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Abstract

Simple and efficient algorithms for orthogonal decomposition of derivatives and antiderivatives of a function with rational Laplace transform are proposed. Based on a new theorem related to Routh $\alpha-\beta$ expansion, they enable direct evaluation of the extended Gram matrix which has proved to be very useful in model-reduction applications.

1 - Introduction

Since the pioneering work of Jain [1, 2] the Gram matrix has proved to be one of the most reliable tools in the field of model reduction [3-6]; its ability to produce good pole positions has motivated a number of papers. Early work has been devoted to the Gram matrix of successive antiderivatives of a time response. Some of the authors [3] have shown that evaluation of this Gram matrix can be achieved in the frequency domain; in a sequel to [3], Lucas [7] and Sreeram and Goddard [8] presented additional simplifications in computation. Sreeram and Agathoklis [5,9] have shown the connection between the Gram matrix and the other Gramians, whereby the Gram matrix can be obtained by solving Lyapunov equations.

In [4,6] the Gram matrix has been successfully extended to include both derivatives and antiderivatives. It is the purpose of this communication to propose a straightforward orthogonal decomposition of these derivatives and antiderivatives, directly obtained from the so-called Routh $\alpha-\beta$ parameters. Owing to this orthogonal representation, an elementary and simple computation of the extended Gram matrix follows readily

2 – Background

Although the Routh $\alpha-\beta$ tables are familiar to model reduction researchers, to make this paper self contained and to state precisely some notations and numberings, the procedure is first outlined below in polynomial form; for a tabular form see [10].

Let $D(s)=\sum_{i=0}^{n} d_i s^i$ denote a strictly Hurwitz polynomial. Starting with its even and odd parts, $D_n(s)=d_n s^n + d_{n-2}s^{n-2}+\ldots$ and $D_{n-1}(s)=d_{n-1}s^{n-1}+d_{n-3}s^{n-3}+\ldots$, let a sequence of polynomials of descending degree be computed recursively by the formula

$$D_{k-1}=D_{k+1}-\alpha_k s D_k\,,\quad k=n-1,\ldots,1 \tag{1}$$

with $\alpha_k = lc(D_{k+1})/lc(D_k),\ \ k=n-1,\ldots,0$, where $lc(P)$ denotes the leading coefficient of the polynomial P. Notice that the numbering as defined in [10] has been modified for convenience. A strictly proper rational Laplace transform $F(s)=N(s)/D(s)$ that is asymptotically stable can always be decomposed into the following form

$$F(s)=\sum_{k=0}^{n-1}\beta_k D_k(s)/D(s) \tag{2}$$

in which the coefficients β_k are uniquely determined by $N(s)=\sum_{k=0}^{n-1}\beta_k D_k(s)$. It is known [11] that the Laplace transforms $\Phi_k(s)\hat{=}D_k(s)/D(s)$, $k=0,...,n-1$, define a set of n orthogonal time functions $\varphi_k(t)$:

$$\langle\varphi_k,\varphi_i\rangle\hat{=}\int_0^\infty \varphi_k(t)\varphi_i(t)dt=\delta_{ki}/(2\alpha_k) \tag{3}$$

As far as the authors are aware, the following theorem, which is the keystone of the two algorithms to be proposed in this paper, has never been mentioned before.

Theorem 1: Define $\varphi_{-1}\hat{=}0$ and $\varphi_n\hat{=}-\varphi_{n-1}$. Then the derivative of $\varphi_k(t)$ satisfies

$$\alpha_k\frac{d\varphi_k(t)}{dt}=\varphi_{k+1}(t)-\varphi_{k-1}(t),\ k=0,...,n-1 \tag{4}$$

Proof of Theorem 1: On the understanding that $D_{-1}=0$, eqn. 1 can be rewritten as

$$sD_k=(D_{k+1}-D_{k-1})/\alpha_k,\ k=0,...,n-1 \tag{5}$$

Assume $k\le n-2$. The initial value theorem for the Laplace transform yields $\varphi_k(+0)=0$, therefore the Laplace transform of $d\varphi_k(t)/dt$ is sD_k/D. Thus, starting from eqn. 5 and dividing throughout by D, eqn. 4 is proved for $k=0,...,n-2$. Now, consider $\varphi_{n-1}(+0)=d_{n-1}/d_n=1/\alpha_{n-1}$, hence the Laplace transform of $d\varphi_{n-1}(t)/dt$ is $(sD_{n-1}/D_n)-1/\alpha_{n-1}$ which, on account of eqn. 5 and $D_n+D_{n-1}=D$ may be written $-(D_{n-1}+D_{n-2})/(\alpha_{n-1}D)$. This achieves the proof of eqn. 4 for $k=n-1$.

