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Non-central convergence of multiple integrals

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Abstract: Fix $\nu > 0$, denote by $G(\nu/2)$ a Gamma random variable with parameter $\nu/2$, and let $n \geq 2$ be an even integer. Consider a sequence $\{F_k\}_{k \geq 1}$ of square integrable random variables, belonging to the n th Wiener chaos of a given Gaussian process and with variance converging to 2ν . We prove that $\{F_k\}_{k \geq 1}$ converges in distribution to $2G(\nu/2) - \nu$, if, and only if, $E(F_k^4) - 12E(F_k^3) \rightarrow 12\nu^2 - 48\nu$. Observe that, if $\nu \geq 1$ is an integer, then $2G(\nu/2) - \nu$ has a centered χ^2 law with ν degrees of freedom. Our approach involves the techniques of Malliavin calculus recently developed by Nualart and Ortiz-Latorre (2007). We also obtain some multidimensional non-central limit theorems, as well as several equivalent conditions in terms of Malliavin derivatives and norms of contraction operators. Our results should be compared with the main findings by Nualart and Peccati (2005), where it is shown that a normalized sequence of multiple Wiener-Itô integrals converges in law to a Gaussian random variable if, and only if, the sequence of their fourth moments converges to 3.

Key words: Gaussian processes; Malliavin calculus; Multiple stochastic integrals; Non-central limit theorems; Weak convergence.

2000 Mathematics Subject Classification: 60F05; 60G15; 60H05; 60H07.

1 Introduction and main results

Let \mathfrak{H} be a real separable Hilbert space and, for $n \geq 1$, let $\mathfrak{H}^{\otimes n}$ (resp. $\mathfrak{H}^{\odot n}$) be the n th tensor product (resp. n th symmetric tensor product) of \mathfrak{H} . In what follows, we write

$$X = \{X(h) : h \in \mathfrak{H}\} \tag{1.1}$$

to indicate a centered isonormal Gaussian process on \mathfrak{H} . For every $n \geq 1$, we denote by I_n the isometry between $\mathfrak{H}^{\odot n}$ (equipped with the norm $\sqrt{n!} \|\cdot\|_{\mathfrak{H}^{\odot n}}$) and the n th Wiener chaos of X . Note that, if \mathfrak{H} is a σ -finite measure space with no atoms, then each random variable $I_n(h)$, $h \in \mathfrak{H}^{\odot n}$, has the form of a multiple Wiener-Itô integral of order n . For $n, m \geq 1$, $f \in \mathfrak{H}^{\odot n}$, $g \in \mathfrak{H}^{\odot m}$ and $p = 0, \dots, n \wedge m$, we denote by $f \otimes_p g \in \mathfrak{H}^{\otimes(n+m-2p)}$ and $f \tilde{\otimes}_p g \in \mathfrak{H}^{\odot(n+m-2p)}$, respectively, the p th contraction and the p th *symmetrized* contraction of f and g (a formal discussion of the properties of the previous objects is deferred to Section 2).

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It is customary to call “Central Limit Theorem” (CLT in the sequel) any result describing the weak convergence of a (normalized) sequence of nonlinear functionals of X towards a Gaussian law. Classic references for CLTs of this type are the works by Breuer and Major (1978), Major (1981), Giraitis and Surgailis (1985) and Chambers and Slud (1990); the reader is also referred to the survey by Surgailis (2003a) and the references therein. More recently, Nualart and Peccati (2005) proved the following result (here, and for the rest of the paper, we shall denote by $\mathcal{N}(0, 1)$ the law of a Gaussian random variable with zero mean and unit variance).

Theorem 1.1 *Fix an integer $n \geq 2$ and a sequence $\{f_k\}_{k \geq 1} \subset \mathfrak{H}^{\odot n}$ such that*

$$\lim_{k \rightarrow \infty} n! \|f_k\|_{\mathfrak{H}^{\otimes n}}^2 = \lim_{k \rightarrow \infty} E[I_n(f_k)^2] = 1. \quad (1.2)$$

Then, the following three conditions are equivalent:

- (i) $\lim_{k \rightarrow \infty} E[I_n(f_k)^4] = 3$;
- (ii) *for every $p = 1, \dots, n-1$, $\lim_{k \rightarrow \infty} \|f_k \otimes_p f_k\|_{\mathfrak{H}^{\otimes 2(n-p)}} = 0$;*
- (iii) *as $k \rightarrow \infty$, the sequence $\{I_n(f_k)\}_{k \geq 1}$ converges in distribution to $N \sim \mathcal{N}(0, 1)$.*

Theorem 1.1 is proved in Nualart and Peccati (2005) by means of a stochastic calculus result, known as the *Dambis, Dubins and Schwarz Theorem* (see e.g. Revuz and Yor (1999, Ch. V)). In particular, Theorem 1.1 implies that the convergence in distribution of a sequence of multiple stochastic integrals towards a Gaussian random variable is completely determined by the asymptotic behavior of their second and fourth moments. As such, Theorem 1.1 can be seen as a drastic simplification of the classic “method of moments and diagrams” (see for instance the previously quoted works by Breuer, Major, Giraitis, Surgailis, Chambers and Slud). In a subsequent paper, Peccati and Tudor (2005) proved a multidimensional version of Theorem 1.1 (see Theorem 5.1 below for a precise statement). Thanks to the chaotic representation property of Gaussian processes, this result has paved the way for the derivation of CLTs involving sequences of general functionals of X (not necessarily with the form of a multiple stochastic integral). Since then, the results by Nualart and Peccati (2005) and Peccati and Tudor (2005) have been applied to different frameworks, such as: p -variations of fractional stochastic integrals (Corcuera *et al.* (2006)), quadratic functionals of bivariate Gaussian processes (Deheuvels *et al.* (2006)), self-intersection local times of fractional Brownian motion (Hu and Nualart (2006)), approximation schemes for scalar fractional differential equations (Neuenkirch and Nourdin (2007)), high-frequency CLTs for random fields on homogeneous spaces (Marinucci and Peccati (2007a,b) and Peccati (2007)), needlets analysis on the sphere (Baldi *et al.* (2007)), and estimation of self-similarity orders (Tudor and Viens (2007)). In Peccati and Taqqu (2006) one can find some generalizations of Theorem 1.1 to the framework of the stable convergence towards mixtures of Gaussian random variables. The recent paper by Nualart and Ortiz-Latorre (2007) contains a crucial methodological breakthrough, showing that one can prove Theorem 1.1 (as well as its multidimensional extensions) by using exclusively results from Malliavin calculus, such as integration by parts formulae and the duality properties of Malliavin derivatives and Skorohod integral operators. In particular, Nualart and Ortiz-Latorre prove that, for every $n \geq 2$ and for every sequence $\{I_n(f_k)\}_{k \geq 1}$ satisfying (1.2), either one of conditions (i)-(iii) in Theorem 1.1 is equivalent to the following: as $k \rightarrow \infty$,

$$\|D[I_n(f_k)]\|_{\mathfrak{H}}^2 \longrightarrow n \text{ in } L^2, \quad (1.3)$$

where D is the usual Malliavin derivative operator (see Section 2).

The principal aim of this paper is to prove several non-central extensions of Theorem 1.1. Our main result is the following, which can be seen as a further simplification of the method of moments and diagrams, as applied to the framework of a non-Gaussian limit law. It should be compared with other non-central limit theorems for non-linear functionals of Gaussian fields, such as the ones proved by Taqqu (1975, 1979), Dobrushin and Major (1979), Fox and Taqqu (1985) and Terrin and Taqqu (1990); see also the survey by Surgailis (2003b) for further references in this direction.

Theorem 1.2 *Let the previous notation prevail, fix $\nu > 0$ and let $F(\nu)$ be a real-valued random variable such that*

$$E\left(e^{i\lambda F(\nu)}\right) = \left(\frac{e^{-i\lambda}}{\sqrt{1-2i\lambda}}\right)^\nu, \quad \lambda \in \mathbb{R}. \quad (1.4)$$

Fix an even integer $n \geq 2$, and define

$$c_n := \frac{1}{(n/2)!\binom{n-1}{n/2-1}^2} = \frac{4}{(n/2)!\binom{n}{n/2}^2}. \quad (1.5)$$

Then, for any sequence $\{f_k\}_{k \geq 1} \subset \mathfrak{H}^{\odot n}$ verifying

$$\lim_{k \rightarrow \infty} n! \|f_k\|_{\mathfrak{H}^{\otimes n}}^2 = \lim_{k \rightarrow \infty} E[I_n(f_k)^2] = 2\nu, \quad (1.6)$$

the following six conditions are equivalent:

- (i) $\lim_{k \rightarrow \infty} E[I_n(f_k)^3] = E[F(\nu)^3] = 8\nu$ and $\lim_{k \rightarrow \infty} E[I_n(f_k)^4] = E[F(\nu)^4] = 48\nu + 12\nu^2$;
- (ii) $\lim_{k \rightarrow \infty} E[I_n(f_k)^4] - 12E[I_n(f_k)^3] = 12\nu^2 - 48\nu$;
- (iii) $\lim_{k \rightarrow \infty} \|f_k \tilde{\otimes}_{n/2} f_k - c_n \times f_k\|_{\mathfrak{H}^{\otimes n}} = 0$ and $\lim_{k \rightarrow \infty} \|f_k \tilde{\otimes}_p f_k\|_{\mathfrak{H}^{\otimes 2(n-p)}} = 0$, for every $p = 1, \dots, n-1$ such that $p \neq n/2$;
- (iv) $\lim_{k \rightarrow \infty} \|f_k \tilde{\otimes}_{n/2} f_k - c_n \times f_k\|_{\mathfrak{H}^{\otimes n}} = 0$ and $\lim_{k \rightarrow \infty} \|f_k \otimes_p f_k\|_{\mathfrak{H}^{\otimes 2(n-p)}} = 0$, for every $p = 1, \dots, n-1$ such that $p \neq n/2$;
- (v) as $k \rightarrow \infty$, $\|D[I_n(f_k)]\|_{\mathfrak{H}}^2 - 2nI_n(f_k) \rightarrow 2n\nu$ in L^2 , where D is the Malliavin derivative operator;
- (vi) as $k \rightarrow \infty$, the sequence $\{I_n(f_k)\}_{k \geq 1}$ converges in distribution to $F(\nu)$.

