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Extended matter coupled to BF theory

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Recently, a topological field theory of membrane-matter coupled to BF theory in arbitrary spacetime dimensions was proposed [1]. In this paper, we discuss various aspects of the four-dimensional theory. Firstly, we study classical solutions leading to an interpretation of the theory in terms of strings propagating on a flat spacetime. We also show that the general classical solutions of the theory are in one-to-one correspondence with solutions of Einstein's equations in the presence of distributional matter (cosmic strings). Secondly, we quantize the theory and present, in particular, a prescription to regularize the physical inner product of the canonical theory. We show how the resulting transition amplitudes are dual to evaluations of Feynman diagrams coupled to three-dimensional quantum gravity. Finally, we remove the regulator by proving the topological invariance of the transition amplitudes.

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I. INTRODUCTION

Based on the seminal results [3] of $2+1$ gravity coupled to point sources, recent developments [4], [5] in the non-perturbative approach to $2+1$ quantum gravity have led to a clear understanding of quantum field theory on a three-dimensional quantum geometrical background spacetime. The idea is to first couple free point particles to the gravitational field *before* going through the second quantization process. In this approach, particles become local conical defects of spacetime curvature and their momenta are recasted as holonomies of the gravitational connection around their worldlines. It follows that momenta become group valued leading to an effective notion of non-commutative spacetime coordinates. The Feynman diagrams of such theories are related via a duality transformation to spinfoam models.

All though conceptually very deep, these results remain three-dimensional. The next step is to probe all possible extensions of these ideas to higher dimensions. Two ideas have recently been put forward. The first is to consider that fundamental matter is indeed pointlike and study the coupling of worldlines to gravity by using the Cartan geometric framework [6] of the McDowell-Mansouri formulation of gravity as a de-Sitter gauge theory [7]. The second is to generalize the description of matter as topological defects of spacetime curvature to higher dimensions. This naturally leads to matter excitations supported by co-dimension two membranes [1], [9]. Before studying the coupling of such sources to quantum gravity, one can consider, as a first step, the BF theory framework as an immediate generalization of the topological character of three-dimensional gravity to higher dimensions.

This paper is dedicated to the second approach, namely the coupling of string-like sources to BF theory in four dimensions. The starting point is the action written in [1] generating a theory of flat connections except at the location of two-dimensional surfaces, where the curvature picks up a singularity, or in other words, where the gauge degrees of freedom become dynamical. The goal of the paper is a two-fold. Firstly, acquire a physical intuition of the algebraic fields involved in the theory which generalize the position and momentum Poincaré coordinates of the particle in three-dimensions. Secondly, provide a complete background independent quantization of the theory in four dimensions, following the work done in [1].

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The organization of the paper is as follows. In section II, we study some classical solutions guided by the three-dimensional example. We show that some specific solutions lead to the interpretation of rigid strings propagating on a flat spacetime. More generally, we prove that the solutions of the theory are in one-to-one correspondence with distributional solutions of general relativity. In section III, we propose a prescription for computing the physical inner product of the theory. This leads us to an interesting duality between the obtained transition amplitudes and Feynman diagrams coupled to three-dimensional gravity. We finally prove in section IV that the transition amplitudes only depend on the topology of the canonical manifold and of the spin network graphs.

II. CLASSICAL THEORY

A. Action principle and classical symmetries

Let G be a Lie group with Lie algebra $\mathfrak{g}$ equipped with an $Ad(G)$-invariant, non degenerate bilinear form noted 'tr' (e.g. the Killing form if G is semi-simple). Consider the principal bundle $\mathcal{P}$ with G as structure group and as base manifold a $d+1$ dimensional, compact, connected, oriented differential manifold M. We will assume that $\mathcal{P}$ is trivial, all though it is not essential, and chose once and for all a global trivialising section. We will be interested in the following first order action principle, describing the interaction between closed membrane-like sources and BF theory [1]:

$$S[A,B;q,p] = S_{BF}[A,B] - \int_{\mathscr{W}} \mathrm{tr}(B + d_A q)p). \tag{1}$$

The action of free BF theory in $d+1$ dimensions is given by

$$S_{BF}[A,B] = \frac{1}{\kappa}\int_M \mathrm{tr}(B \wedge F[A]). \tag{2}$$

Here, B is a $\mathfrak{g}$-valued $(d-1)$-form on M, F is the curvature of a $\mathfrak{g}$-valued one-form A, which is the pull-back to M by the global trivializing section of a connection on $\mathcal{P}$, and $\kappa \in \mathbb{R}$ is a coupling constant.

In the coupling term, $\mathscr{W}$ is the $(d-1)$-brane worldsheet defined by the embedding $\phi : E \subset \mathbb{R}^{d-1} \to M$, d_A is the covariant derivative with respect to the connection A, q is a $\mathfrak{g}$-valued $(d-2)$-form on $\mathscr{W}$ and p is a $\mathfrak{g}$-valued function on $\mathscr{W}$. The physical meaning of the matter variables p and q will be discussed in the following section. Essentially, p is the momentum density of the brane and q is the first integral of the $(d-1)$-volume element; the integral of a line and surface element in in three and four dimensions $(d = 2, 3)$ respectively.

The equations of motion governing the dynamics of the theory are those of a topological field theory:

$$F[A] = \kappa p\ \delta_{\mathscr{W}} \tag{3}$$
$$d_A B = \kappa[p,q]\delta_{\mathscr{W}} \tag{4}$$
$$\phi^*(B + d_A q) = 0 \tag{5}$$
$$d_A p|_{\mathscr{W}} = 0. \tag{6}$$

Here, $\delta_{\mathscr{W}}$ is a distributional two-form, also called current, which has support on the worldsheet $\mathscr{W}$. It is defined such that for all $(d-1)$-form α, $\int_{\mathscr{W}} \alpha = \int_M (\alpha \wedge \delta_{\mathscr{W}})$. The symbol ϕ^* denotes the pull-back of forms on $\mathscr{W}$ by the embedding map ϕ.

We can readily see that the above action describes a theory of local conical defects along brane-like $(d-1)$-submanifolds of M through the first equation. The second states that the obstruction to the vanishing of the torsion is measured by the commutator of p and q. The third equation is crucial. It relates the background field B to the dynamics of the brane. For instance, this equation describes the motion of a particle's position in 3d gravity [2]. The last states that the momentum density is covariantly conserved along the worlsheet. It is in fact a simple consequence of equation (3) together with the Bianchi identity $d_A F = 0$. We will see how this is a sign of the reducibility of the constraints generated by the theory.

The total action is invariant under the following (pull back to M of) vertical automorphisms of $\mathcal{P}$,

$$\begin{aligned} \forall g \in C^\infty(M,G), \qquad B &\mapsto B = gBg^{-1} \\ A &\mapsto A = gAg^{-1} + gdg^{-1} \\ p &\mapsto gpg^{-1} \\ q &\mapsto gqg^{-1} \end{aligned} \tag{7}$$

and the 'topological', or reducible transformations

$$\forall \eta \in \Omega^{d-2}(M,\mathfrak{g}), \qquad \begin{array}{rcl} B & \mapsto & B + d_A\eta \\ A & \mapsto & A \\ p & \mapsto & p \\ q & \mapsto & q - \eta \end{array} \tag{8}$$

where $\Omega^p(M,\mathfrak{g})$ is the space of $\mathfrak{g}$-valued p-forms on M.

B. Physical interpretation: the flat solution

In this section, we discuss some particular solutions of the theory leading to an interpretation of matter propagating on flat backgrounds. We discuss the $d = 2$ and $d = 3$ cases where the gauge degrees of freedom of BF theory become dynamical along one dimensional worldlines and two-dimensional worldsheets respectively.

1. *The point particle in* $2+1$ *dimensions*

We now restrict our attention to the $d = 2$ case with structure group the isometry group $G = \mathrm{SO}(\eta)$ of the diagonal form η of a three-dimensional metric on M; $\eta = (\sigma^2, +, +)$ with $\sigma = \{1, i\}$ in respectively Riemannian ($G = \mathrm{SO}(3)$) and Lorentzian ($G = \mathrm{SO}(1,2)$) signatures. We denote (π, V_η) the vector (adjoint) representation of $\mathfrak{so}(\eta) = \mathbb{R}\{J_a\}_{a=0,1,2}$, i.e., $V_\eta = \mathbb{R}^3$ and $V_\eta = \mathbb{R}^{1,2}$ in Riemannian and Lorentzian signatures respectively. The bilinear form 'tr' is defined such that $\mathrm{tr}(J_aJ_b) = \frac{1}{2}\eta_{ab}$. In this case, the free BF action (2) describes the dynamics of three-dimensional general relativity, where the B field plays the role of the triad e. The matter excitations are 0-branes, that is, particles and the worldsheet $\mathscr{W}$ reduces to a one-dimensional worldline that we will note γ. The degrees of freedom of the particle are encoded in the algebraic variables q and p which are both $\mathfrak{so}(\eta)$-valued functions with support on the world-line γ.

Firstly, we consider the open subset U of M constructed as follows. Consider the three-ball B^3 centered on a point x_0 of the worldline γ and call x and y the two punctures $\partial B^3 \cap \gamma$. Pick two non intersecting paths γ_1 and γ_2 on ∂B^3 both connecting x to y. The open region bounded by the portion of ∂B^3 contained between the two paths and the two arbitrary non intersecting disks contained in B^3 and bounded by the loops $\gamma\gamma_1$ and $\gamma\gamma_2$ defines the open subset $U \subset M$.

Next, we define the coordinate function $X : M \to V_\eta$ mapping spacetime into the 'internal space' $\mathfrak{so}(\eta)$ isomorphic, as a vector space, to its vector representation space V_η. The coordinates are chosen to be centered around a point x in M traversed by the worldline; $X(x) = 0$. Associated to the coordinate function X, there is a natural solution to the equations of motion (3), (4), (6), (5) in U

$$\begin{aligned} e = dX &= \delta \\ A &= 0 \\ q &= -X\mid_\gamma \\ p &= \textit{constant}, \end{aligned} \tag{9}$$

where δ is the unit of $\mathrm{End}(T_pM, V_\eta)$, $\delta(v) = v$ forall v in T_pM and all p in U. The field configuration $e = \delta$ (together with the $A = 0$ solution) provides a natural notion of flat Riemannian or Minkowskian spacetime geometry via its relation to the spacetime metric $g = 2\mathrm{tr}(e \otimes e)$. This flat background is defined in terms of a special gauge (notice that one can make e equal to zero by transformation of the form (8)). From now on, we will call such gauge a flat gauge. The solution for q is obtained through the equation (5) relating the background geometry to the geometry of the worldline. Here, we can readely see that q represents the particle's position X, first integral of the line element defined by the background geometry e. Below we show that equation (4) forces the worldline to be a straight line. Finally, $p = \textit{constant}$ trivially satisfies the conservation equation (6). In fact, the curvature equation of motion (3) constrains p to remain in a fixed adjoint orbit so we can introduce a constant $m \in \mathbb{R}^{*+}$ such that $p = mv$ with $v \in \mathfrak{so}(\eta)$ such that $\mathrm{tr}v^2 = -\sigma^2$. Consequently, p satisfies the mass shell constraints $p^2 := \mathrm{tr}(p^2) = -\sigma^2 m^2$ and acquires the interpretation of the particle's momentum.

We can now relate the position q and momentum p, independent in the first order formulation, by virtue of (4). Indeed, the chosen flat geometry solution $e = \delta$, $A = 0$ leads to a everywhere vanishing torsion d_Ae.

Hence, the commutator $[p,q] = X \times p$, where $\times$ denotes the usual cross product on V_η, vanishes on the worldline. This vanishing of the relativistic angular momentum (which is conserved by virtue of equation (3)) implies, together with the flatness of the background fields, that the the worldline γ of the particle defines a straight line passing through the origin and tangent to its momentum p. Equivalently, we can think of the momentum p as Hodge dual to a bivector $*p$, in which case the worldline is normal to the plane defined by $*p$.

Note that translating γ off the origin, which requires the introduction of spacetime torsion, can be achieved by the gauge transformation $q \to q + C$ with $C = constant$ which leaves all the other fields invariant. In this way we conclude that the previous solution of our theory can be (locally) interpreted as the particle following a geodesic of flat spacetime.

More formally, we can also recover the action of a test particle in flat spacetime by simply 'switching off' the interaction of the particle whith gravity. This can be achieved by evaluating the action (1) on the flat solution and neglecting the interactions between geometry and matter, namely the equations of motion linking the background fields to the matter degrees of freedom (e.g. $e \neq dX$). This formal manipulation leads to the following Hamilton function

$$S[p,X,N] = \int_\gamma \mathrm{tr}(p\dot{X}) + N(p^2 - m^2), \tag{10}$$

which is the standart first order action for a relativistic spinless particle.

2. *The string in* $3+1$ *dimensions*

We now focus on the four dimensional ($d = 3$) extension of the above considerations. Here again we consider the isometry group $G = \mathrm{SO}(\eta)$ of a given four dimensional metric structure $\eta = (\sigma^2, +, +, +)$, in which case the value $\sigma = 1$ leads to the Riemannian group $G = \mathrm{SO}(4)$, while $\sigma = i$ encodes a Lorentzian signature $G = \mathrm{SO}(1,3)$. As in three dimensions, we denote (π, V_η), with $V_\eta = \mathbb{R}\{e_I\}_I$, $I = 0, ..., 3$, the vector representation of $\mathfrak{so}(\eta) = \mathbb{R}\{J_{ab}\}_{a,b=0,...,3}$. Finally, we choose the bilinear form 'tr' such that, forall a, b in $\mathfrak{so}(\eta)$, it is associated to the trace $\mathrm{tr}(ab) = \frac{1}{2}a_{IJ}b^{IJ}$ in the vector representation. We are using the notation $\alpha^{IJ} = \alpha^{ab}\,\pi(J_{ab})^{IJ} := \alpha^{ab}\,J_{ab}^{IJ}$ for the matrix elements of the image of an element $\alpha \in \mathfrak{so}(\eta)$ in $\mathrm{End}(V_\eta)$ under the vector representation. The dynamics of the theory is governed by the action (1) where the matter excitations are string-like and the worldsheet $\mathscr{W}$ is now a two-dimensional submanifold of the four dimensional space time manifold M. The string degrees of freedom are described by an $\mathfrak{so}(\eta)$-valued one-form q and an $\mathfrak{so}(\eta)$-valued function p living on the world-sheet $\mathscr{W}$.

