore, it follows from (5.1) that

$$\mathbf{p}(\vec{X}) \leq \mathbf{q}(\vec{X}) \quad \text{in } \coprod_{i \in I}^0 \text{Id } L_i. \quad \square$$

The following example shows that Theorem 5.2 does not extend to the amalgam $B \amalg_A C$ of two lattices B and C above a common ideal A . The underlying idea can be traced back to Grätzer and Schmidt in [8, Section 5].

Example 5.3. *Lattices B and C with a common ideal A such that the canonical lattice homomorphism $f: (\text{Id } B) \amalg_{\text{Id } A} (\text{Id } C) \rightarrow \text{Id}(B \amalg_A C)$ is not one-to-one.*

Proof. Denote by K the poset represented in Figure 1. We claim that the subsets A , B , and C of K defined by

$$A = \{a_n \mid n < \omega\} \cup \{p_n \mid n < \omega\} \cup \{q_n \mid n < \omega\},$$

$$B = A \cup \{b_n \mid n < \omega\},$$

$$C = A \cup \{c_n \mid n < \omega\}.$$

are as required. Observe that B and C are isomorphic lattices and that A is an ideal of both B and C .

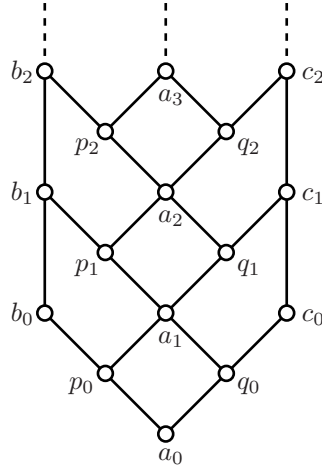


FIGURE 1. The poset K .

The map f is the unique lattice homomorphism that makes the diagram of Figure 2 commute. Unlabeled arrows are the corresponding canonical maps.

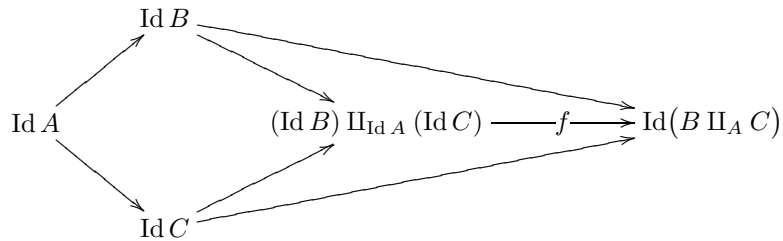


FIGURE 2. The commutative diagram defining the homomorphism f .

Put $D = B \amalg_A C$ and identify B and C with their images in D . Further, we endow $\text{Id } B \cup \text{Id } C$ with its natural structure of partial lattice, that is, the ordering is

the union of the orderings of $\text{Id } B$ and $\text{Id } C$ (remember that $A = B \cap C$ is an ideal of both B and C) and the joins and meets are those taking place in either $\text{Id } B$ or $\text{Id } C$. Observe that $\text{Id } A = \text{Id } B \cap \text{Id } C$ and $(\text{Id } B) \amalg_{\text{Id } A} (\text{Id } C)$ is the free lattice on the partial lattice $(\text{Id } B) \cup (\text{Id } C)$. As the latter is identified with its canonical image in $(\text{Id } B) \amalg_{\text{Id } A} (\text{Id } C)$, the elements A , $B \downarrow b_0$, and $C \downarrow c_0$ belong to $(\text{Id } B) \amalg_{\text{Id } A} (\text{Id } C)$.

We prove by induction that $a_n \leq b_0 \vee c_0$ in D for all $n < \omega$. This is trivial for $n = 0$. Suppose that $a_n \leq b_0 \vee c_0$. Then $a_n \vee b_0 \leq b_0 \vee c_0$, but B is a sublattice of D containing the subset $\{a_n, b_0\}$ with join b_n , thus $b_n \leq b_0 \vee c_0$, and thus $p_n \leq b_0 \vee c_0$. Similarly, $q_n \leq b_0 \vee c_0$, but A is a sublattice of D containing the subset $\{p_n, q_n\}$ with join a_{n+1} , and thus $a_{n+1} \leq b_0 \vee c_0$, which completes the induction step.

