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## EMBEDDING COPRODUCTS OF PARTITION LATTICES

FRIEDRICH WEHRUNG

ABSTRACT. We prove that the lattice  $\text{Eq } \Omega$  of all equivalence relations on an infinite set  $\Omega$  contains, as a 0, 1-sublattice, the 0-coproduct of two copies of itself, thus answering a question by G. M. Bergman. Hence, by using methods initiated by de Bruijn and further developed by Bergman, we obtain that  $\text{Eq } \Omega$  also contains, as a sublattice, the coproduct of  $2^{\text{card } \Omega}$  copies of itself.

## 1. INTRODUCTION

Whitman's Theorem [10] states that every lattice  $L$  can be embedded into the lattice  $\text{Eq } \Omega$  of all equivalence relations on some set  $\Omega$ . The cardinality of  $\Omega$  may be taken equal to  $\text{card } L + \aleph_0$ . There is not much room for improvement of the cardinality bound, as for example,  $\text{Eq } \Omega$  cannot be embedded into its dual lattice. (We believe the first printed occurrence of this result to be Proposition 6.2 in G. M. Bergman's recent preprint [2], although it may have already been known for some time.) Hence the question of embeddability into  $\text{Eq } \Omega$  of lattices of large cardinality (typically,  $\text{card}(\text{Eq } \Omega) = 2^{\text{card } \Omega}$ ) is nontrivial.

In [2], Bergman also extends results of N. G. de Bruijn [3, 4] by proving various embedding results of large powers or copowers of structures such as symmetric groups, endomorphism rings, and monoids of self-maps of an infinite set  $\Omega$ , into those same structures. The nature of the underlying general argument is categorical. The problem whether the lattice  $\text{Eq } \Omega$  contains a coproduct (i.e., "free product" in the language of universal algebra) of two, or more, copies of itself, is left open in that paper. In the present note, we solve this problem in the affirmative.

The idea of our proof is the following. The lattice  $\text{Eq } \Omega$  of all equivalence relations on  $\Omega$  is naturally isomorphic to the ideal lattice  $\text{Id } K$  of the lattice  $K$  of all *finitely generated* equivalence relations, that is, those equivalence relations containing only finitely many non-diagonal pairs. Denote by  $K \amalg^0 K$  the coproduct (amalgamation) of two copies of  $K$  above the common ideal 0. As  $K \amalg^0 K$  has the same cardinality as  $\Omega$ , it follows from Jónsson's proof of Whitman's Embedding Theorem that the lattice  $\text{Id}(K \amalg^0 K)$  embeds into  $\text{Eq } \Omega$ . Finally, we prove that the ideal lattice functor preserves the coproduct  $\amalg^0$  and one-one-ness (Theorem 5.2), in such a way that  $(\text{Id } K) \amalg^0 (\text{Id } K)$  embeds into  $\text{Id}(K \amalg^0 K)$ . Then it is easy to extend this result to the usual coproduct  $\amalg (\text{Id } K) \amalg (\text{Id } K)$ .

We also present an example (Example 5.3) that shows that the result of Theorem 5.2 does not extend to amalgamation above a common (infinite) ideal.

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## 2. BASIC CONCEPTS

We refer to [7] for unexplained lattice-theoretical notions. For any subsets  $Q$  and  $X$  in a poset (i.e., partially ordered set)  $P$ , we put

$$Q \downarrow X = \{p \in Q \mid (\exists x \in X)(p \leq x)\} \quad \text{and} \quad Q \uparrow X = \{p \in Q \mid (\exists x \in X)(p \geq x)\}.$$

We also write  $Q \downarrow x$ , resp.  $Q \uparrow x$  in case  $X = \{x\}$ . A subset  $Q$  of  $P$  is a *lower subset* of  $P$ , if  $Q = P \downarrow Q$ .

A map  $f: K \rightarrow L$  between lattices is *meet-complete*, if for each  $a \in K$  and each  $X \subseteq K$ ,  $a = \bigwedge X$  in  $K$  implies that  $f(a) = \bigwedge f[X]$  in  $L$ . (Observe that we do not require either  $K$  or  $L$  to be a complete lattice.) When this is required only for nonempty  $X$ , we say that  $f$  is *nonempty-meet-complete*. *Join-completeness* and *nonempty-join-completeness* of maps are defined dually. We say that  $f$  is *complete* (resp., *nonempty-complete*), if it is both meet-complete and join-complete (rersp., both nonempty-meet-complete and nonempty-join-complete). We say that  $f$  is *lower bounded*, if  $\{x \in K \mid y \leq f(x)\}$  is either empty or has a least element, for each  $y \in L$ . *Upper bounded* homomorphisms are defined dually. Lower bounded homomorphisms are nonempty-meet-complete and upper bounded homomorphisms are nonempty-join-complete.

An *ideal* of a lattice  $L$  is a nonempty lower subset of  $L$  closed under finite joins. We denote by  $\text{Id } L$  the lattice of all ideals of  $L$ . For a lattice homomorphism  $f: K \rightarrow L$ , the map  $\text{Id } f: \text{Id } K \rightarrow \text{Id } L$  defined by

$$(\text{Id } f)(X) = L \downarrow f[X], \quad \text{for each } X \in \text{Id } L,$$

is a join-complete lattice embedding. If  $L$  is a *0-lattice* (i.e., a lattice with least element), the canonical map  $L \rightarrow \text{Id } L$ ,  $x \mapsto L \downarrow x$  is a 0-lattice embedding. The assignment that to every lattice associates its dual lattice  $L^{\text{op}}$  (i.e., the lattice with the same underlying set as  $L$  but reverse ordering) is a category equivalence—and even a category isomorphism—from the category of all lattices to itself, that sends 0-lattices to 1-lattices. For every lattice  $L$ , we denote by  $L^\circ$  the lattice obtained by adding a new zero element to  $L$ .

A lattice  $L$  is *upper continuous*, if for each  $a \in L$  and each upward directed subset  $\{x_i \mid i \in I\}$  of  $L$  admitting a join, the equality  $a \wedge \bigvee_{i \in I} x_i = \bigvee_{i \in I} (a \wedge x_i)$  holds. Any algebraic lattice is upper continuous, so, for example,  $\text{Id } L$  is upper continuous, for any 0-lattice  $L$ . The lattice  $\text{Eq } \Omega$  of all equivalence relations on a set  $\Omega$ , partially ordered by inclusion, is an algebraic lattice, thus it is upper continuous. Other examples of upper continuous lattices that are not necessarily complete are given in [1]. For example, it follows from [1, Corollary 2.2] that *every finitely presented lattice is upper continuous*.

