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Chen's double sieve, Goldbach's conjecture and the twin prime problem, 2

J. Wu

Abstract. For every even integer N, denote by $D_{1,2}(N)$ the number of representations of N as a sum of a prime and an integer having at most two prime factors. In this paper, we give a new lower bound for $D_{1,2}(N)$.

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§ 1. Introduction

Let $\Omega(n)$ be the number of all prime factors of the integer n with the convention $\Omega(1) = 0$. For each even integer $N \geqslant 4$, we define

$$D(N) := |\{p \leqslant N : \Omega(N-p) = 1\}|,$$

where and in what follows, the letter p, with or without subscript, denotes a prime number. The well known Goldbach conjecture can be stated as $D(N) \geqslant 1$ for all even integers $N \geqslant 4$. A more precise version of this conjecture was proposed by Hardy & Littlewood [10]:

$$D(N) \sim 2\Theta(N) \qquad (N \to \infty), \tag{1.1}$$

where

$$C_N := \prod_{p|N,\, p>2} \frac{p-1}{p-2} \prod_{p>2} \Big(1 - \frac{1}{(p-1)^2}\Big), \qquad \Theta(N) := \frac{C_N N}{(\log N)^2}. \tag{1.2}$$

Certainly, the asymptotic formula (1.1) is extremely difficult. One way of approaching the lower bound problem in (1.1) is to give a non-trivial lower bound for the quantity

$$D_{1,2}(N) := |\{p \leqslant N : \Omega(N-p) \leqslant 2\}|.$$

In this direction, Chen [5] proved, by his system of weights and the switching principle, the following famous theorem: *Every sufficiently large even integer can be written as sum of a prime and an integer having at most two prime factors.* More precisely he established

$$D_{1,2}(N) \geqslant 0.67\,\Theta(N) \tag{1.3}$$

for $N \geqslant N_0$. As Halberstam & Richert indicated in [9], it would be interesting to know whether a more elaborate weighting procedure could be adapted to the purpose of (1.3). This might lead

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to numerical improvements and could be important. Chen's constant 0.67 has been improved by many authors. The historical record is as follows:

0.689 by Halberstam & Richert [9],
0.754 by Chen [6],
0.81 by Chen [7],
0.828 by Cai & Lu [4],
0.836 by Wu [13],
0.867 by Cai [2].

The aim of this paper is to propose a better constant.

Theorem. *For sufficiently large N, we have*

$$D_{1,2}(N) \geqslant 0.899\,\Theta(N).$$

Our improvement comes from a delicate application of Chen's double sieve ([8], [12], [13]), which can be described as follows: With standard notation in theory of sieve method, the linear sieve formulas (see [9], or Lemma 2.2 of [13]) can be stated as

$$XV(z)f\left(\frac{\log Q}{\log z}\right) + \text{error} \leqslant S(\mathcal{A};\mathcal{P},z) \leqslant XV(z)F\left(\frac{\log Q}{\log z}\right) + \text{error}. \tag{1.4}$$

These inequalities are the best possible in the sense that taking

$$\mathcal{A} = \mathcal{B}_\nu := \{n \leqslant x : \Omega(n) \equiv \nu \,(\mathrm{mod}\, 2)\} \qquad (\nu = 1, 2),$$

the upper and lower bounds in (1.4) are respectively attained by $\nu = 1$ and $\nu = 2$ (see [9], page 239). Aiming at a better Bombieri-Davenport's upper bound [1]

$$D(N) \leqslant \{8 + o(1)\}\Theta(N),$$

Chen [8] found improvement for (1.4) for some special sequences $\mathcal{A}$. Roughly speaking, for the sequence

$$\mathcal{A} = \{N - p : p \leqslant N\}$$

he narrowed down the gap in (1.4) by introducing two functions $h(s)$ and $H(s)$ such that the functions $sf(s)/(2e^\gamma)$ and $sF(s)/(2e^\gamma)$ are replaced by $sf(s)/(2e^\gamma)+h(s)$ and $sF(s)/(2e^\gamma)-H(s)$ respectively, where γ is the Euler constant. The key point is thus to prove $h(s) > 0$ and $H(s) > 0$. Chen's proof is very long and somewhat difficult to follow, but his innovative idea is clear (see [11] for example). In [13], we gave a more comprehensive treatment on this method and name it as Chen's double sieve. Indeed, our treatment is not only simpler but even more powerful than Chen's. Our approach improved Chen's upper estimate $D(N) \leqslant 7.8342\Theta(N)$ to $D(N) \leqslant 7.8209\Theta(N)$. It is worth to indicate that Chen's record stood for 26 years before our work [13].

To prove our Theorem, we first simplify and improve Chen's weight system (compare (12) of [7] and Lemma 2.2 below), and then apply Chen's double sieve, as the classical linear sieve,

to handle terms such as Υ_2, Υ_3, Υ_4, Υ_5 and Υ_6 (cf. Propositions 4.1, 4.2, 4.3 and 4.4 below). The idea of using Chen's double sieve to treat sums of the type

$$\sum_{\substack{N^{\phi_1}\leqslant p<N^{\phi_2}\\(p,N)=1}} S(\mathcal{A}_p;\mathcal{P}(N),N^{\kappa}) \tag{1.5}$$

was first appeared in [12]. However, due to the first condition in (3.1) below, a direct application of our Chen's double sieve can only handle the initial part of the sum over small p in (1.5) (i.e. $p\leqslant N^{1/4}$). On the other hand, very recently Cai [2] used a similar idea to control the sum over large p in (1.5). Actually his method can be viewed as a simplified version of Chen's double sieve (see Proposition 4.4 below and the comments before it). Here we shall combine both versions and refine them to obtain our result. Apparently from the proof, we shall see that the first version gives a saving of 0.0211 while the second saves 0.0078. Without Chen's double sieve technique, we still obtain 0.870 in place of 0.899, which is slightly better than Cai's 0.867.

Clearly our method can be used to refine the corresponding constants in the conjugate problems ([2] and [3]). The proofs are very similar and even easier and simpler. Hence we omit the relevant discussion. Maybe this is a good exercise for senior graduate students in analytic number theory.

§ 2. Chen's system of weights

This section is devoted to discuss the weighted sieve of Chen type. Let

$$\mathcal{A}:=\{N-p: p\leqslant N\}\qquad\text{and}\qquad \mathcal{P}(N):=\{p:(p,N)=1\}.$$

The sieve function is defined as

$$S(\mathcal{A};\mathcal{P}(N),z):=|\{a\in\mathcal{A}:(a,P(z))=1\}|,$$

where $P(z):=\prod_{p\leqslant z,\,p\in\mathcal{P}(N)}p$.

Lemma 2.1. *Let $0<\kappa<\sigma\leqslant\frac13$. Then we have*

$$2D_{1,2}(N)\geqslant 2S(\mathcal{A};\mathcal{P}(N),N^{\kappa})-S_1(\kappa,\sigma)-2S_2(\kappa,\sigma)-S_3(\kappa,\sigma)+S_4(\kappa,\sigma)+O(N^{1-\kappa}), \tag{2.1}$$

where

$$\begin{aligned}
S_1(\kappa,\sigma)&:=\sum_{\substack{N^{\kappa}\leqslant p<N^{\sigma}\\(p,N)=1}} S(\mathcal{A}_p;\mathcal{P}(N),N^{\kappa}),\\
S_2(\kappa,\sigma)&:=\sum_{\substack{N^{\sigma}\leqslant p_1<p_2<(N/p_1)^{1/2}\\(p_1p_2,N)=1}}\sum S(\mathcal{A}_{p_1p_2};\mathcal{P}(Np_1),p_2),\\
S_3(\kappa,\sigma)&:=\sum_{\substack{N^{\kappa}\leqslant p_1<N^{\sigma}\leqslant p_2<(N/p_1)^{1/2}\\(p_1p_2,N)=1}}\sum S(\mathcal{A}_{p_1p_2};\mathcal{P}(Np_1),p_2),\\
S_4(\kappa,\sigma)&:=\sum_{\substack{N^{\kappa}\leqslant p_1<p_2<p_3<N^{\sigma}\\(p_1p_2p_3,N)=1}}\sum\sum S(\mathcal{A}_{p_1p_2p_3};\mathcal{P}(Np_1),p_2).
\end{aligned}$$

The inequality (2.1) first appeared in [7] (page 479, (11)) with $(\kappa,\sigma)=(\frac{1}{12},\frac{1}{3.047}),(\frac{1}{9.2},\frac{1}{3.41})$ without proof. Cai & [Lu] [4] gave a proof with an extra assumption $3\sigma+\kappa>1$. In [13], we

proved (2.1) under the hypothesis $0 < \kappa < \sigma < \frac{1}{3}$. Clearly the proof there is also valid for $\sigma = \frac{1}{3}$. Very recently Cai [2] gave another proof for Lemma 2.1.

