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Persistence and stability of solutions of Hamilton–Jacobi equations

Jean-Paul Penot* and Constantin Zălinescu†

Abstract

Given a convergent sequence of Hamiltonians (H_n) and a convergent sequence of initial data (g_n) , we look for conditions ensuring that the sequences (u_n) and (v_n) of Lax solutions and Hopf solutions respectively converge. The convergences we deal with are variational convergences. We take advantage of several recent results giving criteria for the continuity of usual operations.

Key words: asymptotic function, bounded convergence, bounded-Hausdorff convergence, convergence, epiconvergence, Hamilton–Jacobi equation, Mosco convergence, variational convergence.

Mathematics Subject Classification 2000: 35F20, 49N25, 49N15, 54A20

1 Introduction

The question of stability of solutions to the Hamilton–Jacobi equations is treated in several references (see [1], [15], [16], [17], [26]...), usually in the sense of local uniform convergence. It is our purpose to study it from the point of view of variational convergences. The reason justifying such an approach lies in the good behavior of these convergences with respect to minimization and their increasing importance in analysis (see [23], [27], [28], [30], [35], [56]). In view of the links of Hamilton–Jacobi equations with optimal control theory (see [15], [29], [31], [37], [40], [42], [60] among many other references), such a reason is sensible. Also variational convergences are compatible with important operations (under some qualification conditions) and are adapted to extended real-valued functions. Recent studies allow such a generality ([1], [50] and their references). We also make use of continuity results of the Legendre–Fenchel transform for these convergences.

Given a Banach space X with dual X^* and functions $g : X \rightarrow \mathbb{R}_\infty := \mathbb{R} \cup \{+\infty\}$, $H : X^* \rightarrow \mathbb{R}_\infty$, the Hamilton–Jacobi equation is

$$\forall (x, t) \in X \times \mathbb{R}_+ \quad \frac{\partial}{\partial t} u(x, t) + H(Du(x, t)) = 0, \quad (1)$$

$$\forall x \in X \quad u(x, 0) = g(x), \quad (2)$$

where $u : X \times \mathbb{R}_+ \rightarrow \mathbb{R}_\infty$ is the unknown function (extended by $\infty := +\infty$ on $X \times (-\mathbb{P})$ where \mathbb{P} denotes the set of positive real numbers), and Du (resp. $\frac{\partial}{\partial t} u$) denotes the derivative of u with respect to its first (resp. second) variable. Usually one considers the following question: if (g_n) and (H_n) converge to functions g and H respectively, does a sequence (u_n) of solutions to the

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Hamilton–Jacobi equations associated with (g_n) and (H_n) converges to a solution of the Hamilton–Jacobi equation associated with g and H ? Other questions arise. For instance one may wonder whether any solution u of the Hamilton–Jacobi equation associated with g and H is the limit of a sequence (u_n) of solutions of the Hamilton–Jacobi equations associated with (g_n) and (H_n) . These two different questions amount to the upper semicontinuity and the lower semicontinuity of the solution multifunction $(g, H) \rightrightarrows S(g, H)$ respectively. Corresponding questions arise for the subsolution and the supersolution multifunctions too. The involved convergence is often taken to be the uniform convergence on compact subsets and the space X is supposed to be finite dimensional ([13], [15], [16], [17], [57]...).

Here we tackle different questions. We look for conditions ensuring that the explicit Hopf–Lax and Lax–Oleinik solutions converge to the corresponding explicit solutions associated with g and H ([14], [38], [39], [41], [48], [49], [50], [58], [59]...). Recall that these solutions are defined respectively by

$$\begin{aligned} v(x, t) &:= (g^* + tH)^*(x) \text{ for } (x, t) \in X \times \mathbb{R}_+, \infty \text{ else,} \\ u(x, t) &:= (g \square (tH)^*)(x) \text{ for } (x, t) \in X \times \mathbb{R}_+, \infty \text{ else,} \end{aligned}$$

where f^* (resp. h^*) denotes the (Legendre–Fenchel) conjugate of a function f (resp. h) on X (resp. X^*):

$$\begin{aligned} f^*(x^*) &:= \sup\{\langle x^*, x \rangle - f(x) : x \in X\} \\ \text{(resp. } h^*(x^*) &:= \sup\{\langle x^*, x \rangle - h(x^*) : x^* \in X^*\}) \end{aligned}$$

and \square stands for the infimal convolution operation given by

$$(g \square h)(x) := \inf\{g(x - y) + h(y) : y \in X\}.$$

In this paper the product $0H(x^*)$ is interpreted as 0 if $H(x^*) < \infty$, and ∞ if $H(x^*) = \infty$, i.e. $0H = \iota_{\text{dom } H}$ where $\text{dom } H := H^{-1}(\mathbb{R})$ is the domain of H and ι_S denotes the indicator function of a subset S of some space Y , given by $\iota_S(y) = 0$ if $y \in S$, ∞ else.

The novelty of our approach lies in the fact that we use epiconvergence, Mosco convergence and related convergences. These convergences, which are briefly described in the next section, have proved to be of interest for variational inequalities, optimization problems and duality questions (see [3], [8], [21], [30], [35], [56] for comprehensive treatments). Since here the Legendre–Fenchel duality is involved, it is natural to use them. Moreover, since the data functions and the solutions may take the value ∞ , local uniform convergence is not appropriate, without speaking of the lack of local compactness of X when we do not assume X is finite dimensional.

One of the interests of our results lies in the fact that it enables to use regularization processes. It is known that, for a lower semicontinuous (for short l.s.c.) proper convex function f on a Hilbert space, its Moreau–Yosida regularization f_ε is of class C^1 and converges to f as $\varepsilon \rightarrow 0$ for the Mosco convergence and for the bounded convergence ([21, Th. 7.3.8]); using our stability results one can approach the solution of the Hamilton–Jacobi equation by the solutions of the equations obtained by regularizing g or H . That would not be possible without using variational convergences, unless one requires stringent assumptions.

Throughout X is a Banach space; we endow a product of normed vector spaces (for short n.v.s.) with the box norm except in the case it is a dual space, in which case we take the dual norm. We denote by $\text{cl } A$ or \bar{A} the closure of a subset A of X and by $\Gamma(X)$ (resp. $\Gamma^*(X^*)$) the set of l.s.c. proper convex functions on X (resp. X^* which are weak* l.s.c.). Given a function f and $r \in \mathbb{R}$, we set $[f \leq r] := f^{-1}([-\infty, r])$. The distance of $x \in X$ to a subset E of X is $d(x, E) := \inf\{d(x, w) : w \in E\}$, with the usual convention that $\inf \emptyset = \infty$. The *remoteness* of E is $d(0, E)$. We denote by U_X (resp. B_X) the open (resp. closed) unit ball of X .

2 Preliminaries: variational convergences

Since variational convergences will play a crucial role in the present article, let us recall some basic facts; for more information, see [3], [10], [21], [32], [44], [45], [56]...

A sequence (C_n) of subsets of X is said to converge in the sense of *Painlevé–Kuratowski* to a subset C if $C = \limsup_n C_n = \liminf_n C_n$. Here $\liminf_n C_n$ is the set of limits of sequences (x_n) such that $x_n \in C_n$ for each $n \in \mathbb{N}$ large enough and $\limsup_n C_n$ is the set of cluster points of sequences $(x_k)_{k \in K}$ such that $x_k \in C_k$ for k in an infinite subset K of \mathbb{N} . The sequence (C_n) converges in the sense of *Mosco* to $C \subset X$ if $C = w\text{-}\limsup_n C_n = \liminf_n C_n$. Here $w\text{-}\limsup_n C_n$ is the set of weak cluster points of bounded sequences $(x_k)_{k \in K}$ such that $x_k \in C_k$ for k in an infinite subset K of \mathbb{N} . If X is a dual space and if in the preceding definition the convergence of $(x_k)_{k \in K}$ is taken with respect to the weak* topology on X , we write $w^*\text{-}\limsup_n C_n$. When the closed unit ball B_X of X is w^* -sequentially compact (in particular when X is reflexive) $w^*\text{-}\limsup_n C_n$ coincides with the w^* -sequential limsup.

A sequence (f_n) of functions on X (with values in $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$) is said to *epiconverge* to some function f if $(\text{epi } f_n)_n$ converges in the sense of Painlevé–Kuratowski to $\text{epi } f$, where the epigraph of f is given by

$$\text{epi } f := \{(x, r) \in X \times \mathbb{R} : r \geq f(x)\}.$$

Then, we write $(f_n) \xrightarrow{e} f$ or $f = e\text{-}\lim_n f_n$. If $(\text{epi } f_n)_n$ Mosco-converges to $\text{epi } f$, we write $(f_n) \xrightarrow{M} f$. We write $f = e_w\text{-}\liminf_n f_n$ (resp. $f = e_{w^*}\text{-}\liminf_n f_n$) to mean that $\text{epi}_s f \subset \mathcal{T}\text{-}\limsup_n (\text{epi } f_n) \subset \text{epi } f$ where \mathcal{T} is the weak topology (resp. \mathcal{T} is the weak* topology on X when X is a dual space) and $\text{epi}_s f$ is the strict epigraph of f defined by

$$\text{epi}_s f := \{(x, r) \in X \times \mathbb{R} : r > f(x)\};$$

$e\text{-}\limsup_n f_n$ stands for the function whose epigraph is $\liminf_n (\text{epi } f_n)$.

Other convergences are of interest, in particular for what concerns the continuity of usual operations such as sums of sets or functions, intersections of sets... It is not our purpose to review here the many possible variants. We just consider the main instances. Recall that for two nonempty subsets A, B of X the *excess* of A over B is given by

$$e(A, B) := \sup_{a \in A} d(a, B), \quad e(\emptyset, B) = 0 \quad \text{and} \quad e(A, \emptyset) = \infty.$$

Then, for $p \in \mathbb{P}$, we set

$$e_p(A, B) := e(A \cap pU_X, B), \quad d_p(A, B) := \max(e_p(A, B), e_p(B, A)).$$

We write symbolically $A \subset b\text{-}\liminf_n A_n$ if $(e_p(A, A_n))_n \rightarrow 0$ for each $p \in \mathbb{P}$ and $A \supset b\text{-}\limsup_n A_n$ if $(e_p(A_n, \overline{A}))_n \rightarrow 0$ for each $p \in \mathbb{P}$. Let us note that $A \subset \liminf_n A_n$ whenever $A \subset b\text{-}\liminf_n A_n$ and that $\overline{A} \supset \limsup_n A_n$ whenever $A \supset b\text{-}\limsup_n A_n$. If X is finite dimensional, the reverse implications hold. We write $(A_n) \xrightarrow{b} A$ and we say that (A_n) *boundedly converges* to A , or that (A_n) converges to A for the bounded Hausdorff topology, if $b\text{-}\liminf_n A_n \supset A \supset b\text{-}\limsup_n A_n$. The choice of the open unit ball of X in what precedes, rather than the closed unit ball, enables one to use the equalities

$$e_p(\overline{A}, B) = e_p(A, B) = e_p(A, \overline{B}) = e_p(\overline{A}, \overline{B}).$$

Again, one can pass from these convergences of sets to convergences of functions. Accordingly, for a sequence (f_n) of functions on X and a function f on X , we write $f \geq b\text{-}\limsup_n f_n$ if

$\text{epi } f \subset b\text{-}\liminf_n(\text{epi } f_n)$ and $f \leq b\text{-}\liminf_n f_n$ if $\text{epi } f \supset b\text{-}\limsup_n(\text{epi } f_n)$. Of course, writing $(f_n) \xrightarrow{b} f$ when $(\text{epi } f_n) \xrightarrow{b} \text{epi } f$ means that $f \leq b\text{-}\liminf_n f_n$ and $f \geq b\text{-}\limsup_n f_n$. This type of convergence we call bounded (or bounded-Hausdorff) convergence has been thoroughly studied in [7], [10] [21], [22], [54]; it is also called the Attouch–Wets convergence or the epi-distance convergence. The terminology we use is motivated by the fact that for a family of linear continuous forms f, f_n ($n \in \mathbb{N}$) on X one has $(f_n) \xrightarrow{b} f$ if and only if $(\|f_n - f\|) \rightarrow 0$. Using these definitions in terms of epigraphs and the elementary observations made above (see also [51, Section 2]) we note the following implications:

$$f \geq b\text{-}\limsup_n f_n \implies f \geq e\text{-}\limsup_n f_n$$

and, if f is weakly l.s.c.,

$$\begin{aligned} f \leq b\text{-}\liminf_n f_n &\implies f \leq e_w\text{-}\liminf_n f_n, \\ f = b\text{-}\lim_n f_n &\implies f = M\text{-}\lim_n f_n. \end{aligned}$$

3 Continuity of the Fenchel transform

In the present section we gather some results which have been established elsewhere, in particular in [3], [8], [11], [20], [32], [51], [52], concerning continuity of the Legendre–Fenchel transform and continuity properties of operations such as the addition and the infimal convolution. A scheme like this has already been used for interpreting initial conditions (see [17], [48]).