We denote by  $\mathfrak{P}(\Omega)$  the powerset of a set  $\Omega$ , and by  $\omega$  the set of all natural numbers.

## 3. THE FREE LATTICE ON A PARTIAL LATTICE

We recall Dean's description of the free lattice on a partial lattice, see [5] or [6, Section XI.9]. A *partial lattice* is a poset  $(P, \leq)$  endowed with partial functions  $\bigvee$  and  $\bigwedge$  from the nonempty finite subsets of  $P$  to  $P$  such that if  $p = \bigvee X$  (resp.,  $p = \bigwedge X$ ), then  $p$  is the greatest lower bound (resp., least upper bound) of  $X$  in  $P$ . An *o-ideal* of  $P$  is a lower subset  $A$  of  $P$  such that  $p = \bigvee X$  and  $X \subseteq A$  implies that  $p \in A$ , for each  $p \in P$  and each nonempty finite subset  $X$  of  $P$ . The set  $\overline{\text{Id}} P$

of all o-ideals of  $P$ , partially ordered by inclusion, is an algebraic lattice. Observe that  $\overline{\text{Id}} P = (\text{Id } P) \cup \{\emptyset\}$  in case  $P$  is a lattice. O-filters are defined dually; again, the lattice  $\overline{\text{Fil}} P$  of all o-filters of  $P$ , partially ordered by inclusion, is algebraic. We denote by  $\mathcal{J}(A)$  (resp.,  $\mathcal{F}(A)$ ) the least o-ideal (resp., o-filter) of  $P$  containing a subset  $A$  of  $P$ .

The *free lattice*  $F_{\mathbf{L}}(P)$  on  $P$  is generated, as a lattice, by an isomorphic copy of  $P$ , that we shall identify with  $P$ . (The subscript  $\mathbf{L}$  in  $F_{\mathbf{L}}(P)$  stands for the variety of all lattices, as the “free lattice on  $P$ ” construction can be carried out in any other variety of lattices.) We define the *complexity* of an element  $x \in F_{\mathbf{L}}(P)$  as the minimal length of all lattice terms with parameters from  $P$  representing  $x$ . In particular, the elements of  $P$  are the elements of  $F_{\mathbf{L}}(P)$  of complexity zero. For each  $x \in F_{\mathbf{L}}(P)$ , the following subsets of  $P$ ,

$$\mathcal{J}(x) = P \downarrow x = \{p \in P \mid p \leq x\} \quad \text{and} \quad \mathcal{F}(x) = P \uparrow x = \{p \in P \mid x \leq p\}$$

are, respectively, an o-ideal and an o-filter of  $P$ , which can also be evaluated by the following rules:

$$\mathcal{J}(x \vee y) = \mathcal{J}(x) \vee \mathcal{J}(y) \text{ in } \overline{\text{Id}} P, \quad \mathcal{F}(x \vee y) = \mathcal{F}(x) \cap \mathcal{F}(y); \quad (3.1)$$

$$\mathcal{J}(x \wedge y) = \mathcal{J}(x) \cap \mathcal{J}(y), \quad \mathcal{F}(x \wedge y) = \mathcal{F}(x) \vee \mathcal{F}(y) \text{ in } \overline{\text{Fil}} P, \quad (3.2)$$

for all  $x, y \in F_{\mathbf{L}}(P)$ . The natural partial ordering on  $F_{\mathbf{L}}(P)$  satisfies the following “Whitman-type” condition:

$$\begin{aligned} x_0 \wedge x_1 \leq y_0 \vee y_1 &\iff \text{either } (\exists p \in P)(x_0 \wedge x_1 \leq p \leq y_0 \vee y_1) \\ &\text{or there is } i < 2 \text{ such that either } x_i \leq y_0 \vee y_1 \text{ or } x_0 \wedge x_1 \leq y_i, \end{aligned} \quad (3.3)$$

which is also the basis of the inductive definition of that ordering.

#### 4. THE 0-COPRODUCT OF A FAMILY OF LATTICES WITH ZERO

Our presentation of the 0-coproduct of a family of lattices with zero bears some similarities with the presentation of coproducts (called there *free products*) given in [7, Chapter VI]. Nevertheless, as we use the known results about the free lattice on a partial lattice (outlined in Section 3), our presentation becomes significantly shorter.

Let  $(L_i \mid i \in I)$  be a family of lattices with zero. Modulo the harmless set-theoretical assumption that  $L_i \cap L_j = \{0\}$  for all distinct indices  $i, j \in I$ , the *coproduct* (often called *free product* by universal algebraists) of  $(L_i \mid i \in I)$  can be easily described as  $F_{\mathbf{L}}(P)$ , where  $P$  is the partial lattice whose underlying set is the union  $\bigcup_{i \in I} L_i$ , whose underlying partial ordering is the one generated by the partial orders on all the  $L_i$ s, and whose partial lattice structure consists of all existing joins and meets of nonempty finite subsets in each “component”  $L_i$ . We denote this lattice by  $L = \coprod_{i \in I}^0 L_i$ , the superscript 0 meaning that the coproduct of the  $L_i$ s is evaluated in the category of all 0-lattices and 0-preserving homomorphisms, which we shall often emphasize by saying “0-coproduct” instead of just coproduct. We shall also identify each  $L_i$  with its canonical copy in  $L$ . Of course, the coproduct of  $(L_i \mid i \in I)$  in the variety of all lattices is the sublattice of  $\coprod_{i \in I}^0 (L_i)^\circ$  generated by the union of the images of the  $L_i$ s.

Now we shall analyze further the structure of the 0-coproduct  $L$ , in a fashion similar to the one presented in [7, Chapter VI]. We add a new largest element,

denoted by  $\infty$ , to  $L$ , and we set  $\overline{L}_i = L_i \cup \{\infty\}$ , for each  $i \in I$ . The following lemma is an analogue, for 0-coproducts instead of coproducts, of [7, Theorem VI.1.10].