As in [7], we shall apply (2.1) with two different pairs of parameters (κ, σ) to take advantage of $S_4(\kappa, \sigma)$. Our weighted sieve is simpler and more poweful than those of Chen ([7], (12)) and Cai ([2], Lemma 6).

Lemma 2.2. *Let $\kappa_2 > \kappa_1 \geqslant 1/18$ such that $3\kappa_1 + \kappa_2 < 1/2$ and $3\kappa_1 - \kappa_2 < 1/6$. Then we have*

$$4D_{1,2}(N) \geqslant 3\Upsilon_1 + \Upsilon_2 - \Upsilon_3 - \Upsilon_4 + \Upsilon_5 + \Upsilon_6 - 2\Upsilon_7 - \Upsilon_8 - \Upsilon_9 - \Upsilon_{10} - \Upsilon_{11} + O(N^{1-\kappa_1}), \tag{2.2}$$

where

$$\Upsilon_i := S(\mathcal{A}; \mathcal{P}(N), N^{\kappa_i}) \quad (i = 1, 2),$$

$$\Upsilon_3 := \sum_{\substack{N^{\kappa_1} \leqslant p < N^{1/3} \\ (p,N)=1}} S(\mathcal{A}_p; \mathcal{P}(N), N^{\kappa_1}),$$

$$\Upsilon_4 := \sum_{\substack{N^{\kappa_1} \leqslant p < N^{1/2-3\kappa_1} \\ (p,N)=1}} S(\mathcal{A}_p; \mathcal{P}(N), N^{\kappa_1}),$$

$$\Upsilon_5 := \sum_{\substack{N^{\kappa_1} \leqslant p_1 < p_2 < N^{\kappa_2} \\ (p_1p_2,N)=1}} \sum S(\mathcal{A}_{p_1p_2}; \mathcal{P}(N), N^{\kappa_1}),$$

$$\Upsilon_6 := \sum_{\substack{N^{\kappa_1} \leqslant p_1 < N^{\kappa_2} \leqslant p_2 < N^{1/2-3\kappa_1} \\ (p_1p_2,N)=1}} \sum S(\mathcal{A}_{p_1p_2}; \mathcal{P}(N), N^{\kappa_1}),$$

$$\Upsilon_7 := \sum_{\substack{N^{1/2-3\kappa_1} \leqslant p_1 < p_2 < (N/p_1)^{1/2} \\ (p_1p_2,N)=1}} \sum S(\mathcal{A}_{p_1p_2}; \mathcal{P}(Np_1), p_2),$$

$$\Upsilon_8 := \sum_{\substack{N^{\kappa_1} \leqslant p_1 < N^{1/3} \leqslant p_2 < (N/p_1)^{1/2} \\ (p_1p_2,N)=1}} \sum S(\mathcal{A}_{p_1p_2}; \mathcal{P}(Np_1), p_2),$$

$$\Upsilon_9 := \sum_{\substack{N^{\kappa_2} \leqslant p_1 < N^{1/2-3\kappa_1} \leqslant p_2 < (N/p_1)^{1/2} \\ (p_1p_2,N)=1}} \sum S\big(\mathcal{A}_{p_1p_2}; \mathcal{P}(Np_1), (N/p_1p_2)^{1/2}\big),$$

$$\Upsilon_{10} := \sum_{\substack{N^{\kappa_1} \leqslant p_1 < p_2 < p_3 < p_4 < N^{\kappa_2} \\ (p_1p_2p_3p_4,N)=1}} \sum\sum\sum\sum S(\mathcal{A}_{p_1p_2p_3p_4}; \mathcal{P}(N), p_2),$$

$$\Upsilon_{11} := \sum_{\substack{N^{\kappa_1} \leqslant p_1 < p_2 < p_3 < N^{\kappa_2} \leqslant p_4 < N^{1/2-2\kappa_1}/p_3 \\ (p_1p_2p_3p_4,N)=1}} \sum\sum\sum\sum S(\mathcal{A}_{p_1p_2p_3p_4}; \mathcal{P}(N), p_2).$$

Proof. By noticing that our hypothesis implies $\kappa_2 < 1/2 - 3\kappa_1 \leqslant 1/3$, we can apply (2.1) with $(\kappa, \sigma) = (\kappa_2, 1/2 - 3\kappa_1)$ to obtain

$$2D_{1,2}(N) \geqslant 2\Upsilon_2 - S_1(\kappa_2, 1/2 - 3\kappa_1) - 2\Upsilon_7 - S_3(\kappa_2, 1/2 - 3\kappa_1) + O(N^{1-\kappa_2}), \tag{2.3}$$

where the term $S_4(\kappa_2, 1/2 - 3\kappa_1)$ is dropped by non-negativity.

Buchstab's identity, when applied three times, gives the equality

$$\Upsilon_2 = \Upsilon_1 - \sum_{\substack{N^{\kappa_1} \leqslant p < N^{\kappa_2} \\ (p,N)=1}} S(\mathcal{A}_p; \mathcal{P}(N), N^{\kappa_1}) + \Upsilon_5 - \sum_{\substack{N^{\kappa_1} \leqslant p_1 < p_2 < p_3 < N^{\kappa_2} \\ (p_1p_2p_3,N)=1}} \sum\sum\sum S(\mathcal{A}_{p_1p_2p_3}; \mathcal{P}(N), p_1).$$

Similarly, a twice application of Buchstab's identity yields

$$S_1(\kappa_2, 1/2-3\kappa_1) = \sum_{\substack{N^{\kappa_2}\leqslant p<N^{1/2-3\kappa_1}\\(p,N)=1}} S(\mathcal{A}_p;\mathcal{P}(N),N^{\kappa_1}) - \Upsilon_6 + \sum_{\substack{N^{\kappa_1}\leqslant p_1<p_2<N^{\kappa_2}\leqslant p_3<N^{1/2-3\kappa_1}\\(p_1p_2p_3,N)=1}}\sum\sum\sum S(\mathcal{A}_{p_1p_2p_3};\mathcal{P}(N),p_1).$$

By Buchstab's identity, we can prove

$$S_3(\kappa_2, 1/2-3\kappa_1) \leqslant \Upsilon_9 + \sum_{\substack{N^{\kappa_2}\leqslant p_1<N^{1/2-3\kappa_1}\leqslant p_2<p_3<(N/p_1p_2)^{1/2}\\(p_1p_2p_3,N)=1}}\sum\sum\sum S(\mathcal{A}_{p_1p_2p_3};\mathcal{P}(Np_1),p_3).$$

Inserting them into (2.3), we find that

$$2D_{1,2}(N) \geqslant \Upsilon_1+\Upsilon_2-\Upsilon_4+\Upsilon_5+\Upsilon_6-2\Upsilon_7-\Upsilon_9-\Delta_1+O(N^{1-\kappa_2}), \tag{2.4}$$

where

$$\begin{aligned}\Delta_1 :=& \sum_{\substack{N^{\kappa_1}\leqslant p_1<p_2<p_3<N^{\kappa_2}\\(p_1p_2p_3,N)=1}}\sum\sum\sum S(\mathcal{A}_{p_1p_2p_3};\mathcal{P}(N),p_1)\\ &+\sum_{\substack{N^{\kappa_1}\leqslant p_1<p_2<N^{\kappa_2}\leqslant p_3<N^{1/2-3\kappa_1}\\(p_1p_2p_3,N)=1}}\sum\sum\sum S(\mathcal{A}_{p_1p_2p_3};\mathcal{P}(N),p_1)\\ &+\sum_{\substack{N^{\kappa_2}\leqslant p_1<N^{1/2-3\kappa_1}\leqslant p_2<p_3<(N/p_1p_2)^{1/2}\\(p_1p_2p_3,N)=1}}\sum\sum\sum S(\mathcal{A}_{p_1p_2p_3};\mathcal{P}(Np_1),p_3).\end{aligned}$$

The inequality (2.1) with $(\kappa,\sigma)=(\kappa_1,1/3)$ gives

$$2D_{1,2}(N) \geqslant 2\Upsilon_1-\Upsilon_3-\Upsilon_8+S_4(\kappa_1,1/3)+O(N^{1-\kappa_1}), \tag{2.5}$$

where we have used the fact that $S_2(\kappa_1,1/3)=0$.