Giving assumptions ensuring a form of continuity of the Legendre–Fenchel transform will be crucial for the sequel. Assertions (a) and (b) below are well known and elementary. Assertion (c) and (d) are sequential versions of [52, Th. 2, Th. 3]. Assertion (e) is proved in [61] and [63, Prop. 9] for the slice convergence under the stronger assumption that there exists $\bar{x} \in X$ such that $(e\text{-}\limsup_n f_n)(\bar{x}) < \infty$ but without the reflexivity hypothesis; here reflexivity allows to deduce it from assertion (c). Assertion (f) is a classical result of Mosco [43].

Lemma 3.1 (a) For any sequence (f_n) of functions on X one has $(e\text{-}\limsup_n f_n)^* \leq e_{w^*}\text{-}\liminf_n f_n^*$.

(b) For any sequence (g_n) of functions on X^* one has $(e\text{-}\limsup_n g_n)^* \leq e_w\text{-}\liminf_n g_n^*$.

(c) Let (f_n) be a sequence of $\Gamma(X)$ such that $(d((0,0), \text{epi } f_n^*))$ is bounded. Then $e\text{-}\limsup_n f_n = (e_{w^*}\text{-}\liminf_n f_n^*)^*$.

(d) Let (f_n) be a sequence of $\Gamma(X)$. Assume that there exists $\bar{x} \in X$ such that $(e\text{-}\limsup_n f_n)(\bar{x})$ and $(e\text{-}\liminf_n f_n)(\bar{x})$ are finite. Then $e\text{-}\limsup_n f_n = (e_{w^*}\text{-}\liminf_n f_n^*)^*$.

(e) Let f, f_n ($n \in \mathbb{N}$) in $\Gamma(X)$, X being reflexive. Assume that $(d((0,0), \text{epi } f_n))$ is bounded. If $f \leq e_w\text{-}\liminf_n f_n$ then $e\text{-}\limsup_n f_n^* \leq f^*$.

(f) Let f, f_n ($n \in \mathbb{N}$) in $\Gamma(X)$, X being reflexive. If $(f_n) \xrightarrow{M} f$, then $(f_n^*) \xrightarrow{M} f^*$.

In the following statement, we recall the fact that the Fenchel transform is continuous for the b -convergence ([5], [20], [21], [44], [52]). The assertions (a), (b), (c) are just the corresponding assertions of [52, Th. 14],

Theorem 3.2 Let $f, f_n : X \rightarrow \mathbb{R}_\infty$ ($n \in \mathbb{N}$) be proper functions and let $h, h_n \in \Gamma^*(X^*)$ ($n \in \mathbb{N}$).

(a) If $f \geq b\text{-}\limsup_n f_n$, and if f is convex, then one has $f^* \leq b\text{-}\liminf_n f_n^*$.

(b) If $f \leq b\text{-}\liminf_n f_n$, if f_n is convex for every n and if $(d((0,0), \text{epi } f_n))$ is bounded, then one has $f^* \geq b\text{-}\limsup_n f_n^*$.

(c) If $(f_n) \xrightarrow{b} f$, if f_n is convex for every n , then one has $(f_n^*) \xrightarrow{b} f^*$.

(d) If $h \leq b\text{-}\liminf_n h_n$, and $(d((0,0), \text{epi } h_n))$ is bounded, then one has $h^* \geq b\text{-}\limsup_n h_n^*$.

Let us note that in assertion (b) the assumption that $(d((0,0), \text{epi } f_n))$ is bounded cannot be dropped, as shown by the example $f_n = n$, f arbitrary with nonempty domain. In assertion (c) this condition is ensured by the convergence of (f_n) to f and the assumption that f has a nonempty domain.

Another continuity result can be obtained by replacing the convexity assumption by a coercivity assumption (see [52, Cor. 20]). Here we say that a function $f : X \rightarrow \mathbb{R}_\infty$ is *hypercoercive* if $f(x)/\|x\| \rightarrow \infty$ when $\|x\| \rightarrow \infty$ and we say that a family $(f_i)_{i \in I}$ is *equi-hypercoercive* if $\lim_{\|x\| \rightarrow \infty} f_i(x)/\|x\| = \infty$ uniformly for $i \in I$.

Theorem 3.3 *Let (f_n) be a family of functions from X to \mathbb{R}_∞ which is equi-hypercoercive. Suppose $(f_n) \xrightarrow{b} f$, where f is bounded below on bounded subsets. Then $(f_n^*) \xrightarrow{b} f^*$. Moreover, f^* is bounded on bounded sets and $(f_n^*) \rightarrow f^*$ uniformly on bounded sets.*

4 Convergence of Lax–Oleinik solutions

We devote this section to convergence results of Lax–Oleinik solutions. Given g, H and sequences $(g_n), (H_n)$, we look for conditions ensuring that the sequence (u_n) of Lax–Oleinik solutions u_n associated with (g_n, H_n) converges to the Lax–Oleinik solution u associated with (g, H) when (g_n, H_n) converges to (g, H) . We deduce these results from the continuity results of the preceding section and from general convergence properties for infimal convolutions. For the proofs of these last properties, we refer to [51]. Here we use the (sequential) *asymptotic function* of a function $h : X \rightarrow \overline{\mathbb{R}}$ with respect to a topology \mathcal{T} on X , defined by

$$h_\infty(x) := \inf\{\liminf_n t_n^{-1} h(t_n x_n) : (t_n) \rightarrow \infty, (x_n) \xrightarrow{\mathcal{T}} x\}.$$

We say that a set C is (sequentially) *asymptotically compact* for the topology \mathcal{T} on X if for any sequence $(c_n)_{n \in \mathbb{N}}$ in C with $(\|c_n\|) \rightarrow \infty$ there exists an infinite set $P \subset \mathbb{N}$ such that $(c_n/\|c_n\|)_{n \in P}$ \mathcal{T} -converges to some $u \neq 0$. We say that a function h is (sequentially) *asymptotically compact* for the topology \mathcal{T} if its epigraph is asymptotically compact.

In the sequel, we will take for \mathcal{T} either the weak topology or the weak* topology. The following example which refines [51, Example 3] will be used in a proof below; it shows that the notion of asymptotic compactness is present in some interesting cases.

Example 1 Let X be a reflexive Banach space, let Y be a finite dimensional normed vector space and let $\varphi : X \times Y \rightarrow \overline{\mathbb{R}}$, $a, b, c \in \mathbb{P}$ be such that $\varphi(x, y) \geq a\|x\| - b\|y\| - c$ for any $(x, y) \in X \times Y$ with sufficiently large norm. Then $E := \text{epi } \varphi$ is weakly asymptotically compact. In fact, for any sequence $((x_n, y_n, \lambda_n))_n$ of $\text{epi } \varphi$ with $(r_n) \rightarrow \infty$ for $r_n := \|(x_n, y_n, \lambda_n)\| := \max(\|x_n\|, \|y_n\|, |\lambda_n|)$, taking a subsequence, we may assume that $(r_n^{-1}(x_n, y_n, \lambda_n))_n$ has a weak limit (u, v, μ) . When $(v, \mu) \neq (0, 0)$, the conclusion $(u, v, \mu) \neq (0, 0, 0)$ holds. When $(v, \mu) = (0, 0)$, we have $r_n = \|x_n\|$ for n large enough and then $\lambda_n \geq ar_n - b\|y_n\| - c$; dividing by r_n and observing that $(r_n^{-1}\|y_n\|) \rightarrow 0$, we obtain a contradiction.

We first consider upper epilimits of solutions. We rely on a general result for infimal convolutions; while for the usual upper epilimits the convergence result is simple and immediate, for the b -lim sup a qualification condition is needed.

Lemma 4.1 ([51, Prop. 6, Example 4]) *Let $f, g, f_n, g_n : X \rightarrow \overline{\mathbb{R}}$ ($n \in \mathbb{N}$) be proper functions.*

(a) *If $f \geq e\text{-lim sup}_n f_n$ and $g \geq e\text{-lim sup}_n g_n$, then $f \square g \geq e\text{-lim sup}_n (f_n \square g_n)$.*

(b) Assume that $f \geq b\text{-}\limsup_n f_n$ and $g \geq b\text{-}\limsup_n g_n$. Then $f \square g \geq b\text{-}\limsup_n (f_n \square g_n)$ whenever one of the following conditions holds:

(b₁) for any sequences $(w_n), (x_n)$ such that (w_n) is bounded and $(f(x_n) + g(w_n - x_n))$ is majorized, the sequence (x_n) is bounded, or, more generally,

$$\forall p \in \mathbb{P}, \exists q \in \mathbb{P}, \forall w \in [f \square g < p] \cap pU_X : (f \square g)(w) = \inf\{f(x) + g(w - x) : x \in qU_X\}; \quad (3)$$

(b₂) $X = X_1 \times X_2$, $f(x_1, x_2) \geq a_1 \|x_1\| + a_2 \|x_2\| + c$, $g(x_1, x_2) \geq b_1 \|x_1\| + b_2 \|x_2\| + d$ for all $x = (x_1, x_2) \in X$, with $\|x\|$ large enough, where $a_1, a_2, b_1, b_2, c, d \in \mathbb{R}$ with $a_1 + b_1 > 0$, $a_2 + b_2 > 0$;

(b₃) X is reflexive, $f_\infty(0) \geq 0$, $g_\infty(0) \geq 0$, $f_\infty(-u) > -g_\infty(u)$ for each $u \in X \setminus \{0\}$, the asymptotic functions being taken with respect to the weak topology, and f is weakly asymptotically compact.

An application of Lemma 3.1 and of these criteria to our problem yields the following result. In the sequel we set, for $(x, x^*, t) \in X \times X^* \times \mathbb{R}$,

$$F(x^*, t) := \iota_{\text{epi } H}(x^*, -t), \quad (4)$$

$$G(x, t) := g(x) + \iota_{\{0\}}(t), \quad (5)$$

and we define F_n and G_n in a similar way by changing H and g into H_n and g_n , respectively. It has been observed in [38], [48] (and in a special case in [55]) that

$$F^*(x, s) = (sH)^*(x) \text{ for } (x, s) \in X \times \mathbb{R}_+, F^*(x, s) = \infty \text{ otherwise,}$$

and

$$u = F^* \square G.$$

Proposition 4.2 Let $g, g_n : X \rightarrow \overline{\mathbb{R}}$ and let $H, H_n \in \Gamma^*(X^*)$.

(a) Suppose $g \geq e\text{-}\limsup_n g_n$, $H \leq e_{w^*}\text{-}\liminf_n H_n$, and either X is reflexive and there exists $\bar{x}^* \in X^*$ such that $(e\text{-}\limsup_n H_n)(\bar{x}^*) < \infty$, or there exists $\bar{x} \in X$ such that

$$-\infty < (e\text{-}\liminf_n H_n^*)(\bar{x}) \leq (e\text{-}\limsup_n H_n^*)(\bar{x}) < \infty. \quad (6)$$

Then $u \geq e\text{-}\limsup_n u_n$.