**Lemma 4.1.** *For each  $x \in L$  and each  $i \in I$ , there are a largest element of  $L_i$  below  $x$  and a least element of  $\overline{L}_i$  above  $x$  with respect to the ordering of  $L \cup \{\infty\}$ . Furthermore, if we denote these elements by  $x_{(i)}$  and  $x^{(i)}$ , respectively, then the following formulas hold:*

$$\begin{aligned}
p_{(i)} &= p^{(i)} = p, \text{ if } p \in L_i; \\
p_{(i)} &= 0 \text{ and } p^{(i)} = \infty, \text{ if } p \in P \setminus L_i; \\
(x \vee y)_{(i)} &= x_{(i)} \vee y_{(i)} \text{ and } (x \wedge y)_{(i)} = x_{(i)} \wedge y_{(i)}; \\
(x \vee y)^{(i)} &= x^{(i)} \vee y^{(i)}; \\
(x \wedge y)^{(i)} &= \begin{cases} 0, & \text{if } x^{(j)} \wedge y^{(j)} = 0 \text{ for some } j \in I, \\ x^{(i)} \wedge y^{(i)}, & \text{otherwise,} \end{cases}
\end{aligned} \tag{4.1}$$

for each  $x, y \in L$  and each  $i \in I$ .

*Proof.* For an element  $x$  of  $L$ , abbreviate by “ $x_{(i)}$  exists” (resp., “ $x^{(i)}$  exists”) the statement that  $L_i \downarrow x$  is a principal ideal in  $L_i$  (resp.,  $\overline{L}_i \uparrow x$  is a principal filter in  $\overline{L}_i$ ), and then denote by  $x_{(i)}$  (resp.,  $x^{(i)}$ ) the largest element of  $L_i \downarrow x$  (resp., the least element of  $\overline{L}_i \uparrow x$ ). Denote by  $K$  the set of all  $x \in L$  such that both  $x_{(i)}$  and  $x^{(i)}$  exist for each  $i \in I$ . It is clear that  $K$  contains  $P$  and that both  $p_{(i)}$  and  $p^{(i)}$  are given by the first two formulas of (4.1), for any  $p \in P$ . Furthermore, it follows immediately from the definition of  $K$  that

$$\mathcal{J}(x) = P \downarrow \{x_{(i)} \mid i \in I\} \quad \text{and} \quad \mathcal{F}(x) = P \uparrow \{x^{(i)} \mid i \in I\}, \quad \text{for each } x \in K. \tag{4.2}$$

We shall establish that  $K$  is a sublattice of  $L$ . So let  $x, y \in K$ , put  $u = x \wedge y$  and  $v = x \vee y$ . Fix  $i \in I$ . It is straightforward that both  $u_{(i)}$  and  $v^{(i)}$  exist, and

$$u_{(i)} = x_{(i)} \wedge y_{(i)}, \quad v^{(i)} = x^{(i)} \vee y^{(i)}. \tag{4.3}$$

Now we shall prove that  $v_{(i)}$  exists and is equal to  $x_{(i)} \vee y_{(i)}$ . As  $x_{(i)} \vee y_{(i)}$  is a lower bound of  $v$  belonging to  $L_i$ , it suffices to verify that each  $p \in \mathcal{J}(v) \cap L_i$  lies below  $x_{(i)} \vee y_{(i)}$ . As the partial lattice operations on  $P$  are “internal” to each  $L_j$ , the subset

$$U = P \downarrow \{x_{(j)} \vee y_{(j)} \mid j \in I\}$$

is an o-ideal of  $P$ . As, by (4.2),  $U$  contains both  $\mathcal{J}(x)$  and  $\mathcal{J}(y)$ , it also contains  $\mathcal{J}(x) \vee \mathcal{J}(y) = \mathcal{J}(v)$  (see (3.1)). In particular,  $p \in U$ , and so  $p \leq x_{(j)} \vee y_{(j)}$  for some  $j \in I$ . If  $j \neq i$ , then, as  $p \in L_i$ , we get  $p = 0$ ; so, in any case,  $p \leq x_{(i)} \vee y_{(i)}$ . Hence we have proved that  $v_{(i)}$  exists and

$$v_{(i)} = x_{(i)} \vee y_{(i)}. \tag{4.4}$$

Finally, we shall prove that  $u^{(i)}$  exists and is given by the last two lines of (4.1). Suppose first that  $x^{(j)} \wedge y^{(j)} = 0$  for some  $j \in I$ . As  $x \leq x^{(j)}$  and  $y \leq y^{(j)}$ , we get that  $u = x \wedge y = 0$ , thus  $u^{(i)} = 0$ , as required. Now assume that  $x^{(j)} \wedge y^{(j)} \neq 0$  for all  $j \in I$ . As  $x^{(i)} \wedge y^{(i)}$  is an upper bound of  $u$  belonging to  $\overline{L}_i$ , it suffices to verify that any  $p \in \mathcal{F}(u) \cap L_i$  lies above  $x^{(i)} \wedge y^{(i)}$ . Again as the partial lattice operations on  $P$  are “internal” to each  $L_j$ , the subset

$$F = P \uparrow \{x^{(j)} \wedge y^{(j)} \mid j \in I\}$$

is an o-filter of  $P$ . As, by (4.2),  $F$  contains both  $\mathcal{F}(x)$  and  $\mathcal{F}(y)$ , it also contains  $\mathcal{F}(x) \vee \mathcal{F}(y) = \mathcal{F}(u)$  (see (3.2)). In particular,  $p \in F$ , so  $x^{(j)} \wedge y^{(j)} \leq p$  for some  $j \in I$ . As  $x^{(j)} \wedge y^{(j)} \in L_j \setminus \{0\}$  and  $p \in L_i$ , it follows that  $i = j$  and  $x^{(i)} \wedge y^{(i)} \leq p$ . Hence we have verified that  $u^{(i)}$  exists and

$$u^{(i)} = x^{(i)} \wedge y^{(i)}. \quad (4.5)$$

This completes the proof that  $K = L$ . Furthermore, the formulas (4.1) follow from (4.3), (4.4), and (4.5).  $\square$

We obtain immediately the following corollary, which is an analogue for  $\coprod^0$  of the well-known preservation of lattice embeddings by  $\coprod$ , see [7, Corollary VI.1.13].

