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Orthogonal decomposition of derivatives and antiderivatives for easy evaluation of extended Gram matrix

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Abstract
Simple and efficient algorithms for orthogonal decomposition of derivatives and antiderivatives of a function with rational Laplace transform are proposed. Based on a new theorem related to Routh $\alpha - \beta$ expansion, they enable direct evaluation of the extended Gram matrix which has proved to be very useful in model-reduction applications.

1 - Introduction
Since the pioneering work of Jain [1, 2] the Gram matrix has proved to be one of the most reliable tools in the field of model reduction [3-6]; its ability to produce good pole positions has motivated a number of papers. Early work has been devoted to the Gram matrix of successive antiderivatives of a time response. Some of the authors [3] have shown that evaluation of this Gram matrix can be achieved in the frequency domain; in a sequel to [3], Lucas [7] and Sreeram and Goddard [8] presented additional simplifications in computation. Sreeram and Agathoklis [5,9] have shown the connection between the Gram matrix and the other Gramians, whereby the Gram matrix can be obtained by solving Lyapunov equations.

In [4,6] the Gram matrix has been successfully extended to include both derivatives and antiderivatives. It is the purpose of this communication to propose a straightforward orthogonal decomposition of these derivatives and antiderivatives, directly obtained from the so-called Routh $\alpha - \beta$ parameters. Owing to this orthogonal representation, an elementary and simple computation of the extended Gram matrix follows readily.

2 – Background
Although the Routh $\alpha - \beta$ tables are familiar to model reduction researchers, to make this paper self contained and to state precisely some notations and numberings, the procedure is first outlined below in polynomial form; for a tabular form see [10].

Let $D(s) = \sum_{i=0}^{n} d_i s^i$ denote a strictly Hurwitz polynomial. Starting with its even and odd parts, $D_{e}(s) = d_{n}s^n + d_{n-2}s^{n-2} + \ldots$ and $D_{o}(s) = d_{n-1}s^{n-1} + d_{n-3}s^{n-3} + \ldots$, let a sequence of polynomials of descending degree be computed recursively by the formula

$$D_{k-1}=D_{k+1}-\alpha_k s D_k, \quad k = n-1, \ldots, 1$$

with $\alpha_k = lc(D_{k+1})/lc(D_k)$, $k = n-1, \ldots, 0$, where $lc(P)$ denotes the leading coefficient of the polynomial $P$. Notice that the numbering as defined in [10] has been modified for convenience. A strictly proper rational Laplace transform $F(s) = N(s)/D(s)$ that is asymptotically stable can always be decomposed into the following form.
\[ F(s) = \sum_{k=0}^{n-1} \beta_k D_k(s) / D(s) \]  

(2)

in which the coefficients \( \beta_k \) are uniquely determined by \( N(s) = \sum_{k=0}^{n-1} \beta_k D_k(s) \). It is known [11] that the Laplace transforms \( \Phi_k(s) = D_k(s) / D(s) \), \( k = 0, \ldots, n-1 \), define a set of \( n \) orthogonal time functions \( \phi_k(t) \):

\[ \langle \phi_k, \phi_i \rangle = \int_{0}^{\infty} \phi_k(r) \phi_i(r) \, dr = \delta_{ki} / (2 \alpha_k) \]

(3)

As far as the authors are aware, the following theorem, which is the keystone of the two algorithms to be proposed in this paper, has never been mentioned before.

**Theorem 1:** Define \( \phi_{-1} \equiv 0 \) and \( \phi_n \equiv -\phi_{n-1} \). Then the derivative of \( \phi_k(t) \) satisfies

\[ \alpha_k \frac{d\phi_k(t)}{dt} = \phi_{k+1}(t) - \phi_{k-1}(t), \quad k = 0, \ldots, n-1 \]

(4)

**Proof of Theorem 1:** On the understanding that \( D_{-1} = 0 \), eqn. 1 can be rewritten as

\[ sD_k = (D_{k+1} - D_{k-1}) / \alpha_k, \quad k = 0, \ldots, n-1 \]

(5)

Assume \( k \leq n-2 \). The initial value theorem for the Laplace transform yields \( \phi_k(0^+) = 0 \), therefore the Laplace transform of \( d\phi_k(t) / dt \) is \( sD_k / D \). Thus, starting from eqn. 5 and dividing throughout by \( D \), eqn. 4 is proved for \( k = 0, \ldots, n-2 \). Now, consider \( \phi_{n-1}(0^+) = d_{n-1} / d_n = 1 / \alpha_{n-1} \), hence the Laplace transform of \( d\phi_{n-1}(t) / dt \) is \( (sD_{n-1} / D_n) - 1 / \alpha_{n-1} \) which, on account of eqn. 5 and \( D_n + D_{n-1} = D \) may be written \( -(D_{n-1} + D_{n-2}) / (\alpha_{n-1} D) \). This achieves the proof of eqn. 4 for \( k = n - 1 \).