So we have established the inequality

$$f(A) \leq f(B \downarrow b_0) \vee f(C \downarrow c_0) \quad \text{in } \text{Id}(B \amalg_A C) = \text{Id } D. \quad (5.2)$$

Now observe that $\{B \downarrow x \mid x \in B\} \cup \{C \downarrow y \mid y \in C\}$ is an \mathfrak{o} -ideal of the partial lattice $(\text{Id } B) \cup (\text{Id } C)$, containing $\{B \downarrow b_0, C \downarrow c_0\}$ and to which A does not belong. Hence, $A \notin \mathcal{J}(\{B \downarrow b_0, C \downarrow c_0\})$, which means that $A \not\leq (B \downarrow b_0) \vee (C \downarrow c_0)$ in $(\text{Id } B) \amalg_{\text{Id } A} (\text{Id } C)$. Therefore, by (5.2), f is not an embedding. \square

As observed before, this example shows that Lemma 4.4 fails to extend to the amalgam of two lattices over a common ideal. Indeed, while $A = \bigvee_n (A \downarrow a_n)$ in $\text{Id } B$, the same equality fails in $(\text{Id } B) \amalg_{\text{Id } A} (\text{Id } C)$. The reason for this is that $A \downarrow a_n \leq (B \downarrow b_0) \vee (C \downarrow c_0)$ for each n , while $A \not\leq (B \downarrow b_0) \vee (C \downarrow c_0)$.

6. EMBEDDING COPRODUCTS OF INFINITE PARTITION LATTICES

Whitman's Embedding Theorem states that every lattice embeds into $\text{Eq } \Omega$, for some set Ω . We shall use a proof of Whitman's Theorem due to B. Jónsson [9], see also [7, Section IV.4]. The following result is proved there.

Lemma 6.1. *For every lattice L with zero, there are an infinite set Ω and a map $\delta: \Omega \times \Omega \rightarrow L$ satisfying the following properties:*

- (1) $\delta(x, y) = 0$ iff $x = y$, for all $x, y \in \Omega$.
- (2) $\delta(x, y) = \delta(y, x)$, for all $x, y \in \Omega$.
- (3) $\delta(x, z) \leq \delta(x, y) \vee \delta(y, z)$, for all $x, y, z \in L$.
- (4) For all $x, y \in \Omega$ and all $a, b \in L$ such that $\delta(x, y) \leq a \vee b$, there are $z_1, z_2, z_3 \in \Omega$ such that $\delta(x, z_1) = a$, $\delta(z_1, z_2) = b$, $\delta(z_2, z_3) = a$, and $\delta(z_3, y) = b$.

Observe, in particular, that the map δ is *surjective*. Furthermore, a straightforward Löwenheim-Skolem type argument ("keeping only the necessary elements in Ω ") shows that one may take $\text{card } \Omega = \text{card } L + \aleph_0$.

The following is the basis for Jónsson's proof of Whitman's Embedding Theorem.

Corollary 6.2. *For every lattice L with zero and every set Ω such that $\text{card } \Omega = \text{card } L + \aleph_0$, there exists a complete lattice embedding from $\text{Id } L$ into $\text{Eq } \Omega$.*

Proof. Any map δ as in Lemma 6.1 gives rise to a map $\varphi: \text{Id } L \rightarrow \text{Eq } \Omega$ defined by the rule

$$\varphi(A) = \{(x, y) \in \Omega \times \Omega \mid \delta(x, y) \in A\}, \quad \text{for each } A \in \text{Id } L, \quad (6.1)$$

and conditions (1)–(4) above imply that φ is a complete lattice embedding. \square

Theorem 6.3. *Let Ω be an infinite set. Then there exists a 0, 1-lattice embedding from $(\text{Eq } \Omega) \amalg^0 (\text{Eq } \Omega)$ into $\text{Eq } \Omega$.*

Proof. Denote by K the sublattice of $\text{Eq } \Omega$ consisting of all compact equivalence relations of Ω . Thus the elements of K are exactly the equivalence relations containing only finitely many non-diagonal pairs. In particular, $\text{Eq } \Omega$ is canonically isomorphic to $\text{Id } K$.

Now we apply Corollary 6.2 to $L = K \amalg^0 K$. As $\text{card } L = \text{card } \Omega$, we obtain a complete lattice embedding $\varphi: \text{Id } L \hookrightarrow \text{Eq } \Omega$. However, $\text{Id } L = \text{Id}(K \amalg^0 K)$ contains, by Theorem 5.2 and the last sentence of the first paragraph of Section 5, a 0, 1-sublattice isomorphic to $(\text{Id } K) \amalg^0 (\text{Id } K)$, thus to $(\text{Eq } \Omega) \amalg^0 (\text{Eq } \Omega)$. \square