(b) Assume that $(d((0, 0), \text{epi } H_n))$ is bounded, $g \geq b\text{-}\limsup_n g_n$ and $H \leq b\text{-}\liminf_n H_n$. Then $u \geq b\text{-}\limsup_n u_n$ whenever one of the following conditions holds:

(b₁) for any $p \in \mathbb{P}$ there exists $q \in \mathbb{P}$ such that for all $(w, t) \in [u < p] \cap pU_{X \times \mathbb{R}}$ with $t \geq 0$ one has

$$u(w, t) = \inf\{(tH)^*(x) + g(w - x) : x \in qU_X\};$$

(b₂) there exist $\bar{x}^* \in X^*$, $b, c \in \mathbb{R}$, $r \in \mathbb{R}_+$ with $\|\bar{x}^*\| < b + r$ such that $g \geq b\|\cdot\| + c$ and H is bounded above on $\bar{x}^* + rB_{X^*}$;

(b₃) X is reflexive, $g_\infty(0) \geq 0$, $g_\infty(-u) > -(\iota_{\text{dom } H})^*(u)$ for each $u \in X \setminus \{0\}$, and H is bounded above on a neighborhood of 0.

Let us observe that assumption (b₂) is satisfied when $g \geq b\|\cdot\| + c$ for some $b, c \in \mathbb{R}$, and either $b > d(0, \text{dom } H)$ or $b \leq 0$ and H is bounded above on rU_{X^*} for some $r > -b$. In the first case one takes $r = 0$ and one picks some $\bar{x}^* \in \text{dom } H$ such that $\|\bar{x}^*\| < b$; in the second case one takes $\bar{x}^* = 0$.

Proof. (a) Since $\text{epi } G = \{(x, 0, r) : (x, r) \in \text{epi } g\}$ and a similar relation for G_n , one has $G \geq e\text{-}\limsup_n G_n$ and since $w^*\text{-}\limsup_n (\text{epi } H_n) \subset \text{epi } H$ one has $\iota_{\text{epi } H} \leq e_{w^*}\text{-}\liminf_n \iota_{\text{epi } H_n}$. Let

us first suppose X is reflexive and there exists $\bar{x}^* \in X^*$ such that $e\text{-}\limsup_n H_n(\bar{x}^*) < \infty$. Then, there exist $r \in \mathbb{R}$ and a sequence $(x_n^*) \rightarrow \bar{x}^*$ such that for all n one has $(x_n^*, r) \in \text{epi } H_n$, hence $F_n(x_n^*, -r) = 0$ and $e\text{-}\limsup_n F_n(\bar{x}^*, -r) < \infty$. By Lemma 3.1 (c), we have $e\text{-}\limsup_n F_n^* \leq F^*$.

Let us get the same inequality when condition (6) holds. Let us consider $f := F^*$ and $f_n := F_n^*$ for $n \in \mathbb{N}$, so that, for every $x \in X$ one has

$$H^*(x) = \sup_{(x^*, r^*) \in \text{epi } H} \langle (x, -1), (x^*, r^*) \rangle = (\iota_{\text{epi } H})^*(x, -1) = f(x, 1)$$

and, similarly, $H_n^*(x) = f_n(x, 1)$. Then there exist $r, s \in \mathbb{R}$ and a sequence $(\bar{x}_n) \rightarrow \bar{x}$ such that $\limsup_n H_n^*(\bar{x}_n) \leq s$ and for every sequence $(x_n) \rightarrow \bar{x}$ one has $r \leq \liminf_n H_n^*(x_n)$. Thus $(e\text{-}\limsup_n f_n)(\bar{x}, 1)$ and $(e\text{-}\liminf_n f_n)(\bar{x}, 1)$ are finite. Then, since $f_n^* = F_n$, and since for a sequence (E_n) of subsets of $X^* \times \mathbb{R}$ one has $e_{w^*}\text{-}\liminf_n \iota_{E_n} = \iota_E$ for $E := w^*\text{-}\limsup_n E_n$, Lemma 3.1 (d), yields $e\text{-}\limsup_n f_n = (e_{w^*}\text{-}\liminf_n F_n)^* \leq F^*$ or $e\text{-}\limsup_n F_n^* \leq F^*$. Therefore, in both cases, assertion (a) is a direct consequence of the corresponding assertion of the preceding lemma: $u = F^* \square G \geq e\text{-}\limsup_n F_n^* \square G_n = e\text{-}\limsup_n u_n$.

(b) It is clear that $G \geq b\text{-}\limsup_n G_n$ and $F \leq b\text{-}\liminf_n F_n$ (since $\text{epi } H \supset b\text{-}\limsup_n (\text{epi } H_n)$). Using Theorem 3.2 (d), we get $b\text{-}\limsup_n F_n^* \leq F^*$.

(b₁) Since (3) is obtained by transcribing the present condition in terms of F and G , we obtain that

$$u = F^* \square G \geq b\text{-}\limsup_n F_n^* \square G_n = b\text{-}\limsup_n u_n.$$

(b₂) We apply criterion (b₂) of the preceding lemma with $X_1 = X$, $X_2 = \mathbb{R}$. Let $\bar{x}^* \in X^*$, $b, c \in \mathbb{R}$, $m, r \in \mathbb{R}_+$ be such that $g \geq b \|\cdot\| + c$, $\|\bar{x}^*\| < b + r$ and $\sup H(\bar{x}^* + rB_{X^*}) \leq m$. Since $F^*(x, t) = \infty$ for $t < 0$, let us consider $(x, t) \in X \times \mathbb{R}_+$. Then we have

$$F^*(x, t) \geq \sup \{ \langle x^*, x \rangle + \langle \bar{x}^*, x \rangle - tH(\bar{x}^* + x^*) : x^* \in rB_{X^*} \} \geq (r - \|\bar{x}^*\|) \|x\| - m|t|.$$

Since $G(x, t) \geq b \|x\| + (m+1) \cdot |t| + c$ for every $(x, t) \in X \times \mathbb{R}$, the mentioned criterion is satisfied.

(b₃) Since F^* is l.s.c. and sublinear, $(F^*)_\infty = F^* > -\infty$, $(F^*)_\infty(0) \geq 0$, $G_\infty(u, t) = g_\infty(u) + \iota_{\{0\}}(t)$ and $F^*(u, 0) = (\iota_{\text{dom } H})^*(u) > -g_\infty(-u)$ for all $u \in X \setminus \{0\}$, the condition $F^*(-u, -t) > -G_\infty(u, t)$ for any $(u, t) \neq (0, 0)$ is fulfilled. Since H is bounded above by some $m \in \mathbb{R}_+$ on some ball rB_{X^*} with $r > 0$, we have $F^*(x, t) \geq r \|x\| - m|t|$ for all $(x, t) \in X \times \mathbb{R}$, so that Example 1 above shows that F^* is weakly asymptotically compact. \square

Now we turn to the lower epilimits of sequences of solutions. Again, we recall a general result about infimal convolutions of functions.

Lemma 4.3 ([51, Prop. 17, Cor. 18]) *Let $f, g, f_n, g_n : X \rightarrow \overline{\mathbb{R}}$ ($n \in \mathbb{N}$) be proper functions.*

(a) *If X is reflexive, $f \leq e_w\text{-}\liminf_n f_n$ and $g \leq e_w\text{-}\liminf_n g_n$, then $f \square g \leq e_w\text{-}\liminf_n (f_n \square g_n)$ whenever the following condition holds: for any infinite subset K of \mathbb{N} , for any sequences $(w_k)_{k \in K}$, $(x_k)_{k \in K}$ such that $(w_k)_{k \in K}$ is bounded and $(f_k(x_k) + g_k(w_k - x_k))_{k \in K}$ is majorized, the sequence $(x_k)_{k \in K}$ is bounded, or, more generally,*

$$\begin{aligned} \forall p \in \mathbb{P}, \exists q \in \mathbb{P}, \exists m \in \mathbb{N}, \forall n \geq m, \forall w \in [f_n \square g_n < p] \cap pU_X : \\ (f_n \square g_n)(w) = \inf \{ f_n(x) + g_n(w - x) : x \in qU_X \}. \end{aligned} \quad (7)$$

(b) *Assume $f \leq b\text{-}\liminf_n f_n$, $g \leq b\text{-}\liminf_n g_n$ and condition (7) holds. Then $f \square g \leq b\text{-}\liminf_n (f_n \square g_n)$.*

(c) If $X = X_1 \times X_2$ with X_1, X_2 reflexive Banach spaces, $f \leq e_w\text{-}\liminf_n f_n$ and $g \leq e_w\text{-}\liminf_n g_n$, then $f \square g \leq e_w\text{-}\liminf_n (f_n \square g_n)$ whenever the following condition holds:

there exist $a_1, a_2, b_1, b_2, c, d \in \mathbb{R}$ with $a_1 + b_1 > 0$, $a_2 + b_2 > 0$ such that for all $x = (x_1, x_2) \in X$

$$\inf_n f_n(x_1, x_2) \geq a_1 \|x_1\| + a_2 \|x_2\| + c, \quad \inf_n g_n(x_1, x_2) \geq b_1 \|x_1\| + b_2 \|x_2\| + d. \quad (8)$$

(d) Assume $f \leq b\text{-}\liminf_n f_n$ and $g \leq b\text{-}\liminf_n g_n$ and condition (8) holds. Then $f \square g \leq b\text{-}\liminf_n f_n \square g_n$.

(e) If X is reflexive, $f \leq e_w\text{-}\liminf_n f_n$ and $g \leq e_w\text{-}\liminf_n g_n$, then $f \square g \leq e_w\text{-}\liminf_n f_n \square g_n$ whenever the following condition holds:

there exist $\hat{f}, \hat{g}: X \rightarrow \overline{\mathbb{R}}$ with \hat{f} w -asymptotically compact such that

$$\inf_n f_n \geq \hat{f}, \quad \inf_n g_n \geq \hat{g}, \quad \hat{f}_\infty(0) \geq 0, \quad \hat{g}_\infty(0) \geq 0, \quad \hat{f}_\infty(-u) > -\hat{g}_\infty(u) \text{ for all } u \in X \setminus \{0\}. \quad (9)$$

(f) Assume $f \leq b\text{-}\liminf_n f_n$ and $g \leq b\text{-}\liminf_n g_n$ and condition (9) holds. Then $f \square g \leq b\text{-}\liminf_n f_n \square g_n$.

Note that when $\hat{f} := a \|\cdot\| + c$ with $a \geq 0$ or, more generally, when \hat{f} is bounded below by a continuous affine function, in particular when $\hat{f} \in \Gamma(X)$, the condition $\hat{f}_\infty(0) = 0$ is automatically satisfied.

An application of these criteria to Lax–Oleinik solutions yields the following result.

Proposition 4.4 (a) Suppose X is reflexive, $g \leq e_w\text{-}\liminf_n g_n$ and $H \geq e\text{-}\limsup_n H_n$, where $H, H_n \in \Gamma^*(X^*)$ for every $n \in \mathbb{N}$. Then $u \leq e_w\text{-}\liminf_n u_n$ provided one of the following conditions holds:

$$\begin{aligned} \forall p \in \mathbb{P}, \exists q \in \mathbb{P}, \exists m \in \mathbb{N}, \forall n \geq m, \forall (w, t) \in [u_n < p] \cap pU_{X \times \mathbb{R}}, t \geq 0 : \\ u_n(w, t) = \inf\{(tH_n)^*(x) + g_n(w - x) : x \in qU_X\}; \end{aligned} \quad (10)$$

$$\begin{aligned} \exists a, b, c \in \mathbb{R}, \forall n \in \mathbb{N}, \exists \bar{x}_n^* \in X^*, \exists m, r_n \in \mathbb{R}_+ : r_n \geq \|\bar{x}_n^*\| + a, a + b > 0, \\ \inf_n g_n \geq b \|\cdot\| + c \text{ and } \sup_n \sup H_n(\bar{x}_n^* + r_n B_{X^*}) \leq m; \end{aligned} \quad (11)$$

$$\begin{aligned} \exists r \in \mathbb{P}, \exists \hat{g}: X \rightarrow \overline{\mathbb{R}}, \exists \hat{H}: X^* \rightarrow \overline{\mathbb{R}} : \inf_n g_n \geq \hat{g}, \sup_n H_n \leq \hat{H}, \\ \sup \hat{H}(rU_{X^*}) < \infty, \hat{g}_\infty(0) \geq 0, \hat{g}_\infty(-u) > -(\iota_{\text{dom } \hat{H}})^*(u) \quad \forall u \in X \setminus \{0\}. \end{aligned} \quad (12)$$

(b) Assume that $g \leq b\text{-}\liminf_n g_n$, $H \geq b\text{-}\limsup_n H_n$ and $H \in \Gamma^*(X^*)$. Then $u \leq b\text{-}\liminf_n u_n$ whenever one of the conditions (10), (11), (12) holds.