Remark 1.3 1. The limit random variable $F(\nu)$ appearing in formula (1.4) is such that $F(\nu) \stackrel{\text{Law}}{=} 2G(\nu/2) - \nu$, where $G(\nu/2)$ has a Gamma law with parameter $\nu/2$, that is, $G(\nu/2)$ is a (a.s. strictly positive) random variable with density

$$g(x) = \frac{x^{\frac{\nu}{2}-1} e^{-x}}{\Gamma(\nu/2)} \mathbf{1}_{(0, \infty)}(x),$$

where Γ is the usual Gamma function. Note that the following elementary relations have been implicitly used:

$$E(F(\nu)) = 0, \quad E(F(\nu)^2) = 2\nu, \quad E(F(\nu)^3) = 8\nu, \quad E(F(\nu)^4) = 48\nu + 12\nu^2. \quad (1.7)$$

2. When $\nu \geq 1$ is an integer, then $F(\nu)$ has a centered χ^2 law with ν degrees of freedom. That is,

$$F(\nu) \stackrel{\text{Law}}{=} \sum_{i=1}^{\nu} (N_i^2 - 1), \quad (1.8)$$

where (N_1, \dots, N_ν) is a ν -dimensional vector of i.i.d. $\mathcal{N}(0, 1)$ random variables.

3. When $n \geq 1$ is an *odd* integer, there does not exist any sequence $\{I_n(f_k)\}_{k \geq 1}$, with $\{f_k\}_{k \geq 1} \subset \mathfrak{H}^{\odot n}$, such that $I_n(f_k)$ has bounded variances and $I_n(f_k)$ converges in distribution to $F(\nu)$ as $k \rightarrow \infty$. This is a consequence of the fact that any multiple integral of odd order has a third moment equal to zero, whereas $E(F(\nu)^3) = 8\nu > 0$.
4. The only difference between point (iii) and point (iv) of the above statement is the symmetrization of the contractions of order $p \neq n/2$. We will see that one cannot dispense with the symmetrization of the contraction of order $n/2$. Note also that (iii) and (iv) do not depend on ν ; this means that, when applying either one of conditions (iii) and (iv), the dependence on ν is completely encoded by the normalization assumption (1.6).
5. In Section 4 we will specialize Theorem 1.2 to the case of sequences belonging to the second chaos of X . In particular, we will show (see Corollary 4.3) that Theorem 1.2 yields the following implication: if F is a random variable in the second chaos of X and if $F \stackrel{\text{Law}}{=} 2G(\nu/2) - \nu$ with $\nu > 0$, then necessarily ν is an integer, and therefore F has a centered χ^2 distribution with ν degrees of freedom. This characterization should be compared with the well-known fact that a chaos of order ≥ 3 does not contain any centered Gamma random variable. See e.g. Janson (1997, Ch. VI), as well as Corollary 5.6 below. The reader is also referred to Neveu (1968) and Dellacherie *et al.* (1992) for further characterizations of the Wiener-Itô chaoses associated with general Gaussian processes.
6. In Proposition 3.2 we will use Theorem 1.1 in order to provide simple examples of sequences $\{I_n(f_k)\}_{k \geq 1}$ verifying both (1.6) and either one of the equivalent conditions (i)–(vi) of Theorem 1.2, for a given even integer $n \geq 4$ and a given *integer* $\nu \geq 1$.
7. As shown in Section 5, Theorem 1.2 admits several multidimensional extensions, that can be used to obtain weak convergence results towards more general laws in the second Wiener chaos of X .

Before going into details, we shall provide a short outline of the techniques used in the proof of Theorem 1.2. We will prove the following implications

$$(vi) \rightarrow (i) \rightarrow (ii) \rightarrow (iii) \leftrightarrow (iv) \rightarrow (v) \rightarrow (vi).$$

The double implication $(vi) \rightarrow (i) \rightarrow (ii)$ is trivial. The implication $(ii) \rightarrow (iii)$ is obtained by combining a standard version of the multiplication formula between multiple integrals with a result based on the integration by parts formulae of Malliavin calculus (see Lemma 2.1 below). The proof of $(iii) \leftrightarrow (iv)$ is purely combinatorial, whereas that of $(iv) \rightarrow (v)$ relies once again on multiplication formulae. Finally, to show $(v) \rightarrow (vi)$ we will adopt an approach similar to the one by Nualart and Ortiz-Latorre (2007). Our argument goes as follows. Let us first observe that a sequence of random variables $\{I_n(f_k)\}_{k \geq 1}$ verifying (1.6) is tight and therefore, by Prokhorov's

Theorem, it is relatively compact. As a consequence, to show the implication (v) \rightarrow (vi) it is sufficient to prove that any subsequence $\{I_n(f_{k'})\}$, converging in distribution to some random variable F_∞ , must be necessarily such that $F_\infty \stackrel{\text{Law}}{=} F(\nu)$. This last property will be established by means of Malliavin calculus, by proving that condition (v) implies that the characteristic function ϕ_∞ of F_∞ always solves the linear differential equation

$$(1 - 2i\lambda)\phi'_\infty(\lambda) + 2\lambda\nu\phi_\infty(\lambda) = 0, \quad \lambda \in \mathbb{R}, \quad \phi_\infty(0) = 1. \quad (1.9)$$

Since the solution of (1.9) is given by the application $\lambda \mapsto E\{e^{i\lambda F(\nu)}\}$, the desired conclusion will follow immediately.

The paper is organized as follows. In Section 2 we present some preliminary results about Malliavin calculus and second chaos random variables. Section 3 contains the proof of Theorem 1.2. Section 4 focuses on sequences of random variables in the second chaos of X . Section 5 is devoted to multidimensional results, examples and further refinements.

2 Preliminaries

2.1 Wiener chaos and Malliavin calculus

The reader is referred to the monograph by Nualart (2006) for any unexplained notion or result discussed in this section. Let \mathfrak{H} be a real separable Hilbert space. As in formula (1.1), we denote by X an isonormal Gaussian process over \mathfrak{H} . Recall that, by definition, X is a collection of centered and jointly Gaussian random variables indexed by the elements of \mathfrak{H} , defined on some probability space (Ω, \mathcal{F}, P) and such that, for every $h, g \in \mathfrak{H}$,

$$E[X(h)X(g)] = \langle h, g \rangle_{\mathfrak{H}}. \quad (2.10)$$

We will systematically assume that \mathcal{F} is generated by X . It is well-known (see e.g. Nualart (2006, Ch. 1)) that any random variable F belonging to $L^2(\Omega, \mathcal{F}, P)$ admits the following chaotic expansion:

$$F = \sum_{n=0}^{\infty} I_n(f_n), \quad (2.11)$$

where $I_0(f_0) := E[F]$, the series converges in L^2 and the kernels $f_n \in \mathfrak{H}^{\odot n}$, $n \geq 1$, are uniquely determined by F . Observe that a random variable of the type $I_n(f)$, $f \in \mathfrak{H}^{\odot n}$, has finite moments of all orders (see e.g. Janson (1997, Ch. VI)). As already pointed out, in the particular case where $\mathfrak{H} = L^2(A, \mathcal{A}, \mu)$, where (A, \mathcal{A}) is a measurable space and μ is a σ -finite and non-atomic measure, one has that $\mathfrak{H}^{\odot n} = L^2_s(A^n, \mathcal{A}^{\otimes n}, \mu^{\otimes n})$ is the space of symmetric and square integrable functions on A^n . Moreover, for every $f \in \mathfrak{H}^{\odot n}$, $I_n(f)$ coincides with the multiple Wiener-Itô integral (of order n) of f with respect to X (see again Nualart (2006, Ch. 1)). For every $n \geq 0$, we write J_n to indicate the orthogonal projection operator on the n th Wiener chaos associated with X . In particular, if $F \in L^2(\Omega, \mathcal{F}, P)$ is as in (2.11), then $J_n F = I_n(f_n)$ for every $n \geq 0$.

Let $\{e_k, k \geq 1\}$ be a complete orthonormal system in \mathfrak{H} . Given $f \in \mathfrak{H}^{\odot n}$ and $g \in \mathfrak{H}^{\odot m}$, for every $p = 0, \dots, n \wedge m$, the p th contraction of f and g is the element of $\mathfrak{H}^{\otimes(n+m-2p)}$ defined as

$$f \otimes_p g = \sum_{i_1, \dots, i_p=1}^{\infty} \langle f, e_{i_1} \otimes \dots \otimes e_{i_p} \rangle_{\mathfrak{H}^{\otimes p}} \otimes \langle g, e_{i_1} \otimes \dots \otimes e_{i_p} \rangle_{\mathfrak{H}^{\otimes p}}. \quad (2.12)$$

Note that, in the particular case where $\mathfrak{H} = L^2(A, \mathcal{A}, \mu)$ (with μ non-atomic), one has that

$$f \otimes_p g = \int_{A^p} f(t_1, \dots, t_{n-p}, s_1, \dots, s_p) \overline{g(t_{n-p+1}, \dots, t_{m+n-2p}, s_1, \dots, s_p)} d\mu(s_1) \dots d\mu(s_p).$$

Moreover, $f \otimes_0 g = f \otimes g$ equals the tensor product of f and g while, for $n = m$, $f \otimes_n g = \langle f, g \rangle_{\mathfrak{H}^{\otimes n}}$. Note that, in general (and except for trivial cases), the contraction $f \otimes_p g$ is not a symmetric element of $\mathfrak{H}^{\otimes(n+m-2p)}$. As indicated in the Introduction, the canonical symmetrization of $f \otimes_p g$ is written $f \widetilde{\otimes}_p g$.

Let \mathcal{S} be the set of all smooth cylindrical random variables of the form

$$F = g(X(\phi_1), \dots, X(\phi_n))$$

where $n \geq 1$, $g : \mathbb{R}^n \rightarrow \mathbb{R}$ is a smooth function with compact support and $\phi_i \in \mathfrak{H}$. The Malliavin derivative of F with respect to X is the element of $L^2(\Omega, \mathfrak{H})$ defined as

$$DF = \sum_{i=1}^n \frac{\partial g}{\partial x_i}(X(\phi_1), \dots, X(\phi_n)) \phi_i.$$

In particular, $DX(h) = h$ for every $h \in \mathfrak{H}$. By iteration, one can define the m th derivative $D^m F$ (which is an element of $L^2(\Omega, \mathfrak{H}^{\otimes m})$) for every $m \geq 2$.