Adding (2.4) to (2.5) yields

$$4D_{1,2}(N) \geqslant 3\Upsilon_1+\Upsilon_2-\Upsilon_3-\Upsilon_4+\Upsilon_5+\Upsilon_6-2\Upsilon_7-\Upsilon_8-\Upsilon_9+\Delta_2+O(N^{1-\kappa_1}), \tag{2.6}$$

where

$$\Delta_2 := \sum_{\substack{N^{\kappa_1}\leqslant p_1<p_2<p_3<N^{1/3}\\(p_1p_2p_3,N)=1}}\sum\sum\sum S(\mathcal{A}_{p_1p_2p_3};\mathcal{P}(N),p_2)-\Delta_1.$$

Clearly all the summation ranges in the three triple sums of Δ_1 are distinct and the first two are covered in the range of the triple sum in Δ_2 (since our hypothesis on κ_1 and κ_2 implies $\max\{\kappa_2,1/2-3\kappa_1\}\leqslant 1/3$). On the other hand, we easily see that the range of summation in the third triple sum of Δ_1 is equivalent to $N^{\kappa_2}\leqslant p_1<N^{1/2-3\kappa_1}\leqslant p_2\leqslant (N/p_1)^{1/3}$ and $p_2<p_3<(N/p_1p_2)^{1/2}$. From this we deduce that $(N/p_1p_2)^{1/2}\leqslant N^{(1/2+3\kappa_1-\kappa_2)/2}\leqslant N^{1/3}$,

since $3\kappa_1-\kappa_2<1/6$. Thus this range is also contained in the triple sum of Δ_2. Therefore we have

$$\begin{aligned}\Delta_2 \geqslant & -\sum_{\substack{N^{\kappa_1}\leqslant p_1<p_2<p_3<N^{\kappa_2}\\(p_1p_2p_3,N)=1}}\sum\sum \big\{S(\mathcal{A}_{p_1p_2p_3};\mathcal{P}(N),p_1)-S(\mathcal{A}_{p_1p_2p_3};\mathcal{P}(N),p_2)\big\}\\ & -\sum_{\substack{N^{\kappa_1}\leqslant p_1<p_2<N^{\kappa_2}\leqslant p_3<N^{1/2-2\kappa_1}/p_2\\(p_1p_2p_3,N)=1}}\sum\sum \big\{S(\mathcal{A}_{p_1p_2p_3};\mathcal{P}(N),p_1)-S(\mathcal{A}_{p_1p_2p_3};\mathcal{P}(N),p_2)\big\}\\ & +\sum_{\substack{N^{\kappa_2}\leqslant p_1<N^{1/2-3\kappa_1}\leqslant p_2<p_3<(N/p_1p_2)^{1/2}\\(p_1p_2p_3,N)=1}}\sum\sum \big\{S(\mathcal{A}_{p_1p_2p_3};\mathcal{P}(N),p_2)-S(\mathcal{A}_{p_1p_2p_3};\mathcal{P}(N),p_3)\big\}\\ \geqslant & -\Upsilon_{10}-\Upsilon_{11}+O(N^{1-\kappa_1}).\end{aligned}$$

Combining with (2.6), we obtain the required result. □

Remark 1. Apparently from the proof, we have choosen $(\kappa,\sigma)=(\kappa_1,1/2-3\kappa_1)$, $(\kappa_2,1/3)$ in the application of Lemma 2.1. It is possible to optimize the choice of σ. But this augments the number of terms of (2.2) and the numeric improvement for Theorem is quite small.

§ 3. Chen's double sieve

In this section, we recall Chen's double sieve described in [13] and give numeric lower bounds for $H(s)$ and $h(s)$ for later use.

For any large even integer N, we write

$$\mathcal{A}:=\{N-p:p\leqslant N\},\qquad \mathcal{P}(N):=\{p:(p,N)=1\}.$$

Let $\delta>0$ be a sufficiently small number [(*)] and $k\in\mathbb{Z}$. Put

$$Q:=N^{1/2-\delta},\qquad \underline{d}:=Q/d,\qquad \mathcal{L}:=\log N,\qquad W_k:=N^{\delta^{1+k}}.$$

Denote by $\pi_{[Y,Z)}$ the characteristic function of the set $\mathcal{P}(N)\cap[Y,Z)$. For $k\in\mathbb{Z}^+$ and $N\geqslant 2$, let $\mathfrak{U}_k(N)$ be the set of all arithmetical functions σ which can be written as the form

$$\sigma=\pi_{[V_1/\Delta,V_1)}*\cdots*\pi_{[V_i/\Delta,V_i)},$$

where Δ is a real number with $1+\mathcal{L}^{-4}\leqslant\Delta<1+2\mathcal{L}^{-4}$, i is an integer with $0\leqslant i\leqslant k$, and $V_1,\dots,V_i$ are real numbers satisfying

$$\text{(3.1)}\qquad \begin{cases} V_1^2\leqslant Q,\\ V_1V_2^2\leqslant Q,\\ \cdots\cdots\cdots\cdots\\ V_1\cdots V_{i-1}V_i^2\leqslant Q,\\ V_1\geqslant V_2\geqslant\cdots\geqslant V_i\geqslant W_k.\end{cases}$$

We adopt the convention that σ is the characteristic function of the set $\{1\}$ if $i=0$.

(*) In numerical computation, we can formally take $\delta=0$.

Let F and f be defined by

$$\begin{aligned} F(s) &= 2e^{\gamma}/s, \qquad & f(s) &= 0 \qquad (0 < s \leqslant 2), \\ (sF(s))' &= f(s-1), & (sf(s))' &= F(s-1) \quad (s > 2), \end{aligned} \tag{3.2}$$

where γ is Euler's constant. Moreover we take

$$A(s) := sF(s)/2e^{\gamma}, \qquad a(s) := sf(s)/2e^{\gamma}, \tag{3.3}$$

and introduce the notation

$$\Phi(N, \sigma, s) := \sum_{d} \sigma(d) S(\mathcal{A}_d; \mathcal{P}(dN), \underline{d}^{1/s}), \tag{3.4}$$

$$\Theta(N, \sigma) := 4\mathrm{li}(N) \sum_{d} \frac{\sigma(d) C_{dN}}{\varphi(d) \log \underline{d}}, \tag{3.5}$$

where $\varphi(d)$ is the Euler function.

For $k \in \mathbb{Z}^+$, $N_0 \geqslant 2$ and $s \in [1, 10]$, we define $H_{k,N_0}(s)$ and $h_{k,N_0}(s)$ to be the supremum of $h \geqslant -\infty$ such that for all $N \geqslant N_0$ and $\sigma \in \mathfrak{U}_k(N)$, the inequalities

$$\begin{aligned} \Phi(N, \sigma, s) &\leqslant \{A(s) - h\} \Theta(N, \sigma), \\ \Phi(N, \sigma, s) &\geqslant \{a(s) + h\} \Theta(N, \sigma) \end{aligned}$$

hold true respectively. Obviously $H_{k,N_0}(s)$ and $h_{k,N_0}(s)$ are decreasing in N_0, as well as decreasing in k by Lemma 3.1. Hence their limits at infinity exist (in the extended real line), and we write

$$\begin{aligned} H_k(s) &:= \lim_{N_0 \to \infty} H_{k,N_0}(s), \qquad & h_k(s) &:= \lim_{N_0 \to \infty} h_{k,N_0}(s), \\ H(s) &:= \lim_{k \to \infty} H_k(s), & h(s) &:= \lim_{k \to \infty} h_k(s). \end{aligned}$$

The next lemma collects the concerned properties of these functions (see [13], Lemma 3.2, Propositions 1 & 2 and Corollary 1).

Lemma 3.1. *(i) For $k \in \mathbb{Z}^+, N \geqslant N_0, s \in [1, 10]$ and $\sigma \in \mathfrak{U}_k(N)$, we have*

$$\Phi(N, \sigma, s) \leqslant \{A(s) - H_{k,N_0}(s)\} \Theta(N, \sigma), \tag{3.6}$$

$$\Phi(N, \sigma, s) \geqslant \{a(s) + h_{k,N_0}(s)\} \Theta(N, \sigma). \tag{3.7}$$

(ii) For $k \in \mathbb{Z}^+$ and $s \in [1, 10]$, we have $H_k(s) \geqslant 0$ and $h_k(s) \geqslant 0$.

(iii) For $2 \leqslant s \leqslant s' \leqslant 10$, we have

$$h(s) \geqslant h(s') + \int_{s-1}^{s'-1} \frac{H(t)}{t}\, \mathrm{d}t \qquad \text{and} \qquad H(s) \geqslant H(s') + \int_{s-1}^{s'-1} \frac{h(t)}{t}\, \mathrm{d}t. \tag{3.8}$$

(iv) The function $H(s)$ is decreasing on $[1, 10]$. The function $h(s)$ is increasing on $[1, 2]$ and is decreasing on $[2, 10]$.