**Corollary 4.2.** *Let  $K_i$  be a 0-sublattice of a lattice  $L_i$ , for each  $i \in I$ . Then the canonical 0-lattice homomorphism  $f: \coprod_{i \in I}^0 K_i \rightarrow \coprod_{i \in I}^0 L_i$  is an embedding.*

*Proof.* Put  $K = \coprod_{i \in I}^0 K_i$  and  $L = \coprod_{i \in I}^0 L_i$ . An immediate application of (4.1) shows that the set of all  $x \in K$  such that  $f(x)_{(i)} = x_{(i)}$  and  $f(x)^{(i)} = x^{(i)}$  holds for all  $i \in I$  is a sublattice of  $K$  containing the union of the images of the  $K_i$ s, hence it is equal to  $K$ .

Now we prove by induction on the sum of the complexities of  $x$  and  $y$  that  $f(x) \leq f(y)$  implies that  $x \leq y$ , for each  $x, y \in K$ . The case where either  $x$  or  $y$  belongs to the union of the images of the  $K_i$ s is taken care of by the paragraph above. By the induction hypothesis, the only nontrivial remaining case is where  $x$  is a proper meet,  $y$  is a proper join, and there exists  $p \in \bigcup_{i \in I} L_i$  such that  $f(x) \leq p \leq f(y)$ . Pick  $i \in I$  such that  $p \in L_i$ . Then  $x^{(i)} = f(x)^{(i)} \leq p$  in  $L_i$  and  $p \leq f(y)_{(i)} = y_{(i)}$  in  $L_i$ , thus  $x^{(i)} \leq y_{(i)}$  in  $K_i$ , and thus  $x \leq x^{(i)} \leq y_{(i)} \leq y$  in  $K$ .  $\square$

We shall call the adjoint maps  $\alpha_i: x \mapsto x_{(i)}$  and  $\beta_i: x \mapsto x^{(i)}$  the canonical lower, resp. upper adjoint of  $L$  onto  $L_i$ , resp.  $\overline{L}_i$ . The following result is an immediate consequence of well-known general properties of adjoint maps.

**Corollary 4.3.** *The canonical embedding from  $L_i$  into  $L$  is both lower bounded and upper bounded, for each  $i \in I$ . In particular, it is a nonempty-complete lattice homomorphism. Furthermore, the lower adjoint  $\alpha_i$  is meet-complete while the upper adjoint  $\beta_i$  is nonempty-join-complete.*

In the following lemma, we shall represent the elements of  $L = \coprod_{i \in I}^0 L_i$  in the form  $\mathbf{p}(\vec{a})$ , where  $\mathbf{p}$  is a lattice term with variables from  $I \times \omega$  and the ‘‘vector’’  $\vec{a} = (a_{i,n} \mid (i,n) \in I \times \omega)$  is an element of the cartesian product  $\Pi = \prod_{(i,n) \in I \times \omega} L_i$ . Define a *support* of  $\mathbf{p}$  as a subset  $J$  of  $I$  such that  $\mathbf{p}$  involves only variables from  $J \times \omega$ . Obviously,  $\mathbf{p}$  has a finite support. It is straightforward from (4.1) that  $\mathbf{p}(\vec{a})_{(i)} = 0$  and either  $\mathbf{p}(\vec{a}) = 0$  or  $\mathbf{p}(\vec{a})^{(i)} = \infty$ , for each  $i$  outside a support of  $\mathbf{p}$ .

**Lemma 4.4.** *Let  $\Lambda$  be an upward directed poset, let  $(\vec{a}^\lambda \mid \lambda \in \Lambda)$  be an isotone family of elements of  $\Pi$  with supremum  $\vec{a}$  in  $\Pi$ , and let  $\mathbf{p}$  be a lattice term. If all the lattices  $L_i$  are upper continuous, then  $\mathbf{p}(\vec{a}) = \bigvee_{\lambda \in \Lambda} \mathbf{p}(\vec{a}^\lambda)$  in  $L$ .*

*Proof.* As  $\mathbf{p}(\vec{a})$  is clearly an upper bound for all elements  $\mathbf{p}(\vec{a}^\lambda)$ , it suffices to prove that for each lattice term  $\mathbf{q}$  on  $I \times \omega$  and each  $\vec{b} \in \Pi$  such that  $\mathbf{p}(\vec{a}^\lambda) \leq \mathbf{q}(\vec{b})$  for all  $\lambda \in \Lambda$ , the inequality  $\mathbf{p}(\vec{a}) \leq \mathbf{q}(\vec{b})$  holds. We argue by induction on the sums of the lengths of  $\mathbf{p}$  and  $\mathbf{q}$ . The case where  $\mathbf{p}$  is a projection follows immediately from Corollary 4.3. The case where either  $\mathbf{p}$  is a join or  $\mathbf{q}$  is a meet is straightforward.

Now suppose that  $\mathbf{p} = \mathbf{p}_0 \wedge \mathbf{p}_1$  and  $\mathbf{q} = \mathbf{q}_0 \vee \mathbf{q}_1$ . We shall make repeated uses of the following easily established principle, which uses only the assumption that  $\Lambda$  is upward directed:

*For every positive integer  $n$  and every  $X_0, \dots, X_{n-1} \subseteq \Lambda$ , if  $\bigcup_{i < n} X_i$  is cofinal in  $\Lambda$ , then one of the  $X_i$ s is cofinal in  $\Lambda$ .*

Now we use (3.3). If there exists a cofinal subset  $\Lambda'$  of  $\Lambda$  such that

$$(\forall \lambda \in \Lambda')(\exists i < 2)(\text{either } \mathbf{p}_i(\vec{a}^\lambda) \leq \mathbf{q}(\vec{b}) \text{ or } \mathbf{p}(\vec{a}^\lambda) \leq \mathbf{q}_i(\vec{b})),$$

then there are  $i < 2$  and an even smaller cofinal subset  $\Lambda''$  of  $\Lambda'$  such that

$$\begin{aligned} \text{either } & (\forall \lambda \in \Lambda'')(\mathbf{p}_i(\vec{a}^\lambda) \leq \mathbf{q}(\vec{b})) \\ \text{or } & (\forall \lambda \in \Lambda'')(\mathbf{p}(\vec{a}^\lambda) \leq \mathbf{q}_i(\vec{b})). \end{aligned}$$