For any nonempty set Ω , form $\overline{\Omega} = \Omega \cup \{\infty\}$ for an outside point ∞ . As there exists a retraction $\rho: \overline{\Omega} \rightarrow \Omega$ (pick $p \in \Omega$ and send ∞ to p), we can form a meet-complete, nonempty-join-complete lattice embedding $\eta: \text{Eq } \Omega \hookrightarrow \text{Eq } \overline{\Omega}$ by setting

$$\eta(\theta) = \{(x, y) \in \overline{\Omega} \times \overline{\Omega} \mid (\rho(x), \rho(y)) \in \theta\}, \quad \text{for each } \theta \in \text{Eq } \Omega,$$

and η sends the zero element of $\text{Eq } \Omega$ to a nonzero element of $\text{Eq } \overline{\Omega}$. Hence, in case Ω is infinite, $(\text{Eq } \Omega)^\circ$ completely embeds into $\text{Eq } \Omega$. As $(\text{Eq } \Omega) \amalg (\text{Eq } \Omega)$ is the sublattice of $(\text{Eq } \Omega)^\circ \amalg^0 (\text{Eq } \Omega)^\circ$ generated by the union of the images of $\text{Eq } \Omega$ under the two canonical coprojections, it follows from Theorem 6.3 and Lemma 4.2 that $(\text{Eq } \Omega) \amalg (\text{Eq } \Omega)$ has a 1-lattice embedding into $\text{Eq } \Omega$. If we denote by θ the image of zero under this embedding, then $(\text{Eq } \Omega) \amalg (\text{Eq } \Omega)$ has a 0, 1-lattice embedding into $\text{Eq}(\Omega/\theta)$, and thus, as $\text{card}(\Omega/\theta) \leq \text{card } \Omega$, into $\text{Eq } \Omega$. Hence we obtain

Theorem 6.4. *Let Ω be an infinite set. Then there exists a 0, 1-lattice embedding from $(\text{Eq } \Omega) \amalg (\text{Eq } \Omega)$ into $\text{Eq } \Omega$.*

By applying the category equivalence $L \mapsto L^{\text{op}}$ to Theorems 6.3 and 6.4 and denoting by \amalg^1 the coproduct of 1-lattices, we obtain the following result.

Theorem 6.5. *Let Ω be an infinite set. Then there are 0, 1-lattice embeddings from $(\text{Eq } \Omega)^{\text{op}} \amalg^1 (\text{Eq } \Omega)^{\text{op}}$ into $(\text{Eq } \Omega)^{\text{op}}$ and from $(\text{Eq } \Omega)^{\text{op}} \amalg (\text{Eq } \Omega)^{\text{op}}$ into $(\text{Eq } \Omega)^{\text{op}}$.*

By using the results of [2], we can now fit the copower of the optimal number of copies of $L = \text{Eq } \Omega$ into itself. The variety \mathbf{V} to which we apply those results is, of course, the variety of all lattices with zero. The functor to be considered sends every set I to $F(I) = \coprod_I^0 L$, the 0-coproduct of I copies of L . If we denote by $e_i^I: L \hookrightarrow F(I)$ the i -th coprojection, then, for any map $f: I \rightarrow J$, $F(f)$ is the unique 0-lattice homomorphism from $F(I)$ to $F(J)$ such that $F(f) \circ e_i^I = e_{f(i)}^J$ for all $i \in I$. Observe that even in case both I and J are finite, $F(f)$ does not preserve the unit unless f is surjective. The condition labeled (9) in [2, Section 3], stating that every element of $F(I)$ belongs to the range of $F(a)$ for some $a: n \rightarrow I$, for some positive integer n , is obviously satisfied. Hence, by [2, Theorem 3.1], $F(\mathfrak{P}(\Omega))$ has a 0-lattice embedding into $F(\omega)^\Omega$. Furthermore, it follows from [2, Lemma 3.3] that $F(\omega)$ has a 0-lattice embedding into $\prod_{1 \leq n < \omega} F(n)$. By Lemma 4.2 and Theorem 6.3, each $F(n)$ has a 0, 1-lattice embedding into L . As, by the final paragraph of [2, Section 2], L^Ω has a 0-lattice embedding into L , we obtain the following theorem.

Theorem 6.6. *Let Ω be an infinite set. Then the following statements hold:*

- (1) $\prod_{\mathfrak{P}(\Omega)}^0 \text{Eq } \Omega$ has a 0-lattice embedding into $\text{Eq } \Omega$.

- (2) $\coprod_{\mathfrak{P}(\Omega)} \text{Eq } \Omega$ has a lattice embedding into $\text{Eq } \Omega$.
- (3) $\coprod_{\mathfrak{P}(\Omega)}^1 (\text{Eq } \Omega)^{\text{op}}$ has a 1-lattice embedding into $(\text{Eq } \Omega)^{\text{op}}$.
- (4) $\coprod_{\mathfrak{P}(\Omega)} (\text{Eq } \Omega)^{\text{op}}$ has a lattice embedding into $(\text{Eq } \Omega)^{\text{op}}$.

This raises the question whether $(\text{Eq } \Omega) \amalg^1 (\text{Eq } \Omega)$ embeds into $\text{Eq } \Omega$, which the methods of the present paper do not seem to settle in any obvious way. More generally, we do not know whether, for a sublattice A of $\text{Eq } \Omega$, the amalgam $(\text{Eq } \Omega) \amalg_A (\text{Eq } \Omega)$ of two copies of $\text{Eq } \Omega$ over A embeds into $\text{Eq } \Omega$.

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