We cannot give explicit expressions for $H(s)$ and $h(s)$. But it is tractable to obtain numeric lower bounds for these two functions. Let

$$s_i := 2 + 0.1 \times i \quad (i \geqslant 0). \tag{3.9}$$

By ([13], § 7), we have the numeric lower bounds of $H(s_i)$ for $2 \leqslant i \leqslant 10$. Next we shall consider the case of $11 \leqslant i \leqslant 29$ and the lower bounds of $h(s_i)$ for $0 \leqslant i \leqslant 29$. These will be used in the proof of Theorem.

Let $\mathbf{1}_{[a,b]}(t)$ be the characteristic function of the interval $[a,b]$ and

$$\sigma(a,b,c) := \int_a^b \log\left(\frac{c}{t-1}\right)\frac{\mathrm{d}t}{t}, \qquad \sigma_0(t) := \frac{\sigma(3,t+2,t+1)}{1-\sigma(3,5,4)}.$$

From (6.2) of [13] and the decreasing property of $H(s)$, we deduce

$$H(s_j) \geqslant \sum_{2\leqslant i\leqslant 10} c_{i,j} H(s_i), \tag{3.10}$$

for $11 \leqslant j \leqslant 29$, where

$$\begin{aligned} c_{2,j} &:= \int_1^{s_2} \left\{\frac{\sigma_0(t)}{t}\log\left(\frac{4}{s_j-1}\right) + \frac{\mathbf{1}_{[s_j-2,3]}(t)}{t}\log\left(\frac{t+1}{s_j-1}\right)\right\}\mathrm{d}t, \\ c_{i,j} &:= \int_{s_{i-1}}^{s_i} \left\{\frac{\sigma_0(t)}{t}\log\left(\frac{4}{s_j-1}\right) + \frac{\mathbf{1}_{[s_j-2,3]}(t)}{t}\log\left(\frac{t+1}{s_j-1}\right)\right\}\mathrm{d}t \quad (3 \leqslant i \leqslant 10). \end{aligned}$$

From the first inequality of (3.8) and the fact that $h(s) \geqslant 0$, we also derive

$$\begin{aligned} h(s_j) &\geqslant \int_{s_j-1}^{5} \frac{H(t)}{t}\,\mathrm{d}t \\ &\geqslant H(s_2)\log\left(\frac{s_{\max\{2,j-10\}}}{s_j-1}\right) + \sum_{\max\{3,j-9\}\leqslant i\leqslant 29} H(s_i)\log\left(\frac{s_i}{s_{i-1}}\right) \end{aligned} \tag{3.11}$$

for $0 \leqslant j \leqslant 29$.

Using the numeric lower bounds of $H(s_i)$ for $2 \leqslant i \leqslant 10$ given in ([13], § 7), (3.10) and (3.11), we get via a numerical computation the following results.

i	s_i	$H(s_i) \geqslant$	i	s_i	$H(s_i) \geqslant$	i	s_i	$H(s_i) \geqslant$
			10	3.0	0.0072943	20	4.0	0.0010835
			11	3.1	0.0061642	21	4.1	0.0008451
2	2.2	0.0223939	12	3.2	0.0052233	22	4.2	0.0006482
3	2.3	0.0217196	13	3.3	0.0044073	23	4.3	0.0004882
4	2.4	0.0202876	14	3.4	0.0036995	24	4.4	0.0003602
5	2.5	0.0181433	15	3.5	0.0030860	25	4.5	0.0002592
6	2.6	0.0158644	16	3.6	0.0025551	26	4.6	0.0001803
7	2.7	0.0129923	17	3.7	0.0020972	27	4.7	0.0001187
8	2.8	0.0100686	18	3.8	0.0017038	28	4.8	0.0000702
9	2.9	0.0078162	19	3.9	0.0013680	29	4.9	0.0000313

Table 1. Numeric lower bounds for $H(s_i)$

i	s_i	$h(s_i)\geqslant$	i	s_i	$h(s_i)\geqslant$	i	s_i	$h(s_i)\geqslant$
0	2.0	0.0232385	10	3.0	0.0077162	20	4.0	0.0010120
1	2.1	0.0211041	11	3.1	0.0066236	21	4.1	0.0008099
2	2.2	0.0191556	12	3.2	0.0055818	22	4.2	0.0006440
3	2.3	0.0173631	13	3.3	0.0046164	23	4.3	0.0005084
4	2.4	0.0157035	14	3.4	0.0037529	24	4.4	0.0003980
5	2.5	0.0141585	15	3.5	0.0030123	25	4.5	0.0003085
6	2.6	0.0127132	16	3.6	0.0023901	26	4.6	0.0002365
7	2.7	0.0113556	17	3.7	0.0018997	27	4.7	0.0001791
8	2.8	0.0100756	18	3.8	0.0015336	28	4.8	0.0001336
9	2.9	0.0088648	19	3.9	0.0012593	29	4.9	0.0000981

Table 2. Numeric lower bounds for $h(s_i)$

Remark 2. It is possible to get better numeric lower bounds for $H(s_i)$ and $h(s_i)$ by applying (3.8) repeatedly. But the improvement will be small.

§ 4. Application of Chen's double sieve

In this section, we apply Chen's double sieve to estimate the terms Υ_3, Υ_4, Υ_5 and Υ_6 in (2.2). Propositions 4.1, 4.2, 4.3 and 4.4 below are results in general context. These estimates are better than those obtained by the classical linear sieve, since $H(s)>0$ and $h(s)>0$.

Proposition 4.1. *Let $0<\phi_1<\phi_2<1/4$ and $\kappa>0$ such that $\phi_2+\kappa\leqslant 1/2$. Then for $N\to\infty$, we have*

$$\sum_{\substack{N^{\phi_1}\leqslant p<N^{\phi_2}\\(p,N)=1}} S(\mathcal{A}_p;\mathcal{P}(N),N^{\kappa})\leqslant\Big\{8\int_{(1/2-\phi_2)/\kappa}^{(1/2-\phi_1)/\kappa}\frac{A(t)-H(t)}{t(1-2\kappa t)}\,\mathrm{d}t+o(1)\Big\}\Theta(N).$$

Proof. We keep use of the previous notation. Denote by S the sum in the proposition. Let $\alpha_j:=N^{\phi_1}\Delta^j$ and J be the integer such that $\alpha_J\leqslant N^{\phi_2}<\alpha_{J+1}$. We write

$$S=\sum_{1\leqslant j\leqslant J}\sum_{p}\pi_{[\alpha_{j-1},\alpha_j)}(p)S\big(\mathcal{A}_p;\mathcal{P}(pN),\underline{p}^{1/\tau_p}\big)+R_1, \tag{4.1}$$

where $\tau_p:=(\log\underline{p})/(\kappa\log N)$ and

$$R_1:=\sum_{\alpha_J\leqslant p<N^{\phi_2}}S(\mathcal{A}_p;\mathcal{P}(N),N^{\kappa})\ll\sum_{\alpha_J\leqslant p<N^{\phi_2}}N/p\ll\Theta(N)\mathcal{L}^{-3}. \tag{4.2}$$

Introducing

$$\tau_j:=(\log\underline{\alpha_j})/(\kappa\log N),$$

we easily see that $\pi_{[\alpha_{j-1},\alpha_j)}(p) \neq 0 \Rightarrow \tau_j \leqslant \tau_p \leqslant \tau_{j-1}$. Thus we can deduce from (4.1) and (4.2) that

$$S \leqslant \sum_{1\leqslant j\leqslant J}\sum_{p}\pi_{[\alpha_{j-1},\alpha_j)}(p)S\big(\mathcal{A}_p;\mathcal{P}(pN),\underline{p}^{1/\tau_j}\big)+O\big(\Theta(N)\mathcal{L}^{-3}\big), \tag{4.3}$$

where we have used the following estimates:

$$\begin{aligned}
&\sum_{1\leqslant j\leqslant J}\sum_{p}\pi_{[\alpha_{j-1},\alpha_j)}(p)\big\{S\big(\mathcal{A}_p;\mathcal{P}(pN),\underline{p}^{1/\tau_p}\big)-S\big(\mathcal{A}_p;\mathcal{P}(pN),\underline{p}^{1/\tau_j}\big)\big\}\\
&\quad\leqslant \sum_{1\leqslant j\leqslant J}\sum_{\alpha_{j-1}\leqslant p<\alpha_j}\sum_{\underline{p}^{1/\tau_p}\leqslant p'<\underline{p}^{1/\tau_j}} N/(pp')\\
&\quad\ll N\mathcal{L}^{-5}\sum_{1\leqslant j\leqslant J}\sum_{\alpha_{j-1}\leqslant p<\alpha_j}1/p\\
&\quad\ll \Theta(N)\mathcal{L}^{-3}.
\end{aligned}$$