As usual, for $m \geq 1$, $\mathbb{D}^{m,2}$ denotes the closure of \mathcal{S} with respect to the norm $\|\cdot\|_{m,2}$, defined by the relation

$$\|F\|_{m,2}^2 = E[F^2] + \sum_{i=1}^m E[\|D^i F\|_{\mathfrak{H}^{\otimes i}}^2].$$

The Malliavin derivative D verifies the following *chain rule*: if $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$ is in \mathcal{C}_b^1 and if $\{F_i\}_{i=1, \dots, n}$ is a vector of elements of $\mathbb{D}^{1,2}$, then $\varphi(F_1, \dots, F_n) \in \mathbb{D}^{1,2}$ and

$$D \varphi(F_1, \dots, F_n) = \sum_{i=1}^n \frac{\partial \varphi}{\partial x_i}(F_1, \dots, F_n) DF_i.$$

We denote by δ the adjoint of the operator D , also called the *divergence operator*. A random element $u \in L^2(\Omega, \mathfrak{H})$ belongs to the domain of δ , noted $\text{Dom} \delta$, if, and only if, it verifies

$$|E\langle DF, u \rangle_{\mathfrak{H}}| \leq c_u \|F\|_{L^2} \quad \text{for any } F \in \mathcal{S},$$

where c_u is a constant depending uniquely on u . If $u \in \text{Dom} \delta$, then the random variable $\delta(u)$ is defined by the duality relationship (called “integration by parts formula”):

$$E(F \delta(u)) = E\langle DF, u \rangle_{\mathfrak{H}}, \quad (2.13)$$

which holds for every $F \in \mathbb{D}^{1,2}$. We will moreover need the following property: for every $F \in \mathbb{D}^{1,2}$ and every $u \in \text{Dom}\delta$ such that Fu and $F\delta(u) + \langle DF, u \rangle_{\mathfrak{H}}$ are square integrable, one has that $Fu \in \text{Dom}\delta$ and

$$\delta(Fu) = F\delta(u) - \langle DF, u \rangle_{\mathfrak{H}}. \quad (2.14)$$

The operator L is defined through the projection operators $\{J_n\}_{n \geq 0}$ as $L = \sum_{n=0}^{\infty} -nJ_n$, and is called the *infinitesimal generator of the Ornstein-Uhlenbeck semigroup*. It verifies the following crucial property: a random variable F is an element of $\text{Dom}L$ ($= \mathbb{D}^{2,2}$) if, and only if, $F \in \text{Dom}\delta D$ (i.e. $F \in \mathbb{D}^{1,2}$ and $DF \in \text{Dom}\delta$), and in this case:

$$\delta DF = -LF.$$

Note that a random variable F as in (2.11) is in $\mathbb{D}^{1,2}$ if, and only if,

$$\sum_{n=1}^{\infty} n \|f_n\|_{\mathfrak{H}^{\odot n}}^2 < \infty,$$

and, in this case, $E[\|DF\|_{\mathfrak{H}}^2] = \sum_{n \geq 1} n \|f_n\|_{\mathfrak{H}^{\odot n}}^2$. Of course, if $\mathfrak{H} = L^2(A, \mathcal{A}, \mu)$ (with μ non-atomic), then the derivative of a random variable F as in (2.11) can be identified with the element of $L^2(A \times \Omega)$ given by

$$D_a F = \sum_{n=1}^{\infty} n I_{n-1}(f_n(\cdot, a)), \quad a \in A. \quad (2.15)$$

The following Lemma will be used in Section 3.

Lemma 2.1 *Fix an integer $n \geq 2$ and set $F = I_n(f)$, with $f \in \mathfrak{H}^{\odot n}$. Then, for every integer $s \geq 0$, we have*

$$E(F^s \|DF\|_{\mathfrak{H}}^2) = \frac{n}{s+1} E(F^{s+2}).$$

Proof. We can write:

$$\begin{aligned} E(F^s \|DF\|_{\mathfrak{H}}^2) &= E(F^s \langle DF, DF \rangle_{\mathfrak{H}}) = \frac{1}{s+1} E(\langle DF, D(F^{s+1}) \rangle_{\mathfrak{H}}) \\ &= \frac{1}{s+1} E(\delta DF \times F^{s+1}) \quad \text{by integration by parts (2.13)} \\ &= \frac{n}{s+1} E(F^{s+2}) \quad \text{by the property } \delta D = -L \text{ (which implies } \delta DF = nF). \end{aligned}$$

□

2.2 Second chaos random variables

In this subsection we focus on elements of the second chaos of X , that is, random variables of the type $F = I_2(f)$, with $f \in \mathfrak{H}^{\odot 2}$. These results will play an important role in Section 4. Observe that, if $f = h \otimes h$, where $h \in \mathfrak{H}$ is such that $\|h\|_{\mathfrak{H}} = 1$, then $I_2(f) = I_1(h)^2 - 1 \stackrel{\text{Law}}{=} N^2 - 1$, where $N \sim \mathcal{N}(0, 1)$. To every kernel $f \in \mathfrak{H}^{\odot 2}$ we associate two objects:

1. The Hilbert-Schmidt operator

$$H_f : \mathfrak{H} \mapsto \mathfrak{H}; \quad g \mapsto f \otimes_1 g. \quad (2.16)$$

In other words, H_f transforms an element g of \mathfrak{H} into the contraction $H_f g = f \otimes_1 g \in \mathfrak{H}$ (as defined in formula (2.12)). We write $\{\lambda_{f,j}\}_{j \geq 1}$ and $\{e_{f,j}\}_{j \geq 1}$, respectively, to indicate the eigenvalues of H_f and the corresponding eigenvectors (forming an orthonormal system in \mathfrak{H}).

2. The sequence of auxiliary kernels

$$\left\{ f \otimes_1^{(p)} f : p \geq 1 \right\} \subset \mathfrak{H}^{\odot 2} \quad (2.17)$$

defined as follows: $f \otimes_1^{(1)} f = f$, and, for $p \geq 2$,

$$f \otimes_1^{(p)} f = \left(f \otimes_1^{(p-1)} f \right) \otimes_1 f. \quad (2.18)$$

In particular, with the notation of the previous section, $f \otimes_1^{(2)} f = f \otimes_1 f$.

The relations between the objects introduced in (2.16) and (2.18) are explained in the next proposition (the proof is standard and omitted, see e.g. Section 6.2 in Hirsch and Lacombe (1999)).

Proposition 2.2 *Let $f \in \mathfrak{H}^{\odot 2}$.*

- (1) *The series $\sum_{j=1}^{\infty} \lambda_{f,j}^p$ converges for every $p \geq 2$, and f admits the expansion*

$$f = \sum_{j=1}^{\infty} \lambda_{f,j} e_{f,j} \otimes e_{f,j}, \quad (2.19)$$

where the convergence takes place in $\mathfrak{H}^{\odot 2}$.

- (2) *For $p \geq 2$, one has the relations*

$$\mathbf{Tr}(H_f^p) = \langle f \otimes_1^{(p-1)} f, f \rangle_{\mathfrak{H}^{\odot 2}} = \sum_{j=1}^{\infty} \lambda_{f,j}^p, \quad (2.20)$$

where $\mathbf{Tr}(H_f^p)$ stands for the trace of the p th power of H_f .

In the following statement we collect some facts concerning the law of a real-valued random variable of the type $I_2(f)$. We recall that, given a random variable F such that $E|F|^p < \infty$, $\forall p \geq 1$, the sequence of the *cumulants* of F , written $\{\kappa_p(F)\}_{p \geq 1}$, is defined by the relation (in the sense of *formal power series*):

$$\ln E(e^{i\mu F}) = \sum_{p=1}^{\infty} \frac{(i\mu)^p}{p!} \kappa_p(F), \quad \mu \in \mathbb{R}, \quad (2.21)$$

(see e.g. Shiriyayev (1984), or Rota and Shen (2000)). For instance, κ_1 is the mean, and κ_2 is the variance.

Proposition 2.3 *Let $F = I_2(f)$, with $f \in \mathfrak{H}^{\odot 2}$.*

- (1) *The random variable F admits some exponential moments, hence the law of F is determined by its moments or, equivalently, by its cumulants.*
- (2) *The following equality holds:*

$$F = \sum_{j=1}^{\infty} \lambda_{f,j} (N_j^2 - 1), \quad (2.22)$$

where $\{N_j\}_{j \geq 1}$ is a sequence of i.i.d. $\mathcal{N}(0,1)$ random variables, and the series converges in L^2 .

- (3) *For every $p \geq 2$,*

$$\kappa_p(F) = 2^{p-1} (p-1)! \times \mathbf{Tr}(H_f^p) = 2^{p-1} (p-1)! \sum_{j=1}^{\infty} \lambda_{f,j}^p. \quad (2.23)$$

Proof. Part (1) of the statement is proved e.g. in Slud (1993). Relation (2.22) is an immediate consequence of (2.19), of the identity $I_2(e_{f,j} \otimes e_{f,j}) = I_1(e_{f,j})^2 - 1$, and of the fact that the $\{e_{f,j}\}$ are orthonormal (implying that the sequence $\{I_1(e_{f,j})\}_{j \geq 1}$ is i.i.d. $\mathcal{N}(0,1)$). To prove (2.23), first observe that (2.22) implies that

$$E(e^{i\mu F}) = \prod_{j=1}^{\infty} \frac{e^{-i\mu \lambda_{f,j}}}{\sqrt{1 - 2i\mu \lambda_{f,j}}}.$$

Thus, standard computations give

$$\begin{aligned} \ln E(e^{i\mu F}) &= -i\mu \sum_{j=1}^{\infty} \lambda_{f,j} - \frac{1}{2} \sum_{j=1}^{\infty} \ln(1 - 2i\mu \lambda_{f,j}) \\ &= \frac{1}{2} \sum_{p=2}^{\infty} \frac{(2i\mu)^p}{p} \sum_{j=1}^{\infty} \lambda_{f,j}^p = \sum_{p=2}^{\infty} 2^{p-1} \frac{(i\mu)^p}{p} \sum_{j=1}^{\infty} \lambda_{f,j}^p. \end{aligned} \quad (2.24)$$