In the first case, it follows from the induction hypothesis that  $\mathbf{p}_i(\vec{a}) \leq \mathbf{q}(\vec{b})$ . In the second case, it follows from the induction hypothesis that  $\mathbf{p}(\vec{a}) \leq \mathbf{q}_i(\vec{b})$ . In both cases,  $\mathbf{p}(\vec{a}) \leq \mathbf{q}(\vec{b})$ . There remains to consider the case where there exists a cofinal subset  $\Lambda'$  of  $\Lambda$  such that

$$(\forall \lambda \in \Lambda')(\exists c_\lambda \in P)(\mathbf{p}(\vec{a}^\lambda) \leq c_\lambda \leq \mathbf{q}(\vec{b})).$$

Fix a common finite support  $J$  of  $\mathbf{p}_0, \mathbf{p}_1, \mathbf{q}_0, \mathbf{q}_1$ . It follows from the induction hypothesis that

$$\mathbf{p}_\ell(\vec{a}) = \bigvee_{\lambda \in \Lambda'} \mathbf{p}_\ell(\vec{a}^\lambda), \quad \text{for all } \ell < 2. \quad (4.6)$$

Each  $c_\lambda$  belongs to  $L_i$ , for some  $i$  in the given support  $J$ . By using the finiteness of  $J$  and by extracting a further cofinal subset of  $\Lambda'$ , we may assume that all those  $i$  are equal to the same index  $j \in J$ . Hence we have reduced the problem to the case where

$$(\forall \lambda \in \Lambda')(\mathbf{p}(\vec{a}^\lambda) \leq c_\lambda \leq \mathbf{q}(\vec{b})), \quad \text{where } c_\lambda = \mathbf{p}(\vec{a}^\lambda)^{(j)} \in L_j. \quad (4.7)$$

If  $\mathbf{p}_0(\vec{a})^{(i)} \wedge \mathbf{p}_1(\vec{a})^{(i)} = 0$  for some  $i \in I$ , then  $\mathbf{p}(\vec{a}) = 0 \leq \mathbf{q}(\vec{b})$  and we are done. Now suppose that  $\mathbf{p}_0(\vec{a})^{(i)} \wedge \mathbf{p}_1(\vec{a})^{(i)} \neq 0$  for all  $i \in I$ . By using (4.6), the finiteness of  $J$ , and the upper continuity of  $L_i$ , we obtain that there exists a cofinal subset  $\Lambda''$  of  $\Lambda'$  such that

$$(\forall \lambda \in \Lambda'')(\forall i \in J)(\mathbf{p}_0(\vec{a}^\lambda)^{(i)} \wedge \mathbf{p}_1(\vec{a}^\lambda)^{(i)} \neq 0).$$

In particular, both  $\mathbf{p}_0(\vec{a}^\lambda)$  and  $\mathbf{p}_1(\vec{a}^\lambda)$  are nonzero for each  $\lambda \in \Lambda''$ . As  $J$  is a common support of  $\mathbf{p}_0$  and  $\mathbf{p}_1$ , the equality  $\mathbf{p}_0(\vec{a}^\lambda)^{(i)} \wedge \mathbf{p}_1(\vec{a}^\lambda)^{(i)} = \infty$  holds for all  $\lambda \in \Lambda''$  and all  $i \in I \setminus J$ , hence

$$(\forall \lambda \in \Lambda'')(\forall i \in I)(\mathbf{p}_0(\vec{a}^\lambda)^{(i)} \wedge \mathbf{p}_1(\vec{a}^\lambda)^{(i)} \neq 0).$$

Thus it follows from (4.1) that  $c_\lambda = \mathbf{p}(\vec{a}^\lambda)^{(j)} = \mathbf{p}_0(\vec{a}^\lambda)^{(j)} \wedge \mathbf{p}_1(\vec{a}^\lambda)^{(j)}$  for each  $\lambda \in \Lambda''$ . Hence, by the upper continuity of  $L_j$  (and thus of  $\bar{L}_j$ ), (4.6), and the previously observed fact that the upper adjoint  $\beta_j$  is nonempty-join-complete,  $\{c_\lambda \mid \lambda \in \Lambda''\}$  has a join in  $L_j$ , which is equal to  $\mathbf{p}_0(\vec{a})^{(j)} \wedge \mathbf{p}_1(\vec{a})^{(j)} = \mathbf{p}(\vec{a})^{(j)}$ . Therefore, it follows from (4.7) that  $\mathbf{p}(\vec{a}) \leq \mathbf{p}(\vec{a})^{(j)} \leq \mathbf{q}(\vec{b})$ .  $\square$

## 5. IDEAL LATTICES AND 0-COPRODUCTS

In this section we fix again a family  $(L_i \mid i \in I)$  of lattices with zero pairwise intersecting in  $\{0\}$ , and we form  $L = \coprod_{i \in I}^0 L_i$ . We denote by  $\varepsilon_i: \text{Id } L_i \hookrightarrow \text{Id } L$  the 0-lattice homomorphism induced by the canonical embedding  $L_i \hookrightarrow L$ , for each  $i \in I$ . By the universal property of the coproduct, there exists a unique 0-lattice homomorphism  $\varepsilon: \coprod_{i \in I}^0 \text{Id } L_i \rightarrow \text{Id } L$  such that  $\varepsilon_i = \varepsilon \upharpoonright_{\text{Id } L_i}$  for each  $i \in I$ . Observe that in case  $I$  is finite, the lattice  $\coprod_{i \in I}^0 \text{Id } L_i$  has  $\bigvee_{i \in I} L_i$  as a largest element, and this element is sent by  $\varepsilon$  to  $L$  (because every element of  $L$  lies below some join of elements of the  $L_i$ s). Hence, *if the index set  $I$  is finite, then the map  $\varepsilon$  preserves the unit as well.*

**Lemma 5.1.** *Let  $\mathbf{p}$  be a lattice term on  $I \times \omega$  and let  $\vec{X} = (X_{i,n} \mid (i,n) \in I \times \omega)$  be an element of  $\prod_{(i,n) \in I \times \omega} \text{Id } L_i$ . We put  $\vec{\varepsilon}\vec{X} = (\varepsilon_i(X_{i,n}) \mid (i,n) \in I \times \omega) \in (\text{Id } L)^{I \times \omega}$ . Then the following equality holds.*

$$\mathbf{p}(\vec{\varepsilon}\vec{X}) = L \downarrow \{\mathbf{p}(\vec{x}) \mid \vec{x} \vec{\varepsilon} \vec{X}\},$$

where “ $\vec{x} \vec{\varepsilon} \vec{X}$ ” stands for  $(\forall (i,n) \in I \times \omega)(x_{i,n} \in X_{i,n})$ .