Next we treat the inner sum (over p) in (4.3). Clearly for each $j\in\{1,\dots,J\}$, our hypothesis on ϕ_1,ϕ_2 and κ assures that the function $\pi_{[\alpha_{j-1},\alpha_j)}\in\mathfrak{U}_k(N)$ for all $k\geqslant 0$, $N_0\geqslant 2$ and $N\geqslant N_0$, and $\tau_j\geqslant 1$. Thus we can apply (3.6) of Lemma 3.1 to estimate the sum over p (which is $\Phi(N,\pi_{[\alpha_{j-1},\alpha_j)},\tau_j)$) :

$$\begin{aligned}
S&\leqslant \sum_{1\leqslant j\leqslant J}\{A(\tau_j)-H_{k,N_0}(\tau_j)\}\Theta(N,\pi_{[\alpha_{j-1},\alpha_j)})+O\big(\Theta(N)\mathcal{L}^{-3}\big)\\
&\leqslant 4\mathrm{li}(N)\frac{C_N}{\log\underline{1}}\sum_{\alpha_0\leqslant p<\alpha_J}\frac{A(\tau_p)-H_{k,N_0}(\tau_p)}{(p-2)(1-\log p/\log\underline{1})}+O\big(\Theta(N)\mathcal{L}^{-3}\big)\\
&\leqslant 4\mathrm{li}(N)\frac{C_N}{\log\underline{1}}\sum_{N^{\phi_1}\leqslant p<N^{\phi_2}}\frac{A(\tau_p)-H_{k,N_0}(\tau_p)}{(p-2)(1-\log p/\log\underline{1})}+O\big(\Theta(N)\mathcal{L}^{-3}\big),
\end{aligned}$$

where we have used the fact that $A(s)-H_{k,N_0}(s)$ is increasing in s. An integration by parts with the prime number theorem shows that

$$\sum_{N^{\phi_1}\leqslant p<N^{\phi_2}}\frac{A(\tau_p)-H_{k,N_0}(\tau_p)}{(p-2)(1-\log p/\log\underline{1})}=\int_{(1/2-\phi_2)/\kappa}^{(1/2-\phi_1)/\kappa}\frac{A(t)-H_{k,N_0}(t)}{t(1-2\kappa t)}\,\mathrm{d}t+O_{\delta,k}(\varepsilon).$$

Hence

$$S\leqslant 8\bigg\{\int_{(1/2-\phi_2)/\kappa}^{(1/2-\phi_1)/\kappa}\frac{A(t)-H_{k,N_0}(t)}{t(1-2\kappa t)}\,\mathrm{d}t+O_{\delta,k}(\varepsilon)\bigg\}\Theta(N)$$

for $N\geqslant N_0$. From this, we infer that

$$\limsup_{N\to\infty}\frac{S}{\Theta(N)}\leqslant 8\int_{(1/2-\phi_2)/\kappa}^{(1/2-\phi_1)/\kappa}\frac{A(t)-H_{k,N_0}(t)}{t(1-2\kappa t)}\,\mathrm{d}t+O_{\delta,k}(\varepsilon),$$

which implies, by taking $N\to\infty$, $k\to\infty$ and $\varepsilon\to 0$,

$$\limsup_{N\to\infty}\frac{S}{\Theta(N)}\leqslant 8\int_{(1/2-\phi_2)/\kappa}^{(1/2-\phi_1)/\kappa}\frac{A(t)-H(t)}{t(1-2\kappa t)}\,\mathrm{d}t.$$

Clearly this is equivalent to the required inequality. □

In a similar fashion we can prove the following results.

Proposition 4.2. *Let $0 < \phi_1 < \phi_2 < 1/6$ and $\kappa > 0$ such that $2\phi_2 + \kappa \leqslant 1/2$. Then for $N \to \infty$, we have*

$$\sum_{\substack{N^{\phi_1} \leqslant p_1 < p_2 < N^{\phi_2} \\ (p_1p_2, N)=1}} \sum S(\mathcal{A}_{p_1p_2}; \mathcal{P}(N), N^\kappa) \geqslant \left\{ 8 \int_{\phi_1}^{\phi_2} \int_{(1/2-\phi_2-t)/\kappa}^{(1/2-2t)/\kappa} \frac{a(u)+h(u)}{t(1-2t-2\kappa u)} \,\mathrm{d}t\,\mathrm{d}u + o(1) \right\} \Theta(N).$$

Proposition 4.3. *Let $0 < \phi_1 < \phi_2 \leqslant \phi_3 < \phi_4 < 1/4$ and $\kappa > 0$ such that $2\phi_2 + \phi_4 < 1/2$ and $\phi_2 + \phi_4 + \kappa \leqslant 1/2$. Then for $N \to \infty$, we have*

$$\sum_{\substack{N^{\phi_1} \leqslant p_1 < N^{\phi_2} \\ (p_1p_2, N)=1}} \sum_{N^{\phi_3} \leqslant p_1 < N^{\phi_4}} S(\mathcal{A}_{p_1p_2}; \mathcal{P}(N), N^\kappa) \geqslant \left\{ 8 \int_{\phi_1}^{\phi_2} \int_{(1/2-\phi_4-t)/\kappa}^{(1/2-\phi_3-t)/\kappa} \frac{a(u)+h(u)}{tu(1-2t-2\kappa u)} \,\mathrm{d}t\,\mathrm{d}u + o(1) \right\} \Theta(N).$$

Finally we estimate the sum of the type in (1.5) with $\phi_1 \geqslant 1/4$. In this case, we cannot directly apply our delicate Chen's double sieve because of the first condition of (3.1). As what Cai [2] remarked, it is possible to use a simplified version of Chen's double sieve. This approach will give a result better than using the classic linear sieve but weaker than Proposition 4.1, since, without iteration, $\Psi_1(s)$ or $\Psi_2(s)$ are principal contributions of $H(s)$. (See Lemmas 5.1 and 5.2 of [13] and compare Proposition 4.4 below and Proposition 4.1.)

Proposition 4.4. *Let $\kappa > 0$, $\phi > 0$ and $2 \leqslant s \leqslant 3 \leqslant s' \leqslant 5$ such that $1/4 \leqslant 1/2 - s\kappa < \phi$. Then for $N \to \infty$, we have*

$$\sum_{\substack{N^{1/2-s\kappa} \leqslant p < N^\phi \\ (p,N)=1}} S(\mathcal{A}_p; \mathcal{P}(N), N^\kappa) \leqslant \left\{ 8 \int_{(1/2-\phi)/\kappa}^{s} \frac{A(t) - \Psi_1(s)}{t(1-2\kappa t)} \,\mathrm{d}t + o(1) \right\} \Theta(N),$$

where

$$\Psi_1(s) := -\int_2^{s'-1} \frac{\log(t-1)}{t} \,\mathrm{d}t + \frac{1}{2} \int_{1-1/s}^{1-1/s'} \frac{\log(s't-1)}{t(1-t)} \,\mathrm{d}t - \max_{\phi \geqslant 2} \iiint\limits_{1/s' \leqslant t \leqslant u \leqslant v \leqslant 1/s} \omega\left(\frac{\phi - t - u - v}{u} \right) \frac{\mathrm{d}t\,\mathrm{d}u\,\mathrm{d}v}{tu^2v}$$

and $\omega(u)$ is Buchstab's function. The same result also holds if we replace $\Psi_1(s)$ by $\Psi_2(s)$, where the function $\Psi_2(s)$ is defined as in Lemma 5.2 of [13].

Proof. For simplicity, we denote the sum by S. Since $N^\kappa \geqslant \underline{p}^{1/s}$ for $p \geqslant N^{1/2-s\kappa}$, we can write

$$\begin{aligned} S &\leqslant \sum_{\substack{N^{1/2-s\kappa} \leqslant p < N^\phi \\ (p,N)=1}} S\big(\mathcal{A}_p; \mathcal{P}(N), \underline{p}^{1/s}\big) \\ &\leqslant \sum_{1 \leqslant j \leqslant J} \sum_p \pi_{[\alpha_{j-1}, \alpha_j)}(p) S\big(\mathcal{A}_p; \mathcal{P}(N), \underline{p}^{1/s}\big), \end{aligned}$$

where $\alpha_j := N^{1/2-s\kappa}\Delta^j$ and J is the integer such that $\alpha_{J-1} \leqslant N^{\phi} < \alpha_J$.