As before, we construct an open subset $U \subset M$ by cutting out a section of the four-ball B^4, and define the coordinate function $X : M \to V_\eta$, centered around a point x in $M \cap \mathscr{W}$. Consider the following field configurations which define a flat solution to the equations of motion (3), (4), (6), (5) in U.:

$$\begin{aligned} B &= *(e \wedge e), \quad \text{with } e = dX = \delta \\ A &= 0 \\ q &= -*XdX \\ p &= constant, \end{aligned} \tag{11}$$

where the star '$*$' is the Hodge operator $* : \Omega^p(V_\eta) \to \Omega^{4-p}(V_\eta)$ acting on the internal space; $(*\alpha)_{IJ} = \frac{1}{2}\epsilon_{IJ}{}^{KL}\alpha_{KL}$, with the totally antisymmetric tensor ϵ normalized such that $\epsilon^{0123} = +1$.

The solutions $B = *(\delta\wedge\delta)$ ($A = 0$), leads to a natural notion of flat Riemannian or Minkowski background geometry through the standard construction of a metric out of B when B is a simple bivector; $B = *(e \wedge e)$ with $e = \delta$. We can readily see that the q one-form is the first integral of the area element defined by the background field B. As in 3d, the equations of motion constrain p to remain in a fixed adjoint orbit so that we can introduce a constant $\tau \in \mathbb{R}^{*+}$ such that $p = \tau v$ and $v \in \mathfrak{so}(\eta)$ has a fixed norm; $\mathrm{tr}v^2 = -\sigma^2$. We call τ the string tension, or mass per unit length, and p the momentum density which satisfies a generalized mass shell constraint.

This momentum density p is related to the q field by analysis of equation (4). The solution ($B = *\delta\wedge\delta, A = 0$) has zero torsion $d_A B$. Accordingly, the commutator $[p,q] = [*XdX, p]$ vanishes on the worldsheet. This leads to the constraint $X^I p_{IJ} = 0$. Putting everything together, we see that the flat solution in the open subset U leads to the picture of a locally flat worldsheet (a locally straight, rigid string) in flat spacetime,

dual [1], as a two-surface, to the momentum density bivector p (if p is simple, namely if it defines a two-plane). If we consider more general solutions admitting torsion, the plane can be translated off the origin. Indeed, the equation (5) determines the field q in terms of the geometry of the B field up to the addition of an exact one-form $\beta = d\alpha$ which encodes the translational information. For instance, the translation $X \mapsto X + C$ of the plane yields $q \mapsto q - *CdX$ and consequently corresponds to a function α defined by $d\alpha = - * CdX$. This potential is in turn determined by the torsion $T = d_A B$ of the B field via the equation (4). More general solutions can be found for arbitrary α's, as discussed below.

Following the same path as in the case of the particle case, we can 'turn off' the interaction between the topological BF background and the string by evaluating the action on the flat solution (this implies, here again, that we ignore the equations of motion of the coupled theory, i.e. the relation between the matter and geometrical degrees of freedom). We obtain the following Hamilton function

$$S[p, X, N] = \int_{\mathscr{W}} \mathrm{tr}(*p\, dX \wedge dX) + N(p^2 - \tau^2), \tag{12}$$

up to a constant. This is the Polyakov action on a non trivial background with metric $G_{\mu\nu} = 0$ and antisymmetric field $b = v \in \mathfrak{so}(\eta)$.

Now, the previous action leads to trivial equations of motion that are satisfied by arbitrary X (because $p = constant$ and so the Lagrangian is a total differential). This is to be expected, from the string theory viewpoint, this would be a charged string moving on a constant potential, so the field strength is zero. This seems in sharp contrast to the particle case where the effective action leads to straight line solutions. Here any string motion is allowed; however, from the point of view of the full theory, all these possibilities are pure gauge. The reason for this is that in $2+1$ dimensions the flat gauge condition $e = \delta$ fixes the freedom (8) up to a global translation, and hence gauge considerations are not necessary in interpreting the effective action. In the string case, $B = *(\delta \wedge \delta)$ partially fixes the gauge; the remaining freedom being encoded in $\eta = d\alpha$ for any α.

C. Geometrical interpretation: cosmic strings and topological defects

The above discussion shows that particular solutions of the theory in a particular open subset lead to the standard propagation of matter degrees of freedom on a flat (or degenerate) background spacetime. In fact, we can go further in the physical interpretation by considering other solutions, defined everywhere, which are in one-to-one correspondence with solutions of four dimensional general relativity in the presence of distributional matter. These solutions are called cosmic strings.

1. *Cosmic strings*

It is well known that the metric associated to a massive and spinning particle coupled to three-dimensional gravity is that of a locally flat spinning cone. The lift of this solution to $3+1$ dimensions corresponds to a spacetime around an infinitely thin and long straight string (see for instance [15] and references therein). Let us endow our spacetime manifold M with a Riemannian structure (M, g) and let $x \in M$ label a point traversed by the string. We can choose as a basis of the tangent space $T_x M$ the coordinate basis $\{\partial_t, \partial_r, \partial_\varphi, \partial_z\}$ associated to local cylindrical coordinates such that the string is lying along the z axis and goes through the origin. The embedding of the string is given by $\phi(t, z) = (t, 0, 0, z)$. Let τ and s respectively denote the mass and intrinsic (spacetime) spin per unit length of the string. Note that τ is the string tension. Solving Einstein's field equations for a such stationary string carrying the above mass and spin distribution produces a two-parameter (τ, s) family of solutions described by the following line element written in the specified cylindrical coordinates

$$\begin{aligned} ds^2 &= g_{\mu\nu} dx^\mu \otimes dx^\nu \\ &= \sigma^2 \left(dt + \beta d\varphi\right)^2 + dr^2 + (1-\alpha)^2 r^2 d\varphi^2 + dz^2, \end{aligned} \tag{13}$$

[1] Note that this is exactly the same result than the one obtained for the point particle, if we think of the 3d momentum as Hodge dual to a bivector.

where $\beta = 4Gs$ and $\alpha = (1 - 4G\tau)$, G is the Newton constant. In fact this family of metrics is the general solution to Einstein's equations describing a spacetime outside any matter distribution in a bounded region of the plane (r, φ) and having a cylindrical symmetry. Exploiting the absence of structure along the z axis, by simply suppressing the z direction, reduces the theory to that of a point particle coupled to gravity in $2 + 1$ dimensions, where the location of the particle is given by the point where the string punctures the $z = 0$ plane. We will come across a such duality again in the quantization process of the next sections. The dual co-frame for the above metric is written

$$\begin{aligned} e^0 &= dt + \beta d\varphi \\ e^1 &= \cos\varphi dr - \alpha r \sin\varphi d\varphi \\ e^2 &= \sin\varphi dr + \alpha r \cos\varphi d\varphi \\ e^3 &= dz, \end{aligned} \tag{14}$$

such that $ds^2 = e^I \otimes e^J \eta_{IJ}$.If we assume that the connection A associated to the above metric is Riemannian, it is straight-forward to calculate its components by exploiting Cartan's first structure equation ($d_A e = 0$). The result reads

$$A = A_\mu^{IJ}\, \sigma_{IJ} dx^\mu = 4G\tau\, \sigma_{12}\, d\varphi, \tag{15}$$

where $\{\sigma_{IJ}\}_{I,J}$ is a basis of $\Omega^2(V_\eta) \simeq \mathfrak{so}(\eta)$.

Using the distributional identity $dd\varphi = 2\pi\delta^2(r)dxdy$ ($x = r\cos\varphi$, $y = r\sin\varphi$, and $dxdy$ is a wedge product), it is immediate to compute the torsion $T = T^0 e_0$ and curvature $F = F^{12}\, \sigma_{12}$ of the cosmic string induced metric :

$$T^0 = 8\pi G s\, \delta^2(r)\, dxdy, \quad F^{12} = 8\pi G\tau\, \delta^2(r)\, dxdy. \tag{16}$$

These equations state that the torsion and curvature associated to the cosmic string solution are zero everywhere except when the radial coordinate r vanishes, i.e. at the location of the string worlsheet lying in the $z - t$ plane. If we now focus on the spinless cosmic string case $s = 0$, we can establish a one-to-one correspondence between the above solutions of general relativity and the following solutions of BF theory coupled to string sources:

$$\begin{aligned} B^{01} &= \sin\varphi dr dz + \alpha r \cos\varphi d\varphi dz, & B^{02} &= -(\cos\varphi dz dr - \alpha r \sin\varphi dz d\varphi) \\ B^{03} &= \alpha r dr d\varphi, & B^{12} &= -\sigma^2 dz dt \\ B^{13} &= -\sigma^2(\sin\varphi dt dr + \alpha r \cos\varphi dt d\varphi), & B^{23} &= \sigma^2(\cos\varphi dt dr - \alpha r \sin\varphi dt d\varphi), \end{aligned}$$

$$\begin{aligned} A^{12} &= 4G\tau d\varphi, \\ q^{12} &= \sigma^2(z dt - t dz), \quad p^{12} = \tau, \end{aligned} \tag{17}$$

where only the non vanishing components have been written and the coupling constant κ in (1) has been set to $8\pi G$.

In this way, solutions of our theory are in one-to-one correspondence to solutions of Einstein's equations. The converse is obviously not true as our model does not allow for physical local excitations such us gravitational waves. However, augmenting the action (1) with a Plebanski term constraining the B field to be simple, would lead to the full Einstein equations in the presence of distributional matter,

$$\epsilon_{IJKL} e^J \wedge F^{KL} = 8\pi G\tau\, \epsilon_{IJKL} e^J J_{12}^{KL}\, \delta_{\mathscr{W}}, \tag{18}$$

where $J_{12}^{KL} = \delta_1^{[K}\delta_2^{L]}$, starting from the theory considered in this paper.

2. *Many-strings-solution*

One can also construct a many string solution by 'superimposing' solutions of the previous kind at different locations. Here we explicitly show this for two strings. We do this as the example will illustrate the geometric meaning of torsion in our model. Assume that we have two worlsheets $\mathscr{W}_1$ and $\mathscr{W}_2$ respectively traversing the points p_1 and p_2. We will work with two open patches $U_i \subset M$, $i = 1, 2$, such that p_1 and p_2 both belong to the overlap $U_1 \cap U_2$. The cylindrical coordinates $(t_i, r_i, \varphi_i, z_i)$ associated to the charts $(U_i \subset M, X_i^\mu : U_i \to \mathbb{R}^4)$ are chosen such that the strings lie along the z axis, are separated by a distance

x_0 in the x-direction, and are such that $r_i(p_i) = 0$. The coordinate transform occurring in the overlap $U_1 \cap U_2$ is immediate; it yields $t_i = t$, $x_2 = x_1 + x_0$ $y_i = y$ and $z_i = z$, for $i = 1, 2$. The two embeddings are given consequently by $\phi_1(t, z) = (t, 0, 0, z)$ and $\phi_2(t, z) = (t, x_0, 0, z)$. Our notations are such that a field ϕ expressed in the coordinate system associated to the open subset U_i is noted ϕ_{U_i}.

Our strategy to construct the two-string-solution is the following. We need to realize the fact that, regarded from a particular coordinate frame, one of the two strings is translated off the origin. We will choose to observe the translation of $\mathscr{W}_2$ from the coordinate frame 1. Now, the study of the flat solution discussed in the previous section has showed that translations of the worlsheet are related to the torsion T of the B field. In particular, we know how to recognize a translation of the form $X \to X + C$, with $C = x_0 e_1$. It corresponds to a torsion of the form $T = \kappa[p, d\alpha]$, with $d\alpha = - * CdX$. Hence, the two-string-solution is based on the tetrad field which leads to the desired value of the B field torsion taking into account the separation of the two worldsheets. For simplicity, here we assume that the two strings are parallel, hence that they have same momentum density

$$p_{U_1} = p_{U_2} = \tau\sigma_{12}, \tag{19}$$

and accordingly create the same curvature singularity in both coordinate frames 1 and 2. The associated connection yields

$$A_{U_i} = 4G\tau\, d\varphi_i\, \sigma_{12}, \quad \forall i = 1, 2. \tag{20}$$

The dual co-frame $e_{U_i} = e^I_{U_i} \otimes e_{IU_i}$ is defined by the following components

$$\begin{aligned} e^0_{U_i} &= dt \\ e^1_{U_i} &= \cos\varphi_i dr_i - \alpha r_i \sin\varphi_i d\varphi_i \\ e^2_{U_i} &= \sin\varphi_i dr_i + (\alpha r_i \cos\varphi_i + \delta_{i2}\,\frac{\kappa}{4\pi}\tau x_0) d\varphi_i \\ e^3_{U_i} &= dz. \end{aligned} \tag{21}$$

By integrating the $B = *e \wedge e$ solution with e given by (21), we can now calculate the q field, up to the addition of an exact form $\beta = d\alpha$

$$q_{U_i} = \sigma^2(zdt - tdz)\,\sigma_{12} + d\alpha^{IJ}_i \sigma_{IJ}. \tag{22}$$

The potential α is derived from the equation of motion (4) relating the commutator of p and q to the B field torsion three-form $T = d_A B = *d_A e \wedge e + *e \wedge d_A e$:

$$T_{U_i} = \delta_{i2}\,\frac{1}{2}\,\kappa\,\tau x_0\,\delta(r)(dx_2\,dy_2\,dz\,\sigma_{01} + \sigma^2\,dt\,dx_2\,dy_2\,\sigma_{13}). \tag{23}$$

This torsion indeed corresponds to a two-string-solution since it yields the desired value $- * CdX$ for the form $d\alpha$,

$$d\alpha_i = \delta_{i2}\,\frac{1}{2}\,x_0\,(dz\,\sigma_{02} + \sigma^2\,dt\,\sigma_{23}). \tag{24}$$

One can add more than one string in a similar fashion, leading to multiple cosmic string solutions. It is interesting to notice that torsion of the mutiple string solution is related to the distance x_0 separating the world sheets. Of course this is a distance defined in the flat-gauge where $B = *\delta \wedge \delta$. This concludes our discussion on the physical aspects of the action (1) of string-like sources coupled to BF theory. We now turn toward the quantization of the theory.

III. QUANTUM THEORY

For the entire quantization process to be well defined, we will restrict our attention to the case where the symmetry group G is compact. For instance, we can think of G as being SO(4). We will also concentrate on the four-dimensional theory and set the coupling constant κ to one. Also, to rely on the canonical analysis performed in [1], we will work with a slightly different theory where the momentum p is replaced by the string field $\lambda \in C^\infty(\mathscr{W}, G)$. This new field enters the action only through the conjugation $\tau Ad_\lambda(v)$ of a fixed unit element v in $\mathfrak{g}$, and the theory is consequently defined by the action (1) with p set to $\tau\lambda v\lambda^{-1}$.