An extended Gram matrix involves inner products of signals \( f_i(t) \), \( i \in \mathbb{Z} \), recursively defined by \( f_0(t) = f(t) \), \( f_{i+1}(t) = df_i(t) / dt \) and \( f_{i-1}(t) = \int f_i(r) \, dr \). Applying the inverse Laplace transform to eqn. 2 yields the orthogonal decomposition \( f_0(t) = \sum_{k=0}^{n-1} \beta_k \phi_k(t) \). More generally, owing to the pole preserving property of operators \( d/dt \) and \( \int \), any \( f_i(t) \) admits an orthogonal decomposition of the following form

\[ f_i(t) = \sum_{k=0}^{n-1} \beta_k \phi_k(t) \]

(6)

Starting from \( \beta_0^i \equiv \beta_k \), the coefficients \( \beta_k^i \) can be efficiently computed for \( i = 1, 2, \ldots \) and \( i = -1, -2, \ldots \) by algorithms \( D^+ \) and \( D^- \) proposed below.

**Algorithm \( D^+ \):** Given \( \alpha_k \), \( \beta_k^i \) and \( \theta_k^i \equiv \beta_k^i / \alpha_k \) relative to \( f_i(t) \), let \( \theta_{-1}^i \equiv 0 \) and \( \theta_{n-1}^i \equiv \theta_{n-1}^i \); then the following algorithm computes the \( \beta^i \)'s and \( \theta^i \)'s relative to the derivative \( f_{i+1}(t) \):

For \( k = 0, \ldots, n-1 \)

\[ \beta_k^{i+1} := \theta_{k-1}^i - \theta_{k+1}^i \]
\[ \theta_k^{i+1} := \beta_k^{i+1} / \alpha_k \]

**Proof of Algorithm \( D^+ \):** Differentiating eqn. 6 with respect to \( t \), using eqn. 4 and rearranging yields \( f_{i+1}(t) = \sum_{k=0}^{n-1} (\theta_k^i - \theta_{k+1}^i) \phi_k(t) \) which achieves the proof. This algorithm takes only \( (n-1) \) additive operations \( (+, -) \) and \( n \) multiplicative operations \( (x, \div) \). For comparison, the standard way to compute \( \beta_k^i \) is to evaluate the Laplace transform of \( f_{i+1}(t) \) from that of \( f_i(t) \) and then to construct the related \( \beta^- \)-table. The first step takes exactly the same number of operations as algorithm \( D^+ \). Hence, the orthogonal decomposition of \( f_{i+1}(t) \) via algorithm \( D^+ \) can be obtained as cheaply as its Laplace transform.
In short, algorithm $D^+$ saves the $\left[\frac{n}{2}\right]\left[\frac{n}{2}\right]+1$ multiplicative operations and $\left[\frac{n}{2}\right]\left[\frac{n}{2}\right]-1$ additive operations required by the $\beta-$ table, where $\lfloor x \rfloor$ denotes the integer part of $x$ and $\lceil x \rceil$ denotes the smallest integer greater than or equal to $x$ .

Algorithm $D^-$: Given $\alpha_k$, $\beta_k^i$ and $\theta_k^i = \beta_k^i / \alpha_k$ relative to $f_i(t)$, define $\theta_k^{-1} \equiv 0$; then the following algorithm computes $\beta$’s and $\theta$’s relative to the antiderivative $f_{i-1}(t)$:

For $k = 0, \ldots, \left\lfloor \frac{n-2}{2} \right\rfloor$
\[ \theta_{2k+1}^{-1} := \theta_{2k-1}^{-1} - \beta_{2k}^i \]
if $n$ is odd then \[ \theta_{n-1}^{-1} := \theta_{n-2}^{-1} - \beta_{n-1}^i \]
else \[ \theta_{n-2}^{-1} := \theta_{n-1}^{-1} + \beta_{n-1}^i \]

For $k = \left\lfloor \frac{n-1}{2} \right\rfloor, \ldots, 1$
\[ \theta_{2k-2}^{-1} := \theta_{2k-1}^{-1} + \beta_{2k-1}^i \]

For $k = 0, \ldots, n-1$
\[ \beta_k^{i+1} := \alpha_k \theta_k^{-1} \]

Proof of Algorithm $D^-$: Replacing $i$ by $i-1$ in algorithm $D^+$ we have $\beta_k^i = \theta_k^{-1} - \theta_k^{-1} - \beta_k^i = \beta_k^i / \alpha_k$.