We can now identify the coefficients in the series (2.21) and (2.24), so to deduce that $\kappa_1(F) = E(F) = 0$, and

$$\frac{2^{p-1}}{p} \sum_{j=1}^{\infty} \lambda_{f,j}^p = \frac{\kappa_p(F)}{p!},$$

thus obtaining the desired conclusion. \square

Proposition 2.3(2) is also proved in Janson (1997, Ch. VI, Theorem 6.1). In particular, Proposition 2.3(2) shows that centered χ^2 random variables are the basic building blocks of any element of the second Wiener chaos of X (see also Neveu (1968, Ch. VII)). Moreover, by using (2.20) it is easily seen that (2.23) coincides with the usual “diagram formulae” for the cumulants of double Wiener-Itô integrals: in particular, for $p \geq 3$ the coefficient $2^{p-1}(p-1)!$ counts the number of *non-flat, Gaussian and connected diagrams* over the partition of $\{1, \dots, 2p\}$ given by $\{\{1, 2\}, \{3, 4\}, \dots, \{2p-1, 2p\}\}$ (see Surgailis (2003a)). An alternate (combinatorial) proof of (2.23) can be found in Fox and Taqqu (1985). As a particular case, (2.23) gives immediately that, if $N \sim \mathcal{N}(0, 1)$, then

$$\kappa_p(N^2 - 1) = 2^{p-1}(p-1)!, \quad p \geq 2. \quad (2.25)$$

Moreover, since the random variable $F(1)$ (as defined in (1.4)) is such that $F(1) \stackrel{\text{Law}}{=} N^2 - 1$, one can combine (2.21) and (2.25) to deduce that, for every $\nu > 0$,

$$\kappa_p(F(\nu)) = 2^{p-1}(p-1)! \times \nu, \quad p \geq 2. \quad (2.26)$$

3 Proof of Theorem 1.2

Throughout this section, $n \geq 2$ is an even integer, and $\{I_n(f_k)\}_{k \geq 1}$ is a sequence of multiple stochastic Wiener-Itô integrals of order n , such that condition (1.6) is satisfied for some $\nu > 0$.

3.1 Proof of (vi) \rightarrow (i) \rightarrow (ii)

Since the sequence $\{I_n(f_k)\}_{k \geq 1}$ lives inside the n th chaos of X , and since condition (1.6) is in order, we deduce that, for every $p > 0$,

$$\sup_{k \geq 1} E[|I_n(f_k)|^p] < \infty \quad (3.27)$$

(see e.g. Janson (1997, Ch. V)). This implies immediately that, if $\{I_n(f_k)\}_{k \geq 1}$ converges in distribution to $F(\nu)$, then, for every integer $p \geq 3$, $E(I_n(f_k)^p) \rightarrow E(F(\nu)^p)$. The implications (vi) \rightarrow (i) \rightarrow (ii) are therefore a direct consequence of (1.7).

3.2 Proof of (ii) \rightarrow (iii)

Suppose that (ii) holds. We start by observing that, due to the multiplication formulae between stochastic integrals (see Proposition 1.1.3 in Nualart (2006)), we have

$$I_n(f_k)^2 = n! \|f_k\|_{\mathfrak{H}^{\otimes n}}^2 + \sum_{p=0}^{n-1} p! \binom{n}{p}^2 I_{2(n-p)}(f_k \tilde{\otimes}_p f_k), \quad (3.28)$$

and

$$\|D[I_n(f_k)]\|_{\mathfrak{H}}^2 = nn! \|f_k\|_{\mathfrak{H}^{\otimes n}}^2 + n^2 \sum_{p=1}^{n-1} (p-1)! \binom{n-1}{p-1}^2 I_{2(n-p)}(f_k \tilde{\otimes}_p f_k) \quad (3.29)$$

(see also Nualart and Ortiz-Latorre (2007, Lemma 2)). Relation (3.28) gives immediately that

$$E [I_n(f_k)^3] = n! (n/2)! \binom{n}{n/2}^2 \langle f_k, f_k \tilde{\otimes}_{n/2} f_k \rangle_{\mathfrak{H}^{\otimes n}}. \quad (3.30)$$

On the other hand, we deduce from Lemma 2.1 (specialized to the case $s = 2$) that

$$E [I_n(f_k)^4] = \frac{3}{n} E [I_n(f_k)^2 \|D[I_n(f_k)]\|_{\mathfrak{H}}^2], \quad (3.31)$$

and therefore, thanks to (3.28)–(3.29),

$$\begin{aligned} E [I_n(f_k)^4] &= 3 \left[n! \|f_k\|_{\mathfrak{H}^{\otimes n}}^2 \right]^2 + \\ &\quad \frac{3}{n} \sum_{p=1}^{n-1} n^2 (p-1)! \binom{n-1}{p-1}^2 p! \binom{n}{p}^2 (2n-2p)! \|f_k \tilde{\otimes}_p f_k\|_{\mathfrak{H}^{\otimes 2(n-p)}}^2. \end{aligned} \quad (3.32)$$

In what follows, given two (deterministic) sequences $a(k)$ and $b(k)$, we write $a(k) \approx b(k)$ whenever $a(k) - b(k) \rightarrow 0$. Since (ii) and (1.6) hold, we deduce from (3.30)–(3.32) and condition (1.6), that

$$\begin{aligned} &E [I_n(f_k)^4] - 12 E [I_n(f_k)^3] \\ &\approx [12\nu^2 - 48\nu] + \frac{3}{n} \sum_{\substack{p=1, \dots, n-1 \\ p \neq n/2}} n^2 (p-1)! \binom{n-1}{p-1}^2 p! \binom{n}{p}^2 (2n-2p)! \|f_k \tilde{\otimes}_p f_k\|_{\mathfrak{H}^{\otimes 2(n-p)}}^2 \\ &\quad + 24n! \|f_k\|_{\mathfrak{H}^{\otimes n}}^2 + 3n (n/2 - 1)! \binom{n-1}{n/2-1}^2 (n/2)! \binom{n}{n/2}^2 n! \|f_k \tilde{\otimes}_{n/2} f_k\|_{\mathfrak{H}^{\otimes n}}^2 \\ &\quad - 12n! (n/2)! \binom{n}{n/2}^2 \langle f_k, f_k \tilde{\otimes}_{n/2} f_k \rangle_{\mathfrak{H}^{\otimes n}}. \end{aligned} \quad (3.33)$$

Elementary simplifications give

$$\begin{aligned} &24n! \|f_k\|_{\mathfrak{H}^{\otimes n}}^2 + 3n (n/2 - 1)! \binom{n-1}{n/2-1}^2 (n/2)! \binom{n}{n/2}^2 n! \|f_k \tilde{\otimes}_{n/2} f_k\|_{\mathfrak{H}^{\otimes n}}^2 \\ &\quad - 12n! (n/2)! \binom{n}{n/2}^2 \langle f_k, f_k \tilde{\otimes}_{n/2} f_k \rangle_{\mathfrak{H}^{\otimes n}} \\ &= 24n! \|f_k\|_{\mathfrak{H}^{\otimes n}}^2 + \frac{3}{2} (n!)^2 \binom{n}{n/2}^3 \|f_k \tilde{\otimes}_{n/2} f_k\|_{\mathfrak{H}^{\otimes n}}^2 - 12n! (n/2)! \binom{n}{n/2}^2 \langle f_k, f_k \tilde{\otimes}_{n/2} f_k \rangle_{\mathfrak{H}^{\otimes n}} \\ &= \left\| 2\sqrt{n!} \sqrt{6} f_k - \sqrt{\frac{3}{2}} \frac{(n!)^2 \sqrt{n!}}{[(n/2)!]^3} f_k \tilde{\otimes}_{n/2} f_k \right\|_{\mathfrak{H}^{\otimes n}}^2 = \frac{3}{2} \frac{(n!)^5}{[(n/2)!]^6} \|f_k \tilde{\otimes}_{n/2} f_k - f_k \times c_n\|_{\mathfrak{H}^{\otimes n}}^2, \end{aligned}$$

where c_n is defined in (1.5). This yields the desired conclusion.

3.3 Proof of (iii) \leftrightarrow (iv)

We can assume that $n \geq 4$. We shall introduce some further notation. Fix an integer $M \geq 1$, and denote by \mathfrak{S}_{2M} the group of the $(2M)!$ permutations of the set $\{1, \dots, 2M\}$. We write π_0 to indicate the identity (trivial) permutation. Given a set A and a vector $a = (a_1, \dots, a_{2M}) \in A^{2M}$, for every $\pi \in \mathfrak{S}_{2M}$ we denote by $a_\pi = (a_{\pi(1)}, \dots, a_{\pi(2M)})$ the canonical action of π on a . Note that, with this notation, one has $a = a_{\pi_0}$. For every $r = 0, \dots, M$ and for $\pi, \sigma \in \mathfrak{S}_{2M}$, we write

$$\pi \sim_r \sigma$$

whenever the set $\{\pi(1), \dots, \pi(M)\} \cap \{\sigma(1), \dots, \sigma(M)\}$ contains exactly r elements. For every $\pi \in \mathfrak{S}_{2M}$, there are exactly $M!^2 \binom{M}{r}^2$ permutations σ such that $\pi \sim_r \sigma$. The double implication (iii) \leftrightarrow (iv) in the statement of Theorem 1.2 is a consequence of the following result.