The field λ transforms as $\lambda \to g\lambda$ under gauge transformations of the type (7) and the theory acquires a new invariance under the subgroup $H \subseteq G$ generated by v. The link between the two theories is established by the fact that, as remarked before, the equation of motion $F = p\delta_{\mathscr{W}}$ implies that p remains in the same conjugacy class along the worldsheet. Here, we choose to label the class by τv and to consider λ as dynamical field instead of p.

A. Canonical setting

As a preliminary step, we assume that the spacetime manifold M is diffeomorphic to the canonical split $\mathbb{R} \times \Sigma$, where $\mathbb{R}$ represents time and Σ is the canonical spatial hypersurface. The intersection of Σ with the string worldsheet $\mathscr{W}$ forms a one dimensional manifold $\mathscr{S}$ that we will assume to be closed[2]. We choose local coordinates (t, x^a) for which Σ is given as the hypersurface $\{t = 0\}$. By definition, x^a, $a = 1, 2, 3$, are local coordinates on Σ. We also choose local coordinates (t, s) on the 2-dimensional world-sheet $\mathscr{W}$, where $s \in [0, 2\pi]$ is a coordinate along the one-dimensional string $\mathscr{S}$. We will note $x_{\mathscr{S}} = \phi\,|_{\Sigma}$ the embedding of the string $\mathscr{S}$ in Σ. We pick a basis $\{X_i\}_{i=1,\dots,dim(\mathfrak{g})}$ of the real Lie algebra $\mathfrak{g}$, raise and lower indices with the inner product 'tr', and define structure constants by $[X_i, X_j] = f_{ij}{}^k X_k$. Next, we choose a polarization on the phase space such that the degrees of freedom are encoded in the configuration variable $(A, \lambda) \in \mathcal{A} \times \Lambda$ defined by the couples formed by (the pull-back to Σ of) connections and string momenta.

The canonical analysis of the coupled action (1) shows that the legendre transform from configuration space to phase space is singular : the system is constrained. Essentially [3], the constraints are first class and are given by the following set of equations :

$$G_i \;:=\; D_a E^a_i + f_{ij}{}^k\, q^j_a\, \pi^a_k\, \delta_{\mathscr{S}} \;\approx\; 0 \tag{25}$$

$$H^a_i \;:=\; \epsilon^{abc} F_{ibc} - \pi^a_i\, \delta_{\mathscr{S}} \;\approx\; 0. \tag{26}$$

Here, $E^a_i = \epsilon^{abc} B_{ibc}$ is the momentum canonically conjugate to A^a_i, $\pi^a_i = \partial_s\, x^a_{\mathscr{S}} p_i$ is conjugate to q and satisfies $D_a \pi^a_i = 0$, where $p_i = \mathrm{tr}(X_i p)$ denote the components of the Lie algebra element p in the chosen basis of $\mathfrak{g}$. The symbols D and F denote respectively the covariant derivative and curvature of the spatial connection A.

The first constraint (25), the Gauss law, generates kinematical gauge transformations while the second (26), the curvature constraint, contains the dynamical data of the theory. To quantize the theory, one can follow Dirac's program of quantization of constrained systems which consists in first quantizing the system before imposing the constraints at the quantum level. The idea is to construct an algebra $\mathfrak{A}$ of basic observables, that is, simple phase space functions which admit unambiguous quantum analogues, which is then represented unitarely, as an involutive and unital $\star$-algebra of abstract operators, on an unphysical or auxiliary Hilbert space $\mathcal{H}$. Since the classical constraints are simple functionals of the basic observables, they can be unambiguously quantized, that is, promoted to self-adjoint operators on $\mathcal{H}$. The kernel of these constraint operators are spanned by the physical states of the theory.

The structure of the constraint algebra enables us to solve the constraints in different steps. One can first solve the Gauss law to obtain a quantum kinematical setting. Then, impose the curvature constraint on the kinematical states to fully solve the dynamical sector of the theory. In [1], the kinematical subset of the constraints is solved and a kinematical Hilbert space $\mathcal{H}_{\text{kin}}$ solution to the quantum Gauss law is defined. We first review the kinematical setting of [1] before exploring the dynamical sector of the theory.

B. Quantum kinematics : the Gauss law

The Hilbert space $\mathcal{H}_{\text{kin}}$ of solutions to the Gauss constraint is spanned by so-called string spin network states. String spin network states are the gauge invariant elements of the auxiliary Hilbert space $\mathcal{H}$ of cylindrical functions which is constructed as follows.

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[2] In fact if Σ is compact the equations of motion (3) implies that the string must be closed (or have zero tension).
[3] See the original work [1] for a detailed canonical analysis.

1. Auxiliary Hilbert space $\mathcal{H}$

Firstly, we define the canonical BF states. Let $\Gamma \subset \Sigma$ denote an open graph, that is, a collection of one dimensional oriented sub-manifolds [4] of Σ called edges e_Γ, meeting if at all only on their endpoints called vertices v_Γ. The vertices forming the boundary of a given edge e_Γ are called the source $s(e_\Gamma)$ and target $t(e_\Gamma)$ vertices depending on the orientation of the edge. We will call $n \equiv n_\Gamma$ the cardinality of the set of edges $\{e_\Gamma\}$ of Γ. Let $\phi : G^{\times n} \to \mathbb{C}$ denote a continuous complex valued function on $G^{\times n}$ and $A(e_\Gamma) \equiv g_{e_\Gamma} = P\exp\left(\int_{e_\Gamma} A\right)$ denote the holonomy of the connection A along the edge $e_\Gamma \in \Gamma$. The cylindrical function associated to the graph Γ and to the function ϕ is a complex valued map $\Psi_{\Gamma,\phi} : \mathcal{A} \to \mathbb{C}$ defined by :

$$\Psi_{\Gamma,\phi}[A] = \phi(A(e^1_\Gamma), ..., A(e^n_\Gamma)), \tag{27}$$

forall A in $\mathcal{A}$. The space of such functions is an abelian $\star$-algebra denoted $\mathrm{Cyl}_{BF,\Gamma}$, where the $\star$-structure is simply given by complex conjugation on $\mathbb{C}$. The algebra of all cylindrical functions will be called Cyl_{BF} $= \cup_\Gamma \mathrm{Cyl}_{BF,\Gamma}$.

Next, we define string states. Since the configuration variable is a zero-form, we expect to consider wave functions associated to points $x \in \Sigma$. Accordingly, we define the $\star$-algebra Cyl_S of cylindrical functions on the space Λ of λ fields as follows. An element $\Phi_{X,f}$ of Cyl_S is a continuous map $\Phi_{X,f} : \Lambda \to \mathbb{C}$, where $X = \{x_1, ..., x_n\}$ is a finite set of points in $\mathscr{S}$ and $f : G^{\times n} \to \mathbb{C}$ is a complex valued function on the Cartesian product G^n, defined by

$$\Phi_{X,f}[\lambda] = f(\lambda(x_1), ..., \lambda(x_n)). \tag{28}$$

Both algebras Cyl_{BF} and Cyl_S, regarded as vector spaces, can be given a pre-Hilbert space structure. Fixing a graph $\Gamma \subset \Sigma$ with n edges and a set of m points $X \subset \Sigma$, we define the scalar products respectively on $\mathrm{Cyl}_{BF,\Gamma}$ and $\mathrm{Cyl}_{S,p}$ as

$$< \Psi'_{\Gamma,\phi}, \Phi_{\Gamma,\psi} > = \int_{G^{\times n}} \overline{\phi}\, \psi, \tag{29}$$

and

$$< \Phi'_{X,f}, \Phi_{X,g} > = \int_{G^{\times m}} \overline{f}\, g, \tag{30}$$

where the integration over the group is realized through the Haar measure on G. These scalar products can be extended to the whole of Cyl_{BF} (resp. Cyl_S), i.e. to cylindrical functions defined on different graphs (resp. set of points), by redefining a larger graph (resp. set of points) containing the two different ones. The resulting measure, precisely constructed via projective techniques, is the AL measure. The string Hilbert space was in fact introduced by Thiemann as a model for the coupling of Higgs fields to loop quantum gravity [13] via point holonomies. Completing these two pre-Hilbert spaces in the respective norms induced by the AL measures, one obtains the BF and string auxiliary Hilbert spaces respectively denoted $\mathcal{H}_{BF}$ and $\mathcal{H}_S$. Tensoring the two Hilbert spaces yields the auxiliary Hilbert space $\mathcal{H} = \mathcal{H}_{BF} \otimes \mathcal{H}_S$ of the coupled system.

Using the harmonic analysis on G, one can define an orthonormal basis in $\mathcal{H}_{BF}$ and $\mathcal{H}_S$ the elements of which are respectively denoted (open) spin networks and n-points spin states. Using the isomorphism of Hilbert spaces $L^2(G^{\times n}) \simeq \bigotimes_{e_\Gamma} L^2(G_{e_\Gamma})$, any cylindrical function $\Psi_{\Gamma,\phi}$ in $\mathcal{H}_{BF}$ decomposes according to the Peter-Weyl theorem into the basis of matrix elements of the unitary, irreducible representations of G :

$$\Psi_{\Gamma,\phi}[A] = \sum_{\rho_1,...,\rho_n} \phi_{\rho_1,...,\rho_n}\, \rho_1[A(e^1_\Gamma)] \otimes ... \otimes \rho_n[A(e^n_\Gamma)], \tag{31}$$

where $\rho : G \to \mathrm{Aut}(\mathbb{V}_\rho)$ denotes the unitary, irreducible representation of G acting on the vector space $\mathbb{V}_\rho$ and the mode $\phi_{\rho_1,...,\rho_n} := \otimes_{i=1}^n \phi_{\rho_i}$ is an element of $(\ \mathbb{V}_{\rho_i} \otimes \mathbb{V}_{\rho_i}{}^*)^{\otimes_{i=1}^n}$. The functions appearing in the above sum are called open spin network states.

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[4] More precisely, one usually endows the canonical hypersurface Σ with a real, analytic structure and restricts the edges to be piecewise analytic or semi-analytic manifolds, as a mean to control the intersection points.

Equivalently, the string cylindrical functions decompose as :

$$\Phi_{X,f}[\lambda] = \sum_{\rho_1,...,\rho_m} f_{\rho_1,...,\rho_m}\, \rho_1[\lambda(x_1)] \otimes ... \otimes \rho_m[\lambda(x_m)], \tag{32}$$

and a given element in the sum is called an n-point spin state.

2. *String spin network states*

One can now compute a unitary action of the gauge group $C^\infty(\Sigma, G)$ on $\mathcal{H}$ by using the transformation properties of the holonomies and of the string fields $\lambda \to g\lambda$ under the gauge group and derive the subset of G-invariant states, that is, the states solution to the Gauss constraint. A vectorial basis of the vector space of gauge invariant states can be constructed, in analogy with $3d$ quantum gravity coupled to point particles [12], by tensoring the open spin network basis with the n point spin states elements. Such an tensorial element is required the following consistency conditions to be G-invariant.

The graph Γ of the open spin network has a set of vertices $\mathcal{V}_\Gamma$ including the points $\{x_1, ..., x_n\}$ forming the set X. The vertices of Γ are coloured with a chosen element ι_v of an orthonormal basis of the vector space of intertwining operators

$$\mathrm{Hom}_G \left[\bigotimes_{e_\Gamma | t(e_\Gamma)=v_\Gamma} \mathbb{V}_{\rho_{e_\Gamma}} \,,\, \bigotimes_{e_\Gamma | s(e_\Gamma)=v_\Gamma} \mathbb{V}_{\rho_{e_\Gamma}} \right], \tag{33}$$

if the vertex v_Γ is not on the string. If a vertex v_Γ is on the string, it coincides with some point $x_k \in X$. In this case, we chose an element ι_{v_Γ} in an orthonormal basis of

$$\mathrm{Hom}_G \left[\bigotimes_{e_\Gamma | t(e_\Gamma)=v_\Gamma} \mathbb{V}_{\rho_{e_\Gamma}} \,,\, \left(\bigotimes_{e_\Gamma | s(e_\Gamma)=v_\Gamma} \mathbb{V}_{\rho_{e_\Gamma}} \right) \otimes \mathbb{V}_{\rho_k} \right], \tag{34}$$

where $\mathbb{V}_{\rho_k}$ is the representation space associated to the point x_k.

By finally implementing the invariance under the sub-group $H \subseteq G$ generated by v of the n-point spin states by choosing the modes to be H-invariant, one obtains a vectorial basis in the kinematical Hilbert space $\mathcal{H}_{\mathrm{kin}}$ where the inner product is that of (BF and string) cylindrical functions. The elements of this basis are called *string spin networks states* and are of the form (see fig. 1) :

$$\Psi_{\Gamma,X}[A,\lambda] := (\Psi_\Gamma \otimes \Phi_X)[A,\lambda] = \left[\bigotimes_{e_\Gamma \in \Gamma} \rho_{e_\Gamma}[A(e_\Gamma)] \bigotimes_{x \in X} \rho_x[\lambda(x)] \right] \;.\; \bigotimes_{v_\Gamma \in \Gamma} \iota_{v_\Gamma}, \tag{35}$$

where the dot '.' denotes tensor index contraction.

This concludes the quantum kinematical framework of strings coupled to BF theory performed in [1]. We now solve the curvature constraint and compute the full physical Hilbert space $\mathcal{H}_{\mathrm{phys}}$.

C. Quantum dynamics: the curvature constraint

In this section, we explore the dynamics of the theory by constructing the physical Hilbert space $\mathcal{H}_{\mathrm{phys}}$ solution to the last constraint of the system, that is, the curvature or Hamiltonian constraint (26). Note that the physical states that we construct below are also solutions to the constraints of four dimensional quantum gravity coupled to distributional matter, as in the classical case.

We first underline a crucial property of the curvature constraint of $d+1$-dimensional BF theory with $d > 2$, namely its reducible character which has to be taken into account during the quantization process. We then proceed (as in [12]) à la Rovelli and Reisenberger [10, 11] by building and regularizing a generalized projection operator mapping the kinematical states into the kernel of the curvature constraint operator. This procedure automatically provides the vector space of solutions with a physical inner product and a Hilbert space structure, and leads to an interesting duality with the coupling of Feynman loops to 3d gravity [17], [14] from the covariant perspective.