Similar to Lemma 4.1 of [13], we can prove that there is a constant $\eta > 0$ such that

$$S \leqslant \sum_{1\leqslant j\leqslant J} \sum_{p} \pi_{[\alpha_{j-1},\alpha_j)}(p)\Big(\Omega_1(p) - \frac{1}{2}\Omega_2(p) + \frac{1}{2}\Omega_3(p)\Big) + O\big(N^{1-\eta}\big), \tag{4.4}$$

where

$$\Omega_1(p) := S\big(\mathcal{A}_p; \mathcal{P}(pN), \underline{p}^{1/s'}\big),$$

$$\Omega_2(p) := \sum_{\substack{\underline{p}^{1/s'}\leqslant p_1<\underline{p}^{1/s}\\(p_1,N)=1}} S\big(\mathcal{A}_{pp_1}; \mathcal{P}(pN), \underline{p}^{1/s'}\big),$$

$$\Omega_3(p) := \sum_{\substack{\underline{p}^{1/s'}\leqslant p_1<p_2<p_3<\underline{p}^{1/s}\\(p_1p_2p_3,N)=1}}\sum\sum\sum S\big(\mathcal{A}_{pp_1p_2p_3}; \mathcal{P}(pp_1N), p_2\big).$$

Similar to (5.1), (5.2) and (5.9) of [13], we can prove, uniformly for $N \geqslant 10$ and for $1 \leqslant j \leqslant J$,

$$\sum_{p} \pi_{[\alpha_{j-1},\alpha_j)}(p)\Omega_i(p) \leqslant \big\{\widetilde{\Omega}_i(s,s') + o(1)\big\}\Theta\big(N, \pi_{[\alpha_{j-1},\alpha_j)}\big) \quad (i = 1,2,3),$$

where

$$\widetilde{\Omega}_2(s,s') := A(s'),$$

$$\widetilde{\Omega}_i(s,s') := \int_{1-1/s}^{1-1/s'} \frac{a(s't)}{t(1-t)}\,\mathrm{d}t,$$

$$\widetilde{\Omega}_3(s,s') := 2\max_{\phi\geqslant 2} \iiint\limits_{1/s'\leqslant t\leqslant u\leqslant v\leqslant 1/s} \omega\Big(\frac{\phi-t-u-v}{u}\Big)\frac{\mathrm{d}t\,\mathrm{d}u\,\mathrm{d}v}{tu^2v}.$$

Inserting these into (4.4) and noticing that

$$A(s') = 1 + \int_2^{s'-1} \frac{\log(t-1)}{t}\,\mathrm{d}t, \qquad a(s't) = \log(s't-1),$$

we find that

$$\begin{aligned} S &\leqslant \{1 - \Psi_1(s) + o(1)\} \sum_{1\leqslant j\leqslant J} \Theta\big(N, \pi_{[\alpha_{j-1},\alpha_j)}\big) + O\big(N^{1-\eta}\big),\\ &\leqslant \Big\{8\big(1-\Psi_1(s)\big)\int_{1/2-s\kappa}^{\phi} \frac{\mathrm{d}t}{t(1-2t)} + o(1)\Big\}\Theta(N), \end{aligned}$$

which is equivalent to the required result for the case of $\Psi_1(s)$, since

$$\begin{aligned} \int_{(1/2-\phi)/\kappa}^{s} \frac{A(t)}{t(1-2\kappa t)}\,\mathrm{d}t &= \int_{(1/2-\phi)/\kappa}^{s} \frac{\mathrm{d}t}{t(1-2\kappa t)}\\ &= \int_{1/2-s\kappa}^{\phi} \frac{\mathrm{d}t}{t(1-2t)}. \end{aligned}$$

The case of $\Psi_2(s)$ can be treated in the same way. The main difference is to use Lemma 4.2 of [13] in place of Lemma 4.1 of [13]. We omit the details. □

§ 5. Proof of Theorem

Take

$$\kappa_1 = 1/13.27 \qquad \text{and} \qquad \kappa_2 = 1/8.24, \tag{5.1}$$

which satisfy the hypothesis of Lemma 2.2. Next we estimate all the terms Υ_i in (2.2).

1° *Lower bounds of* Υ_1 *and* Υ_2

Write $N^\kappa = \underline{1}^{\kappa'}$ with $\kappa' := \kappa/(1/2-\delta)$. By using (4.2) with $\sigma := \mathbf{1}_{\{1\}}$ (the charateristic function of $\{1\}$), it follows that

$$\begin{aligned} \Upsilon_i &= \Phi(N, \mathbf{1}_{\{1\}}, 1/\kappa_i') \\ &\geqslant \big\{a(1/\kappa_i') + h_{k,N_0}(1/\kappa_i')\big\}\Theta\big(N, \mathbf{1}_{\{1\}}\big) \\ &\geqslant \{F_i + o(1)\}\,\Theta(N) \end{aligned} \tag{5.2}$$

with

$$F_i := 8a(1/(2\kappa_i)) + 8h(1/(2\kappa_i)) \quad (i = 1, 2).$$

Write

$$G_2 := 8a(1/(2\kappa_i)) + 8h(1/(2\kappa_2)).$$

2° *Upper bounds of* Υ_3 *and* Υ_4

We divide the sum Υ_3 (resp. Υ_4) into subsums according to

(a) $N^{\kappa_1} \leqslant p < N^{1/4}$,
(b) $N^{1/4} \leqslant p < N^{1/2-s_9\kappa_1}$,
(c) $N^{1/2-s_j\kappa_1} \leqslant p < N^{1/2-s_{j-1}\kappa_1}$ $(9 \geqslant j \geqslant 4)$,
(d) $N^{1/2-s_3\kappa_1} \leqslant p < N^{1/3}$

(resp. $N^{\kappa_1} \leqslant p < N^{1/4}$ or $N^{1/4} \leqslant p < N^{1/2-3\kappa_1}$), where s_i is defined by (3.9). The contribution of (a) is estimated by Proposition 4.1 and we evaluate (b) (resp. $N^{1/4} \leqslant p < N^{1/2-3\kappa_1}$) by the classic linear sieve. The remaining subsums are treated by Proposition 4.4. It is worth to point out that the case (b) requires another kind of treatment because $\Psi_1(s_{10}) = 0$ (see Table 3 below). Thus we obtain

$$\Upsilon_i \leqslant \{F_i + o(1)\}\,\Theta(N) \quad (i = 3, 4), \tag{5.3}$$

where

$$F_3 := 8\int_{1/(6\kappa_1)}^{1/(2\kappa_1)-1} \frac{A(t)}{t(1-2\kappa_1 t)}\,\mathrm{d}t - G_3,$$

$$F_4 := 8\int_{3}^{1/(2\kappa_1)-1} \frac{A(t)}{t(1-2\kappa_1 t)}\,\mathrm{d}t - G_4,$$

and

$$G_4 := 8\int_{(1/(4\kappa_1)}^{1/(2\kappa_1)-1} \frac{H(t)}{t(1-2\kappa_1 t)}\,\mathrm{d}t,$$

$$\begin{aligned} G_3 := 8\int_{(1/(4\kappa_1)}^{1/(2\kappa_1)-1} \frac{H(t)}{t(1-2\kappa_1 t)}\,\mathrm{d}t + 8\int_{1/(6\kappa_1)}^{s_3} \frac{\Psi_2(s_3)}{t(1-2\kappa t)}\,\mathrm{d}t \\ + 8\sum_{4\leqslant i\leqslant 5}\int_{s_{i-1}}^{s_i} \frac{\Psi_2(s_i)}{t(1-2\kappa t)}\,\mathrm{d}t + 8\sum_{6\leqslant i\leqslant 9}\int_{s_{i-1}}^{s_i} \frac{\Psi_1(s_i)}{t(1-2\kappa t)}\,\mathrm{d}t \end{aligned}$$

3° *Lower bounds of* Υ_5 *and* Υ_6

Since $\kappa_1 + 2\kappa_2 = 0.318\ldots < 1/2$, Proposition 4.2 yields

$$\Upsilon_5 \geqslant \{F_5 + o(1)\}\,\Theta(N), \tag{5.4}$$

where

$$F_5 := 8\int_{\kappa_1}^{\kappa_2}\int_{(1/2-\kappa_2-t)/\kappa_1}^{(1/2-2t)/\kappa_1} \frac{a(u)\,\mathrm{d}t\,\mathrm{d}u}{tu(1-2t-2\kappa_1 u)} + G_5$$

and

$$G_5 := 8\int_{\kappa_1}^{\kappa_2}\int_{(1/2-\kappa_2-t)/\kappa_1}^{(1/2-2t)/\kappa_1} \frac{h(u)\,\mathrm{d}t\,\mathrm{d}u}{tu(1-2t-2\kappa_1 u)}.$$

We divide the double sum Υ_6 into three subsums according to

(a) $N^{\kappa_1} \leqslant p_1 < N^{\kappa_2} \leqslant p_2 < N^{1/2-2\kappa_2}$,

(b) $N^{\kappa_1} \leqslant p_1 < N^{3\kappa_1/2}$ and $N^{1/2-2\kappa_2} \leqslant p_2 < N^{1/2-3\kappa_1}$,

(c) $N^{3\kappa_1/2} \leqslant p_1 < N^{\kappa_2}$ and $N^{1/2-2\kappa_2} \leqslant p_2 < N^{1/2-3\kappa_1}$.