Proposition 3.1 *Let $n \geq 4$ be an even integer, and let $\{f_k\} \subset \mathfrak{H}^{\odot n}$ be a sequence of symmetric kernels. Then, the following two conditions are equivalent:*

- (A) $\|f_k \widetilde{\otimes}_p f_k\|_{\mathfrak{H}^{\otimes 2(n-p)}} \rightarrow 0, \quad p = 1, \dots, n-1, \quad p \neq n/2;$
- (B) $\|f_k \otimes_p f_k\|_{\mathfrak{H}^{\otimes 2(n-p)}} \rightarrow 0, \quad p = 1, \dots, n-1, \quad p \neq n/2.$

Proof. Since $\|f_k \otimes_p f_k\|_{\mathfrak{H}^{\otimes 2(n-p)}} \geq \|f_k \widetilde{\otimes}_p f_k\|_{\mathfrak{H}^{\otimes 2(n-p)}}$, the implication (B) \Rightarrow (A) is trivial. Moreover, since

$$\|f_k \otimes_p f_k\|_{\mathfrak{H}^{\otimes 2(n-p)}} = \|f_k \otimes_{n-p} f_k\|_{\mathfrak{H}^{\otimes 2p}} \quad (3.34)$$

for every $p = 1, \dots, n-1$, to show that (A) \Rightarrow (B) it is sufficient to prove that (A) implies that, $\forall p = 1, \dots, \frac{n}{2} - 1$,

$$\|f_k \otimes_p f_k\|_{\mathfrak{H}^{\otimes 2(n-p)}} \rightarrow 0. \quad (3.35)$$

Thanks to (3.34), and since $f_k \otimes_{n-1} f_k = f_k \widetilde{\otimes}_{n-1} f_k$, we immediately deduce that (A) implies that (3.35) holds for $p = 1$. This proves the implication (A) \Rightarrow (B) in the case $n = 4$, so that from now on we can suppose that $n \geq 6$. The rest of the proof is done by recurrence. In particular, we shall show that, for every $q = 2, \dots, \frac{n}{2} - 1$, the following implication holds: if (A) is true and if (3.35) holds for $p = 1, \dots, q-1$, then

$$\|f_k \otimes_q f_k\|_{\mathfrak{H}^{\otimes 2(n-q)}} = \|f_k \otimes_{n-q} f_k\|_{\mathfrak{H}^{\otimes 2q}} \rightarrow 0.$$

Now fix $q = 2, \dots, \frac{n}{2} - 1$, suppose (A) is verified, and assume that (3.35) takes place for $p = 1, \dots, q-1$. To simplify the discussion, we shall suppose (without loss of generality) that $\mathfrak{H} = L^2(A, \mu)$, where μ is σ -finite and non-atomic. Start by writing

$$\begin{aligned} \|f_k \widetilde{\otimes}_{n-q} f_k\|_{\mathfrak{H}^{\otimes 2q}}^2 &= \langle f_k \otimes_{n-q} f_k, f_k \widetilde{\otimes}_{n-q} f_k \rangle_{\mathfrak{H}^{\otimes 2q}} \\ &= \frac{1}{(2q)!} \sum_{\pi \in \mathfrak{S}_{2q}} \int_{A^{2q}} f_k \otimes_{n-q} f_k(a_{\pi_0}) \times f_k \otimes_{n-q} f_k(a_\pi) \mu^{2q}(da). \end{aligned}$$

Now, if $\pi \sim_0 \pi_0$ or $\pi \sim_q \pi_0$, one has that

$$\int_{A^{2q}} f_k \otimes_{n-q} f_k(a_{\pi_0}) \times f_k \otimes_{n-q} f_k(a_\pi) \mu^{2q}(da) = \|f_k \otimes_{n-q} f_k\|_{\mathfrak{H}^{\otimes 2q}}^2.$$

On the other hand, if $\pi \sim_p \pi_0$ for some $p = 1, \dots, q-1$, then

$$\int_{A^{2q}} f_k \otimes_{n-q} f_k(a_{\pi_0}) \times f_k \otimes_{n-q} f_k(a_\pi) \mu^{2q}(da) \quad (3.36)$$

$$= \int_{A^{2(n-p)}} f_k \otimes_p f_k(a_{\pi^{[2(n-p)]}}) f_k \otimes_p f_k(a_{\sigma^{[2(n-p)]}}) \mu^{2(n-p)}(da), \quad (3.37)$$

where $(\pi^{[2(n-p)]}, \sigma^{[2(n-p)]}) \subset \mathfrak{S}_{2(n-p)}$ is any pair of permutations of $\{1, \dots, 2(n-p)\}$ such that

$$\pi^{[2(n-p)]} \sim_{(q-p)} \sigma^{[2(n-p)]}.$$

Now, thanks to the recurrence assumption, and by Cauchy-Schwarz, we deduce that the expression in (3.37) converges to zero as $k \rightarrow \infty$, thus yielding that

$$\begin{aligned} 0 &= \lim_{k \rightarrow \infty} \|f_k \tilde{\otimes}_{n-q} f_k\|_{\mathfrak{H}^{\otimes 2q}}^2 = \lim_{k \rightarrow \infty} \frac{2q!^2}{(2q)!} \|f_k \otimes_{n-q} f_k\|_{\mathfrak{H}^{\otimes 2q}}^2 \\ &= \lim_{k \rightarrow \infty} \frac{2}{\binom{2q}{q}} \|f_k \otimes_q f_k\|_{\mathfrak{H}^{\otimes 2(n-q)}}^2. \end{aligned}$$

This concludes the proof. \square

3.4 Proof of (iv) \rightarrow (v)

Suppose that (iv) holds. By using (3.29), we infer that $E[\|D[I_n(f_k)]\|_{\mathfrak{H}}^2] = nn! \|f_k\|_{\mathfrak{H}^{\otimes n}}^2 \rightarrow 2n\nu$. Moreover, by taking into account the orthogonality between multiple stochastic integrals of different orders and by using the multiplication formulae for multiple Wiener-Itô integrals, we have

$$E[I_n(f_k) \|D[I_n(f_k)]\|_{\mathfrak{H}}^2] = n^2(n/2 - 1)! \binom{n-1}{n/2-1}^2 n! \langle f_k \tilde{\otimes}_{n/2} f_k, f_k \rangle_{\mathfrak{H}^{\otimes n}}$$

and

$$E[\|D[I_n(f_k)]\|_{\mathfrak{H}}^4] = n^4 \sum_{p=1}^n (p-1)!^2 \binom{n-1}{p-1}^4 (2n-2p)! \|f_k \tilde{\otimes}_p f_k\|_{\mathfrak{H}^{\otimes 2(n-p)}}^2.$$

Now define c_n according to (1.5), and observe that (iv) and (1.6) imply that

$$\lim_{k \rightarrow \infty} \|f_k \tilde{\otimes}_{\frac{n}{2}} f_k\|_{\mathfrak{H}^{\otimes n}}^2 = \lim_{k \rightarrow \infty} \|c_n f_k\|_{\mathfrak{H}^{\otimes n}}^2 = (2\nu c_n^2)/n! \quad \text{and} \quad \lim_{k \rightarrow \infty} \langle f_k \tilde{\otimes}_{\frac{n}{2}} f_k, f_k \rangle_{\mathfrak{H}^{\otimes n}} = (2\nu c_n)/n!.$$

Thus, under (iv) one has that, as $k \rightarrow \infty$,

$$\begin{aligned} &E(\|D[I_n(f_k)]\|^2 - 2nI_n(f_k) - 2n\nu)^2 \\ &= E[\|D[I_n(f_k)]\|^4 - 4nE[I_n(f_k) \|D[I_n(f_k)]\|^2] \\ &\quad + 4n^2E[I_n(f_k)^2] + 4n^2\nu^2 - 4n\nu E[\|D[I_n(f_k)]\|^2]] \\ &\longrightarrow 4\nu^2 n^2 + 2c_n^2 \nu n^4 (n/2 - 1)!^2 \binom{n-1}{n/2-1}^4 \\ &\quad - 8c_n \nu n^3 (n/2 - 1)! \binom{n-1}{n/2-1}^2 + 8n^2\nu + 4n^2\nu^2 - 8n^2\nu^2 = 0. \end{aligned}$$

3.5 Proof of (v) \rightarrow (vi)

Now we assume that (v) holds. We start by observing that condition (1.6) implies that the sequence of the laws of the random variables $\{I_n(f_k)\}_{k \geq 1}$ is tight (since it is bounded in L^2). By Prokhorov's Theorem, we deduce that $\{I_n(f_k)\}_{k \geq 1}$ is relatively compact so that, to prove our claim, it is sufficient to show that any subsequence $\{I_n(f_{k'})\}$ converging in distribution to some random variable F_∞ is necessarily such that $F_\infty \stackrel{\text{Law}}{=} F(\nu)$, where the law of $F(\nu)$ is defined by formula (1.4). From now on, and only for notational convenience, we assume that $\{I_n(f_k)\}$ itself converges to F_∞ . Also, for any $k \geq 1$, we let $\phi_k(\lambda) = E(e^{i\lambda I_n(f_k)})$ denote the characteristic function of $I_n(f_k)$, so that $\phi'_k(\lambda) = iE(I_n(f_k) e^{i\lambda I_n(f_k)})$. On the one hand, by the Continuous Mapping Theorem, we have that

$$I_n(f_k) e^{i\lambda I_n(f_k)} \xrightarrow{\text{Law}} F_\infty e^{i\lambda F_\infty}.$$

Since boundedness in L^2 implies convergence of the expectations, we also deduce that $\phi'_k(\lambda) \rightarrow \phi'_\infty(\lambda)$ for any $\lambda \in \mathbb{R}$. On the other hand, we can write

$$\begin{aligned} \phi'_k(\lambda) &= \frac{i}{n} E(\delta D[I_n(f_k)] \times e^{i\lambda I_n(f_k)}) \quad \text{since } \delta D = -L, \\ &= \frac{i}{n} E(\langle D[I_n(f_k)], D(e^{i\lambda I_n(f_k)}) \rangle_{\mathfrak{H}}) \quad \text{by integration by parts (2.13),} \\ &= -\frac{\lambda}{n} E(e^{i\lambda I_n(f_k)} \|D[I_n(f_k)]\|_{\mathfrak{H}}^2). \end{aligned}$$

Since (v) is in order, we deduce that

$$\phi'_k(\lambda) + 2\lambda E(e^{i\lambda I_n(f_k)} I_n(f_k)) + 2\lambda\nu E(e^{i\lambda I_n(f_k)}) \rightarrow 0.$$

As a consequence, ϕ_∞ must necessarily solve the linear differential equation (1.9), yielding

$$\phi_\infty(\lambda) = \left(\frac{e^{-i\lambda}}{\sqrt{1 - i2\lambda}} \right)^\nu = E(e^{i\lambda F(\nu)}), \quad \lambda \in \mathbb{R}.$$

This concludes the proof of Theorem 1.2.

3.6 Further remarks

(I) When $\nu \geq 1$ is an integer, one can use Theorem 1.1 in order to obtain examples of sequences of multiple integrals $\{I_n(f_k)\}_{k \geq 1}$, where $n \geq 4$ is even, satisfying either one of conditions (i)–(vi) in Theorem 1.2. This fact is summarized in the following statement.