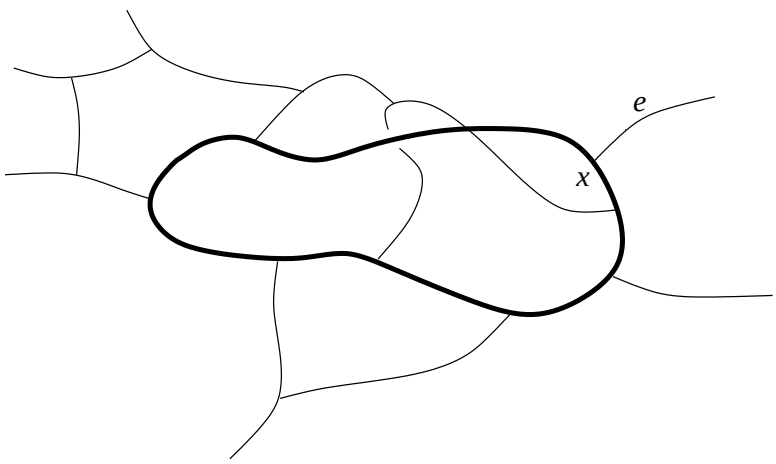

FIG. 1: A typical string spin network (the string is represented by the bold line).

1. *The reducibility of the curvature constraint*

A naive imposition of the curvature constraint on the kinematical states leads to severe divergences. This is due to the fact that there is a redundancy in the implementation of the constraint; the components of the curvature constraint of $(d+1)$-dimensional BF theory are not linearly independent, they are said to be reducible, if $d > 2$. The same is true for the theory coupled to sources under study here. As an illustration of this fact, let us simply count the degrees of freedom of source free ($\tau = 0$) BF theory in $d+1$ dimensions.

The configuration variable of the theory A_a^i is a $\mathfrak{g}$-valued connection one-form, thus containing $d \times \dim(\mathfrak{g})$ independent components for each space point of Σ. In turn, the number of constraints is given by the $\dim(\mathfrak{g})$ components of the Gauss law (25) plus the $d \times \dim(\mathfrak{g})$ components of the Hamiltonian constraint (26) for each space point $x \in \Sigma$. Hence, we have $N_C = (d+1) \times \dim(\mathfrak{g})$ constraints per space point. This leads to a negative number of degrees of freedom. What is happening [5] ? The point is that the N_C constraints are not independent: the Bianchi identity ($D^{(2)}F = d^{(2)}F + [A, F] = 0$, where the superscript $^{(p)}$ indicates the degree of the form acted upon) imply the reducibility equation

$$D_a H_i^a = 0. \tag{36}$$

In the case where sources are present, the reducibility equation remains valid because the curvature constraint $F = p\delta_{\mathscr{S}}$ together with the Bianchi identity automatically implements the momentum density conservation $Dp = 0$. We will come back to this reducibility of the matter sector of the theory. The system is said to be $(d-2)$-th stage reducible in the first class curvature constraints. This designation is due to the fact that the operator $d^{(2)}$ is himself reducible since $d^{(3)}d^{(2)} \equiv 0$. In turn, $d^{(3)}$ is reducible and so on. The chain stops after precisely $d-2$ steps since the action of the $d^{(d)}$ de-Rham differential operator on d-forms is trivial. Accordingly, the $N_R = \dim(\mathfrak{g})$ reducibility equations (36) imply a linear relation between the components of the curvature constraint. The number N_I of independent constraints is thus given by $N_C - N_R = d \times \dim(\mathfrak{g})$. Using N_I to count the number of degrees of freedom leads to the correct answer, namely zero degrees of freedom for topological BF theory.

The standard procedure to quantize systems with such reducible constraints consists in selecting a subset $H\ |_{irr}$ of constraints which are linearly independent and impose *solely* this subset of constraints on the auxiliary states of $\mathcal{H}$.

Keeping this issue in mind, we now proceed to the definition and regularization of the generalized projector on the physical states and construct the Hilbert space $\mathcal{H}_{\text{phys}}$ of solutions to all of the constraints of the theory.

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[5] We thank Merced Montesinos for pointing out this property of BF theory's constraints.

2. Physical projector : formal definition - the particle/string duality

We start by introducing the rigging map

$$\begin{aligned} \eta_{\rm phys} : {\rm Cyl} &\rightarrow {\rm Cyl}^* \\ \Psi &\mapsto \delta(\hat{H}\mid_{irr})\,\Psi, \end{aligned} \tag{37}$$

where Cyl^* is the (algebraic) dual vector space of $\mathrm{Cyl} = \mathrm{Cyl}_{BF} \otimes \mathrm{Cyl}_S$. The range of the rigging map $\eta_{\rm phys}$ formally lies in the kernel of the Hamiltonian constraint of the coupled model. The power of the rigging map technology is that it automatically provides the vector space $\eta_{\rm phys}(\mathrm{Cyl}) = \mathrm{Cyl}^*_{\rm phys} \subset \mathrm{Cyl}^*$ of solutions to the Hamiltonian constraint with a pre-Hilbert space structure encoded in the physical inner product

$$< \eta_{\rm phys}(\Psi_1), \eta_{\rm phys}(\Psi_2) >_{\rm phys} = [\eta_{\rm phys}(\Psi_2)](\Psi_1) := < \Psi_1,\, \delta(\hat{H}\mid_{irr})\,\Psi_2 >, \tag{38}$$

for any two string spin network states $\Psi_1,\, \Psi_2 \in \mathcal{H}_{\rm kin}$. The scalar product used in the last equality is the kinematical inner product (29), (30). The physical Hilbert space $\mathcal{H}_{\rm phys}$ is then obtained by the associated Cauchy completion of the quotient of $\mathcal{H}_{\rm kin}$ by the Gel'fand ideal defined by the set of zero norm states.

Accordingly, the construction of the physical inner product can explicitly be achieved if we can rigorously make sense of the formal expression $\delta(\hat{H}\mid_{irr})$. This task is greatly simplified by virtue of the following duality. Indeed, we can re express the above formal quantity as follows

$$\delta(\hat{H}\mid_{irr}) = \prod_{x\in\Sigma} \delta(\hat{H}\mid_{irr}(x)) = \int_{\mathcal{N}} \mathcal{D}\mu[N]\, \exp\left(i \int_\Sigma {\rm tr}(N\wedge \hat{H}) \right). \tag{39}$$

Here, $\mathcal{N} \ni N$ is the space of regular $\mathfrak{g}$-valued one-forms on Σ and $\mathcal{D}\mu[N]$ denotes a formal functional measure on $\mathcal{N}$ imposing constraints on the test one-form N to remove the redundant delta functions on H. Simply plugging in the explicit expression of the exponent in (39) leads to

$$\begin{aligned} H[N] = \int_\Sigma {\rm tr}(N\wedge H) &= \int_\Sigma {\rm tr}(N\wedge F) + \int_{\mathscr{S}} {\rm tr}(Np) \\ &= S^{3d}_{BF+part}[N, A], \end{aligned} \tag{40}$$

which, in the case where $G = SO(\eta)$, where η is a three-dimensional metric, is the action of 3d gravity coupled to a (spinless) point particle [2], [14] : the role of the triad is played by N, the mass and the worldline of the particle are respectively given by the string tension τ (hidden in the string variable $p = \tau Ad_\lambda(v)$) and $\mathscr{S}$. Finally, the role of the Cartan subalgebra generator J_0 is played by v (also hidden in p). This relation is reminiscent to the link between cosmic strings in 4d and point particles in three-dimensional gravity discussed in the first sections. More generally, we have in fact the following duality :

$$P(\Omega)^{d+1}_{BF+(d-2)-branes} = \mathcal{Z}^d_{BF+(d-3)-branes}, \tag{41}$$

where Ω denotes the ($d+1$-dimensional) no-spin-network vacuum state, $\mathcal{Z}$ is the path integral of BF theory in d spacetime dimensions and we have introduced the linear form [6] P on $\mathrm{Cyl} \subset \mathfrak{A}$ defined by

$$\begin{aligned} \forall \Psi \in {\rm Cyl}, \qquad P(\Psi) &= < \eta_{\rm phys}(\Omega), \eta_{\rm phys}(\Psi) >_{\rm phys} \\ &= < \Omega\, , \delta(\hat{H}\mid_{irr})\, \Psi > . \end{aligned} \tag{42}$$

Furthermore, when $d = 3$, the formal functional measure $\mathcal{D}\mu[N]$ introduced above to take into account the reducible character of the four-dimensional theory corresponds to the Fadeev-Popov determinant gauge-fixing the translational topological symmetry of the 3d theory; the reducibility of the 4d theory is mapped via this duality onto the gauge redundancies of the three-dimensional theory.

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[6] More precisely, the linear form P, once normalized by the evaluation $P(1)$, is a state

$$P/P(1) : {\rm Cyl} \subset \mathfrak{A} \rightarrow \mathbb{C},$$

whose associated GNS construction leads equivalently to the physical Hilbert space $\mathcal{H}_{\rm phys}$. The associated Gel'fand ideal $\mathcal{I}$ is immense by virtue of the topological nature of the theory under consideration. Indeed, one can show that any element of Cyl based on a contractible graph is equivalent to a complex number. The associated physical representation $\pi_{\rm phys} : \mathrm{Cyl} \rightarrow \mathrm{End}(\mathcal{H}_{\rm phys})$ is defined such that forall cylindrical function $a_\Gamma \in \mathrm{Cyl}$ defined on a contractible graph Γ, $\pi_{\rm phys}(a_\Gamma[A])\Psi = a_\gamma[0]\Psi$, forall Ψ in $\mathcal{H}_{\rm phys}$.

Now because of the above duality (41), regularizing the formal expression (39) is, roughly speaking, equivalent to regularizing the path integral for 3d gravity coupled to point particles, up to the insertion of spin network observables. The physical inner product in our theory will therefore be related to amplitudes computed in [14], [27], although they would have here a quite different physical interpretation. Following [12, 19], we will regularize the Hamiltonian constraint at the classical level by defining a lattice-like discretization of Σ and by constructing holonomies around the elementary plaquettes of the discretization as a first order approximation of the curvature.

However, there are two major obstacles to the direct and naive implementation of such a program. The first is the reducible character of the curvature constraint and the second is the presence of spin network edges ending on the string. We will use the above duality to treat the first issue while the second will be dealt with by introducing an appropriate regularization scheme.

3. Physical projector : regularization

Throughout this section, we will concentrate on the definition of the linear form (42) evaluated on the most general string spin network state $\Psi \in \mathcal{H}_{\rm kin}$, since it contains all the necessary information to compute transition amplitudes between any two arbitrary elements of the kinematical Hilbert space. We will consider a string spin network basis elements Ψ of $\mathcal{H}_{\rm kin}$ defined on the (open) graph Γ. The set of end points of the graph living on the string $\mathscr{S}$ will be denoted X.

We follow the natural generalization of the regularization defined in [12] for $2+1$ gravity coupled to point particles. In order to deal with the curvature singularity at the string location, we thicken the smooth curve $\mathscr{S}$ to a torus topology, smooth, non-intersecting tube T_η of constant radius $\eta > 0$ centered on the string $\mathscr{S}$. The radius η is defined in terms of the local arbitrary coordinate system. If the string is disconnected, we blow up each string component in a similar fashion.

Next, we remove the tube T_η from the spatial manifold Σ. We are left with a three-manifold with torus boundary $\Sigma \setminus T_\eta$ noted Σ_η. For instance, if Σ has the topology of S^3, we know by Heegard's splitting, that the resulting manifold has the topology of a solid torus whose boundary surface is the Heegard surface defined by the string tube. In this way we construct a new three-manifold with boundary where each boundary component is in one-to-one correspondence with a string component and has the topology of a torus. Finally, the open graph Γ is embedded in the bulk manifold and its endpoints lie on the boundary torus.

The next step is to choose a simplicial decomposition [7] of Σ_η or more generally any cellular decomposition, i.e., a homeomorphism $\phi : \Sigma_\eta \to \Delta$ from our spatial bulk manifold Σ_η to a cellular complex Δ. The discretized manifold $\Delta \equiv \Delta_\epsilon$ depends on a parameter $\epsilon \in \mathbb{R}^+$ controlling the characteristic (coordinate) 'length scale' of the cellular complex. We will see that, by virtue of the three-dimensional equivalence between smooth, topological and piecewise-linear (PL) categories, together with the background independent nature of our theory, no physical quantities will depend on this extra parameter. We will note Δ_k the k-cells of Δ. To make contact with the literature, we will in fact work with the dual cellular decomposition Δ^*. The dual cellular complex Δ^* is obtained from Δ by placing a vertex v in the center of each three-cell Δ_3, linking adjacent vertices with edges e topologically dual to the two-cells Δ_2 of Δ, and defining the dual faces f, punctured by the one-cells Δ_1, as closed sequences of dual edges e. The intersection between Δ^* and the boundary tube T_η induces a closed, oriented (trivalent if Δ is simplicial) graph which is the one-skeleton of the cellular complex $\partial\Delta^* = (\overline{v}, \overline{e}, \overline{f})$ dual to the cellular decomposition $\partial\Delta$ of the 2d boundary T_η induced by the bulk complex Δ. We will note $\mathcal{F}$ the set of faces f of the cellular pair $(\Delta^*, \partial\Delta^*)$ and require that each dual face of $\mathcal{F}$ admits an orientation (induced by the orientation of Σ_η) and a distinguished vertex.

Finally, among all possible cellular decompositions, we select a subsector of two-complexes which are *adapted* to the graph Γ. Namely, we consider dual cellular complexes $(\Delta^*, \partial\Delta^*)$ whose one-skeletons admit the graph Γ as a subcomplex. In particular, the open edges of Γ end on the vertices $\overline{v}$ of the boundary two-complex $\partial\Delta^*$.

The meaning of the curvature constraint $F = p\,\delta_{\mathscr{S}}$ is that the physical states have support on the space of connections which are flat everywhere except at the location of the string where they are singular. In other words, the holonomy $g_\gamma = A(\gamma)$ of an infinitesimal loop γ circling an empty, simply connected region

[7] Note that in dimension $d \leq 3$, each topological d-manifold admits a piecewise-linear-structure (this is the so-called 'triangulation conjecture').

yields the identity, while the holonomy g_γ circling the string around a point $x \in \mathscr{S}$ is equal to $\exp p(x)$, the image of the fixed group element $u = e^{\tau v}$ under the inner automorphism $Ad_\lambda : G \to G;\ u \mapsto \lambda(x)\, u\, \lambda^{-1}(x)$, with the string field λ evaluated at the point x. The integration over the string field λ appearing in the computation of the physical inner product then forces the holonomy of the connection around the string to lie in the same conjugacy class $Cl(u)$ than the group element u.