*Proof.* We argue by induction on the length of the term  $\mathbf{p}$ . If  $\mathbf{p}$  is a projection, then the result follows immediately from the definition of the maps  $\varepsilon_i$ . If  $\mathbf{p}$  is either a join or a meet, then the result follows immediately from the expressions for the join and the meet in the ideal lattice of  $L$ , in a fashion similar to the end of the proof of [7, Lemma I.4.8].  $\square$

**Theorem 5.2.** *The canonical map  $\varepsilon: \coprod_{i \in I}^0 \text{Id } L_i \rightarrow \text{Id}(\coprod_{i \in I}^0 L_i)$  is a 0-lattice embedding.*

*Proof.* We put again  $L = \coprod_{i \in I}^0 L_i$ . Let  $\mathbf{p}, \mathbf{q}$  be lattice terms in  $I \times \omega$  and let  $\vec{X} \in \prod_{(i,n) \in I \times \omega} \text{Id } L_i$  such that  $\mathbf{p}(\vec{\varepsilon}\vec{X}) \leq \mathbf{q}(\vec{\varepsilon}\vec{X})$  in  $\text{Id } L$ , we must prove that  $\mathbf{p}(\vec{X}) \leq \mathbf{q}(\vec{X})$  in  $\coprod_{i \in I}^0 \text{Id } L_i$ . For each  $\vec{x} \vec{\varepsilon} \vec{X}$ , the inequalities  $L \downarrow \mathbf{p}(\vec{x}) \leq \mathbf{p}(\vec{\varepsilon}\vec{x}) \leq \mathbf{q}(\vec{\varepsilon}\vec{x})$  hold in  $\text{Id } L$ , thus, by Lemma 5.1, there exists  $\vec{y} \vec{\varepsilon} \vec{X}$  such that  $L \downarrow \mathbf{p}(\vec{x}) \leq L \downarrow \mathbf{q}(\vec{y})$  in  $\text{Id } L$ , that is,  $\mathbf{p}(\vec{x}) \leq \mathbf{q}(\vec{y})$  in  $L$ . Therefore, by applying the canonical map from  $L = \coprod_{i \in I}^0 L_i$  to  $\coprod_{i \in I}^0 \text{Id } L_i$  and putting  $\vec{L} \downarrow \vec{x} = (L_i \downarrow x_{i,n} \mid (i,n) \in I \times \omega)$ , we obtain

$$\mathbf{p}(\vec{L} \downarrow \vec{x}) \leq \mathbf{q}(\vec{L} \downarrow \vec{y}) \leq \mathbf{q}(\vec{X}) \quad \text{in } \coprod_{i \in I}^0 \text{Id } L_i. \quad (5.1)$$

As  $\vec{X}$  is equal to the directed join  $\bigvee_{\vec{x} \vec{\varepsilon} \vec{X}} (\vec{L} \downarrow \vec{x})$  in  $\prod_{(i,n) \in I \times \omega} \text{Id } L_i$  and each  $\text{Id } L_i$  is upper continuous, it follows from Lemma 4.4 that

$$\mathbf{p}(\vec{X}) = \bigvee \left( \mathbf{p}(\vec{L} \downarrow \vec{x}) \mid \vec{x} \vec{\varepsilon} \vec{X} \right) \quad \text{in } \coprod_{i \in I}^0 \text{Id } L_i.$$

Therefore, it follows from (5.1) that

$$\mathbf{p}(\vec{X}) \leq \mathbf{q}(\vec{X}) \quad \text{in } \coprod_{i \in I}^0 \text{Id } L_i. \quad \square$$

The following example shows that Theorem 5.2 does not extend to the pushout of two lattices above a common ideal. The underlying idea can be traced back to Grätzer and Schmidt in [8, Section 5].

**Example 5.3.** *Lattices  $B$  and  $C$  with a common ideal  $A$  such that the canonical lattice homomorphism  $f: (\text{Id } B) \amalg_{\text{Id } A} (\text{Id } C) \rightarrow \text{Id}(B \amalg_A C)$  is not one-to-one.*



*Proof.* Denote by  $K$  the poset represented in Figure 1. We claim that the subsets  $A$ ,  $B$ , and  $C$  of  $K$  defined by

$$\begin{aligned} A &= \{a_n \mid n < \omega\} \cup \{p_n \mid n < \omega\} \cup \{q_n \mid n < \omega\}, \\ B &= A \cup \{b_n \mid n < \omega\}, \\ C &= A \cup \{c_n \mid n < \omega\}. \end{aligned}$$

are as required. Observe that  $B$  and  $C$  are isomorphic lattices and that  $A$  is an ideal of both  $B$  and  $C$ .

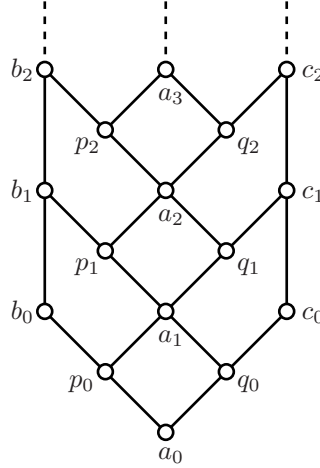


FIGURE 1. The poset  $K$ .

The map  $f$  is the unique lattice homomorphism that makes the diagram of Figure 2 commute. Unlabeled arrows are the corresponding canonical maps.

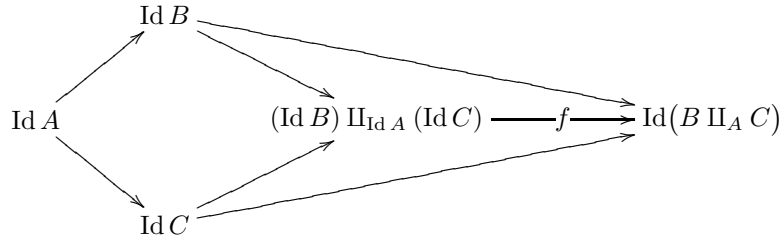


FIGURE 2. The commutative diagram defining the homomorphism  $f$ .