Proposition 3.2 *Let $m \geq 2$ and $\nu \geq 1$ be integers and, for $i = 1, \dots, \nu$, let $\{g_k^i\}_{k \geq 1} \subset \mathfrak{H}^{\odot m}$, be a sequence of kernels such that: (i) as $k \rightarrow \infty$, $m! \|g_k^i\|_{\mathfrak{H}^{\otimes m}}^2 \rightarrow 1$, (ii) $\{I_m(g_k^i)\}_{k \geq 1}$ converges in distribution to $N \sim \mathcal{N}(0, 1)$, and (iii) for every fixed k , the random variables $I_m(g_k^i)$, $i = 1, \dots, \nu$, are mutually independent (of course, condition (iii) is immaterial when $\nu = 1$). Then, the sequence $\{I_{2m}(f_k)\}_{k \geq 1}$, where $f_k = \sum_{i=1}^{\nu} g_k^i \tilde{\otimes} g_k^i \in \mathfrak{H}^{\odot 2m}$, converges in distribution to $F(\nu) \stackrel{\text{Law}}{=} \sum_{i=1}^{\nu} (N_i^2 - 1)$, where (N_1, \dots, N_ν) is a vector of i.i.d. $\mathcal{N}(0, 1)$ random variables.*

Proof. Since $I_m(g_k^i)$ converges in law to N , we deduce from Theorem 1.1 that, for every $p = 1, \dots, m-1$, $\lim_{k \rightarrow \infty} \|g_k^i \otimes_p g_k^i\|_{\mathfrak{H}^{\otimes 2(m-p)}} = 0$. The conclusion is once again a consequence of the multiplication formulae for multiple Wiener-Itô integrals, yielding

$$\sum_{i=1}^{\nu} I_m(g_k^i)^2 = \sum_{i=1}^{\nu} m! \|g_k^i\|_{\mathfrak{H}^{\otimes m}}^2 + \sum_{i=1}^{\nu} \sum_{p=1}^{m-1} p! \binom{m}{p}^2 I_{2(m-p)}(g_k^i \tilde{\otimes}_p g_k^i) + \sum_{i=1}^{\nu} I_{2m}(g_k^i \tilde{\otimes} g_k^i).$$

Since $\sum_{i=1}^{\nu} I_{2m}(g_k^i \tilde{\otimes} g_k^i) = I_{2m}(f_k)$ (by linearity) and $\|g_k^i \tilde{\otimes}_p g_k^i\|_{\mathfrak{H}^{\otimes 2(m-p)}} \leq \|g_k^i \otimes_p g_k^i\|_{\mathfrak{H}^{\otimes 2(m-p)}}$, the conclusion is immediately obtained. \square

Remark 3.3 When k is fixed, a necessary and sufficient condition implying that the variables $I_m(g_k^i)$, $i = 1, \dots, \nu$, are mutually independent is that $g_k^i \otimes_1 g_k^j = 0$ for every $i \neq j$. See e.g. Üstünel and Zakai (1989).

(II) For $n \geq 4$, it is not possible to replace the condition $\lim_{k \rightarrow \infty} \|f_k \tilde{\otimes}_{n/2} f_k - c_n \times f_k\|_{\mathfrak{H}^{\otimes n}} = 0$ in Points (iii) and (iv) of Theorem 1.2 with its “non-symmetrized” version

$$\lim_{k \rightarrow \infty} \|f_k \otimes_{n/2} f_k - c_n \times f_k\|_{\mathfrak{H}^{\otimes n}} = 0. \quad (3.38)$$

Suppose indeed that $n \geq 4$ and that the kernels $\{f_k\}_{k \geq 1}$ verify (3.38), and moreover

$$n! \|f_k\|_{\mathfrak{H}^{\otimes n}}^2 \rightarrow 2 \quad (3.39)$$

$$\|f_k \otimes_p f_k\|_{\mathfrak{H}^{\otimes 2(n-p)}} \rightarrow 0, \quad \text{for } p = 1, \dots, n-1 \text{ with } p \neq n/2. \quad (3.40)$$

Then, by reasoning as in Section 3.3, it is not difficult to show that conditions (3.38)–(3.40) necessarily imply that

$$2 \|f_k \otimes_{n/2} f_k\|_{\mathfrak{H}^{\otimes n}} - \binom{n}{n/2} \|f_k \tilde{\otimes}_{n/2} f_k\|_{\mathfrak{H}^{\otimes n}} \rightarrow 0,$$

which is absurd, since one would also obtain $\|f_k \otimes_{n/2} f_k\|_{\mathfrak{H}^{\otimes n}} - c_n \|f_k\|_{\mathfrak{H}^{\otimes n}} \rightarrow 0$, $\|f_k \tilde{\otimes}_{n/2} f_k\|_{\mathfrak{H}^{\otimes n}} - c_n \|f_k\|_{\mathfrak{H}^{\otimes n}} \rightarrow 0$ and $\|f_k\|_{\mathfrak{H}^{\otimes n}} \rightarrow \sqrt{2\nu/n!}$.

(III) Plainly, relation (3.32) holds for multiple integrals of any order $n \geq 2$ (with n not necessarily even). Such an expression should be compared with the following formula, proved in Nualart and Peccati (2005, p. 183): for every $n \geq 2$ and every $f \in \mathfrak{H}^{\odot n}$,

$$\begin{aligned} E \left[I_n(f)^4 \right] &= 3n!^2 \|f\|_{\mathfrak{H}^{\otimes n}}^4 + \sum_{p=1}^{n-1} \frac{n!^4}{p!^2 (n-p)!^2} \left[\|f \otimes_p f\|_{\mathfrak{H}^{\otimes 2(n-p)}}^2 \right. \\ &\quad \left. + \binom{2n-2p}{n-p} \|f \tilde{\otimes}_p f\|_{\mathfrak{H}^{\otimes 2(n-p)}}^2 \right]. \end{aligned} \quad (3.41)$$

Note that it would be much more difficult to use (3.41) in order to prove the implication (ii) \rightarrow (iii) of Theorem 1.2, since (3.41) involves the norm $\|f \otimes_{n/2} f\|_{\mathfrak{H}^{\otimes n}}$ of the non-symmetrized contraction $f \otimes_{n/2} f$.

4 Second chaos sequences

In this section, we specialize the content of Theorem 1.2 to the case of sequences belonging to the second Wiener chaos of X . The first result provides a further characterization of the non-central convergence of the elements of the second chaos, based on the discussion contained in Section 2.2.

Theorem 4.1 *Let $\{I_2(f_k)\}_{k \geq 1}$, with $f_k \in \mathfrak{H}^{\odot 2}$, be such that $\lim_{k \rightarrow \infty} \|f_k\|_{\mathfrak{H}^{\otimes 2}}^2 = \nu$. Then, as $k \rightarrow \infty$, the six following conditions are equivalent:*

- (i) $I_2(f_k)$ converges in distribution to $F(\nu)$, as given by (1.4);
- (ii) $E[I_2(f_k)^4] \rightarrow 48\nu + 12\nu^2$, and $E[I_2(f_k)^3] \rightarrow 8\nu$.
- (iii) $E[I_2(f_k)^4] - 12E[I_2(f_k)^3] \rightarrow 12\nu^2 - 48\nu$;
- (iv) $\|f_k \tilde{\otimes}_1 f_k - f_k\|_{\mathfrak{H}^{\otimes 2}} = \|f_k \otimes_1 f_k - f_k\|_{\mathfrak{H}^{\otimes 2}} \rightarrow 0$;
- (v) $\sum_{j=1}^{\infty} (\lambda_{f_k,j} - \lambda_{f_k,j}^2)^2 \rightarrow 0$, where, for $k \geq 1$, $\{\lambda_{f_k,j}\}_{j \geq 1}$ stands for the sequence of the eigenvalues of the operator H_{f_k} , as defined in formula (2.16);
- (vi) for every $p \geq 3$, $\sum_{j=1}^{\infty} \lambda_{f_k,j}^p \rightarrow \nu$.

Proof. The equivalence of (i), (ii), (iii) and (iv) is a direct consequence of Theorem 1.2 and of the fact that $c_2 = 1$ (see formula (1.5)). To prove the equivalence of (iv) and (v), we use relation (2.19) to deduce that

$$f_k \otimes_1 f_k = \sum_{j=1}^{\infty} \lambda_{f_k,j}^2 e_{f_k,j} \otimes e_{f_k,j},$$

where $\{e_{f_k,j}\}_{j \geq 1}$ is the sequence of the eigenvectors of H_{f_k} . It is therefore immediate that

$$\|f_k \otimes_1 f_k - f_k\|_{\mathfrak{H}^{\otimes 2}}^2 = \sum_{j=1}^{\infty} (\lambda_{f_k,j}^2 - \lambda_{f_k,j})^2, \quad (4.42)$$

which yields the desired implication. We are left with the double implication (v) \leftrightarrow (vi). To prove this, suppose first that (v) holds and observe that, by assumption, $\sum_{j=1}^{\infty} \lambda_{f_k,j}^2 = \|f_k\|_{\mathfrak{H}^{\otimes 2}}^2 \rightarrow \nu$. The proof of (v) \rightarrow (vi) can now be achieved by recurrence. Indeed, if $\sum_{j=1}^{\infty} \lambda_{f_k,j}^{2+m} \rightarrow \nu$ for some $m \geq 0$, then

$$\begin{aligned} & \left| \sum_{j=1}^{\infty} \lambda_{f_k,j}^{2+m+1} - \sum_{j=1}^{\infty} \lambda_{f_k,j}^{2+m} \right| = \left| \sum_{j=1}^{\infty} \lambda_{f_k,j}^{m+1} (\lambda_{f_k,j}^2 - \lambda_{f_k,j}) \right| \\ & \leq \sqrt{\sum_{j=1}^{\infty} \lambda_{f_k,j}^{2m+2}} \sqrt{\sum_{j=1}^{\infty} (\lambda_{f_k,j}^2 - \lambda_{f_k,j})^2} \rightarrow 0, \end{aligned}$$

thus yielding that (iv) \rightarrow (v). To see that (vi) \rightarrow (v), just write

$$\sum_{j=1}^{\infty} (\lambda_{f_k,j} - \lambda_{f_k,j}^2)^2 = \sum_{j=1}^{\infty} \lambda_{f_k,j}^2 + \sum_{j=1}^{\infty} \lambda_{f_k,j}^4 - 2 \sum_{j=1}^{\infty} \lambda_{f_k,j}^3.$$

□

A first consequence of Theorem 4.1 is the following simple characterization of centered χ^2 random variables (with any number of degrees of freedom) in the second chaos of X .