Proof. (a) In view of the constructions of F and G , we have $G \leq e_w\text{-}\liminf_n G_n$ and $\iota_{\text{epi } H} \geq e\text{-}\limsup_n \iota_{\text{epi } H_n}$ (since $\text{epi } H \subset \liminf_n (\text{epi } H_n)$), hence $F \geq e\text{-}\limsup_n F_n$. Using Lemma 3.1 (b), we get $e_w\text{-}\liminf_n F_n^* \geq F^*$.

Assume that condition (10) holds. After transcribing it we observe that condition (7) is satisfied for $f_n := F_n^*$ and $g_n := G_n$; the conclusion is a consequence of assertion (a) of the preceding lemma: $u = F^* \square G \leq e_w\text{-}\liminf_n F_n^* \square G_n = e_w\text{-}\liminf_n u_n$.

Assume now that condition (11) holds. We use an argument similar to the one used in the proof of Proposition 4.2 (b₂). Let $\bar{x}_n^* \in X^*$, $b, c \in \mathbb{R}$, $m, r_n \in \mathbb{R}_+$ be such that $g_n \geq b \|\cdot\| + c$, $\|\bar{x}_n^*\| < b + r_n$ and $\sup H_n(\bar{x}_n^* + r_n B_{X^*}) \leq m$. Then, for any $(x, t) \in X \times \mathbb{R}_+$, we have

$$\begin{aligned} F_n^*(x, t) &\geq \sup\{\langle x^*, x \rangle + \langle \bar{x}_n^*, x \rangle - tH_n(\bar{x}_n^* + x^*) : x^* \in r_n B_{X^*}\} \\ &\geq (r_n - \|\bar{x}_n^*\|) \|x\| - m|t| \geq a \|x\| - m|t|, \end{aligned}$$

and so $F_n^*(x, t) \geq a \|x\| - m |t|$, $G_n(x, t) \geq b \|x\| + (m + 1) \cdot |t| + c$ for all $(x, t) \in X \times \mathbb{R}$; hence the criterion (c) of the preceding lemma with $X_1 = X$, $X_2 = \mathbb{R}$ can be applied.

Finally, assume that condition (12) holds. This time we apply criteria (e) of the preceding lemma, observing that setting $\widehat{F}(x^*, t) := \iota_{\text{epi } \widehat{H}}(x^*, -t)$ for $(x^*, t) \in X^* \times \mathbb{R}$, $\widehat{G}(x, t) = \widehat{g}(x) + \iota_{\{0\}}(t)$, we have $F_n^* \geq \widehat{F}^*$ and $G_n \geq \widehat{G}$ for every $n \in \mathbb{N}$ with $(\widehat{F}^*)_\infty(0, 0) \geq 0$ (since \widehat{F}^* is the support function of a nonempty set), $\widehat{G}_\infty(0) \geq 0$, $(\widehat{F}^*)_\infty(-u) > -\widehat{G}_\infty(u)$ for each $u \in X \times \mathbb{R} \setminus \{(0, 0)\}$, \widehat{F}^* being w -asymptotically compact by Example 1.

(b) Clearly, $G \leq b\text{-}\liminf_n G_n$ and $\iota_{\text{epi } H} \geq b\text{-}\limsup_n \iota_{\text{epi } H_n}$ (since $\text{epi } H \subset b\text{-}\liminf_n (\text{epi } H_n)$). Using Theorem 3.2 (a) and the fact that T is an isometry, we get $b\text{-}\liminf_n F_n^* \geq F^*$.

In the proof of (a) we obtained that conditions (10), (11) and (12) imply the corresponding assumptions of Lemma 4.3. Thus, Lemma 4.3 ensures that $u = F^* \square G \leq b\text{-}\liminf_n F_n^* \square G_n = b\text{-}\liminf_n u_n$. \square

When $H_n = H$ for all $n \in \mathbb{N}$, the preceding conditions can be simplified. For simplicity, we limit ourselves to condition (11).

Corollary 4.5 *Assume that X is reflexive, $g \leq b\text{-}\liminf_n g_n$, $H_n = H$ for all n , with H proper and*

$$\exists b, c \in \mathbb{R}, \exists \bar{x}^* \in X^*, \exists m, r \in \mathbb{P} : r > \|\bar{x}^*\| - b, \inf_n g_n \geq b \|\cdot\| + c \text{ and } \sup H(\bar{x}^* + rB_{X^*}) \leq m. \quad (13)$$

Then $u \leq b\text{-}\liminf_n u_n$.

Proof. Since $H_n = H$ for all n , we do not need to use Theorem 3.2 (a), so that we can drop the convexity assumption on H . \square

Gathering the assertions of Propositions 4.2 (a) and 4.4 (a) and observing that condition (6) is satisfied when $(H_n^*) \xrightarrow{M} H^*$, what is the case when $(H_n) \xrightarrow{M} H$ with $H_n \in \Gamma^*(X^*)$ and X is reflexive, we get the following result about Mosco convergence of Lax–Oleinik solutions.

Theorem 4.6 *Suppose X is reflexive, $(g_n) \xrightarrow{M} g$, $(H_n) \xrightarrow{M} H$, with $H, H_n \in \Gamma^*(X^*)$, $\text{dom } g_n \neq \emptyset$ for each $n \in \mathbb{N}$. Suppose one of the conditions (10), (11), (12) holds. Then $(u_n) \xrightarrow{M} u$.*

Similarly, gathering the assertions of Propositions 4.2 (b), (d), (f) and 4.4 (b), and using the following result about the b -convergence of infimal convolutions, we will get a b -convergence result for Lax–Oleinik solutions.

Lemma 4.7 ([51, Prop. 19 (a)]) *Suppose X is reflexive. Let $f, g, f_n, g_n : X \rightarrow \overline{\mathbb{R}}$ be proper functions, f and g being w -l.s.c. If $(f_n) \xrightarrow{b} f$, $(g_n) \xrightarrow{b} g$ and if condition (7) holds, then $(f_n \square g_n) \xrightarrow{b} f \square g$.*

Theorem 4.8 *Suppose $(g_n) \xrightarrow{b} g$ and $(H_n) \xrightarrow{b} H$, with g, g_n proper and $H, H_n \in \Gamma^*(X^*)$ for each $n \in \mathbb{N}$. Then $(u_n) \xrightarrow{b} u$ whenever one of the following conditions holds*

- (a) (10) is satisfied, X is reflexive and g is weakly l.s.c.;
- (b) (11) is satisfied;
- (c) (12) is satisfied and X is reflexive.

Proof. Defining F, F_n, G, G_n as in (4), (5) and using Theorem 3.2 (c), we have $(F_n^*) \xrightarrow{b} F^*$ and, obviously, $(G_n) \xrightarrow{b} G$. For each of the assertions (a), (b), (c) we have $u \leq b\text{-}\liminf_n u_n$ by Proposition 4.4.

(a) Let us first assume (10). Then condition (7) is satisfied for the sequences (G_n) and (F_n^*) and the conclusion follows from the preceding lemma.

(b) Now let us assume (11) holds. Let us check the assumptions of assertion (b₂) of Proposition 4.2. Let $b, c \in \mathbb{R}$, $m, r \in \mathbb{P}$, $r > -b$ be as in (11). Since $\text{epi } g \subset \liminf_n \text{epi } g_n \subset \text{epi } \hat{g}$ with $\hat{g} := b\|\cdot\| + c$ we have $g \geq \hat{g}$. When $b > \sup_n d(0, [H_n \leq m])$, taking b' between these two numbers we get a sequence $(x_n) \in b'U_X$ such that $(x_n, m) \in \text{epi } H_n$ for each n . It follows that for n large enough there exists $(x'_n, m'_n) \in \text{epi } H$ with $\|x'_n - x_n\| < b - b'$ and $d(0, \text{dom } H) < b$. When $\sup_n \sup H_n(rU_{X^*}) \leq m$ we have

$$rU_{X^*} \times \{m\} \subset \bigcap_n \text{epi } H_n \subset \limsup_n (\text{epi } H_n) \subset \text{epi } H,$$

so that $\sup H(rU_{X^*}) \leq m$.

(c) Finally let us assume (12) holds. Using the fact that the asymptotic function of a function is equal to the asymptotic function of its l.s.c. hull (for the norm topology), we may assume that \hat{g} is l.s.c. Then $\text{epi } g \subset \liminf_n (\text{epi } g_n) \subset \text{epi } \hat{g}$, hence $g \geq \hat{g}$. On the other hand, $\text{epi } \hat{H} \subset \bigcap_n \text{epi } H_n \subset \limsup_n (\text{epi } H_n) \subset \text{epi } H$, hence $H \leq \hat{H}$. Thus, the assumptions of assertion (b₃) of Proposition 4.2 are satisfied. \square

The preceding results have consequences on regularity issues. Recall that if X is a Hilbert space, and if $g \in \Gamma(X)$, then $(g \square_n \|\cdot\|^2) \xrightarrow{b} g$ and $g \square_n \|\cdot\|^2$ is continuously differentiable. Several extensions can be given, either by taking for X a more general Banach space ([3, Th. 3.24], [25]) or by requiring that g just satisfies a quadratic minorization ([4], [25]). One can also take a more general regularizing kernel ([9], [21, Th. 7.3.8], [25]). Here, we say that a family (j_n) of nonnegative convex functions (j_n) is a *regularizing kernel* if $(j_n) \xrightarrow{b} \iota_{\{0\}}$ and $j_n(0) = 0$ for all n . In [21, Def. 7.3.5] the functions j_n are supposed to be lower semicontinuous and continuous at 0. Let us show that we can obtain the convergence $(g \square j_n) \xrightarrow{b} g$ for every $g \in \Gamma(X)$ without these additional assumptions made in ([21, Th. 7.3.8]).

Note that for a sequence (j_n) of nonnegative functions null at 0 one has $(j_n) \xrightarrow{b} \iota_{\{0\}}$ if and only if for every bounded sequence $((x_n, r_n))$ with $(x_n, r_n) \in \text{epi } j_n$ for large n one has that $(x_n) \rightarrow 0$.

Lemma 4.9 *Let (j_n) be a sequence of nonnegative convex functions null at 0 such that $(j_n) \xrightarrow{b} \iota_{\{0\}}$ and let g be a proper function for which there exist $b, c \in \mathbb{R}$ such that $g \geq b\|\cdot\| + c$. Let $g_n := g \square j_n$. Then*

- (a) *for every $c' < c$ there exists some $k \in \mathbb{N}$ such that for all $n \geq k$ one has $g_n \geq b\|\cdot\| + c'$;*
- (b) *one has $(g_n) \xrightarrow{b} g$.*

Proof. Since $(j_n) \xrightarrow{b} \iota_{\{0\}}$, given $q > 1$, we can find $k := k(q) \in \mathbb{N}$ such that $e_q(\text{epi } j_n, \text{epi } \iota_{\{0\}}) < q^{-1}$ for $n \geq k$, hence $j_n(u) \geq q$ for all $n \geq k$, $u \in X$ satisfying $\|u\| = q^{-1}$ since if we had $j_n(u) < q$ for such an u , we could find $t \in \mathbb{R}_+$ satisfying $\|(u, j_n(u)) - (0, t)\| < q^{-1}$, what is impossible. By convexity, we get $j_n(v) \geq q^2 \|v\|$ for $v \in X \setminus q^{-1}U_X$, $n \geq k$.