To impose the $F = 0$ part of the curvature constraint, we will require that the holonomy

$$\begin{aligned} A : \mathcal{F} &\to G \\ \partial f &\mapsto g_f = \prod_{e \subset \partial f} A(e), \end{aligned} \tag{43}$$

around all the oriented boundaries of the faces f of $\mathcal{F}$ be equal to one [8]. Each such flat connection defines a monodromy representation of the fundamental group $\pi_1(\Sigma_\eta)$ in G. Concretely, the holonomies are computed by taking the edges in the boundary ∂f of the face f in cyclic order, following the chosen orientation, starting from the distinguished vertex. Reversing the orientation maps the associated group element to its inverse.

It is here crucial to take into account the reducible character of the curvature constraint to avoid divergences due to redundancies in the implementation of the constraints (i.e. divergences coming from the incorrect product of redundant delta functions). As discussed above, the reducibility equation induced by the Bianchi identity implies that the components of the curvature are not independent. In the discretized framework, we know [18] that forall set of faces f forming a closed surface $\mathcal{S}$ with the topology of a two-sphere,

$$\prod_{f \in \mathcal{S}} g_f = \mathbb{1}, \tag{44}$$

modulo orientation and some possible conjugations depending on the base points of the holonomies. Accordingly, there is, for each three-cell of the dual cellular complex Δ^*, one group element g_f, among the finite number of group variables attached to the faces bounding the bubble, which is completely determined by the others. It follows that imposing $g_f = \mathbb{1}$ on *all* faces of the cellular complex Δ^* is redundant and would create divergences in the computation of the physical inner product. The proper way [22], [14] to address the reducibility issue, or over determination of the holonomy variables, is to pick a maximal tree T of the cellular decomposition Δ and impose $F = 0$ *only* on the faces of Δ^* that are not dual to any one simplex contained in T. A tree T of a cellular decomposition Δ is a sub-complex of the one-skeleton of Δ which never closes to form a loop. A tree T of Δ is said to be maximal if it is connected and goes through all vertices of Δ. The fact that T is a maximal tree implies that one is only removing redundant flatness constraints taking consistently into account the reducibility of the flatness constraints.

Finally, we need to impose the $F = p$ part of the curvature constraint. The idea is to require that the holonomy g_γ around any loop γ in $\partial\Delta^*$ based at a point x, belonging to the homology class of loops of the boundary torus T_η normal to the string $\mathscr{S}$ (these loops are the ones wrapping around the cycle of the torus circling the string, i.e., the non-contractible loops in Σ_η), be equal to the image of the group element u under the adjoint automorphism $\lambda(x)u\lambda(x)^{-1}$, i.e., belong to $Cl(u)$. Intuitively, this could be achieved by picking a finite set $\{\gamma_i\}_i$ of such homologous paths all along the tube T_η and imposing $g_{\gamma_i} = Ad_{\lambda_i}(u)$, with the field λ_i evaluated at the base point of the holonomy g_{γ_i}. However, here again, care must be taken in addressing the reducibility issue induced by the equation $D_a H_i^a = 0$. In the presence of matter, the reducibility implies that the curvature constraint $F = p\delta_{\mathscr{S}}$ together with the Bianchi identity $DF = 0$ induce the momentum density conservation $Dp = 0$. In our setting, this is reflected in the fact that the holonomies g_{γ_1} and g_{γ_2} associated to two distinct homologous loops γ_1 an γ_2 circling the string satisfy the property $Cl(g_{\gamma_1}) = Cl(g_{\gamma_2})$ on shell. This is due to the Bianchi identity in the interior of the cylindrically shaped section of the torus T_η bounded by γ_1 and γ_2, and the flatness constraint $F = 0$ imposing the holonomies around all the dual faces on the boundary of the cylindrical section to be trivial (see e.g. (44)). Accordingly, imposing $g_{\gamma_1} \in Cl(u)$ naturally implies that g_{γ_2} belongs to the conjugacy class labeled by u. In other words, choosing *one* arbitrary closed path circling the string, say γ_1 based at a point x_1, and imposing $F = p$ *only along that path* naturally propagates via the Bianchi identity and the flatness constraint and forces the holonomy g_{γ_2} around any other homologous loop γ_2 based at a point x_2 to be of the form

[8] Note that the blow up of the string, reflected here in the presence of a flatness constraint on the boundary torus, gives us the opportunity to impose that the connection is flat also on the string.

$g_{\gamma_2} = huh^{-1} \in Cl(u)$, for some $h \in G$. This shows that imposing $F = p$ more than once, e.g. also around γ_2, would lead to divergences which can be traced back to the reducibility of the constraints.

However, for the prescription to be complete [9], it is not sufficient to have g_{γ_2} in $Cl(u)$; we need to recover the fact that the holonomy along the loop γ_2 is the conjugation of the group element u under the *dynamical field* λ evaluated at the base point x_2 of the holonomy, namely $\lambda_2 = \lambda(x_2)$. This suggests an identification of the group element h conjugating u with the value of the string field λ_2, which leads to a relation between the holonomy g_β along a path β connecting the points x_1 and x_2 and the value of the string field λ at x_1 and x_2:

$$g_\beta = \lambda_{s(\beta)} {\lambda_{t(\beta)}}^{-1},$$

stating that λ is covariantly constant along the string. We have seen that the Bianchi identity together with the full curvature constraint induces the momentum density conservation. Our treatment of the reducibility issue consists in truncating the curvature constraint, i.e., in imposing $F = p$ only once, and using the Bianchi identity supplemented with the *momentum conservation* $Dp = 0$ to recover the truncated components of the curvature constraint without any loss of information.

Accordingly, the full prescription is defined via a choice of a closed, oriented path α and a finite set C of open, oriented paths β in $\partial\Delta^*$. The closed path α circles the string (it is non-contractible in the three-manifold Σ_η). This loop is based at a point $x \in X$ lying on a dual vertex $\overline{v} \in \partial\Delta^*$ supporting a spin network endpoint. The open paths $\beta \in C$ are defined as follows. Let $\overline{\gamma} \in \partial\Delta^*$ be an oriented loop based at x, non-homologous to α (along the cycle of T_η contractible in Σ_η) and connecting all the spin network endpoints $x_k \in X$. Define the open path γ by erasing the segment of $\overline{\gamma}$ supported by the edge $\bar{e}$ which is such that $x = t(\bar{e})$. The paths β of C are 1d sub-manifolds of γ each connecting x to a vertex $\bar{v}$ traversed by γ. If the graph Γ is closed, one reiterates the same prescription simply dropping the requirements on the spin network endpoints $x_k \in X$, in particular, the base point x is chosen arbitrarily. We then impose $g_\alpha = \exp p$ with p evaluated at the point x, and $g_\beta = \lambda_{s(\beta)} {\lambda_{t(\beta)}}^{-1}$, where $x = s(\beta)$, on each open path β of C.

To summarize, we choose a regulator $\mathcal{R}_{(\eta,\epsilon)} = (T_\eta, (\Delta_\epsilon, \partial\Delta_\epsilon), T, \alpha, C)$ consisting in a thickening T_η of the string, a cellular decomposition $(\Delta_\epsilon, \partial\Delta_\epsilon)$ of the manifold (Σ_η, T_η) adapted to the graph Γ, a maximal tree T of Δ, a closed path α in $\partial\Delta^*$, and a collection C of open paths β in $\partial\Delta^*$. The associated regularized physical scalar product is then given by

$$P[\Psi] := \lim_{\eta,\epsilon\to 0} P[\mathcal{R}_{(\eta,\epsilon)}; \Psi], \tag{45}$$

with

$$P[\mathcal{R}_{(\eta,\epsilon)}; \Psi] =< \ \Omega \, , \left[\prod_{f \notin T} \delta\left(g_f\right) \prod_{\alpha} \delta\left(g_\alpha \exp p\right) \prod_{\beta \in C} \delta(g_\beta \, \lambda_{t(\beta)} \, {\lambda_{s(\beta)}}^{-1}) \right] \Psi \ >, \tag{46}$$

where the product over α is to take into account the possible multiple connected components of the string.

It is important to point out that, in addition to the expression of the generalized projection above, we can use the regularization to give an explicit expression of the regularized constraint corresponding to $H[N]$ in equation (40). With the notation introduced so far the regulated quantum curvature constraint becomes

$$\hat{H}_{\eta,\epsilon}[N] = \sum_{f \in \Delta^*} \mathrm{Tr}[N(x_f) g_f] + \sum_{\alpha} \mathrm{Tr}[N(x_p) g_\alpha \exp p], \tag{47}$$

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[9] To understand these last points, consider two loops γ_1 and γ_2 belonging to the same homology class circling the string at two neighboring points, say x_1 and x_2. These two loops define a section of the torus T_η homeomorphic to a cylinder. Suppose that the dual cellular complex Δ^* is such that the cylinder is discretized by a single face with two opposite edges glued along an dual edge β in $\partial\Delta^*$ connecting x_1 and x_2. The flatness constraint on the boundary of the cylinder implies the following presentation of the cylinder's fundamental polygon:

$$g_{\gamma_1} g_\beta g_{\gamma_2}^{-1} g_\beta^{-1} = \mathbb{1},$$

which relates the holonomies g_{γ_1} and g_{γ_2} by virtue of the Bianchi identity in the interior of the tube. Hence, imposing $F = p$ only along one of the two loops, say γ_1, naturally leads to the constraint $g_{\gamma_2} \in Cl(u)$. Finally, plugging the relation $g_\beta = \lambda_1 \lambda_2^{-1}$ in the value of the holonomy g_{γ_2} leads to the required constraint $g_{\gamma_2} = \lambda_2 u \lambda_2^{-1}$.

where x_f is an arbitrary point in the interior of the face f and x_p and arbitrary point on the string dual to the loop α (the sum over α is over all the string components). It is easy to check that the regulated quantum curvature constraints satisfy off-shell anomaly freeness condition. For instance

$$U[g]\hat{H}_{\eta,\epsilon}[N]U^{\dagger}[g] = \hat{H}_{\eta,\epsilon}[gNg^{-1}], \tag{48}$$

where $U[g]$ is the unitary generator of G-gauge transformations. Therefore the regulator does not break the algebraic structure of the classical constraints. The quantization is consistent. The quantum constraint operator is defined as the limit where η and ϵ are taking to zero. Instead of doing this in detail we shall simply concentrate on the regulator independence of the physical inner product in the following section.

The inner product is computed using the AL measure, that is, by Haar integration along all edges of the graph $(\Delta^*, \partial\Delta^*) \ni \Gamma$ and along all endpoints $x_k \in X \subset \partial\Delta^*$ of the graph Γ.

We can now promote the classical delta function on the lattice phase space to a multiplication operator on $\mathcal{H}_{\text{kin}}$ by using its expansion in irreducible unitary representations :

$$\forall g \in G, \;\; \delta(g) = \sum_{\rho} \dim(\rho)\chi_{\rho}(g), \tag{49}$$

where $\chi_\rho \equiv \text{tr}\, \rho : G \to \mathbb{C}$ is the character of the representation ρ. Each χ_ρ is then promoted to a self-adjoint Wilson loop operator $\hat{\chi}_\rho$ on $\mathcal{H}_{\text{kin}}$ creating loops in the ρ representation around each plaquette defined by our regularization, which is charged for the face bounded by the loop α. To summarize, we have, for each face f of the regularization, a sum over the unitary, irreducible representations ρ_f of G, a weight given by the dimension $\dim(\rho_f)$ of the representation ρ_f summed over and a loop around the oriented boundary of the face in the representation ρ_f. See for instance [19], [20],[21] for details. This concludes our regularization of the transition amplitudes of string-like sources coupled to four dimensional BF theory.

Now, the physical inner product that we have constructed above depends manifestly on the regulating structure $\mathcal{R}_{(\eta,\epsilon)}$. To complete the procedure, we have to calculate the limit in which the regulating parameters η and ϵ go to zero.

IV. REGULATOR INDEPENDENCE

Throughout this section we will suppose that the cellular complex $(\Delta, \partial\Delta)$ is simplicial, i.e. a triangulation of (Σ_η, T_η). We will also make the simplifying assumption that $G = \text{SU}(2)$, the unitary irreducible representations of which will be noted $(\overset{j}{\pi}, \mathbb{V}_j)$, with the spin j in $\mathbb{N}/2$. The generalization to arbitrary cellular decompositions and arbitrary compact Lie groups can be achieved by using the same techniques that we develop below. If Δ is a now a regular triangulation, it cannot be adpated to any graph Γ. It can only be so for graphs with three-or-four-valent vertices. Now, we can always decompose a n-valent intertwiner, with $n > 3$, into a three-valent intertwiner by using repeatedly the complete reducibility of the tensor product of two representations [10]. We will therefore decompose all n-valent interwiners ι_v with $n > 3$ into three-valent vertices.

We now show how to remove the regulator $\mathcal{R}_{(\eta,\epsilon)}$ from the regularized scalar product (45). Instead of computing the $\eta, \epsilon \to 0$ limit, we demonstrate that the transition amplitudes are in fact independent of the regulator. To prove such a statement, we show that the expression (45) does not depend on any component of the regulator. The transition amplitudes are proven to be invariant under any finite combination of elementary moves called regulator moves $\mathcal{R} : \mathcal{R}_{(\eta,\epsilon)} \mapsto \mathcal{R}'_{(\eta,\epsilon')}$, where each regulator move is a combination of elementary moves acting on the components of the regulator :

- Bulk and boundary (adapted) Pachner moves $(\Delta, \partial\Delta) \mapsto (\Delta', \partial\Delta')$,
- Elementary maximal tree moves $T \mapsto T'$,
- Elementary curve moves $\gamma \mapsto \gamma'$.

[10] One can show that the amplitudes obtained using the three-valent decomposition (where the virtual edges are assumed to be real) are identical to the ones obtained using a suitable non-simplicial cellular decomposition adapted to spin networks with arbitrary valence vertices.

We will see how the invariance under the above moves also implies an invariance under dilatation/contraction of the string thickening radius η.

To conclude on the topological invariance of the amplitudes from the above elementary moves, we will furthermore prove that the transition amplitudes are invariant under elementary moves acting on the string spin network graph $\mathcal{G} : \Gamma \mapsto \Gamma'$ which map ambient isotopic PL-graphs into ambient isotopic PL-graphs.

We now detail the regulator and graph topological moves.

A. Elementary regulator moves

The regulator moves are finite combinations of the following elementary moves acting on the simplical complex $(\Delta, \partial\Delta)$, the maximal tree T and the paths α and $\beta \in C$.