Corollary 4.2 *Fix an integer $\nu \geq 1$. Let $I_2(f)$, with $f \in \mathfrak{H}^{\odot 2}$, be such that $E(I_2(f)^2) = 2\|f\|_{\mathfrak{H}^{\otimes 2}}^2 = 2\nu$. Then, the following conditions are equivalent:*

- (i) $I_2(f) \stackrel{\text{Law}}{=} F(\nu)$, where $F(\nu)$ has a centered χ^2 distribution with ν degrees of freedom (see (1.4));
- (ii) $E[I_2(f)^4] - 12E[I_2(f)^3] = 12\nu^2 - 48\nu$;
- (iii) $f = f \otimes_1 f$;
- (iv) *there exists $h^i \in \mathfrak{H}$, $i = 1, \dots, \nu$, such that $\|h^i\|_{\mathfrak{H}} = 1$, $\langle h^i, h^j \rangle_{\mathfrak{H}} = 0$ ($i \neq j$), and $I_2(f) = \sum_{i=1}^{\nu} I_2(h^i \otimes h^i) = \sum_{i=1}^{\nu} (I_1(h^i)^2 - 1)$.*

Proof. In view of the previous discussion, the only non-immediate implication is (iii) \rightarrow (iv). To prove this, we can use (4.42) to deduce that, if (iii) is verified, then, for every $j \geq 1$, $\lambda_{f,j} = \lambda_{f,j}^2$. Since $\sum_{j=1}^{\infty} \lambda_{f,j}^2 = \nu$ by assumption, we deduce immediately that there exist indices j_1^*, \dots, j_{ν}^* such that $\lambda_{f,j_a^*} = 1$, for every $a = 1, \dots, \nu$, and $\lambda_{f,j} = 0$ for every $j \notin \{j_1^*, \dots, j_{\nu}^*\}$. The conclusion is obtained by using formula (2.19). □

For instance, Corollary 4.2 yields that two centered χ^2 random variables in the second chaos of X (even with a different number of degrees of freedom) are independent if, and only if, they are non-correlated. Indeed, any two centered χ^2 random variables F and G in the second chaos of X have necessarily the form

$$F = I_2 \left(\sum_{i=1}^M f^i \otimes f^i \right) = \sum_{i=1}^M (I_1(f^i)^2 - 1) \quad \text{and} \quad G = I_2 \left(\sum_{j=1}^K g^j \otimes g^j \right) = \sum_{j=1}^K (I_1(g^j)^2 - 1),$$

where $M, K \geq 1$ are integers. As a consequence,

$$E(FG) = 2 \left\langle \sum_{i=1}^M f^i \otimes f^i, \sum_{j=1}^K g^j \otimes g^j \right\rangle_{\mathfrak{H}^{\otimes 2}} = 2 \sum_{i=1}^M \sum_{j=1}^K \langle f^i, g^j \rangle_{\mathfrak{H}}^2,$$

and the last sum is zero if, and only if, $I_1(f^i)$ and $I_1(g^j)$ are independent for every i, j such that $1 \leq i \leq M$ and $1 \leq j \leq K$.

The next result, which is a consequence of Theorem 4.1, implies that any centered Gamma random variable in the second chaos of X must necessarily be a multiple of a centered χ^2 .

Corollary 4.3 *Let $I_2(f)$, with $f \in \mathfrak{H}^{\odot 2}$, be such that $E(I_2(f)^2) = 2\|f\|_{\mathfrak{H}^{\otimes 2}}^2 = 2\nu$, for some $\nu > 0$. If $I_2(f) \stackrel{\text{Law}}{=} 2G(\nu/2) - \nu$, where $G(\nu/2)$ is a Gamma random variable with parameter $\nu/2$, then necessarily ν is an integer, and $I_2(f)$ has a χ^2 distribution with ν degrees of freedom.*

Proof. By arguing as in the proof of Corollary 4.2, one sees immediately that the relation $I_2(f) \stackrel{\text{Law}}{=} 2G(\nu/2) - \nu$ implies that, for every $j \geq 1$, $\lambda_{f,j} = \lambda_{f,j}^2$, that is, $\lambda_{f,j}$ is either 0 or 1. Since, by assumption,

$$\sum_{j=1}^{\infty} \lambda_{f,j}^2 = \|f\|_{\mathfrak{H}^{\otimes 2}}^2 = \nu,$$

we deduce that ν must necessarily be an integer. The conclusion follows. \square

Further results on second chaos sequences are contained in the next section.

5 Multidimensional results

For the sake of completeness, we start by stating the following multidimensional generalization of Theorem 1.1, proved in Peccati and Tudor (2005):

Theorem 5.1 *For $d \geq 2$, fix d integers $1 \leq n_1 \leq \dots \leq n_d$. Consider a sequence of random vectors of the form*

$$F_k = (I_{n_1}(f_k^1), \dots, I_{n_d}(f_k^d))$$

where $f_k^i \in \mathfrak{H}^{\otimes n_i}$. For every $1 \leq i, j \leq d$, assume that $\lim_{k \rightarrow \infty} E[I_{n_j}(f_k^j)I_{n_i}(f_k^i)] = \delta_{ij}$, where δ_{ij} is the Kronecker symbol. Then, the following two conditions are equivalent:

- (i) as $k \rightarrow \infty$, the sequence $\{F_k\}_{k \geq 1}$ converges in distribution to a standard centered Gaussian vector $\mathcal{N}_d(0, I_d)$ (I_d is the $d \times d$ identity matrix),
- (ii) for every $j = 1, \dots, d$, as $k \rightarrow \infty$, the sequence $\{I_{n_j}(f_k^j)\}_{k \geq 1}$ converges in distribution to a standard Gaussian random variable $\mathcal{N}(0, 1)$.

The content of Theorem 1.2 can be extended to the multidimensional case, as shown in the following statement. The proof is obtained by simply mimicking the arguments displayed in Nualart and Ortiz-Latorre (2007, proof of Th. 7), and it is therefore omitted.

Theorem 5.2 *Let $r \geq 1$ and $s \geq 1$ be integers, and fix $r+s$ integers $n_1, \dots, n_r, n_{r+1}, \dots, n_{r+s} \geq 1$ such that n_{r+u} is even and ≥ 2 for every $u = 1, \dots, s$. Let also $\nu_1, \dots, \nu_s > 0$ be strictly positive real numbers. Consider a sequence of random vectors of the form*

$$I_k = (I_{n_1}(f_k^1), \dots, I_{n_r}(f_k^r), I_{n_{r+1}}(f_k^{r+1}), \dots, I_{n_{r+s}}(f_k^{r+s}),)$$

where $f_k^i \in \mathfrak{H}^{\otimes n_i}$, $i = 1, \dots, r+s$. We suppose the following: (A) for every i , $\lim_{k \rightarrow \infty} n_i! \|f_k^i\|_{\mathfrak{H}^{\otimes n_i}}^2 = \epsilon_i$, where ϵ_i equals 1 or $2\nu_i$, according as $1 \leq i \leq r$, or $r+1 \leq i \leq r+s$, and (B) for every pair (i, j) such that $i \neq j$

$$\lim_{k \rightarrow \infty} E\langle D[I_{n_i}(f_k^i)], D[I_{n_j}(f_k^j)] \rangle_{\mathfrak{H}}^2 = 0. \quad (5.43)$$

Then, the following two conditions are equivalent, as $k \rightarrow \infty$:

(i) the sequence $\{I_k\}_{k \geq 1}$ converges in distribution to

$$(N_1, \dots, N_r, F(\nu_1), \dots, F(\nu_s)),$$

where $(N_1, \dots, N_r, F(\nu_1), \dots, F(\nu_s))$ denotes a $(r + s)$ -dimensional vector of independent random variables, such that each N_ℓ , $\ell = 1, \dots, r$, has the law $\mathcal{N}(0, 1)$ and each $F(\nu_\ell)$, $\ell = 1, \dots, s$, has the law specified by formula (1.4) in the case $\nu = \nu_\ell$;

(ii) for every $j = 1, \dots, r$, the sequence $\{I_{n_j}(f_k^j)\}_{k \geq 1}$ converges in distribution to $N \sim \mathcal{N}(0, 1)$, while, for every $\ell = r + 1, \dots, r + s$, the sequence $\{I_{n_\ell}(f_k^\ell)\}_{k \geq 1}$ converges in distribution to $F(\nu_\ell)$.

Remark 5.3 In Lemma 6 of Nualart and Ortiz-Latorre (2007), it is proved that, if one assumes that condition (ii) in the previous statement is true for every $1 \leq i \leq r$, and if moreover $E(I_{n_i}(f_k^i)I_{n_j}(f_k^j)) \rightarrow \delta_{ij}$, $1 \leq i \neq j \leq r$, where δ_{ij} is the Kronecker symbol, then (5.43) is automatically verified for every $1 \leq i \neq j \leq r$.

Our applications of Theorem 5.2 will be mainly based on the following relation, proved in Nualart and Ortiz-Latorre (2007, Lemma 6): for every $n, m \geq 1$ and every $f \in \mathfrak{H}^{\odot n}$, $g \in \mathfrak{H}^{\odot m}$,

$$\begin{aligned} E\langle D[I_n(f)], D[I_m(g)] \rangle_{\mathfrak{H}}^2 &= \sum_{r=1}^{n \wedge m} \frac{m!^2 n!^2 (n+m-2r)!}{(n-r)!^2 (m-r)!^2 (r-1)!^2} \|f \tilde{\otimes}_r g\|_{\mathfrak{H}^{\otimes(n+m-2r)}}^2 \\ &\leq \sum_{r=1}^{n \wedge m} \frac{n!^2 m!^2 (n+m-2r)!}{(n-r)!^2 (m-r)!^2 (r-1)!^2} \|f \otimes_r g\|_{\mathfrak{H}^{\otimes(n+m-2r)}}^2. \end{aligned} \quad (5.44)$$

The first consequence of Theorem 5.2 is a ‘‘stable convergence’’ generalization of Theorem 1.2 in the case of integrals of even order $n \geq 4$.