(a) Without loss of generality, we may assume that $c' := c - 1$. Let $q > |b| + 1$ and let $n \geq k := k(q)$. For $w, x \in X$ with $\|w - x\| \geq q^{-1}$, we have, when $b < 0$,

$$g(w) + j_n(x - w) \geq b\|w\| + c + q^2 \|w - x\| \geq c - |b|\|w\| + |b|\|w - x\| \geq c - |b|\|x\| = b\|x\| + c,$$

while for $b \geq 0$ we have

$$g(w) + j_n(x - w) \geq b \|w\| + c + q^2 \|w - x\| \geq b \|x\| + c.$$

When $\|w - x\| < q^{-1}$ we have for $b < 0$

$$g(w) + j_n(x - w) \geq c - |b| \|w\| \geq c - |b| q^{-1} - |b| \|x\| \geq b \|x\| + c - 1,$$

while for $b \geq 0$ we have

$$g(w) + j_n(x - w) \geq b \|w\| + c \geq b \|x\| - bq^{-1} + c \geq b \|x\| + c - 1.$$

Taking the infimum over $w \in X$, we get $g_n(x) \geq b \|x\| + c - 1$.

(b) Let $a := \max(|b|, |c|)$, so that $g(x) \geq -a(\|x\| + 1)$ for all $x \in X$. Since $j_n(0) = 0$, we have $g_n \leq g$, hence $b\text{-}\limsup_n g_n \leq g$. Let $p \in \mathbb{P}$ and let $((x_n, t_n))$ be a sequence of $pU_{X \times \mathbb{R}}$ with $(x_n, t_n) \in \text{epi } g_n$ for all n . Let us show that $d((x_n, t_n), \text{epi } g) \rightarrow 0$. Given $\varepsilon \in (0, 1)$, let us take $q > \max(1, a + \varepsilon^{-1}(p + 1)(a + 1))$. Let us pick $w_n \in X$ such that

$$g(w_n) + j_n(x_n - w_n) < g_n(x_n) + \varepsilon.$$

If for some $n \geq k := k(q)$ we had $\|x_n - w_n\| \geq \varepsilon$ (hence $\|x_n - w_n\| > q^{-1}$), as $q^2 \|x_n - w_n\| \leq j_n(x_n - w_n)$ by the first part of the proof, we would get

$$\begin{aligned} -a(\|w_n\| + 1) + q^2 \|x_n - w_n\| &< g_n(x_n) + \varepsilon < p + 1, \\ (q - a)\varepsilon &\leq (q^2 - a) \|x_n - w_n\| < p + 1 + a(\|x_n\| + 1) \leq (a + 1)(p + 1), \end{aligned}$$

a contradiction with the choice of q . Thus $\|x_n - w_n\| < \varepsilon$. Since $g(w_n) < g_n(x_n) + \varepsilon \leq t_n + \varepsilon$, for $n \geq k$ we have

$$d((x_n, t_n), \text{epi } g) \leq \|(x_n, t_n) - (w_n, t_n + \varepsilon)\| \leq \varepsilon.$$

That shows that $(e_p(\text{epi } g_n, \text{epi } g)) \rightarrow 0$, i.e. $g \leq b\text{-}\liminf_n g_n$ and so $(g_n) \xrightarrow{b} g$. \square

Corollary 4.10 *Assume that X is reflexive, g, H are proper and (j_n) is a family of regularizing kernels. Suppose there exist $\bar{x}^* \in X^*$, $b, c \in \mathbb{R}$, $r > 0$ such that $r \geq \|\bar{x}^*\| + a$, $a + b > 0$, $g \geq b \|\cdot\| + c$ and $\sup H(\bar{x}^* + rU_{X^*}) < \infty$. Let $g_n := g \square j_n$ and let u_n be the Lax-Oleinik solution associated with g_n and H . Then $(u_n) \xrightarrow{b} u$.*

Proof. The preceding lemma yields some $k \in \mathbb{N}$ such that for $n \geq k$ one has $g_n \geq b \|\cdot\| + c - 1$. Moreover $(g_n) \xrightarrow{b} g$. Corollary 4.5 ensures that $u \leq b\text{-}\liminf_n u_n$. Now, since $g_n \leq g$ for all n , we have $u_n \leq u$, hence $b\text{-}\limsup_n u_n \leq u$. Thus $(u_n) \xrightarrow{b} u$. \square

Remark 4.1 In the preceding corollary one has $u_n = F^* \square G_n$, where F is given by (4) and $G_n(x, t) := g_n(x) + \iota_{\{0\}}(t) = (G \square K_n)(x, t)$ with $K_n(x, t) := j_n(x) + \iota_{\{0\}}(t)$. By the associativity of the infimal convolution, we have $u_n = (F^* \square G) \square K_n = u \square K_n$; u_n can be seen as the regularization of u by the regularizing kernel K_n . For the smoothing of u , a more standard regularizing kernel is given by $J_n(x, t) := j_n(x) + nt^2$. In order to obtain the convergence of

$$w_n := u \square J_n = (F^* \square G) \square J_n = F^* \square (G \square J_n)$$

to u we need to check the assumptions of Lemma 4.9 with g replaced by u . Let us assume again that there exist $\bar{x}^* \in X^*$, $b, c, m \in \mathbb{R}$, $r > 0$ such that $r > \|\bar{x}^*\| - b$, $g \geq b \|\cdot\| + c$ and

$\sup H(\bar{x}^* + rU_{X^*}) < m$. As in the proof of Proposition 4.2 (b₂), taking $a > -b$ with $r > \|\bar{x}^*\| + a$, we have $F^*(y, s) \geq -b\|y\| - m|s|$ for all $(y, s) \in X \times \mathbb{R}$ and $G(z, t) \geq b\|z\| + (m+1) \cdot |t| + c$, for all $(z, t) \in X \times \mathbb{R}$. Thus, for $(x, q) \in X \times \mathbb{R}$,

$$(F^* \square G)(x, q) \geq -|b|\|x\| - m|q| + c$$

and the preceding lemma shows that $(w_n) \xrightarrow{b} u$. However, for $(x, t) \in X \times \mathbb{R}$, one has $(G \square J_n)(x, t) = (g \square j_n)(x) + nt^2$. Thus, in general, $w_n = (F^* \square G) \square J_n \neq u_n$.

5 Convergence of Hopf–Lax solutions

Let g_n, H_n ($n \in \mathbb{N}$), g, H be given functions and let v_n (resp. v) be the Hopf solution associated with (g_n, H_n) (resp. (g, H)). In order to establish convergence results for the Hopf solutions v_n , v associated with those pairs, we recall from [38], [39], that

$$v(x, t) = (F + G^*)^*(x, t),$$

with F, G given by (4), (5) provided that

$$\text{dom } g^* \cap \text{dom } H \neq \emptyset. \quad (14)$$

Throughout this section this assumption will be in force as will be the similar one with (g_n, H_n) . We recall that when H is in $\Gamma^*(X^*)$, the Fenchel conjugate v^* of v is given by

$$v^*(x^*, t^*) = F(x^*, t^*) + G^*(x^*, t^*) = \iota_{\text{epi } H}(x^*, -t^*) + g^*(x^*).$$

The main feature of the following convergence result is that no convexity assumption is required on the Hamiltonians. However, the convergence of the sequence (H_n) is not of the types we have used previously. It is a one-sided form of continuous convergence. We recall (or make precise, as the case of extended real-valued functions is not classical) that a sequence (H_n) of functions from a topological space Y to $\overline{\mathbb{R}}$ converges continuously to H if, for any $y \in H^{-1}(\mathbb{R})$ and any sequence $(y_n) \rightarrow y$, the sequence $(H_n(y_n))$ converges to $H(y)$. For a continuous function H with finite values, continuous convergence of (H_n) to H is equivalent to uniform convergence on compact subsets of X^* . We say that (H_n) converges upper continuously if, for any $y \in H^{-1}(\mathbb{R})$ and any sequence $(y_n) \rightarrow y$, we have $\limsup_n H_n(y_n) \leq H(y)$. This property is obviously satisfied when H is finite, upper semicontinuous and $(H_n) \rightarrow H$ locally uniformly. It is also satisfied when the family (H_n) is equicontinuous at any $y \in H^{-1}(\mathbb{R})$ and $(H_n) \rightarrow H$ pointwise (or even $e\text{-}\limsup_n H_n \leq H$). Let us note that, when H is proper, (H_n) converges upper continuously to H if, and only if $-H \leq e\text{-}\liminf_n (-H_n)$.

Proposition 5.1 *Let $g, g_n \in \Gamma(X)$ ($n \in \mathbb{N}$), and let H, H_n ($n \in \mathbb{N}$) be proper. Suppose X is reflexive, $g \leq e_w\text{-}\liminf_n g_n$, there exists $\bar{x} \in X$ such that $e\text{-}\limsup_n g_n(\bar{x}) < \infty$ and (H_n) converges upper continuously to H . Then, one has $v \leq e_w\text{-}\liminf_n v_n$.*

Proof. Let us show that $F + G^* \geq e\text{-}\limsup_n (F_n + G_n^*)$. Then Lemma 3.1 (a) will yield $v \leq e_w\text{-}\liminf_n v_n$. Let $(x^*, t^*, r) \in \text{epi}(F + G^*)$. Thus $(x^*, -t^*) \in \text{epi } H$ and $(x^*, r) \in \text{epi } g^*$. By Lemma 3.1 (e) we have $g^* \geq e\text{-}\limsup_n g_n^*$; hence, there exists $((x_n^*, r_n)) \rightarrow (x^*, r)$ with $(x_n^*, r_n) \in \text{epi } g_n^*$ for all n . Since (H_n) converges upper continuously to H , we have $\limsup_n H_n(x_n^*) \leq H(x^*)$. Setting $t_n^* := \min(t^*, -H_n(x_n^*))$, we have $t_n^* > -\infty$, $(x_n^*, t_n^*, r_n) \in \text{epi}(F_n + G_n^*)$ and $((x_n^*, t_n^*, r_n)) \rightarrow (x^*, t^*, r)$. This shows that $F + G^* \geq e\text{-}\limsup_n (F_n + G_n^*)$. \square

Let us turn to $e\text{-}\limsup_n v_n$; here a convexity assumption is made on the family (H_n) . The proof relies on the following lemma of independent interest.

Lemma 5.2 *Let h_n, k_n be functions on X^* such that (h_n) and (k_n) are equi-bounded from below on bounded subsets. Then*

$$e_{w^*}\text{-}\liminf_n h_n + e_{w^*}\text{-}\liminf_n k_n \leq e_{w^*}\text{-}\liminf_n (h_n + k_n).$$

A similar result holds when h_n and k_n are functions on X .