1. *Adapted Pachner moves*

The first invariance property that we will need is that under moves acting on the simplicial-pair $(\Delta, \partial\Delta)$, leaving the one-complex Γ *invariant* and mapping a Γ-adapted triangulation into a PL-homeomorphic Γ-adapted simplicial structure. We call these moves *adapted* Pachner moves. There are two types of moves to be considered : the bistellar moves [28], acting on the bulk triangulation Δ and leaving the boundary simpicial structure unchanged, and the elementary shellings [29], deforming the boundary triangulation $\partial\Delta$ with induced action in the bulk.

a. Bistellar moves. There are four bistellar moves in three dimensions: the $(1, 4)$, the $(2, 3)$ and their inverses. In the first, one creates four tetrahedra out of one by placing a point p in the interior of the original tetrahedron whose vertices are labeled p_i, $i = 1, ..., 4$, and by adding the four edges (p, p_i), the six triangles $(p, p_i, p_j)_{i\neq j}$, and the four tetrahedra $(p, p_i, p_j, p_k)_{i\neq j\neq k}$. The $(2, 3)$ move consists in the splitting of two tetrahedra into three : one replaces two tetrahedra (u, p_1, p_2, p_3) and (d, p_1, p_2, p_3) (u and d respectively refer to 'up' and 'down') glued along the (p_1, p_2, p_3) triangle with the three tetrahedra $(u, d, p_i, p_j)_{i\neq j}$. The dual moves, that is the associated moves in the dual triangulation, follow immediately. See FIG. 2.

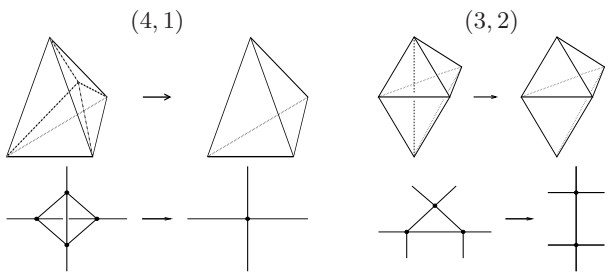


FIG. 2: The $(4, 1)$ and $(3, 2)$ bistellar moves.

b. Elementary shellings. Since the manifold (Σ_η, T_η) has non-empty boundary, extra topological transformations have to be taken into account to prove discretization independence. These operations, called elementary shellings, involve the cancellation of one 3-simplex at a time in a given triangulation $(\Delta, \partial\Delta)$. In order to be deleted, the tetrahedron must have some of its two-dimensional faces lying in the boundary $\partial\Delta$. The idea is to remove three-simplices admitting boundary components such that the boundary triangulation admits, as new triangles after the move, the faces along which the given tetrahedron was glued to the bulk simplices. Moreover, for each elementary shelling there exists an inverse move which corresponds to the attachment of a new three-simplex to a suitable component in $\partial\Delta$. These moves correspond to bistellar moves on the boundary $\partial\Delta$ and there are accordingly three distinct moves for a three-manifold with boundary, the $(3, 1)$, its inverse and the $(2, 2)$ shellings, where the numbers (p, q) here correspond to the number of two-simplices of a given tetrahedron lying on the boundary triangulation. In the first, one considers a tetrahedron admitting three faces lying in $\partial\Delta$ and erases it such that the remaining boundary component is the unique triangle which did not belong to the boundary before the move. The inverse move follows immediately. The $(2, 2)$ shelling consists in removing a three-simplex intersecting the boundary along two of its triangles such that, after the move, $\partial\Delta$ contains the two remaining faces of the given tetrahedron. These shellings and the associated boundary bistellars are depicted in FIG.3.

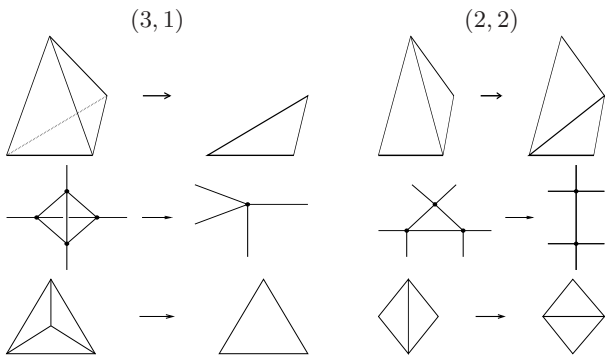


FIG. 3: The $(3,1)$ and $(2,2)$ shellings, their dual moves and the associated boundary bistellars.

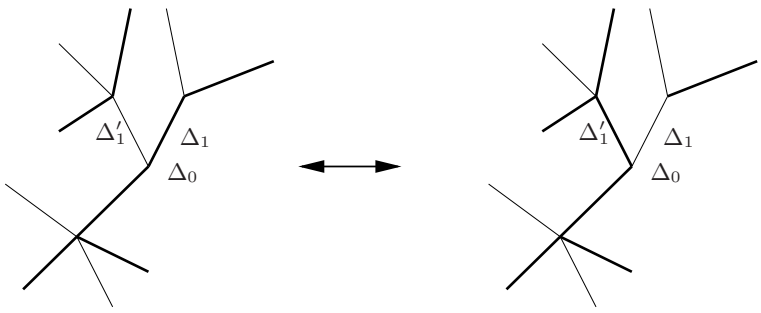
The subset of bistellar moves and shellings which map Γ-adapted triangulations into Γ-adapted triangulations will be called adapted Pachner moves, and, considering a local simplicial structure $\mathcal{T}_k = \cup_{n=1}^{k} \Delta_3^n$, $k = 1, ..., 4$, an adapted (p,q) Pachner move $\mathcal{T}_p \mapsto \mathcal{T}_q$ will be noted $\mathcal{P}_{(p,q)}$, or more generally $\mathcal{P}$. Any two PL-homeomorphic, Γ-adapted triangulations $(\Delta, \partial\Delta)$ of the PL-pair (Σ_η, T_η) are related by a finite sequence of such moves.

Here it is important to take into account the PL-embedding of the string spin network graph Γ in the (dual) triangulation $(\Delta, \partial\Delta)$. We will call $\Gamma_k = \Gamma \cap \mathcal{T}_k^*$ the restrictions of the graph Γ to the local simplex configurations $\mathcal{T}_k$ appearing in the moves. If the graph Γ_k is not the null graph, we will consider that it is open and does not contain any loop. If this was not the case, the set of adapted moves would reduce to the identity move, under which the transition amplitudes are obviously invariant. Hence, Γ_k, if at all, can only be either an edge, or more generally a collection of edges, either a (three-valent) vertex. The associated string spin network functional Ψ_{Γ_k} will be represented by a group function ϕ_k, which is the constant map $\phi_k = 1$ if Γ_k is the null graph. We will make sure to check that, under a (p,q) Pachner move, ϕ_k transforms as $\phi_p \mapsto \phi_q$.

2. *Maximal tree moves*

It is also necessary to define topological moves for the trees [22], [26]. Any two homologous trees T_1 and T_2 are related by a finite sequence of the following elementary tree moves $\mathcal{T} : T_1 \mapsto T_2$.

Definition 1 (Tree move) *Considering a vertex Δ_0 belonging to a tree T, choose a pair of edges Δ_1, Δ_1' in Δ touching the vertex Δ_0 such that Δ_1 is in T, Δ_1' is not in T and such that Δ_1' combined to the other edges of T does not form a loop. The move $\mathcal{T}$ consists in erasing the edge Δ_1 from T and replacing it by Δ_1'.*



There is another operation on trees that we need to define. When acting on the simplicial complex $(\Delta, \partial\Delta)$ with a bistellar move or a shelling, one can possibly map $(\Delta, \partial\Delta)$ into a simplicial complex $(\Delta', \partial\Delta')$ with a different number of vertices. Hence, a maximal tree T of Δ is not necessarily a maximal tree of Δ'; the Pachner moves have a residual action on the trees. This leads us to define the notion of maximal tree extension (or reduction) accompanying Pachner moves modifying the number of vertices of the associated simplicial complex.

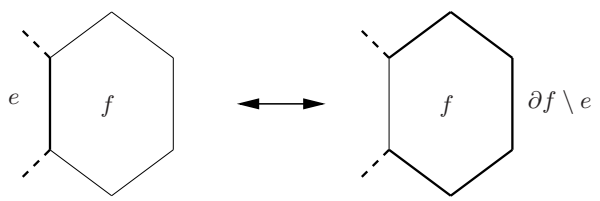
Definition 2 (Tree extension or reduction) *An extended (or reduced) tree T, associated to a Pachner move $\mathcal{P} : \Delta \mapsto \Delta'$ modifying locally the number of vertices of a simplicial complex Δ, is a maximal tree of Δ' obtained from a maximal tree T of Δ by adding (or removing) the appropriate number of edges to T as a mean to transform T into a maximal tree $T_{\mathcal{P}}$ of Δ'.*

Obviously, there is an ambiguity in the operation of tree extension or reduction. But, because of the fact that the regularized physical inner product will turn out to be independent of a choice of maximal tree, there will be no trace of this ambiguity in the computations of the transition amplitudes.

3. Curve moves

Finally, we define the PL analogue of the Reidemeister moves, which where in fact a crucial ingredient in the proof of Reidemeister's theorem. Any two ambient isotopic PL embeddings γ_1 and γ_2 of a curve γ in the dual complex $(\Delta^*, \partial\Delta^*)$ are related by a finite sequence of the following elementary topological moves $\mathcal{C} : \gamma_1 \mapsto \gamma_2$.

Definition 3 (Curve move) *Consider a PL path γ lying along the p boundary edges $e_1, e_2, ..., e_p$ of a two-cell f of the dual pair $(\Delta^*, \partial\Delta^*)$, where f has no other edges nor vertices traversed by the curve γ. Erase the path γ along the edges $e_1, e_2, ..., e_p$ and add a new curve along the complementary $\partial f \setminus \{e_1, e_2, ..., e_p\}$ of the erased segment in f.*



We have now defined all of the elementary regulator moves. To summarize, an elementary regulator move $\mathcal{R} : \mathcal{R}_{(\eta,\epsilon)} \mapsto \mathcal{R}'_{(\eta,\epsilon')}$ is a finite combination of all the above moves : $\mathcal{R}[\mathcal{R}_{(\eta,\epsilon)}] = (T_\eta, \mathcal{P}[(\Delta, \partial\Delta)], \mathcal{T}[T_{\mathcal{P}}], \mathcal{C}[\alpha], \mathcal{C}[C])$. Proving the invariance of the regularized physical inner product (45) under all of these elementary regulator moves is equivalent to showing the independence of the regulating structure $\mathcal{R}_{(\eta,\epsilon)}$. Note however that we have not included contractions or dilatations of the string tube T_η radius η in the regulator moves. This is because the invariance under shellings implies the invariance under increasing or decreasing of η. Indeed, the bistellars and shellings are the simplicial analogues of the action of the homeomorphisms $\mathrm{Homeo}[(\Sigma_\eta, T_\eta)]$. In particular, the topological group $\mathrm{Homeo}[(\Sigma_\eta, T_\eta)]$ contains transformations deforming continuously the boundary T_η, like for instance maps decreasing or increasing the (non-contractible) radius $\eta > 0$ of the boundary torus T_η. Hence, showing the invariance under elementary shellings is sufficient to prove the independence on the string thinckening radius, and the moves defined above are sufficient to conclude on the regulator independence of the definition of the regularized physical inner product.

To push the result further and conclude on the topological invariance of the transition amplitudes, we need extra ingredients that we define in the following section.

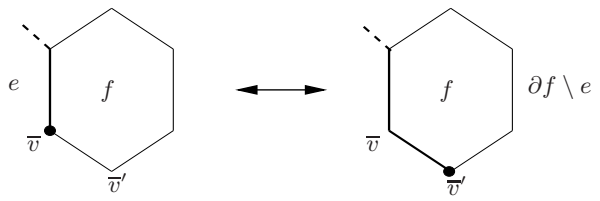
B. String spin network graph moves

We now introduce the following elementary moves respectively acting on the edges and vertices of the open graph Γ. All ambient isotopic PL embeddings of the one complex Γ are related by a finite sequence of the following elementary moves noted $\mathcal{G}$.

Definition 4 (Edge move) *An edge move is a curve move applied to an edge e_Γ of the graph Γ.*

Note that these moves apply also to the open edges of the graph Γ. However, there exists other moves which displace the endpoints.

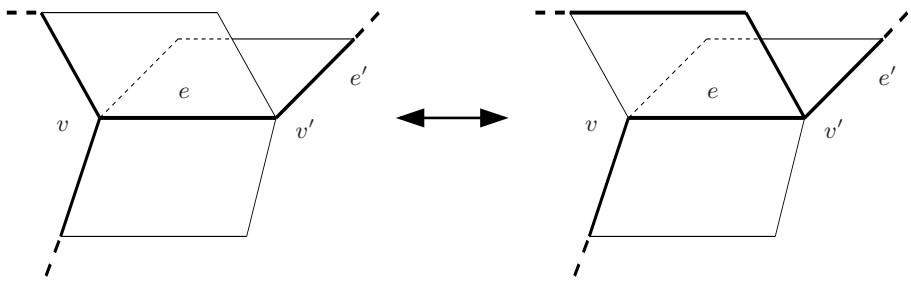
Definition 5 (Endpoint move) *Considering an open string spin network edge e_Γ ending on the point $x_k \in X$ supported by a dual vertex $\overline{v}$ of $\partial\Delta^*$, which is such that its neighbouring vertex $\overline{v}'$ not touched by e_Γ belongs to $\partial\Delta^*$, an endpoint move consist in adding a section to e_Γ connecting $\overline{v}$ to $\overline{v}'$.*



We also need similar moves for the vertices.

Definition 6 (Vertex translation) *Let v_Γ denote a three-valent spin network vertex sitting on the vertex v of the dual complex $(\Delta^*, \partial\Delta^*)$. Choose one edge e_Γ among the three edges emerging from v_Γ and call v' the dual vertex adjacent to v which is traversed by e_Γ. Call $e, e' \subset (\Delta^*, \partial\Delta^*)$ the dual edges locally supporting e_Γ, i.e, such that $\partial e = \{v, v'\}$ and $v' = e_\Gamma \cap (e \cap e')$.*

The move consists in translating the vertex v_Γ along e from v to v'. This is achieved by choosing one dual face sharing the dual edge e and not containing the dual edge e', and acting upon it with the edge move.



Note that the use of rectangular faces in the above picture is only for the clarity of the picture, the move is defined for faces of arbitrary shape.

It is important to remark that the above moves respect the topological structure of the embedding because no discontinuous transformations are allowed and because the number and nature of the crossings are preserved since the faces used to define the moves are required to have empty intersections with the string or graph a part from the specified ones.

The combination of the adapted Pachner moves and the spin network moves are the simplicial analogues of the action of the homeomorphisms $\mathrm{Homeo}[(\Sigma_\eta, T_\eta)]$ on the triple $((\Sigma_\eta, T_\eta), \Gamma)$.

C. Invariance theorem

We can now prove the following theorem.