An extended Gram matrix involves inner products of signals $f_i(t)$, $i\in Z$, recursively defined by $f_0(t)\hat{=}f(t)$, $f_{i+1}(t)\hat{=}df_i(t)/dt$ and $f_{i-1}(t)\hat{=}\int_\infty^t f_i(\tau)d\tau$. Applying the inverse Laplace transform to eqn. 2 yields the orthogonal decomposition $f_0(t)=\sum_{k=0}^{n-1}\beta_k\varphi_k(t)$. More generally, owing to the pole preserving property of operators d/dt and $\int_\infty^t$, any $f_i(t)$ admits an orthogonal decomposition of the following form

$$f_i(t)=\sum_{k=0}^{n-1}\beta_k^i\varphi_k(t) \tag{6}$$

Starting from $\beta_k^0\hat{=}\beta_k$, the coefficients β_k^i can be efficiently computed for $i=1,2,...$ and $i=-1,-2,...$ by algorithms D^+ and D^- proposed below.

Algorithm D^+: Given α_k, β_k^i and $\theta_k^i\hat{=}\beta_k^i/\alpha_k$ relative to $f_i(t)$, let $\theta_{-1}^i\hat{=}0$ and $\theta_n^i\hat{=}\theta_{n-1}^i$; then the following algorithm computes the β's and θ's relative to the derivative $f_{i+1}(t)$:

For $k=0,...,n-1$

$$\beta_k^{i+1}:=\theta_{k-1}^i-\theta_{k+1}^i$$
$$\theta_k^{i+1}:=\beta_k^{i+1}/\alpha_k$$

Proof of Algorithm D^+: Differentiating eqn. 6 with respect to t, using eqn. 4 and rearranging yields $f_{i+1}(t)=\sum_{k=0}^{n-1}(\theta_{k-1}^i-\theta_{k+1}^i)\varphi_k(t)$ which achieves the proof. This algorithm takes only $(n-1)$ additive operations $(+,-)$ and n multiplicative operations $(\times,\div)$. For comparison, the standard way to compute β_k^{i+1} is to evaluate the Laplace transform of $f_{i+1}(t)$ from that of $f_i(t)$ and then to construct the related $\beta-$table. The first step takes exactly the same number of operations as algorithm D^+. Hence, the orthogonal decomposition of $f_{i+1}(t)$ via algorithm D^+ can be obtained as cheaply as its Laplace transform.

In short, algorithm D^+ saves the $\left\lceil \frac{n}{2} \right\rceil \left(\left\lfloor \frac{n}{2} \right\rfloor + 1 \right)$ multiplicative operations and $\left\lfloor \frac{n}{2} \right\rfloor \left(\left\lceil \frac{n}{2} \right\rceil - 1 \right)$ additive operations required by the $\beta-$table, where $\lfloor x \rfloor$ denotes the integer part of x and $\lceil x \rceil$ denotes the smallest integer greater than or equal to x.

Algorithm D^-: Given α_k, β_k^i and $\theta_k^i \mathrel{\hat{=}} \beta_k^i / \alpha_k$ relative to $f_i(t)$, define $\theta_{-1}^{i-1} \mathrel{\hat{=}} 0$; then the following algorithm computes β's and θ's relative to the antiderivative $f_{i-1}(t)$:

$$
\begin{aligned}
&\text{For } k = 0,\ldots,\left\lfloor \frac{n-2}{2} \right\rfloor \\
&\qquad \theta_{2k+1}^{i-1} := \theta_{2k-1}^{i-1} - \beta_{2k}^{i} \\
&\qquad \text{if } n \text{ is odd then } \theta_{n-1}^{i-1} := \theta_{n-2}^{i-1} - \beta_{n-1}^{i} \\
&\qquad\qquad\qquad \text{else } \theta_{n-2}^{i-1} := \theta_{n-1}^{i-1} + \beta_{n-1}^{i} \\
&\text{For } k = \left\lfloor \frac{n-1}{2} \right\rfloor,\ldots,1 \\
&\qquad \theta_{2k-2}^{i-1} := \theta_{2k}^{i-1} + \beta_{2k-1}^{i} \\
&\text{For } k = 0,\ldots,n-1 \\
&\qquad \beta_k^{i-1} := \alpha_k \theta_k^{i-1}
\end{aligned}
$$

Proof of Algorithm D^-: Replacing i by $i-1$ in algorithm D^+ we have $\beta_k^i = \theta_{k-1}^{i-1} - \theta_{k+1}^{i-1}$, $\theta_k^i = \beta_k^i / \alpha_k$. Letting $k = 0$, with $\theta_{-1}^{i-1} = 0$ in mind, yields $\theta_1^{i-1} = -\beta_0^i$; then it is a simple matter to see that the θ_k^{i-1} can be computed in succession, in the order indicated by algorithm D^-. Once again the orthogonal decomposition is obtained as cheaply as the Laplace transform and all the operations required by the $\beta-$table are saved.