Proof. Let (x^*, r) be a w^* -cluster point of a bounded sequence $((x_n^*, r_n))_{n \in K}$ with $r_n \geq (h_n + k_n)(x_n^*)$ for each n in an infinite subset K of \mathbb{N} . Let $\varphi : I \rightarrow K$ be a filtering map from a directed set I to K such that $(x^*, r) = \lim_{i \in I} (x_{\varphi(i)}^*, r_{\varphi(i)})$. Since (x_n^*) is bounded and (h_n) and (k_n) are equi-bounded from below on bounded subsets, there exists some $c \in \mathbb{R}$ with $h_n(x_n^*) \geq c$, $k_n(x_n^*) \geq c$ for each $n \in K$. As $h_n(x_n^*) + k_n(x_n^*) \leq r_n$ for $n \in K$, the sequences $(h_n(x_n^*))_{n \in K}$, $(k_n(x_n^*))_{n \in K}$ are bounded. Then, we can find a subnet of $(x_{\varphi(i)}^*, h_{\varphi(i)}(x_{\varphi(i)}^*), k_{\varphi(i)}(x_{\varphi(i)}^*))_{i \in I}$ which weak* converges to some (u^*, s, t) . Then we have $u^* = x^*$ and $s + t \leq r$. Then

$$r \geq s + t \geq e_{w^*}\text{-}\liminf_n h_n(x^*) + e_{w^*}\text{-}\liminf_n k_n(x^*).$$

Since (x^*, r) has been taken arbitrarily in $w^*\text{-}\limsup_n \text{epi}(h_n + k_n)$, we get the result. \square

Proposition 5.3 *Let $g, g_n \in \Gamma(X)$, $H, H_n \in \Gamma^*(X^*)$ ($n \in \mathbb{N}$). Suppose X is reflexive, $g \geq e\text{-}\limsup_n g_n$, $H \leq e_{w^*}\text{-}\liminf_n H_n$ and there exist a bounded sequence (\bar{x}_n) in X and a converging sequence (\bar{x}_n^*) in X^* such that $(g_n(\bar{x}_n))$, $(g_n^*(\bar{x}_n^*))$ and $(H_n(\bar{x}_n^*))$ are bounded above. Then one has $v \geq e\text{-}\limsup_n v_n$.*

Proof. Since $(e\text{-}\limsup_n G_n)(x, t) = (e\text{-}\limsup_n g_n)(x) + \iota_{\{0\}}(t)$ we have $G \geq e\text{-}\limsup_n G_n$, hence, by Lemma 3.1 (a), $G^* \leq e_{w^*}\text{-}\liminf_n G_n^*$. Let us show that $F \leq e_{w^*}\text{-}\liminf_n F_n$. Let (x^*, t, r) be a w^* -cluster point of a bounded sequence $((x_n^*, t_n, r_n))_{n \in K}$ with $(x_n^*, t_n, r_n) \in \text{epi } F_n$ for each n in an infinite subset K of \mathbb{N} , i.e. $r_n \geq 0$ and $(x_n^*, -t_n) \in \text{epi } H_n$ for $n \in K$. Since $H \leq e_{w^*}\text{-}\liminf_n H_n$ we get $(x^*, -t) \in \text{epi } H$, i.e. $F(x^*, t) = 0 \leq r$, hence $(x^*, t, r) \in \text{epi } F$. Now let B be a bounded subset of $X^* \times \mathbb{R}$ and let $m := \sup_n g(\bar{x}_n)$. Then, for each $n \in \mathbb{N}$ and each $(x^*, t^*) \in B$ we have

$$G_n^*(x^*, t^*) = g_n^*(x^*) \geq \langle x^*, \bar{x}_n \rangle - g_n(\bar{x}_n) \geq -(\sup_{x^* \in B} \|x^*\|)(\sup_n \|\bar{x}_n\|) - m.$$

Since $F_n \geq 0$, the preceding lemma yields

$$F + G^* \leq e_{w^*}\text{-}\liminf_n F_n + e_{w^*}\text{-}\liminf_n G_n^* \leq e_{w^*}\text{-}\liminf_n (F_n + G_n^*).$$

Since $F_n, G_n^* \in \Gamma^*(X^*)$ and $\text{dom}(F_n + G_n^*)$ is nonempty by our standing assumption corresponding to (14), we have $F_n + G_n^* \in \Gamma^*(X^*)$; applying Lemma 3.1 (c), with the help of our assumption about the sequence (\bar{x}_n^*) we obtain

$$(F + G^*)^* \geq e\text{-}\limsup_n (F_n + G_n^*)^*,$$

or $v \geq e\text{-}\limsup_n v_n$. \square

Theorem 5.4 *Let $g, g_n \in \Gamma(X)$, $H, H_n \in \Gamma^*(X^*)$ ($n \in \mathbb{N}$), X being reflexive. Suppose $(g_n) \xrightarrow{M} g$, $(H_n) \xrightarrow{M} H$ and $-H \leq e\text{-}\liminf_n (-H_n)$. Then one has $(v_n) \xrightarrow{M} v$.*

Proof. The result is a consequence of the preceding two propositions, since for any $\bar{x}^* \in \text{dom } g^* \cap \text{dom } H$ one can find a sequence $(\bar{x}_n^*) \rightarrow \bar{x}^*$ with $(g_n^*(\bar{x}_n^*))$ bounded from above and from the upper continuous convergence of (H_n) one gets that $H_n(\bar{x}_n^*) \leq H(\bar{x}^*) + 1$ for n large enough. \square

A convergence result for the Hopf–Lax solutions can be deduced from a convergence result for the Lax–Oleinik solution.

Theorem 5.5 Suppose X is reflexive, $(g_n) \xrightarrow{M} g$, $(H_n) \xrightarrow{M} H$, with g, g_n convex, $H, H_n \in \Gamma^*(X^*)$, $\text{dom } g_n \neq \emptyset$ for each $n \in \mathbb{N}$. Suppose one of the conditions (10), (11), (12) holds. Then $(v_n) \xrightarrow{M} v$.

Proof. Under our assumptions, we have $v = u^{**} = \bar{u}$, $v_n = u_n^{**} = \bar{u}_n$, the l.s.c. hulls of u and u_n respectively. Since $(u_n) \xrightarrow{M} u$ by Theorem 4.6, we get that $(v_n) = (\bar{u}_n) \xrightarrow{M} \bar{u} = v$. \square

Under reinforced assumptions, we can obtain a stronger conclusion.

Theorem 5.6 Let $g, g_n \in \Gamma(X)$, $H, H_n \in \Gamma^*(X^*)$ ($n \in \mathbb{N}$), X being reflexive. Suppose $(g_n) \xrightarrow{M} g$, $(H_n) \xrightarrow{M} H$ and $-H \leq e\text{-}\liminf_n(-H_n)$. If $u = v$, then one has $(u_n) \xrightarrow{M} u$ and $(v_n) \xrightarrow{M} v$.

Proof. Since $(g_n) \xrightarrow{M} g$, the assumptions $g \leq e_w\text{-}\liminf_n g_n$ and $e\text{-}\limsup_n g_n(\bar{x}) < \infty$ for some $\bar{x} \in X$ (in fact any $\bar{x} \in \text{dom } g$) of Proposition 5.1 are satisfied, so that $v \leq e_w\text{-}\liminf_n v_n$. Now, given $\bar{x}^* \in \text{dom } H$ we have

$$-H(\bar{x}^*) \leq (e\text{-}\liminf_n(-H_n))(\bar{x}^*) \leq -\limsup_n H_n(\bar{x}^*) \leq -(e\text{-}\limsup_n H_n)(\bar{x}^*),$$

so that the assumptions of Proposition 4.2 (a) are satisfied, hence $e\text{-}\limsup_n u_n \leq u$. Since $u = v$ and since $v_n \leq u_n$ for every $n \in \mathbb{N}$, the conclusion follows. \square

In the next results we turn to b-convergence.

Proposition 5.7 Let $g, g_n \in \Gamma(X)$, $H, H_n \in \Gamma^*(X^*)$ ($n \in \mathbb{N}$). Suppose $g \leq b\text{-}\liminf_n g_n$, $H \geq b\text{-}\limsup_n H_n$ and $X^* = \mathbb{R}_+(\text{dom } g^* - \text{dom } H)$. Suppose $(d((0, 0), \text{epi } g_n))_n$ is bounded. Then, one has $v \leq b\text{-}\liminf_n v_n$.

Proof. Clearly, we have $F \geq b\text{-}\limsup_n F_n$, $G \leq b\text{-}\liminf_n G_n$ and $X^* \times \mathbb{R} = \mathbb{R}_+(\text{dom } G^* - \text{dom } F)$. Since $(G_n(x_n, 0))$ is bounded above, we have $G^* \geq b\text{-}\limsup_n G_n^*$ by Theorem 3.2 (b). Using [51, Th. 30 (b)] we get $G^* + F \geq b\text{-}\limsup_n(G_n^* + F_n)$. Taking into account Theorem 3.2 (a), we get $(G^* + F)^* \leq b\text{-}\liminf_n(G_n^* + F_n)^*$. \square

In order to give a result with $b\text{-}\limsup_n v_n$, we need auxiliary results of independent interest. They have to replace Proposition 27 (d) and Theorem 30 (d) in [51], respectively.

Lemma 5.8 Let A, B, A_n, B_n be nonempty subsets of a Banach space X , the sets A, B being closed convex and such that $X = \mathbb{R}_+(A - B)$. Suppose $b\text{-}\limsup_n A_n \subset A$, $b\text{-}\limsup_n B_n \subset B$. Then $b\text{-}\limsup_n (A_n \cap B_n) \subset A \cap B$.

Proof. Corollary 25 in [51] asserts that if $L : X \rightarrow Y$ is a continuous linear map and if D, D_n ($n \in \mathbb{N}$) are nonempty subsets of Y with D closed convex, $b\text{-}\limsup_n D_n \subset D$ and if $Y = \mathbb{R}_+(L(X) - D)$, then $b\text{-}\limsup_n L^{-1}(D_n) \subset L^{-1}(D)$. Taking $Y = X \times X$, $L(x) := (x, x)$ and $D := A \times B$, $D_n := A_n \times B_n$, using [51, Lem. 21 (d)], we get the result, the relation $Y = \mathbb{R}_+(L(X) - D)$ being an easy consequence of the assumption $X = \mathbb{R}_+(A - B)$ (see the proof of [51, Prop. 27]). \square

Lemma 5.9 Let f, g, f_n, g_n be functions on a Banach space X , with $f, g \in \Gamma(X)$. Suppose that $f \leq b\text{-}\liminf_n f_n$, $g \leq b\text{-}\liminf_n g_n$ and $X = \mathbb{R}_+(\text{dom } f - \text{dom } g)$. Then $f + g \leq b\text{-}\liminf_n (f_n + g_n)$.

Proof. We follow the line of the proof of [51, Th. 30]. Again, we introduce B, C, B_n, C_n, E, E_n as the epigraphs of $f, g, f_n, g_n, f + g, f_n + g_n$ respectively and we set

$$D := \{(x, r, x, s) \mid x \in X, r, s \in \mathbb{R}\}.$$

We observe that the epigraph E of $f + g$ is obtained as $E = T(A)$ with $A := (B \times C) \cap D$, where $T : W := (X \times \mathbb{R})^2 \rightarrow V := X \times \mathbb{R}$ is given by $T(x, r, y, s) = (x, r + s)$. The epigraph E_n of $f_n + g_n$ is also given by $E_n = T(A_n)$ with $A_n := (B_n \times C_n) \cap D$. The relation $X = \mathbb{R}_+(\text{dom } f - \text{dom } g)$ yields $X \times \mathbb{R} \times X \times \mathbb{R} = \mathbb{R}_+(B \times C - D)$. By Lemma 5.8 we get $b\text{-}\limsup_n A_n \subset A$. Then we apply Proposition [51, Prop. 8 (d)] by checking its condition (15):

$$\forall p > 0, \exists q > 0, \exists m \in \mathbb{N}, \forall n \geq m : T(A_n) \cap pU_V \subset T(A_n \cap qU_W).$$

Given $p > 0$, we use [51, Lem. 29] and the fact that f and g are bounded below on bounded sets (they are bounded below by continuous affine functions) $f \leq b\text{-}\liminf_n f_n$, $g \leq b\text{-}\liminf_n g_n$ to get $\ell > 0$ and $m \in \mathbb{N}$ such that $-\ell \leq \inf f_n(pU_X)$, $-\ell \leq \inf g_n(pU_X)$ for $n \geq m$. Taking $q := p + \ell$ we obtain the desired condition. Thus, we get $b\text{-}\limsup_n E_n \subset E$, i.e. the conclusion $f + g \leq b\text{-}\liminf_n (f_n + g_n)$. \square

We are now in a position to state and prove a result about $b\text{-}\limsup_n v_n$.

Proposition 5.10 *Let $g \in \Gamma(X)$, $H, H_n \in \Gamma^*(X^*)$ ($n \in \mathbb{N}$). Suppose $g \geq b\text{-}\limsup_n g_n$, $H \leq b\text{-}\liminf_n H_n$ and $X^* = \mathbb{R}_+(\text{dom } g^* - \text{dom } H)$. Suppose there exists a bounded sequence (x_n^*) in X^* such that $(g_n^*(x_n^*))$ and $(H_n(x_n^*))$ are bounded above. Then, one has $v \geq b\text{-}\limsup_n v_n$.*

Proof. Since $g \geq b\text{-}\limsup_n g_n$, we have $G \geq b\text{-}\limsup_n G_n$ and by Theorem 3.2 (a), G being convex, we get $G^* \leq b\text{-}\liminf_n G_n^*$. Now, since $H \leq b\text{-}\liminf_n H_n$ we have $F \leq b\text{-}\liminf_n F_n$. The condition $X^* \times \mathbb{R} = \mathbb{R}_+(\text{dom } G^* - \text{dom } F)$ being again satisfied, Lemma 5.9 yields $F + G^* \leq b\text{-}\liminf_n (F_n + G_n^*)$. Our assumptions guarantee that $F_n + G_n^*$ is convex and $(d(0, \text{epi}(F_n + G_n^*)))$ is bounded. Thus, Theorem 3.2 (d) entails that $v \geq b\text{-}\limsup_n v_n$. \square

The following theorem is our main result.