Theorem 1 (Invariance theorem) *Let Ψ_Γ denote a string spin network element of a given basis of $\mathcal{H}_{kin}$ defined with respect to the one-complex Γ. Choose a regulator $\mathcal{R}_{(\eta,\epsilon)} = (T_\eta, (\Delta_\epsilon, \partial\Delta_\epsilon), T, \alpha, C)$ consisting in a thickening T_η of the string, a cellular decomposition $(\Delta_\epsilon, \partial\Delta_\epsilon)$ of the manifold (Σ_η, T_η) adapted to the graph Γ, a maximal tree T of Δ, a closed path α in $\partial\Delta^*$, and a collection C of open paths β in $\partial\Delta^*$. Let $\mathcal{R} : \mathcal{R}_{(\eta,\epsilon)} \mapsto \mathcal{R}'_{(\eta,\epsilon')}$ (resp. $\mathcal{G} : \Gamma \mapsto \Gamma'$) denote an elementary regulator move (resp. a string spin network move). The evaluated linear form (46) is invariant under the action of $\mathcal{R}$ and $\mathcal{G}$:*

$$\begin{aligned} P[\mathcal{R}_{(\eta,\epsilon)}; \Psi_\Gamma] &= P[\mathcal{R}[\mathcal{R}_{(\eta,\epsilon)}]; \Psi_\Gamma] \\ &= P[\mathcal{R}_{(\eta,\epsilon)}; \Psi_{\mathcal{G}(\Gamma)}]. \end{aligned} \tag{50}$$

Proof. We proceed by separatly showing the invariance under each elementary regulator moves, before proving the invariance under graph moves.

- *Invariance under maximal tree moves.*

  Here, we simply apply to the proof of invariance under maximal tree moves writen in [26]. Firstly, we need to endow the tree T of the left hand side of the move with a partial order. To this aim, we pick a distinguished vertex r of T, chosen to be the other vertex of the edge Δ'_1. The rooted tree (T, r)

thus acquires a partial order $\preceq$: a vertex Δ_0' of (T,r) is *under* a vertex Δ_0, $\Delta_0' \preceq \Delta_0$, if it lies on the unique path connecting r to Δ_0. We can now define the tree T_{Δ_0} to be the subgraph of T connecting all the vertices *above* Δ_0 : Δ_0 is the root of T_{Δ_0}. The second ingredient that we need is the notion of Bianchi identity (44) applied to trees. Indeed, if T is a tree of the (regular) simplicial complex Δ, its tubular neighborhood has the topology of a 3-ball and its boundary has the topology of a 2-sphere. This surface $\mathcal{S}$ can be built as the union of the faces f dual to the edges Δ_1 in Δ touching the vertices of T without belonging to T. Hence, applying the Bianchi identity to the tree T_{Δ_0} yields

$$g_{f_0'} = \left(\prod_{f\in\mathcal{S}_1} g_f\right) g_{f_0} \left(\prod_{f\in\mathcal{S}_2} g_f\right). \tag{51}$$

Here, f_0 and f_0' are the faces dual to the segments Δ_0 and Δ_0' (note that Δ_0 does not belong to T_{Δ_0}). The sets $\mathcal{S}_1$, $\mathcal{S}_2$ are the set of faces dual to the segments Δ_1 touching the vertices of T_{Δ_0} without belonging to T_{Δ_0}, and which are not f_0 nor f_0'. The presence of two different sets $\mathcal{S}_1$ and $\mathcal{S}_2$ is simply to take into account the arbitrary positioning of the group element g_{f_0} amoung the product over all faces. As usual, the group elements are defined up to orientation and conjugation. Next, we apply a delta function to both sides of the above equation and multiply the result as follows :

$$\begin{aligned} \prod_{\substack{f\,\notin\,T\\ f\,\neq\,f_0,\,f_0'}} \delta(g_f)\,\delta(g_{f_0'}) &= \prod_{\substack{f\,\notin\,T\\ f\,\neq\,f_0,\,f_0'}} \delta(g_f)\,\delta\left[\left(\prod_{f\in\mathcal{S}_1} g_f\right) g_{f_0} \left(\prod_{f\in\mathcal{S}_2} g_f\right)\right] \\ \prod_{f\notin T} \delta(g_f) &= \prod_{f\notin T'} \delta(g_f). \end{aligned} \tag{52}$$

In the second step, we have simply used the delta functions with which the expression has been multiplied to set the group elements associated to the sets $\mathcal{S}_1$ and $\mathcal{S}_2$ to the identity (the faces of $\mathcal{S}_1$ and $\mathcal{S}_2$ are dual to segments not belonging to T). One can then check that the various steps of the proof remain valid if the boundaries of the dual faces carry string spin networks. This shows the invariance of the regularized inner product (46) under maximal tree move □

- *Invariance under adapted Pachner moves.*

  To prove the invariance under Pachner moves, we introduce a simplifying lemma [23], [24].

Lemma 1 (Gauge fixing identity) *To each vertex of the dual triangulation Δ^* are associated four group elements $\{g_a\}_{a=1,\ldots,4}$, six unitary irreducible representations $\{j_{ab}\}_{a<b=1,\ldots 4}$ of G and a string spin network function $\phi_1(\{g_a\}_{a=1,\ldots,4})$. If ϕ_1 is the constant map $\phi_1 = 1$, or depends on its group arguments only through monomial combinations $\{g_a g_b\}_{a\neq b}$ of degree two, then the following identity holds :*

$$\prod_{b>a=1}^{4} \int_G dg_a \overset{jab}{\pi}(g_a g_b)\,\phi_1(\{g_d\}_d) = \prod_{b>a=1}^{4} \int_G dg_a\,\delta(g_c) \overset{jab}{\pi}(g_a g_b)\,\phi_1(\{g_d\}_d), \tag{53}$$

for $c = 1,2,3$ or 4.

Proof of Lemma 1. The above equality is trivially proven by using the invariance of the Haar measure and performing the change of variables $\gamma_{cb} = g_c g_b$, for $c < b$ (resp. $\gamma_{bc} = g_b g_c$, for $c > b$) in the left hand side. This translation is always possible since the group function ϕ is either the constant map or depends on the group elements only through monomials of degree two □

Let us comment here on the validity of the hypothesis made on the spin network function ϕ_1 associated to the graph $\Gamma_1 = \Gamma \cap \mathcal{T}_1$, with $\mathcal{T}_1 = \Delta_3$ in the above Lemma. In fact ϕ_1 depends necessarily on combinations of the form $g_a g_b$ locally if the graph Γ_1 is not the null graph. Indeed, Γ_1 can either be a collection of edges, in which case this requirement simply states that the edges are open, either a vertex, where one can always use the invariance of the associated intertwining operator to satisfy the desired assumption. Hence, this requirement is always locally satisfied.

We can now show the invariance under bistellar moves and shellings.

  – Bistellars moves:

* **The** $(4,1)$ **move.** Consider the four simplices configuration $\mathcal{T}_4$ in $(\Delta, \partial\Delta)$ (FIG. 2). Since the amplitudes do not depend on the maximal tree T of Δ, we are free to chose it. The simplest choice consists in considering a maximal tree T whose intersection T_4 with the simplex configuration $\mathcal{T}_4$ reduces to the four external vertices and to a single one-simplex touching the central vertex. We work in the dual picture and label the four external dual edges from one to four. The face dual to the internal tree segment is chosen to be the face 142. We note g_a and h_{ab}, $a, b = 1, ..., 4$, the group elements [11] associated to the external and internal dual edges respectively, while the representations assigned to the dual faces are noted j_{ab}. The general PL string spin network state restricted to the configuration $\mathcal{T}_4^*$ is noted $\phi_4(\{g_a\}_a, \{h_{ab}\}_{a<b})$. Obviously, ϕ_4 is not a function of all of its ten arguments, otherwise it would contain a loop, but can generally depend on any one of these ten group elements, as suggested by the notation. The regularized physical inner product (46) restricted to these four simplices yields

$$\begin{aligned}&\int_{G^{10}} dg_1 dg_2 dg_3 dg_4\, dh_{12} dh_{13} dh_{14} dh_{23} dh_{24} dh_{34} \\ &\overset{j_{12}}{\pi}(g_1 h_{12} g_2)\, \overset{j_{13}}{\pi}(g_1 h_{13} g_3)\, \overset{j_{14}}{\pi}(g_1 h_{14} g_4)\, \overset{j_{23}}{\pi}(g_2 h_{23} g_3)\, \overset{j_{24}}{\pi}(g_2 h_{24} g_4)\, \overset{j_{34}}{\pi}(g_3 h_{34} g_4) \\ &\delta(h_{12} h_{23} h_{31})\, \delta(h_{13} h_{34} h_{41})\, \delta(h_{23} h_{34} h_{24})\ \phi_4(\{g_a\}_a, \{h_{ab}\}_{a<b}),\end{aligned} \tag{54}$$

where we have omitted the sum over representations weighted by the associated dimensions. We start by implementing Lemma 1 at vertices 2, 3 and 4 to eliminate the three variables h_{1b}, $b \neq 1$. We obtain

$$\begin{aligned}&\int_{G^7} dg_1 dg_2 dg_3 dg_4\, dh_{23} dh_{24} dh_{34} \\ &\overset{j_{12}}{\pi}(g_1 g_2)\, \overset{j_{13}}{\pi}(g_1 g_3)\, \overset{j_{14}}{\pi}(g_1 g_4)\, \overset{j_{23}}{\pi}(g_2 h_{23} g_3)\, \overset{j_{24}}{\pi}(g_2 h_{24} g_4)\, \overset{j_{34}}{\pi}(g_3 h_{34} g_4) \\ &\delta(h_{23})\, \delta(h_{34})\, \delta(h_{23} h_{34} h_{24})\ \phi(\{g_a\}_a, \{h_{23}, h_{34}, h_{24}\}) \\ =\ &\int_{G^4} dg_1 dg_2 dg_3 dg_4\ \overset{j_{12}}{\pi}(g_1 g_2)\, \overset{j_{13}}{\pi}(g_1 g_3)\, \overset{j_{14}}{\pi}(g_1 g_4) \\ &\overset{j_{23}}{\pi}(g_2 g_3)\, \overset{j_{24}}{\pi}(g_2 g_4)\, \overset{j_{34}}{\pi}(g_3 g_4)\ \phi_1(\{g_a\}_a),\end{aligned} \tag{55}$$

where we have integrated over the delta functions to eliminate the interior variables in the second step. The right hand side of the above equality corresponds to the one simplex configuration of the $(4,1)$ move with the associated maximal tree reduction (the obvious removal of the internal tree segment; T_1 is given by the four vertices of the resulting tetrahedron). In other words, we have just proved the invariance under the transformation $\mathcal{P}_{(4,1)} : (\mathcal{T}_4, T_4) \mapsto (\mathcal{T}_1, T_1')$, with $T_k = T \cap \mathcal{T}_k$ and $T' = T_{\mathcal{P}_{(4,1)}}$.

* **The** $(3,2)$ **move.** Here, we consider the three simplices configuration $\mathcal{T}_3$ (see FIG. 2) and chose a tree T intersecting $\mathcal{T}_3$ only on its five vertices. Concentrating on the dual graph, we label the three vertices from one to three and respectively note g_a^α, h_{ab}, and $j_{ab}^{\alpha\beta}$, $a, b = 1, ..., 3$, $\alpha, \beta = 1, 2$, the external and internal group elements, and the representation labels. The associated string spin network state is called ϕ_3. The transition amplitude, restricted to these three simplices, yields (omitting the sum over representations and associated dimensions)

$$\begin{aligned}&\int_{G^9} dg_1^1 dg_1^2 dg_2^1 dg_2^2 dg_3^1 dg_3^2\, dh_{12} dh_{13} dh_{23} \\ &\overset{j_{12}^{11}}{\pi}(g_1^1 h_{12} g_2^1)\, \overset{j_{13}^{11}}{\pi}(g_1^1 h_{13} g_3^1)\, \overset{j_{23}^{11}}{\pi}(g_2^1 h_{23} g_3^1)\, \overset{j_{12}^{22}}{\pi}(g_1^2 h_{12} g_2^2)\, \overset{j_{13}^{22}}{\pi}(g_2^1 h_{13} g_3^2)\, \overset{j_{23}^{22}}{\pi}(g_2^2 h_{23} g_3^2) \\ &\overset{j_{11}^{12}}{\pi}(g_1^1 g_1^2)\, \overset{j_{22}^{12}}{\pi}(g_2^1 g_2^2)\, \overset{j_{33}^{12}}{\pi}(g_3^1 g_3^2)\, \delta(h_{12} h_{23} h_{13})\ \phi_3(\{g_a\}_a, \{h_{ab}\}_{a<b}).\end{aligned} \tag{56}$$

Using the gauge fixing identity at the vertices 2 and 3 to eliminate the variables h_{1a}, $a \neq 1$,

[11] The notations takes into account the appropriate orientations.

and solving for the delta function leads to

$$\begin{aligned}
&\int_{G^6} dg_1^1 dg_1^2 dg_2^1 dg_2^2 dg_3^1 dg_3^2 \\
&\overset{j_{12}^{11}}{\pi}(g_1^1 g_2^1)\overset{j_{13}^{11}}{\pi}(g_1^1 g_3^1)\overset{j_{23}^{11}}{\pi}(g_2^1 g_3^1)\overset{j_{12}^{22}}{\pi}(g_1^2 g_2^2)\overset{j_{13}^{22}}{\pi}(g_2^2 g_3^2)\overset{j_{23}^{22}}{\pi}(g_2^2 g_3^2) \\
&\overset{j_{11}^{12}}{\pi}(g_1^1 g_1^2)\overset{j_{22}^{12}}{\pi}(g_2^1 g_2^2)\overset{j_{33}^{12}}{\pi}(g_3^1 g_3^2)\,\phi(\{g_a\}_a) \\
=&\int_{G^7} dg_1^1 dg_1^2 dg_2^1 dg_2^2 dg_3^1 dg_3^2\, dh \\
&\overset{j_{12}^{11}}{\pi}(g_1^1 g_2^1)\overset{j_{13}^{11}}{\pi}(g_1^1 g_3^1)\overset{j_{23}^{11}}{\pi}(g_2^1 g_3^1)\overset{j_{12}^{22}}{\pi}(g_1^2 g_2^2)\overset{j_{13}^{22}}{\pi}(g_2^2 g_3^2)\overset{j_{23}^{22}}{\pi}(g_2^2 g_3^2) \\
&\overset{j_{11}^{12}}{\pi}(g_1^1 h g_1^2)\overset{j_{22}^{12}}{\pi}(g_2^1 h g_2^2)\overset{j_{33}^{12}}{\pi}(g_3^1 h g_3^2)\,\phi_2(\{g_a\}_a, h),
\end{aligned} \tag{57}$$

where we have used the inverse gauge fixing identity in the last step. This expression corresponds to the two simplices configuration $\mathcal{T}_2$ of the $(3,2)$ move.