The first two subsums can be estimated by Proposition 4.3 and the last one by the classic linear sieve. Thus we obtain

$$\Upsilon_6 \geqslant \{F_6 + o(1)\}\,\Theta(N), \tag{5.5}$$

where

$$F_6 := 8\int_{\kappa_1}^{\kappa_2}\int_{(3\kappa_1-t)/\kappa_1}^{(1/2-\kappa_2-t)/\kappa_1} \frac{a(u)\,\mathrm{d}t\,\mathrm{d}u}{tu(1-2t-2\kappa_1 u)} + G_6$$

and

$$G_6 := 8\int_{\kappa_1}^{\kappa_2}\int_{(2\kappa_2-t)/\kappa_1}^{(1/2-\kappa_2-t)/\kappa_1} \frac{h(u)\,\mathrm{d}t\,\mathrm{d}u}{tu(1-2t-2\kappa_1 u)} + 8\int_{\kappa_1}^{3\kappa_1/2}\int_{(3\kappa_1-t)/\kappa_1}^{(2\kappa_2-t)/\kappa_1} \frac{h(u)\,\mathrm{d}t\,\mathrm{d}u}{tu(1-2t-2\kappa_1 u)}.$$

4° *Upper bounds of* Υ_i *for* $i = 7, 8, 9, 10, 11$

Clearly the terms Υ_7, Υ_8, Υ_9, Υ_{10} and Υ_{11} here are those terms Υ_7 (with $\sigma_1 = 1/2 - 3\kappa_1$), Υ_9 (with $\sigma_1 = 1/3$), Υ_{10} (with $\sigma_2 = 1/2 - 3\kappa_1$), Υ_{13} and Υ_{14} of (9.4) in [13]. Thus (10.10), (10.11), (10.12) of [13] give us the estimates

$$\Upsilon_i \leqslant \{F_i + o(1)\}\,\Theta(N) \qquad (i = 7, 8, 9, 10, 11), \tag{5.6}$$

where

$$F_7 := 8\int_2^{2/(1-6\kappa_1)-1} \frac{\log(t-1)}{t}\,\mathrm{d}t,$$

$$F_8 := \frac{36}{5}\int_{\kappa_1}^{1/10} \frac{\log(2-3t)}{t(1-t)^2}\,\mathrm{d}t + 8\int_{1/10}^{1/3} \frac{\log(2-3t)}{t(1-t)}\,\mathrm{d}t,$$

$$F_9 := 8\int_{\kappa_2}^{1/2-3\kappa_1} \frac{\log\{(1+6\kappa_1-2t)/(1-6\kappa_1)\}}{t(1-t)}\,\mathrm{d}t,$$

$$F_{10} := \frac{36}{5}\int_{\kappa_1}^{1/10} \frac{\mathrm{d}t_1}{t_1(1-t_1)}\int_{t_1}^{\kappa_2}\frac{\mathrm{d}t_2}{t_2^2}\int_{t_2}^{\kappa_2}\frac{\mathrm{d}t_3}{t_3}\int_{t_3}^{\kappa_2}\omega\Big(\frac{1-t_1-t_2-t_3-t_4}{t_2}\Big)\frac{\mathrm{d}t_4}{t_4}$$
$$+ 8\int_{1/10}^{\kappa_2} \frac{\mathrm{d}t_1}{t_1}\int_{t_1}^{\kappa_2}\frac{\mathrm{d}t_2}{t_2^2}\int_{t_2}^{\kappa_2}\frac{\mathrm{d}t_3}{t_3}\int_{t_3}^{\kappa_2}\omega\Big(\frac{1-t_1-t_2-t_3-t_4}{t_2}\Big)\frac{\mathrm{d}t_4}{t_4},$$

$$F_{11} := \frac{36}{5}\int_{\kappa_1}^{1/10} \frac{\mathrm{d}t_1}{t_1(1-t_1)}\int_{t_1}^{\kappa_2}\frac{\mathrm{d}t_2}{t_2^2}\int_{t_2}^{\kappa_2}\frac{\mathrm{d}t_3}{t_3}\int_{\kappa_2}^{1/2-2\kappa_1-t_3}\omega\Big(\frac{1-t_1-t_2-t_3-t_4}{t_2}\Big)\frac{\mathrm{d}t_4}{t_4}$$
$$+ 8\int_{1/10}^{\kappa_2} \frac{\mathrm{d}t_1}{t_1}\int_{t_1}^{\kappa_2}\frac{\mathrm{d}t_2}{t_2^2}\int_{t_2}^{\kappa_2}\frac{\mathrm{d}t_3}{t_3}\int_{\kappa_2}^{1/2-2\kappa_1-t_3}\omega\Big(\frac{1-t_1-t_2-t_3-t_4}{t_2}\Big)\frac{\mathrm{d}t_4}{t_4},$$

and $\omega(t)$ is the Buchstab function (see Lemma 2.10 of [13]).

Inserting (5.2)–(5.6) into (2.2), we get the following inequality

$$D_{1,2}(N) \geqslant \{F(\kappa_1,\kappa_2)+o(1)\}\,\Theta(N),$$

where

$$F(\kappa_1,\kappa_2) := \tfrac{1}{4}(3F_1+F_2-F_3-F_4+F_5+F_6-2F_7-F_8-F_9-F_{10}-F_{11}).$$

5° *Numeric computation*

From (3.2) and (3.3), we deduce easily that

$$a(s) := \begin{cases} 0 & \text{if } 0<s\leqslant 2,\\ \log(s-1) & \text{if } 2<s\leqslant 4,\\ \log(s-1)+\displaystyle\int_3^{s-1}\frac{\mathrm{d}t}{t}\int_2^{t-1}\frac{\log(u-1)}{u}\,\mathrm{d}u & \text{if } 4<s\leqslant 6,\\ \log(s-1)+\displaystyle\int_3^{s-1}\frac{\mathrm{d}t}{t}\int_2^{t-1}\frac{\log(u-1)}{u}\,\mathrm{d}u & \\ \quad+\displaystyle\int_5^{s-1}\frac{\mathrm{d}t}{t}\int_4^{t-1}\frac{\mathrm{d}u}{u}\int_3^{u-1}\frac{\mathrm{d}v}{v}\int_2^{v-1}\frac{\log(w-1)}{w}\,\mathrm{d}w & \text{if } 6<s\leqslant 8, \end{cases}$$

and

$$A(s) := \begin{cases} 1 & \text{if } 0<s\leqslant 3,\\ 1+\displaystyle\int_2^{s-1}\frac{\log(t-1)}{t}\,\mathrm{d}t & \text{if } 3<s\leqslant 5,\\ 1+\displaystyle\int_2^{s-1}\frac{\log(t-1)}{t}\,\mathrm{d}t+\int_4^{s-1}\frac{\mathrm{d}t}{t}\int_3^{t-1}\frac{\mathrm{d}u}{u}\int_2^{u-1}\frac{\log(v-1)}{v}\,\mathrm{d}v & \text{if } 5<s\leqslant 7. \end{cases}$$

BY using (3.8), we have

$$G_2 \geqslant 8\biggl(h(s_{22})+\int_{s_{22}-1}^{1/(2\kappa_2)-1}\frac{H(t)}{t}\,\mathrm{d}t\biggr) \geqslant 0.005283.$$

In order to estimate G_4, we use Table 1 and the decreasing property of $H(s)$ to obtain

$$G_4 = 8\int_{1/(4\kappa_1)}^{1/(2\kappa_1)-1}\frac{H(t)}{t(1-2\kappa_1 t)}\,\mathrm{d}t \geqslant 8\sum_{14\leqslant i\leqslant 29} g_4^i H(s_i) \geqslant 0.008860$$

with

$$g_4^{14} := \log\biggl(\frac{2\kappa_1 s_{14}}{1-2\kappa_1 s_{14}}\biggr),$$
$$g_4^i := \log\biggl(\frac{s_i(1-2\kappa_1 s_{i-1})}{s_{i-1}(1-2\kappa_1 s_i)}\biggr) \quad (15\leqslant i\leqslant 29).$$