Proposition 5.4 *Let $n \geq 4$ be an even integer, and let $\{I_n(f_k)\}_{k \geq 1}$ be a sequence of multiple integrals verifying (1.6) and satisfying either one of conditions (i)–(vi) of Theorem 1.2. Then, as $k \rightarrow \infty$, the pair $(I_n(f_k), X)$ converges in law to $(F(\nu), X)$, where $F(\nu) \stackrel{\text{Law}}{=} 2G(\nu/2) - \nu$ is independent of X .*

Proof. According to Theorem 5.2 and the inequality (5.44), it is sufficient to prove that, for every $h \in \mathfrak{H}$,

$$\|f_k \otimes_1 h\|_{\mathfrak{H}^{\otimes(n-1)}} \rightarrow 0.$$

By the definitions of the contractions of order 1 and $n - 1$, one has that

$$\|f_k \otimes_1 h\|_{\mathfrak{H}^{\otimes(n-1)}}^2 = \langle f_k \otimes_{n-1} f_k, h \otimes h \rangle_{\mathfrak{H}^{\otimes 2}} \leq \|f_k \otimes_{n-1} f_k\|_{\mathfrak{H}^{\otimes 2}} \|h\|_{\mathfrak{H}}^2 \rightarrow 0,$$

where the last convergence is a consequence of Point (iv) in Theorem 1.2, and of the fact that $n > 2$. \square

Remark 5.5 When $n = 2$, the statement of Proposition 5.4 is not true in general. As a counterexample, one can consider a sequence $I_2(f_k)$, $k \geq 1$, such that $f_k = h \otimes h$ for every k , and $\|h\|_{\mathfrak{H}} = 1$. The stable convergence of sequences in the second chaos of X is discussed in the forthcoming Proposition 5.7.

Proposition 5.4 can be reformulated by saying that $I_n(f_k)$ converges $\sigma(X)$ -stably to $F(\nu)$ (see e.g. Jacod and Shiryaev (1987) for an exhaustive discussion of stable convergence). Moreover, Proposition 5.4 yields a refinement of a well-known result (see e.g. Janson (1997, Ch. VI, Corollary 6.13)), stating that Wiener chaoses of order $n > 2$ do not contain any Gamma random variable (our refinement consists in a further restriction on moments).

Corollary 5.6 *Fix a real $\nu > 0$ and an even integer $n \geq 4$. Let $I_n(f)$ be such that $E(I_n(f)^2) = 2\nu$. Then, $I_n(f)$ cannot be equal in law to $2G(\nu/2) - \nu$, where $G(\nu/2)$ stands for a Gamma random variable of parameter $\nu/2$, and $E(I_n(f)^4) - 12E(I_n(f)^3) > 12\nu^2 - 48\nu$.*

Proof. According to Proposition 5.4, if $I_n(f)$ was equal in law to $2G(\nu/2) - \nu$ (or if $E(I_n(f)^4) - 12E(I_n(f)^3) = 12\nu^2 - 48\nu$), then $I_n(f)$ would be independent of X . Plainly, this is only possible if $f = 0$, which is absurd, since $\|f\|_{\mathfrak{H}^{\otimes n}}^2 = 2\nu/n!$. The fact that $E(I_n(f)^4) - 12E(I_n(f)^3)$ cannot be less than $12\nu^2 - 48\nu$ derives from a straightforward modification of the calculations following formula (3.33). \square

The following result characterizes the stable convergence of double integrals. The proof (omitted) is analogous to that of Proposition 5.4.

Proposition 5.7 *Fix $\nu > 0$. Let the sequence $\{I_2(f_k)\}_{k \geq 1}$ be such that $E(I_2(f_k)^2) \rightarrow 2\nu$ and either one of conditions (i)–(vi) of Theorem 4.1 are satisfied. Then, for every $h \in \mathfrak{H}$, $(I_1(h), I_2(f_k))$ converges in law to $(I_1(h), F(\nu))$, where $F(\nu) \stackrel{\text{Law}}{=} 2G(\nu/2) - \nu$ is independent of $I_1(h)$, if, and only if,*

$$\langle f_k \otimes_1 f_k, h \otimes h \rangle_{\mathfrak{H}^{\otimes 2}} \rightarrow 0. \quad (5.45)$$

In particular, (5.45) is verified for every $h \in \mathfrak{H}$ if, and only if, $I_2(f_k)$ is asymptotically independent of X .

The next result gives sufficient conditions under which one can have a neat non-CLT, analogous to the central result of Theorem 5.1. Note that, in the following statement, all multiple integrals have different orders, so that no covariance conditions are assumed. Observe also that the second part of condition (5.46) below can be replaced by either one of the equivalent conditions (i)–(vi) of Theorem 1.2.

Proposition 5.8 *For $d \geq 2$, fix d strictly positive reals $\nu_1, \dots, \nu_d > 0$, as well as d even numbers $2 \leq n_1 < \dots < n_d$. Suppose that $2n_i \neq n_j$ for every $i \neq j$. Consider a sequence of random vectors of the form*

$$I_k = (I_{n_1}(f_k^1), \dots, I_{n_d}(f_k^d))$$

where $f_k^i \in \mathfrak{H}^{\otimes n_i}$. Assume finally that, for every $1 \leq i \leq d$,

$$\lim_{k \rightarrow \infty} E[I_{n_i}(f_k^i)^2] = 2\nu_i \quad \text{and} \quad \lim_{k \rightarrow \infty} E[I_{n_i}(f_k^i)^4] - 12E[I_{n_i}(f_k^i)^3] = 12\nu_i^2 - 48\nu_i. \quad (5.46)$$

Then, as $k \rightarrow \infty$, the sequence $\{I_k\}_{k \geq 1}$ converges in distribution to

$$(F(\nu_1), \dots, F(\nu_d)),$$

where $(F(\nu_1), \dots, F(\nu_d))$ denotes a vector of independent random variables such that, for $i = 1, \dots, d$, the variable $F(\nu_i)$ verifies (1.4) for $\nu = \nu_i$.

Proof. The conclusion follows once we show that the asymptotic condition (5.43) holds for every $1 \leq i \neq j \leq d$. Thanks to (5.44) one has that

$$E \langle D[I_{n_i}(f_k^i)], D[I_{n_j}(f_k^j)] \rangle_{\mathfrak{H}}^2 \leq \sum_{r=1}^{n_i} \frac{n_i!^2 n_j!^2 (n_i + n_j - 2r)!}{(n_i - r)!^2 (n_j - r)!^2 (r - 1)!^2} \|f_k^i \otimes_r f_k^j\|_{\mathfrak{H}^{\otimes(n_i+n_j-2r)}}^2.$$

Using the definitions of the contractions, we also have

$$\|f_k^i \otimes_r f_k^j\|_{\mathfrak{H}^{\otimes(n_i+n_j-2r)}}^2 = \langle f_k^i \otimes_{n_i-r} f_k^i, f_k^j \otimes_{n_j-r} f_k^j \rangle_{\mathfrak{H}^{\otimes 2r}}$$

which yields, by the Cauchy-Schwarz inequality:

$$\|f_k^i \otimes_r f_k^j\|_{\mathfrak{H}^{\otimes(n_i+n_j-2r)}}^2 \leq \|f_k^i \otimes_{n_i-r} f_k^i\|_{\mathfrak{H}^{\otimes 2r}} \|f_k^j \otimes_{n_j-r} f_k^j\|_{\mathfrak{H}^{\otimes 2r}}. \quad (5.47)$$

In the case $1 \leq r \leq n_i - 1$, the RHS of equation (5.47) tends to zero as $k \rightarrow \infty$ by Theorem 1.2, because, on one hand, either $n_i - r \in \{1, \dots, n_i - 1\} \setminus \{\frac{n_i}{2}\}$ or $n_j - r \in \{1, \dots, n_j - 1\} \setminus \{\frac{n_j}{2}\}$ (recall that $n_i \neq n_j$ for every $i \neq j$) and, on the other hand, for $\ell \in \{i, j\}$:

$$\sup_{k \geq 1} \|f_k^\ell \otimes_{n_\ell-r} f_k^\ell\|_{\mathfrak{H}^{\otimes 2r}} \leq \sup_{k \geq 1} \|f_k^\ell\|_{\mathfrak{H}^{\otimes n_\ell}}^2 < +\infty.$$

In the case $r = n_i < n_j$, one has that the RHS of equation (5.47) equals

$$\|f_k^i\|_{\mathfrak{H}^{\otimes n_i}}^2 \|f_k^j \otimes_{n_j-n_i} f_k^j\|_{\mathfrak{H}^{\otimes 2n_i}},$$

which tends to zero as $k \rightarrow \infty$, because $n_j - n_i \in \{1, \dots, n_j - 1\} \setminus \{\frac{n_j}{2}\}$ (recall that $2n_i \neq n_j$ for every $i \neq j$). These two facts imply that $E \langle D[I_{n_i}(f_k^i)], D[I_{n_j}(f_k^j)] \rangle_{\mathfrak{H}}^2$ tends to zero as $k \rightarrow \infty$, which yields the announced result. \square

We conclude this section by pointing out that, due to representation (2.22), Theorem 5.2 and Proposition 5.8 can be used to study the convergence of functionals towards more general laws in the second chaos of X . A simple illustration of this fact is the following result.

Proposition 5.9 *Let $n_2 > n_1 \geq 2$ be even integers such that $n_2 \neq 2n_1$. For $i = 1, 2$, consider a sequence $\{I_{n_i}(f_k^i)\}_{k \geq 1}$, with $f_k^i \in \mathfrak{H}^{\otimes n_i}$, such that condition (5.46) is satisfied for $\nu_1 = \nu_2 = 1$. Then, as $k \rightarrow \infty$, the sequence*

$$\frac{1}{2} \times (I_{n_1}(f_k^1) - I_{n_2}(f_k^2)), \quad k \geq 1,$$

converges in law to $N \times N'$ where N, N' are two independent $\mathcal{N}(0, 1)$ random variables.

Proof. Proposition 5.8 implies immediately that $(I_{n_1}(f_k^1) - I_{n_2}(f_k^2))$ converges in law to $N^2 - (N')^2$, and the result is obtained by using the elementary identity in law

$$\frac{1}{2}(N^2 - (N')^2) \stackrel{\text{Law}}{=} N \times N'.$$

\square

Remark 5.10 Suppose that the sequences $\{I_{n_i}(f_k^i)\}_{k \geq 1}$, $i = 1, 2$, are such that (1.2) is verified, and that either one of conditions (i)-(iii) of Theorem 1.1 holds. Then, according to Theorem 5.1, the product $I_{n_1}(f_k^1) \times I_{n_2}(f_k^2)$ also converges in distribution to $N \times N'$.

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