Put  $D = B \amalg_A C$  and identify  $B$  and  $C$  with their images in  $D$ . Further, we endow  $\text{Id } B \cup \text{Id } C$  with its natural structure of partial lattice, that is, the ordering is the union of the orderings of  $\text{Id } B$  and  $\text{Id } C$  (remember that  $A = B \cap C$  is an ideal of both  $B$  and  $C$ ) and the joins and meets are those taking place in either  $\text{Id } B$  or  $\text{Id } C$ . Observe that  $\text{Id } A = \text{Id } B \cap \text{Id } C$  and  $(\text{Id } B) \amalg_{\text{Id } A} (\text{Id } C)$  is the free lattice on the partial lattice  $(\text{Id } B) \cup (\text{Id } C)$ . As the latter is identified with its canonical image in  $(\text{Id } B) \amalg_{\text{Id } A} (\text{Id } C)$ , the elements  $A$ ,  $B \downarrow b_0$ , and  $C \downarrow c_0$  belong to  $(\text{Id } B) \amalg_{\text{Id } A} (\text{Id } C)$ .

We prove by induction that  $a_n \leq b_0 \vee c_0$  in  $D$ , for all  $n < \omega$ . This is trivial for  $n = 0$ . Suppose that  $a_n \leq b_0 \vee c_0$ . Then  $a_n \vee b_0 \leq b_0 \vee c_0$ , but  $B$  is a sublattice of  $D$  containing the subset  $\{a_n, b_0\}$  with join  $b_n$ , thus  $b_n \leq b_0 \vee c_0$ , and thus  $p_n \leq b_0 \vee c_0$ . Similarly,  $q_n \leq b_0 \vee c_0$ , but  $A$  is a sublattice of  $D$  containing the subset  $\{p_n, q_n\}$  with join  $a_{n+1}$ , and thus  $a_{n+1} \leq b_0 \vee c_0$ , which completes the induction step.

So we have established the inequality

$$f(A) \leq f(B \downarrow b_0) \vee f(C \downarrow c_0) \quad \text{in } \text{Id}(B \amalg_A C) = \text{Id } D. \quad (5.2)$$

Now observe that  $\{B \downarrow x \mid x \in B\} \cup \{C \downarrow y \mid y \in C\}$  is an  $\mathfrak{o}$ -ideal of the partial lattice  $(\text{Id } B) \cup (\text{Id } C)$ , containing  $\{B \downarrow b_0, C \downarrow c_0\}$  and to which  $A$  does not belong. Hence,  $A \notin \mathcal{J}(\{B \downarrow b_0, C \downarrow c_0\})$ , which means that  $A \not\leq (B \downarrow b_0) \vee (C \downarrow c_0)$  in  $(\text{Id } B) \amalg_{\text{Id } A} (\text{Id } C)$ . Therefore, by (5.2),  $f$  is not an embedding.  $\square$

## 6. EMBEDDING COPRODUCTS OF INFINITE PARTITION LATTICES

Whitman's Embedding Theorem states that every lattice embeds into  $\text{Eq } \Omega$ , for some set  $\Omega$ . We shall use a proof of Whitman's Theorem due to B. Jónsson [9], see also [7, Section IV.4]. The following result is proved there.

**Lemma 6.1.** *For every lattice  $L$  with zero, there are an infinite set  $\Omega$  and a map  $\delta: \Omega \times \Omega \rightarrow L$  satisfying the following properties:*

- (1)  $\delta(x, y) = 0$  iff  $x = y$ , for all  $x, y \in \Omega$ .
- (2)  $\delta(x, y) = \delta(y, x)$ , for all  $x, y \in \Omega$ .
- (3)  $\delta(x, z) \leq \delta(x, y) \vee \delta(y, z)$ , for all  $x, y, z \in L$ .
- (4) For all  $x, y \in \Omega$  and all  $a, b \in L$  such that  $\delta(x, y) \leq a \vee b$ , there are  $z_1, z_2, z_3 \in \Omega$  such that  $\delta(x, z_1) = a$ ,  $\delta(z_1, z_2) = b$ ,  $\delta(z_2, z_3) = a$ , and  $\delta(z_3, y) = b$ .

Observe, in particular, that the map  $\delta$  is *surjective*. Furthermore, a straightforward Löwenheim-Skolem type argument ("keeping only the necessary elements in  $\Omega$ ") shows that one may take  $\text{card } \Omega = \text{card } L + \aleph_0$ .

The following is the basis for Jónsson's proof of Whitman's Embedding Theorem.

**Corollary 6.2.** *For every lattice  $L$  with zero and every set  $\Omega$  such that  $\text{card } \Omega = \text{card } L + \aleph_0$ , there exists a complete lattice embedding from  $\text{Id } L$  into  $\text{Eq } \Omega$ .*

*Proof.* Any map  $\delta$  as in Lemma 6.1 gives rise to a map  $\varphi: \text{Id } L \rightarrow \text{Eq } \Omega$  defined by the rule

$$\varphi(A) = \{(x, y) \in \Omega \times \Omega \mid \delta(x, y) \in A\}, \quad \text{for each } A \in \text{Id } L, \quad (6.1)$$

and conditions (1)–(4) above imply that  $\varphi$  is a complete lattice embedding.  $\square$

**Theorem 6.3.** *Let  $\Omega$  be an infinite set. Then there exists a 0, 1-lattice embedding from  $(\text{Eq } \Omega) \amalg^0 (\text{Eq } \Omega)$  into  $\text{Eq } \Omega$ .*

*Proof.* Denote by  $K$  the sublattice of  $\text{Eq } \Omega$  consisting of all *compact* equivalence relations of  $\Omega$ . Thus the elements of  $K$  are exactly the equivalence relations containing only finitely many non-diagonal pairs. In particular,  $\text{Eq } \Omega$  is canonically isomorphic to  $\text{Id } K$ .