Theorem 2: Let $B = [b_{ij}]$ and $\Theta = [\theta_{ij}]$ denote $(m+p+1)\times n$ matrices with (i, j) entries respectively given by $b_{ij} = \beta_{j-1}^{i-p-1}$ and $\theta_{ij} = \theta_{j-1}^{i-p-1}$. Then, denoting transposition by T, the extended Gram matrix involving m antiderivatives and p derivatives is given by

$$G(f_{-m},\ldots, f_0,\ldots, f_p) = (1/2)B\Theta^T \tag{7}$$

Proof: Using eqn. 6 and the orthogonality property of eqn. 3, any entry $\langle f_i, f_j \rangle$ of G is readily written as $\langle f_i, f_j \rangle = \sum_{k=0}^{n-1} \beta_k^i \beta_k^j / (2\alpha_k) = \sum_{k=0}^{n-1} \beta_k^i \theta_k^j / 2$ by which eqn. 7 follows

3 – Illustrative examples

We first consider the transfer function given by Krajewski *et al.* in [6]

$$F(s) = \frac{s^2 + 10s + 100}{1.21s^4 + 3s^3 + 110s^2 + 230s + 100}$$

with a view to deriving $G(f_{-1}, f_0, f_1)$ and a second-order reduced model. The entries in rows 2 of B and Θ are readily obtained by the standard $\alpha-\beta$ Routh algorithm. A run of algorithm D^- yields the entries in rows 1 and a run of algorithm D^+ yields the entries in rows 3:

$$G=\frac{1}{2}\begin{bmatrix}-2.026 & -0.076 & -0.174 & -0.403\\ 0.942 & 0.047 & 0.058 & 0.000\\ -0.580 & 0.110 & 0.580 & 0.333\end{bmatrix}\begin{bmatrix}-0.953 & -0.942 & -1.000 & -1.000\\ 0.443 & 0.580 & 0.333 & 0.000\\ -0.273 & 1.354 & 3.333 & 0.826\end{bmatrix}^T$$

$$=\begin{bmatrix}1.290 & -0.5 & -0.232\\ -0.5 & 0.232 & 0\\ -0.232 & 0 & 1.258\end{bmatrix}$$

After the matrix product has been computed, the method described in [4] yields the $(1,-1,2)$ approximant whose second-order denominator matches exactly that of eqn. 44 in [6]. The squared L_2 norm of the error is equal to 1.125×10^{-2} to be compared with 1.95×10^{-2} obtained through balancing as pointed out in [6]. Note that the L_2 optimal value obtained via a Gauss-Newton nonlinear optimisation procedure is 1.099×10^{-2}.

In the case of the transfer function given by Hwang and Chen in [13],

$$F(s)=\frac{9s^3+42s^2+31s+10}{s^4+8s^3+21s^2+22s+8}$$

with a view to deriving $G(f_0,f_1,f_2)$ given by

$$G=\begin{bmatrix}11.40 & -40.50 & 111.7\\ -40.50 & 158.2 & -450.0\\ 111.7 & -450.0 & 0.451\end{bmatrix}$$

and a second order model as in the previous example, the technique described in [4] yields the (0,1,2) approximant with an error norm of 1.5142×10^{-2} while the approximant given by [13] yields 8.2904×10^{-2}.
The optimal L_2 value in this case, calculated by Lucas in [14], is 7.0135×10^{-3}.

We finally refer to Pal's celebrated example [12]:

$$F(s)=\frac{8s^2+6s+2}{s^3+4s^2+5s+2}$$

with the corresponding Gram matrix *G*:

$$G=\begin{bmatrix}0.451 & -0.125 & -1.194\\ -0.125 & 0.694 & -0.5\\ -1.194 & -0.5 & 9.222\end{bmatrix}$$

This time we use Jain's model order reduction method [2] as described in [3] which yields an error norm of 3.007×10^{-2} to be compared to Pal's original model yielding 4.098×10^{-2}. We may observe that our result is in fact very close to the optimal value obtained via Gauss-Newton optimisation providing 2.909×10^{-2}.

4 – Conclusion

Efficient algorithms for orthogonal decomposition of derivatives and antiderivatives of functions with rational Laplace transforms have been presented. A simple method for computing the extended Gram matrix follows, whereby model order reduction very close to the optimal can be carried out without any optimising iterative procedure

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