Theorem 5.11 *Let $g, g_n \in \Gamma(X)$, $H, H_n \in \Gamma^*(X^*)$ ($n \in \mathbb{N}$). Suppose $(g_n) \xrightarrow{b} g$, $(H_n) \xrightarrow{b} H$ and $X^* = \mathbb{R}_+(\text{dom } g^* - \text{dom } H)$. Then $(v_n) \xrightarrow{b} v$.*

Moreover, if X is reflexive, one has $v = u$ and $v_n = u_n$ for n large enough.

Proof. Since $(g_n) \xrightarrow{b} g$ we can find a converging sequence (x_n) such that $(g_n(x_n))$ is bounded above, so that $v \leq b\text{-}\liminf_n v_n$ by Proposition 5.7. Since $X^* = \mathbb{R}_+(\text{dom } g^* - \text{dom } H)$ and since $(F_n) \xrightarrow{b} F$, $(G_n^*) \xrightarrow{b} G^*$, by the Robinson–Ursescu theorem and the cancelation lemma, there exist $r, p > 0$ such that

$$rU_{X^* \times \mathbb{R}} \subset [F_n \leq p] \cap pU_{X^* \times \mathbb{R}} - [G_n^* \leq p] \cap pU_{X^* \times \mathbb{R}} \quad (15)$$

for n large enough ([10], [12, Rem. 1 b]), [53, Lem. 3.5]). Thus, there exists a bounded sequence (x_n^*) in X^* such that $(g_n^*(x_n^*))$ and $(H_n(x_n^*))$ are bounded above. Hence, by Proposition 5.10, $v \geq b\text{-}\limsup_n v_n$, and so the first assertion holds.

Let us prove the second part, assuming X is reflexive. Again we have $X^* \times \mathbb{R} = \mathbb{R}_+(\text{dom } G^* - \text{dom } F)$. Thus, the Attouch–Brézis theorem ([6], [62]) ensures that $v = (G^* + F)^* = G^{**} \square F^* = G \square F^* = u$. On the other hand, since $(G_n^*) \xrightarrow{b} G^*$ and $(F_n) \xrightarrow{b} F$, we have $X^* \times \mathbb{R} = \mathbb{R}_+(\text{dom } G_n^* - \text{dom } F_n)$ for n large enough ([10], [12, Rem. 1 b]), [53, Lem. 3.5]). Thus, as above, we get $v_n = u_n$ for n large enough. \square

Let us compare the qualification condition of the preceding theorem with condition (10). For such a purpose we need the following statement of independent interest.

Proposition 5.12 (a) Let W, X be n.v.s., let $\Phi : W \times X \rightarrow \overline{\mathbb{R}}$ be such that for some $\alpha, \beta, \gamma > 0$, for every $w \in \alpha B_W$ one can find $x \in \beta B_X$ with $\Phi(w, x) \leq \gamma$. Then, for all $w^* \in W^*, x^* \in X^*$, one has

$$\Phi^*(w^*, x^*) \geq \alpha \|w^*\| - \beta \|x^*\| - \gamma. \quad (16)$$

(b) In particular, if W, X are Banach spaces and if $\Phi \in \Gamma(W \times X)$ is such that $W = \mathbb{R}_+ \text{Pr}_W(\text{dom } \Phi)$, then the preceding estimate holds.

(c) Let $f, g \in \Gamma(X)$ be such $X = \mathbb{R}_+(\text{dom } f - \text{dom } g)$. If X is complete then there exist $\alpha, \beta, \gamma > 0$ such that for all $x^*, y^* \in X^*$

$$f^*(x^*) + g^*(y^*) \geq \alpha(\|x^*\| + \|y^*\|) - \beta \|x^* + y^*\| - \gamma.$$

Proof. (a) Given $w^* \in W^*, x^* \in X^*$, setting $p(w) := \inf\{\Phi(w, x) - \langle x^*, x \rangle : x \in X\}$ for $w \in W$, one gets

$$p(w) \leq \inf\{\Phi(w, x) - \langle x^*, x \rangle : x \in \beta B_X\} \leq \gamma + \beta \|x^*\| + \iota_{\alpha B_W}(w).$$

Therefore

$$\Phi^*(w^*, x^*) = p^*(w^*) \geq \sup_{w \in W} (\langle w^*, w \rangle - \gamma - \beta \|x^*\| - \iota_{\alpha B_W}(w)) = \alpha \|w^*\| - \beta \|x^*\| - \gamma.$$

(b) The existence of constants $\alpha, \beta, \gamma > 0$ satisfying the assumption of (a) follows from the hypothesis and the Robinson–Ursescu theorem (see [62, Prop. 2.7.2]).

(c) Let us take $W = X$, and set $\Phi(w, x) = f(x) + g(w + x)$. Then $\Phi \in \Gamma(W \times X)$ and, as easily seen,

$$\Phi^*(w^*, v^*) = f^*(v^* - w^*) + g^*(w^*).$$

Moreover $\text{Pr}_W(\text{dom } \Phi) = \text{dom } g - \text{dom } f$, so that the assumptions of (b) are satisfied. Thus, one can find some positive constants $\alpha', \beta', \gamma' > 0$ such that, for all $x^*, y^* \in X^*$, taking $v^* := x^* + y^*$, $w^* := y^*$ we get

$$f^*(x^*) + g^*(y^*) = \Phi^*(y^*, x^* + y^*) \geq \alpha' \|y^*\| - \beta' \|x^* + y^*\| - \gamma'.$$

Interchanging the roles of f and g , we can find some constants $\alpha'', \beta'', \gamma'' > 0$ such that for all $x^*, y^* \in X^*$

$$g^*(y^*) + f^*(x^*) \geq \alpha'' \|x^*\| - \beta'' \|x^* + y^*\| - \gamma''.$$

Adding side by side these two relations we get the conclusion with $\alpha := \frac{1}{2} \min(\alpha', \alpha'')$, $\beta := \frac{1}{2}(\beta' + \beta'')$, $\gamma := \frac{1}{2}(\gamma' + \gamma'')$. \square

Now let us show that (10) is a consequence of the relation $X^* = \mathbb{R}_+(\text{dom } g^* - \text{dom } H)$ when $g, g_n \in \Gamma(X)$, $H, H_n \in \Gamma^*(X^*)$ ($n \in \mathbb{N}$) are such that $(g_n) \xrightarrow{b} g$, $(H_n) \xrightarrow{b} H$. Let F, F_n and G, G_n be as in relations (4), (5). Then $(F_n) \xrightarrow{b} F$, $(G_n^*) \xrightarrow{b} G^*$. We first observe that the Robinson–Ursescu theorem and these convergences yield $m \in \mathbb{N}$ and constants $\alpha, \beta, \gamma > 0$ such that for every $n \geq m$, $(w^*, s^*) \in \alpha B_{X^* \times \mathbb{R}}$ one can find $(x^*, t^*) \in \beta B_{X^* \times \mathbb{R}}$ with $\Phi_n((w^*, s^*), (x^*, t^*)) := F_n(x^*, t^*) + G_n^*(w^* + x^*, s^* + t^*) \leq \gamma$. The preceding proposition ensures that

$$F_n^*(x, t) + G_n(y, s) \geq \alpha(\|(x, t)\| + \|(y, s)\|) - \beta \|(x + y, s + t)\| - \gamma$$

for all $n \geq m$, $(x, t), (y, s) \in X \times \mathbb{R}$. Taking $t \geq 0, s = 0$, we get for all $n \geq m, w, x \in X, t \in \mathbb{R}_+$

$$(tH_n)^*(x) + g_n(w - x) \geq \alpha(\|(x, t)\| + \|(w - x, 0)\|) - \beta \|(w, t)\| - \gamma.$$

Then, setting $q := \alpha^{-1}(\beta p + p + \gamma)$, for every $n \geq m, (w, t) \in [u_n < p] \cap pU_{X \times \mathbb{R}}, t \geq 0$ for $x \in X \setminus qU_X$ one has $(tH_n)^*(x) + g_n(w - x) \geq p$. Therefore condition (10) is satisfied.

6 Convergence for fixed $t > 0$

Because our proofs involve auxiliary functions F, G, F_n, G_n , the reader may wonder whether one would get simpler results when considering the convergence of $(u_n(\cdot, t))$ and $(v_n(\cdot, t))$ to $u(\cdot, t)$ and $v(\cdot, t)$ respectively for fixed $t > 0$. Let us first observe that, for $f, f_n : X \times \mathbb{R} \rightarrow \overline{\mathbb{R}}$, one has

$$e_w\text{-}\liminf_n f_n \geq f \implies [\forall t \in \mathbb{R} : e_w\text{-}\liminf_n f_n(\cdot, t) \geq f(\cdot, t)], \quad (17)$$

$$[\forall t \in \mathbb{R} : e\text{-}\limsup_n f_n(\cdot, t) \leq f(\cdot, t)] \implies e\text{-}\limsup_n f_n \leq f. \quad (18)$$

Thus Mosco convergence of (f_n) and Mosco convergence for $(f_n(\cdot, t))$ for all t are a priori independent properties.

However, a direct analysis shows that convergence results for $(u_n(\cdot, t))$ and $(v_n(\cdot, t))$ for all $t \in \mathbb{P}$ are not very different from the results above for (u_n) and (v_n) , at least for epiconvergence and Mosco convergence. Let us give details for the convenience of the reader. For the sake of simplicity, here we use assumptions bearing on the sequence (H_n^*) rather than assumptions on the sequence (H_n) . We could also use condition (11) or (12). In view of implication (17) it is only in the case of assumption (20) below that the analysis for fixed t provides a new result. Moreover, since implication (17) with b -convergence instead of epiconvergence is not known to us, in this case one has to devise direct proofs which follow the same type of arguments.

Proposition 6.1 *Let $g, g_n : X \rightarrow \overline{\mathbb{R}}$, let $H, H_n : X \rightarrow \overline{\mathbb{R}}$ and let $t \in \mathbb{P}$.*

(a) *Suppose $g \geq e\text{-}\limsup_n g_n$ and $H^* \geq e\text{-}\limsup_n H_n^*$. Then $u(\cdot, t) \geq e\text{-}\limsup_n u_n(\cdot, t)$.*

(b) *Suppose $g \geq b\text{-}\limsup_n g_n$ and $H^* \geq b\text{-}\limsup_n H_n^*$. Then $u(\cdot, t) \geq b\text{-}\limsup_n u_n(\cdot, t)$ provided the following condition holds:*

$$\forall p \in \mathbb{P}, \exists q \in \mathbb{P}, \forall x \in [u(\cdot, t) < p] \cap pU_X : u(x, t) = \inf\{(tH)^*(w) + g(x - w) : w \in qU_X\}. \quad (19)$$

(c) *Suppose $g \leq e_w\text{-}\liminf_n g_n$, $H^* \leq e_w\text{-}\liminf_n H_n^*$ and X is reflexive. Then $u(\cdot, t) \leq e_w\text{-}\liminf_n u_n(\cdot, t)$ provided the following condition holds:*

$$\forall p \in \mathbb{P}, \exists q \in \mathbb{P}, \exists m \in \mathbb{N}, \forall n \geq m, \forall x \in [u_n(\cdot, t) < p] \cap pU_X : \\ u_n(x, t) = \inf\{g_n(x - z) + (tH_n)^*(z) : z \in qU_X\}. \quad (20)$$

(d) *Suppose $g \leq b\text{-}\liminf_n g_n$ and $H^* \leq b\text{-}\liminf_n H_n^*$. Then $u(\cdot, t) \leq b\text{-}\liminf_n u_n(\cdot, t)$ provided the preceding condition holds.*

Proof. (a) Let $x \in X$ and $r > u(x, t)$. There exists some $w \in X$ such that $g(x - w) + (tH)^*(w) < r$. Since $g \geq e\text{-}\limsup_n g_n$, $H^* \geq e\text{-}\limsup_n H_n^*$, we can find sequences $(w_n) \rightarrow w$, $(z_n) \rightarrow x - w$ such that $\limsup_n g_n(z_n) \leq g(x - w)$ and $\limsup_n H_n^*(t^{-1}w_n) \leq H^*(t^{-1}w)$. Then $(x_n) := (w_n + z_n) \rightarrow x$ and

$$\limsup_n u_n(x_n, t) \leq \limsup_n (g_n(z_n) + tH_n^*(t^{-1}w_n)) \leq g(x - w) + (tH)^*(w) < r.$$

Since r can be taken arbitrarily close to $u(x, t)$, this shows that $u(\cdot, t) \geq e\text{-}\limsup_n u_n(\cdot, t)$.