Now we apply Corollary 6.2 to  $L = K \amalg^0 K$ . As  $\text{card } L = \text{card } \Omega$ , we obtain a complete lattice embedding  $\varphi: \text{Id } L \hookrightarrow \text{Eq } \Omega$ . However,  $\text{Id } L = \text{Id}(K \amalg^0 K)$  contains, by Theorem 5.2, a 0, 1-sublattice isomorphic to  $(\text{Id } K) \amalg^0 (\text{Id } K)$ , thus to  $(\text{Eq } \Omega) \amalg^0 (\text{Eq } \Omega)$ .  $\square$

For any nonempty set  $\Omega$ , form  $\overline{\Omega} = \Omega \cup \{\infty\}$  for an outside point  $\infty$ . As there exists a retraction  $\rho: \overline{\Omega} \rightarrow \Omega$  (pick  $p \in \Omega$  and send  $\infty$  to  $p$ ), we can form a complete lattice embedding  $\eta: \text{Eq } \Omega \hookrightarrow \text{Eq } \overline{\Omega}$  by setting

$$\eta(\theta) = \{(x, y) \in \overline{\Omega} \times \overline{\Omega} \mid (\rho(x), \rho(y)) \in \theta\}, \quad \text{for each } \theta \in \text{Eq } \Omega,$$

and  $\eta$  sends the zero element of  $\text{Eq } \Omega$  to a nonzero element of  $\text{Eq } \overline{\Omega}$ . Hence, in case  $\Omega$  is infinite,  $(\text{Eq } \Omega)^\circ$  completely embeds into  $\text{Eq } \Omega$ . As  $(\text{Eq } \Omega) \amalg (\text{Eq } \Omega)$  is the sublattice of  $(\text{Eq } \Omega)^\circ \amalg^0 (\text{Eq } \Omega)^\circ$  generated by the union of the images of  $\text{Eq } \Omega$  under the two canonical coprojections, it follows from Theorem 6.3 and Corollary 4.2 that  $(\text{Eq } \Omega) \amalg (\text{Eq } \Omega)$  has a 1-lattice embedding into  $\text{Eq } \Omega$ . If we denote by  $\theta$  the image of zero under this embedding, then  $(\text{Eq } \Omega) \amalg (\text{Eq } \Omega)$  has a 0, 1-lattice embedding into  $\text{Eq}(\Omega/\theta)$ , and thus, as  $\text{card}(\Omega/\theta) \leq \text{card } \Omega$ , into  $\text{Eq } \Omega$ . Hence we obtain

**Theorem 6.4.** *Let  $\Omega$  be an infinite set. Then there exists a 0, 1-lattice embedding from  $(\text{Eq } \Omega) \amalg (\text{Eq } \Omega)$  into  $\text{Eq } \Omega$ .*

By applying the category equivalence  $L \mapsto L^{\text{op}}$  to Theorems 6.3 and 6.4 and denoting by  $\amalg^1$  the coproduct of 1-lattices, we obtain the following result.

**Theorem 6.5.** *Let  $\Omega$  be an infinite set. Then there are 0, 1-lattice embeddings from  $(\text{Eq } \Omega)^{\text{op}} \amalg^1 (\text{Eq } \Omega)^{\text{op}}$  into  $(\text{Eq } \Omega)^{\text{op}}$  and from  $(\text{Eq } \Omega)^{\text{op}} \amalg (\text{Eq } \Omega)^{\text{op}}$  into  $(\text{Eq } \Omega)^{\text{op}}$ .*

By using the results of [2], we can now fit the optimal number of copowers of  $L = \text{Eq } \Omega$  into itself. The variety  $\mathbf{V}$  under question is, of course, the variety of all lattices with zero. The functor to be considered sends every set  $I$  to  $F(I) = \amalg_I^0 L$ , the 0-coproduct of  $I$  copies of  $L$ . If we denote by  $e_i^I: L \hookrightarrow F(I)$  the canonical  $i$ -th coprojection, then, for any map  $f: I \rightarrow J$ ,  $F(f)$  is the unique 0-lattice homomorphism from  $F(I)$  to  $F(J)$  such that  $F(f) \circ e_i^I = e_{f(i)}^J$  for all  $i \in I$ . Observe that even in case both  $I$  and  $J$  are finite,  $F(f)$  does not preserve the unit unless  $f$  is surjective. The condition labeled (9) in [2, Section 3], stating that every element of  $F(I)$  belongs to the range of  $F(a)$  for some  $a: n \rightarrow I$ , for some positive integer  $n$ , is obviously satisfied. Hence, by [2, Theorem 3.1],  $F(\mathfrak{P}(\Omega))$  has a 0-lattice embedding into  $F(\omega)^\Omega$ . Furthermore, it follows from [2, Lemma 3.3] that  $F(\omega)$  has a 0-lattice embedding into  $\prod_{1 \leq n < \omega} F(n)$ . By Corollary 4.2 and Theorem 6.3, each  $F(n)$  has a 0, 1-lattice embedding into  $L$ . As, by the final result of [2, Section 2],  $L^\Omega$  has a 0-lattice embedding into  $L$ , we obtain the following theorem.

**Theorem 6.6.** *Let  $\Omega$  be an infinite set. Then the following statements hold:*

- (1)  $\amalg_{\mathfrak{P}(\Omega)}^0 \text{Eq } \Omega$  has a 0-lattice embedding into  $\text{Eq } \Omega$ .
- (2)  $\amalg_{\mathfrak{P}(\Omega)} \text{Eq } \Omega$  has a lattice embedding into  $\text{Eq } \Omega$ .
- (3)  $\amalg_{\mathfrak{P}(\Omega)}^1 (\text{Eq } \Omega)^{\text{op}}$  has a 1-lattice embedding into  $(\text{Eq } \Omega)^{\text{op}}$ .
- (4)  $\amalg_{\mathfrak{P}(\Omega)} (\text{Eq } \Omega)^{\text{op}}$  has a lattice embedding into  $(\text{Eq } \Omega)^{\text{op}}$ .

This raises the question whether  $(\text{Eq } \Omega) \amalg^1 (\text{Eq } \Omega)$  embeds into  $\text{Eq } \Omega$ , that the methods of the present paper do not seem to settle in any obvious way. More generally, we do not know whether, for a sublattice  $A$  of  $\text{Eq } \Omega$ , the amalgam  $(\text{Eq } \Omega) \amalg_A (\text{Eq } \Omega)$  of two copies of  $\text{Eq } \Omega$  over  $A$  embeds into  $\text{Eq } \Omega$ .

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