– Shellings :

∗ **The** $(3,1)$ **move.** Remarkably, writting the amplitudes associated to the left and right hand sides of the $(3,1)$ shelling leads to the same expression than the $(4,1)$ bistellar, even if the geometrical interpretation is obviously different. This is due to the fact that we are imposing the flatness constraint $F=0$ also on the faces of the boundary [12] simplicial complex $\partial\Delta$ and integrating also on the boundary edges. The only difference is in the presence of possible open string spin network edges reflected in the group function $\phi = \phi(\{g_a\}_a, \{\lambda_a\}_a)$, without any incidence an any steps of the proof given for the $(4,1)$ bistellar. Accordingly, the proof of invariance under the $(3,1)$ shelling is the one sketched above.

∗ **The** $(2,2)$ **move.** The same remark applies here, the amplitudes are exactly identical to the ones of the $(3,2)$ bistellar.

Accordingly, we have proven the invariance under adapted Pachner moves □

- *Invariance under curve moves.* We here show that the regularized physical inner product is invariant under curve moves. The proof uses the flatness constraint $F=0$. Consider a particular dual face f of $(\Delta^*, \partial\Delta^*)$ containing n boundary edges positively oriented from vertex 1 to vertex n. Suppose that there are $p<n$ dual edges $e_1, ..., e_p$ supporting a curve positively oriented w.r.t the face f, to which is associated a spin j representation. We want to prove that the associated amplitude is equal to the amplitude corresponding to the curve lying along the $n-p$ edges of $\partial f \setminus \{e_1, ..., e_p\}$ after the edge move. We start from the initial configuration

$$\int_{G^n} \prod_{a=1}^{n} dg_a \overset{j}{\pi}(g_1...g_p)\,\delta(g_1...g_p g_{p+1}...g_n)\,\delta(g_1G_1)\delta(g_1H_1)...\,\delta(g_nG_n)\,\delta(g_nH_n), \tag{58}$$

where the capital letter G_a, H_a, $a=1,...,n$, represent the sequences of group elements associated to the two others faces sharing the edge a. We then simply integrate over the group element g_1 to obtain

$$\begin{aligned}
&\int_{G^{n-1}} \prod_{a=2}^{n} dg_a \overset{j}{\pi}(g_n^{-1}...g_{p+1}^{-1})\,\delta(g_n^{-1}...g_2^{-1}G_1)\,\delta(g_n^{-1}...g_2^{-1}H_1)...\,\delta(g_nG_n)\,\delta(g_nH_n) \\
=&\int_{G^n} \prod_{a=1}^{n} dg_a \overset{j}{\pi}(g_n^{-1}...g_{p+1}^{-1})\,\delta(g_1...g_p g_{p+1}...g_n)\,\delta(g_1G_1)\,\delta(g_1H_1)...\,\delta(g_nG_n)\,\delta(g_nH_n).
\end{aligned} \tag{59}$$

Note the reversal of orientations intrinsic to the move. This closes the proof of invariance under curve move □

We finish the proof of theorem 1 by showing the second part, namely the invariance of the transition amplitudes under string spin network graph moves.

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[12] In this sense, the boundary amplitudes are very different from a $2+1$ quantum gravity model defined on an open manifold.

- *Invariance under edge moves.*

  The proof is the one given for the curve move □

- *Invariance under endpoint moves.*

  Here, we use the momentum conservation $Dp = 0$. Considering a particular dual face f containing n boundary edges, with $p < n$ dual edges $e_1, ..., e_p$ supporting an open string spin network edge (positively oriented w.r.t f) ending on the boundary of the edge p, to which is associated a spin j representation. We call λ_k the string field evaluated at the target of the kth edge. We choose the holonomy starting point x to be on the endpoint of the p edge (we prove below that nothing depends on this choice) and, since nothing depends on the paths β by virtue of the invariance under curve moves, we choose a path β of C along the edge $p+1$. The relevant amplitude is given by

$$\int_{G^n} \prod_{a=1}^{n} dg_a d\lambda_p d\lambda_{p+1} \overset{j}{\pi} (g_1 ... g_p \lambda_p) \delta(g_{p+1} \lambda_{p+1} \lambda_p^{-1}), \tag{60}$$

  where the notations are the same than above. It is immediate to rewrite the above quantity as

$$\int_{G^n} \prod_{a=1}^{n} dg_a d\lambda_p d\lambda_{p+1} \overset{j}{\pi} (g_1 ... g_p g_{p+1} \lambda_{p+1}) \delta(g_{p+1} \lambda_{p+1} \lambda_p^{-1}), \tag{61}$$

  which concludes the proof of endpoint move invariance □

- *Invariance under vertex translations.*

  Here, we consider three dual face f_i, $i = 1, 2, 3$, of $(\Delta^*, \partial\Delta^*)$ each containing n_i boundary edges positively oriented from vertex 1 to vertex n_i. The three faces meet on the common edge e which is such that $1 = t(e)$, i.e., $e = e^i_{n_i}$, forall i. Suppose that there are $p_1 < n_1$ dual edges $e^1_1, ..., e^1_{p_1}$ (resp. $p_2 < n_2$ dual edges $e^2_1, ..., e^2_{p_2}$) of the face f_1 (resp. f_2) supporting a string spin network edge e^1_Γ (resp. e^2_Γ) colored by a spin j_1 (resp. j_2) representation and oriented negatively w.r.t the orientation of f_1 (resp. f_2). Suppose also that the face f_3 contains $n_3 - p_3$ dual edges $e^3_{p_3+1}, ..., e$ along which lies a positively oriented (w.r.t. to the orientation of the face) string spin network edge e^3_Γ colored by a spin j_3 representation. Consider that the three edges meet on the vertex v_Γ supported by the vertex 1 of $(\Delta^*, \partial\Delta^*)$. Noting g the group element associated to the common edge e, one can write the spin network function associated to the three valent vertex v_Γ lying on 1 and use the invariance property of the associated intertwining operator ι to 'slide' the vertex along the edge e :

$$\begin{aligned} & \overset{j_1}{\pi} ((g^1_{p_1})^{-1} (g^1_{p_1-1})^{-1} ... (g^1_1)^{-1}) \overset{j_2}{\pi} ((g^2_{p_2})^{-1} (g^2_{p_2-1})^{-1} ... (g^2_1)^{-1}) \\ & \overset{j_3}{\pi} (g^3_{p_3+1} ... g^3_{n_3-1} g) \ \iota_{j_1 j_2 j_3} \\ = & \overset{j_1}{\pi} ((g^1_{p_1})^{-1} ... (g^1_1)^{-1} g^{-1}) \overset{j_2}{\pi} ((g^2_{p_2})^{-1} ... (g^2_1)^{-1} g^{-1}) \overset{j_3}{\pi} (g^3_{p_3+1} ... g^3_{n_3-1}) \ \iota_{j_1 j_2 j_3}. \end{aligned} \tag{62}$$

  It is then possible to use the flatness constraint $F = 0$ on either of the faces f_1 or f_2 to implement an edge move on e^1_Γ or e^2_Γ thus completing the vertex move □

By virtue of all the above derivations, we have now fully proven theorem 1.

To be perfectly complete, we need to verify that the amplitudes are also independent under orientation and holonomy base point change. This leads to the following proposition.

Proposition 1 *The regularized physical inner product* (46) *is independent of the choice of orientations of the dual edges and faces of* $(\Delta^*, \partial\Delta^*)$*, and does not depend on the choice of holonomy base points.*

Proof of Proposition 1. It is immediate to see that the amplitudes do not depend on the orientations of the dual faces and dual edges of $(\Delta^* \partial\Delta^*)$, nor on the holonomy starting points on the boundaries of the dual faces [26]. Indeed, a dual face and dual edge orientation change correspond respectively to a change $g_e \mapsto g_e^{-1}$ and $g_f \mapsto g_f^{-1}$ which are respectively compensated by the invariance of the Haar measure, $dg_e = dg_e^{-1}$, and of the delta function: $\delta(g_f) = \delta(g_f^{-1})$. A change in the holonomy base point associated to a dual face f will have as a consequence the conjugation of the group element g_f by some element h in G. Since the delta function is central, $\delta(g_f) = \delta(h g_f h^{-1})$, the regularized physical inner product (46) will remain unchanged under a such transformation.

Concerning the base point x used to define the holonomies along the loop α and the paths $\beta \in C$ in (46), the situation is similar. Let x be noted x_1 and suppose that we change the point x_1 to another point x_2 in X neighbouring x_1. Since we have showen the invariance under bistellars and shellings, we are free to choose the simplest discretization [13] of the manifold (Σ_η, T_η). We choose it such that the cylindrical section of T_η between x_1 and x_2 is discretized by a single dual face with two opposite sides glued along a dual edge e linking x_1 to x_2. By virtue of the curve move invariance, we are also free to choose the path β to be along e. The amplitude based on x_2 as a starting point for the paths α and β, restricted to this section of T_η, yields

$$\int_{G^5} \prod_{a=1}^{2} dg_a d\lambda_a dg_\beta \, \delta(g_2\lambda_2 u \lambda_2^{-1}) \, \delta(g_\beta \lambda_1 \lambda_2^{-1}) \, \delta(g_2 g_\beta g_1^{-1} g_\beta^{-1}) \, f(\{g_a\}_a, \{\lambda_a\}_a, g_\beta), \tag{63}$$

where g_a is the holonomy around the disk bounding the tube section at the point x_a and the function f describes the string spin network function together with the other delta functions containing the group elements g_a and g_β. It is immediate to rewrite the above expression as

$$\int_{G^5} \prod_{a=1}^{2} dg_a d\lambda_a dg_\beta \, \delta(g_1\lambda_1 u \lambda_1^{-1}) \, \delta(g_\beta^{-1} \lambda_2 \lambda_1^{-1}) \, \delta(g_1^{-1} g_\beta^{-1} g_2 g_\beta) \, f(\{g_a\}_a, \{\lambda_a\}_a, g_\beta), \tag{64}$$

which is the amplitude based on x_1 as a starting point for the paths α and β $\square$

There are two major consequences due to the above theorem and proposition. First, there is no continuum limit to be taken in (45). Since the transition amplitudes are invariant under elementary regulator moves, the regularized physical inner product (46) is independent of the regulator and the expression (46) is consequently exact, there is no need to take the limits [14] $\epsilon, \eta \to 0$. In particular, we have shown that the amplitudes are invariant under any finite sequence of bistellar moves and shellings which implies, by Pachner's theorem, that the physical inner product is well defined and invariant on the equivalence classes of PL-manifolds [15] $(\Delta, \partial\Delta)$ up to PL-homeomorphisms. Accordingly, the transition amplitudes are invariant under triangulation change and thus under refinement. This leads to the second substantial consequence of theorem 1. The crucial point is that the equivalence classes of PL-manifolds up to PL-homeomorphism are in one-to-one correspondence with those of topological manifolds up to homeomorphism. See for instance [25] for details. Hence, showing the invariance of the regularized physical inner product under triangulation change is equivalent to showing homeomorphism invariance: the discretized expression (46) is in fact a topological invariant of the manifold (Σ_η, T_η). In particular, the amplitudes are invariant on the equivalence classes of boundary torii T_η up to homeomorphisms. It follows that they do not depend on the embedding of the string $\mathscr{S}$.

Combining these results with the second part of theorem 1 stating that the regularized physical inner product is invariant under string spin netwotk graph moves, we obtain the following corollary.

Corollary 1 *The physical inner product* (46) *is a topological invariant of the triple* $((\Sigma_\eta, T_\eta), \Gamma)$ *:*

$$P[\mathcal{R}_{(\eta,\epsilon)}; \Psi_\Gamma] = P[[(\Sigma_\eta, T_\eta)]; \Psi_{[\Gamma]}], \tag{65}$$

where $[(\Sigma_\eta, T_\eta)]$ *and* $[\Gamma]$ *denote the equivalence classes of topological open manifolds and one-complexes up to homeomorphisms and ambient isotopy respectively.*

This corollary concludes our study of the topological invariance of the theory of extended matter coupled to BF theory studied in this paper.

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[13] Anticipating on the next paragraph, we are using the fact that the invariance under Pachner moves implies the topological invariance of the amplitudes. Accordingly, we can use a cellular decomposition which is not necessarily a triangulation.

[14] More precisely, we have shown that (46) is invariant when going from $\mathcal{R}_{(\eta,\epsilon)}$ to $\mathcal{R}_{(\eta',\epsilon')}$ for all $(\eta', \epsilon') \neq (\eta, \epsilon)$ which implies regulator independence.

[15] More precisely, a triangulation Δ is not a PL-manifold. It is a combinatorial manifold which is PL-isomorphic to a PL-manifold.

V. CONCLUSION

In the first part of this paper we have studied the geometrical interpretation of the solutions of the BF theory with string-like conical defects. We showed the link between solutions of our theory and solutions of general relativity of the cosmic string type. We provided a complete geometrical interpretation of the classical string solutions and explained (by analyzing the multiple strings solution) how the presence of strings at different locations induces torsion. In turn torsion can in principle be used to define localization in the theory.

We have achieved the full background independent quantization of the theory introduced in [1]. We showed that the implementation of the dynamical constraints at the quantum level require the introduction of regulators. These regulators are defined by (suitable but otherwise arbitrary) space discretization. Physical amplitudes are independent of the ambiguities associated to the way this regulator is introduced and are hence well defined. There are other regularization ambiguities arising in the quantization process that have not been explicitly treated here. For an account of these as well as for a proof that these have no effect on physical amplitudes see [30].

The results of this work can be applied to the more general type of models introduced in [31], were it is shown that a variety of physically interesting 2-dimensional field theories can be coupled to the string world sheet in an consistent manner. An interesting example is the one where in addition to the degrees of freedom described here, the world sheet carries Yang-Mills excitations.

There is an intriguing connection between this type of topological theories and certain field theories in the 2+1 gravity plus particles case. One would expect a similar connection to exist in this case. However, due to the higher dimensional character of the excitations in this model this relationship allows for the inclusion of more general structures: only spin and mass is allowed in 2+1 dimensions. The study of the case involving Yang-Mills world-sheet degrees of freedom is of special interest. This work provides the basis for the computation of amplitudes in the topological theory. A clear understanding of the properties of string transition amplitudes should shed light on the eventual relation with field theories with infinitely many degrees of freedom.

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