(b) The proof is similar, replacing (x, r) by a bounded sequence $((x_n, r_n))$ such that $r_n > u(x_n, t)$.

(c) Let (x, r) be the weak limit of a bounded sequence $((x_n, r_n))$ such that $(x_n, r_n) \in \text{epi } u_n(\cdot, t)$ for each n in an infinite subset K of \mathbb{N} . Condition (20) yields a bounded sequence (z_n) of X such that $g_n(x_n - z_n) + (tH_n)^*(z_n) < r_n + 2^{-n}$ for all $n \in K$. Taking a subsequence if necessary,

we may assume that (z_n) weakly converges to some $z \in X$ which is reflexive. Then $(x_n - z_n)$ converges to $x - z$, hence

$$r \geq \liminf_n g_n(x_n - z_n) + \liminf_n t H_n^*(t^{-1}z_n) \geq g(x - z) + tH(t^{-1}z) \geq u(x, t)$$

and the conclusion holds.

(d) The proof is similar, avoiding weak convergence and reflexivity. \square

Corollary 6.2 (a) Suppose $(g_n) \xrightarrow{M} g$ and $(H_n) \xrightarrow{M} H$. If X is reflexive and if condition (20) holds, then, for all $t \in \mathbb{P}$ one has $(u_n(\cdot, t)) \xrightarrow{M} u(\cdot, t)$.

(b) Suppose $(g_n) \xrightarrow{b} g$ and $(H_n) \xrightarrow{b} H$. If conditions (19) and (20) hold, then, for all $t \in \mathbb{P}$ one has $(u_n(\cdot, t)) \xrightarrow{b} u(\cdot, t)$.

Let us turn to Hopf–Lax solutions. For Propositions 5.1 and 5.7, the same assumptions give $v(\cdot, t) \leq e\text{-}\liminf_n v_n(\cdot, t)$ and $v(\cdot, t) \leq b\text{-}\liminf_n v_n(\cdot, t)$ for all $t \in \mathbb{P}$, respectively (with similar proofs). Taking into account (17), one expects to have to strengthen the hypotheses of Proposition 5.3 in order to obtain $v(\cdot, t) \geq e\text{-}\limsup_n v_n(\cdot, t)$ for $t \in \mathbb{P}$. Indeed, to obtain this conclusion, besides the hypotheses of Proposition 5.3 we have to assume that (H_n) is equi-bounded from below on bounded sets. On the other hand, under the assumptions of Proposition 5.10, we obtain also that $v(\cdot, t) \geq b\text{-}\limsup_n v_n(\cdot, t)$ for every $t \in \mathbb{P}$. The same hypotheses as in Theorem 5.4 yield $v_n(\cdot, t) \xrightarrow{M} v(\cdot, t)$ for $t \in \mathbb{P}$. Indeed, because $(H_n) \xrightarrow{M} H$ we have that (H_n) is equi-bounded from below on bounded sets.

7 Convergence properties of arbitrary solutions

In this section, we deal with sequences of arbitrary solutions in the sense of viscosity ([33], [34]). For simplicity, we suppose throughout that X is reflexive and we limit ourselves to some cases which may give the flavor of further developments. We will use two comparison results taken from [46], [48] (see also [1], [38] for related results).

Part of our analysis can be given for the equations

$$J(x, t, u(x, t), Du(x, t), \frac{\partial}{\partial t}u(x, t)) = 0, \quad (21)$$

$$J_n(x, t, u(x, t), Du(x, t), \frac{\partial}{\partial t}u(x, t)) = 0, \quad (22)$$

where $J, J_n : X \times \mathbb{R}^2 \times X^* \times \mathbb{R} \rightarrow \overline{\mathbb{R}}$ are given functions. Such equations obviously include (1).

In order to get some flexibility, we formulate notions of solution which involve a general concept of subdifferential. Such an approach can be considered as an enlargement of the notion of viscosity solution, which, in Asplund spaces corresponds to the choice of the Fréchet subdifferential. By a *subdifferential* on a class $\mathcal{F}(Z)$ of functions on a n.v.s. Z we mean a set-valued map $\partial : \mathcal{F}(Z) \times Z \rightrightarrows Z^*$ which assigns to a pair $(f, z) \in \mathcal{F}(Z) \times Z$ a subset $\partial f(z)$ of Z^* which is empty if $f(z)$ is not finite. Moreover we require the following condition:

(C) if f is a convex function finite at z , then $\partial f(z) = \{z^* \in Z^* : f(\cdot) \geq z^*(\cdot) - z^*(z) + f(z)\}$.

In the next statement we also use the following condition which is satisfied by most subdifferentials:

(C') if f is concave, finite at z and if $z^* \in \partial f(z)$, then $-z^* \in \partial(-f)(z)$.

Given a subdifferential ∂ one says that u is a *supersolution* (resp. *subsolution*) to (21) if for all $(x, t) \in X \times \mathbb{P}$ and all $(p, q) \in \partial u(x, t)$ (resp. $(p, q) \in -\partial(-u)(x, t)$) one has $J(x, t, u(x, t), p, q) \geq 0$ (resp. $J(x, t, u(x, t), p, q) \leq 0$). We say that u is a *lower solution* (resp. *upper solution*) to (21) if for all $(x, t) \in X \times \mathbb{P}$ and all $(p, q) \in \partial u(x, t)$ (resp. $(p, q) \in -\partial(-u)(x, t)$) one has $J(x, t, u(x, t), p, q) \leq 0$ (resp. $J(x, t, u(x, t), p, q) \geq 0$). A supersolution which is also a subsolution is called a *viscosity solution* (for ∂). A first comparison between these concepts is inspired by [18], [19]. Its assumption on J is satisfied when $J(x, t, r, p, q) = q + H(p)$ with H convex and l.s.c.

Proposition 7.1 *Let ∂ be the Fréchet subdifferential and let X be reflexive. Suppose there exists a l.s.c. and quasiconvex function K such that $J(x, t, r, p, q) = K(x, t, p, q)$ for all (x, t, r, p, q) . Then a continuous function u is a lower solution to (21), if, and only if, it is a subsolution to (21).*

Proof. Let u be a lower solution. By [24, Thm 3.6.5], given $(x, t) \in X \times \mathbb{P}$ and $(p, q) \in -\partial(-u)(x, t)$ and taking a sequence $(\varepsilon_n) \rightarrow 0_+$, we can find $(p_n, q_n) \in B((p, q), \varepsilon_n)$ which belongs to the convex hull of the set $\partial u(B((x, t), \varepsilon_n)) := \{(p', q') \in \partial u(x', t') : (x', t') \in B((x, t), \varepsilon_n)\}$. For any $(p', q') \in \partial u(x', t')$ with $(x', t') \in B((x, t), \varepsilon_n)$ we have $K(x', t', p', q') \leq 0$ since u is a lower solution. Taking a convex combination, we can find $(x_n, t_n) \in B((x, t), \varepsilon_n)$ such that $K(x_n, t_n, p_n, q_n) \leq 0$. Passing to the limit, we get $K(x, t, p, q) \leq \liminf_n K(x_n, t_n, p_n, q_n) \leq 0$. Thus u is a subsolution.

The converse is similar, using [24, Thm 3.6.6]. \square

When u is convex, another comparison can be made between these concepts.

Proposition 7.2 *Suppose ∂ satisfies (C'). Let $u : X \times \mathbb{R} \rightarrow \overline{\mathbb{R}}$ be a convex function.*

(a) *If u is a lower solution to (21), then u is a subsolution to (21).*

(b) *If u is an supersolution to (21), then u is an upper solution to (21).*

Proof. Let $(p, q) \in -\partial(-u)(x, t)$. Then $u(x, t)$ is finite and by (C'), one has $(p, q) \in \partial u(x, t)$, so that $J(x, t, u(x, t), p, q) \leq 0$. This shows that u is a subsolution to (21). The proof of assertion (b) is similar. \square

Theorem 7.3 *Suppose X is finite dimensional and ∂ satisfies (C) and (C'). Let $J, J_n : X \times \mathbb{R}^2 \times X^* \times \mathbb{R} \rightarrow \overline{\mathbb{R}}$ and let $w_n : X \times \mathbb{R} \rightarrow \overline{\mathbb{R}}$ ($n \in \mathbb{N}$) be l.s.c. and proper.*

(a) *Suppose $J \leq e\text{-}\liminf_n J_n$ and, for all $n \in \mathbb{N}$, w_n is a convex lower solution to equation (22). Then (w_n) has a subsequence which epiconverges to a lower solution to (21).*

(b) *Suppose $-J \leq e\text{-}\liminf_n (-J_n)$ and, for all $n \in \mathbb{N}$, w_n is a convex supersolution to equation (22). Then (w_n) has a subsequence which epiconverges to a supersolution to (21).*

(c) *Suppose $J \leq e\text{-}\liminf_n J_n$ and, for all $n \in \mathbb{N}$, w_n is a convex lower solution of equation (22). Then (w_n) has a subsequence which epiconverges to a lower solution to (21) which is also a subsolution to (21).*

(d) *Suppose $J \leq e\text{-}\liminf_n J_n$ and $-J \leq e\text{-}\liminf_n (-J_n)$. If (w_n) is a sequence of convex viscosity solutions to (22), then (w_n) has a subsequence which epiconverges to a viscosity solution to (21).*

Proof. (a) Transposing to functions the Zarankiewicz theorem ([3, Th. 2.22], [8, Th. 1.1.7]), for any subsequence of (w_n) we can find a further subsequence $(w_{k(n)})$ which epiconverges to some l.s.c. function w . Since w is convex, either it is proper or does not take a finite value. In the later case, there is nothing to prove, so that we may suppose w is proper. Let $(x, t) \in \text{dom } w$ and let $(p, q) \in \partial w(x, t)$. The Attouch theorem ([2], [3]) provides a sequence $((x_n, t_n, p_n, q_n)) \rightarrow$

(x, t, p, q) such that $(w_n(x_n, t_n)) \rightarrow w(x, t)$ and $(p_n, q_n) \in \partial w_{k(n)}(x_n, t_n)$ for all $n \in \mathbb{N}$. Since $J \leq e\text{-}\liminf_n J_n$ and since w_n is a lower solution of (22), we have

$$J(x, t, w(x, t), p, q) \leq \liminf_n J_{k(n)}(x_n, t_n, w_n(x_n, t_n), p_n, q_n) \leq 0$$

so that w is a lower solution.

(b) The proof is similar to that in (a), changing J into $-J$.

(c) Let $(w_{k(n)})$ be a subsequence of (w_n) which epiconverges to some l.s.c. function w . Then, w being convex and a lower solution to (21), it is also a subsolution to (21) by the preceding proposition.

(d) We can apply (b) and (c) to a subsequence $(w_{k(n)})$ which epiconverges. \square

When w satisfies a boundary condition, and when some uniqueness property holds, assertion (d) can produce a convergence result for the whole sequence. See [1], [15], [16], [40], [58] for a sample of uniqueness results.

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