With a simpler calculation, we get

$$\begin{aligned} G_3 &= G_4 + 8\sum_{3\leqslant i\leqslant 5} g_3^i\Psi_2(s_i) + 8\sum_{6\leqslant i\leqslant 9} g_3^i\Psi_1(s_i)\\ &\geqslant 0.039890 \end{aligned}$$

with

$$g_3^3 := \log\left(\frac{4\kappa_1 s_3}{1-2\kappa_1 s_3}\right),$$
$$g_3^i := \log\left(\frac{s_i(1-2\kappa_1 s_{i-1})}{s_{i-1}(1-2\kappa_1 s_i)}\right) \quad (4 \leqslant i \leqslant 9).$$

Here we have used Table 1 of [13] on the lower bounds for $\Psi_2(s_i)$ $(3 \leqslant i \leqslant 5)$ and $\Psi_1(s_i)$ $(6 \leqslant i \leqslant 9)$:

i	s_i	s_i'	$\kappa_{1,i}$	$\kappa_{2,i}$	$\kappa_{3,i}$	$\Psi_1(s_i)$	$\Psi_2(s_i)$
3	2.3	4.50	3.54	2.88	2.43		0.015247971
4	2.4	4.46	3.57	2.87	2.40		0.013898757
5	2.5	4.12	3.56	2.91	2.50		0.011776059
6	2.6	3.58				0.009405211	
7	2.7	3.47				0.006558950	
8	2.8	3.34				0.003536751	
9	2.9	3.19				0.001056651	
10	3.0	3.00				0	

Table 3. Lower bounds for $\Psi_1(s_i)$ and $\Psi_2(s_i)$

Similarly

$$\begin{aligned}
G_5 &= 8\int_{\kappa_1}^{\kappa_2}\int_{(1/2-\kappa_2-t)/\kappa_1}^{(1/2-2t)/\kappa_1} \frac{h(u)\,\mathrm{d}u\,\mathrm{d}t}{tu(1-2t-2\kappa_1 u)} \\
&= 8\int_{(1/2-2\kappa_2)/\kappa_1}^{(1/2-\kappa_1-\kappa_2)/\kappa_1} h(u)\log\left(\frac{2\kappa_2}{1-2\kappa_2-2\kappa_1 u}\right)\frac{\mathrm{d}u}{u(1-2\kappa_1 u)} \\
&\quad + 8\int_{(1/2-\kappa_1-\kappa_2)/\kappa_1}^{(1/2-2\kappa_1)/\kappa_1} h(u)\log\left(\frac{1-2\kappa_1-2\kappa_1 u}{2\kappa_1}\right)\frac{\mathrm{d}u}{u(1-2\kappa_1 u)} \\
&\geqslant 8\sum_{15\leqslant i\leqslant 27} g_5^i h(s_i) \\
&\geqslant 0.001359
\end{aligned}$$

with

$$\begin{aligned}
g_5^{15} &:= \int_{(1/2-2\kappa_2)/\kappa_1}^{s_{15}} \log\left(\frac{2\kappa_2}{1-2\kappa_2-2\kappa_1 u}\right)\frac{\mathrm{d}u}{u(1-2\kappa_1 u)}, \\
g_5^i &:= \int_{s_{i-1}}^{s_i} \log\left(\frac{2\kappa_2}{1-2\kappa_2-2\kappa_1 u}\right)\frac{\mathrm{d}u}{u(1-2\kappa_1 u)} \qquad (16 \leqslant i \leqslant 20), \\
g_5^{21} &:= \int_{s_{20}}^{(1/2-\kappa_1-\kappa_2)/\kappa_1} \log\left(\frac{2\kappa_2}{1-2\kappa_2-2\kappa_1 u}\right)\frac{\mathrm{d}u}{u(1-2\kappa_1 u)} \\
&\quad + \int_{(1/2-\kappa_1-\kappa_2)/\kappa_1}^{s_{21}} \frac{\log(1/(2\kappa_1)-1-u)}{u(1-2\kappa_1 u)}\,\mathrm{d}u, \\
g_5^i &:= \int_{s_{i-1}}^{s_i} \frac{\log(1/(2\kappa_1)-1-u)}{u(1-2\kappa_1 u)}\,\mathrm{d}u \qquad (22 \leqslant i \leqslant 26), \\
g_5^{27} &:= \int_{s_{26}}^{(1/2-2\kappa_1)/\kappa_1} \frac{\log(1/(2\kappa_1)-1-u)}{u(1-2\kappa_1 u)}\,\mathrm{d}u;
\end{aligned}$$

and

$$\begin{aligned}
G_6 &= 8\int_{\kappa_1}^{\kappa_2}\int_{(2\kappa_2-t)/\kappa_1}^{(1/2-\kappa_2-t)/\kappa_1}\frac{h(u)\,\mathrm{d}u\,\mathrm{d}t}{tu(1-2t-2\kappa_1u)} + 8\int_{\kappa_1}^{3\kappa_1/2}\int_{(3\kappa_1-t)/\kappa_1}^{(2\kappa_2-t)/\kappa_1}\frac{h(u)\,\mathrm{d}t\,\mathrm{d}u}{tu(1-2t-2\kappa_1u)}\\
&\geqslant 8\int_{2}^{(1/2-2\kappa_2)/\kappa_1}\log\left(\frac{\kappa_2(1-2\kappa_1-2\kappa_1u)}{\kappa_1(1-2\kappa_2-2\kappa_1u)}\right)\frac{h(u)\,\mathrm{d}u}{u(1-2\kappa_1u)}\\
&\quad + 8\int_{(1/2-2\kappa_2)/\kappa_1}^{(1/2-\kappa_1-\kappa_2)/\kappa_1}\log\left(\frac{(1-2\kappa_1-2\kappa_1u)(1-2\kappa_2-2\kappa_1u)}{4\kappa_1\kappa_2}\right)\frac{h(u)\,\mathrm{d}u}{u(1-2\kappa_1u)}\\
&\geqslant 8\sum_{1\leqslant i\leqslant 21} g_6^i h(s_i)\\
&\geqslant 0.060469
\end{aligned}$$

with

$$\begin{aligned}
g_6^i &:= \int_{s_{i-1}}^{s_i}\log\left(\frac{\kappa_2(1-2\kappa_1-2\kappa_1u)}{\kappa_1(1-2\kappa_2-2\kappa_1u)}\right)\frac{\mathrm{d}u}{u(1-2\kappa_1u)} \qquad (1\leqslant i\leqslant 14),\\
g_6^{15} &:= \int_{s_{14}}^{(1/2-2\kappa_2)/\kappa_1}\log\left(\frac{\kappa_2(1-2\kappa_1-2\kappa_1u)}{\kappa_1(1-2\kappa_2-2\kappa_1u)}\right)\frac{\mathrm{d}u}{u(1-2\kappa_1u)}\\
&\quad + \int_{(1/2-2\kappa_2)/\kappa_1}^{s_{15}}\log\left(\frac{(1-2\kappa_1-2\kappa_1u)(1-2\kappa_2-2\kappa_1u)}{4\kappa_1\kappa_2}\right)\frac{\mathrm{d}u}{u(1-2\kappa_1u)},\\
g_6^i &:= \int_{s_{i-1}}^{s_i}\log\left(\frac{(1-2\kappa_1-2\kappa_1u)(1-2\kappa_2-2\kappa_1u)}{4\kappa_1\kappa_2}\right)\frac{\mathrm{d}u}{u(1-2\kappa_1u)} \qquad (16\leqslant i\leqslant 20),\\
g_6^{21} &:= \int_{s_{20}}^{(1/2-\kappa_1-\kappa_2)/\kappa_1}\log\left(\frac{(1-2\kappa_1-2\kappa_1u)(1-2\kappa_2-2\kappa_1u)}{4\kappa_1\kappa_2}\right)\frac{\mathrm{d}u}{u(1-2\kappa_1u)}.
\end{aligned}$$

To simplify the computation of F_{10} and F_{11}, we make use of the fact that $\omega(t)\leqslant 0.561522$ for $t\geqslant 3.4$.

Finally a numerical computation concludes

$$\begin{aligned}
F(\kappa_1,\kappa_2) &\geqslant \tfrac{1}{4}\big\{3\times 14.900897 + (9.103015+0.005283)\\
&\qquad - (23.652925-0.039890) - (19.643510-0.008860)\\
&\qquad + (1.654808+0.001359) + (3.819092+0.060469)\\
&\qquad - 2\times 0.585179 - 5.279581 - 5.372410 - 0.104305 - 0.543858\big\}\\
&> 0.899.
\end{aligned}$$

This completes the proof of Theorem. □

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