Letting $k = 0$, with $\theta_k^{i-1} = 0$ in mind, yields $\theta_k^{i-1} = -\beta_0^i$; then it is a simple matter to see that the $\theta_k^{i-1}$ can be computed in succession, in the order indicated by algorithm $D^-$. Once again the orthogonal decomposition is obtained as cheaply as the Laplace transform and all the operations required by the $\beta-$table are saved.

Theorem 2: Let $B = [b_{ij}]$ and $\Theta = [\theta_{ij}]$ denote $(m + p + 1) \times n$ matrices with $(i, j)$ entries respectively given by $b_{ij} = \beta_i^{j-1}$ and $\theta_{ij} = \theta_i^{j-1}$. Then, denoting transposition by $T$, the extended Gram matrix involving $m$ antiderivatives and $p$ derivatives is given by

\[ G(f_m, \ldots, f_0, \ldots, f_p) = (1/2)BT \]

Proof: Using eqn. 6 and the orthogonality property of eqn. 3, any entry $\langle f_i, f_j \rangle$ of $G$ is readily written as
\[ \langle f_i, f_j \rangle = \sum_{k=0}^{n-1} \beta_k^i \beta_k^j / (2\alpha_k) = \sum_{k=0}^{n-1} \beta_k^i \theta_k^j / 2 \]
by which eqn. 7 follows.

3 – Illustrative examples

We first consider the transfer function given by Krajewski et al. in [6]

\[ F(s) = \frac{s^2 + 10s + 100}{1.21s^4 + 3s^3 + 110s^2 + 230s + 100} \]

with a view to deriving $G(f_{i-1}, f_0, f_1)$ and a second-order reduced model. The entries in rows 2 of $B$ and $\Theta$ are readily obtained by the standard $\alpha - \beta$ Routh algorithm. A run of algorithm $D^-$ yields the entries in rows 1 and a run of algorithm $D^+$ yields the entries in rows 3:
After the matrix product has been computed, the method described in [4] yields the \((1,-1,2)\) approximant whose second-order denominator matches exactly that of eqn. 44 in [6]. The squared \(L_2\) norm of the error is equal to \(1.125 \times 10^{-2}\) to be compared with \(1.95 \times 10^{-2}\) obtained through balancing as pointed out in [6]. Note that the \(L_2\) optimal value obtained via a Gauss-Newton nonlinear optimisation procedure is \(1.099 \times 10^{-2}\).

In the case of the transfer function given by Hwang and Chen in [13],

\[
F(s) = \frac{9s^3 + 42s^2 + 31s + 10}{s^3 + 8s^2 + 21s^2 + 22s + 8}
\]

with a view to deriving \(G(f_0, f_1, f_2)\) given by

\[
G = \begin{bmatrix}
11.40 & -40.50 & 111.7 \\
-40.50 & 158.2 & -450.0 \\
111.7 & -450.0 & 0.451
\end{bmatrix}
\]

and a second order model as in the previous example, the technique described in [4] yields the \((0,1,2)\) approximant with an error norm of \(1.5142 \times 10^{-2}\) while the approximant given by [13] yields \(8.2904 \times 10^{-2}\).

The optimal \(L_2\) value in this case, calculated by Lucas in [14], is \(7.0135 \times 10^{-3}\).

We finally refer to Pal’s celebrated example [12]:

\[
F(s) = \frac{8s^2 + 6s + 2}{s^3 + 4s^2 + 5s + 2}
\]

with the corresponding Gram matrix \(G:\)

\[
G = \begin{bmatrix}
0.451 & -0.125 & -1.194 \\
-0.125 & 0.694 & -0.5 \\
-1.194 & -0.5 & 9.222
\end{bmatrix}
\]

This time we use Jain’s model order reduction method [2] as described in [3] which yields an error norm of \(3.007 \times 10^{-2}\) to be compared to Pal’s original model yielding \(4.098 \times 10^{-2}\). We may observe that our result is in fact very close to the optimal value obtained via Gauss-Newton optimisation providing \(2.909 \times 10^{-2}\).
4 – Conclusion

Efficient algorithms for orthogonal decomposition of derivatives and antiderivatives of functions with rational Laplace transforms have been presented. A simple method for computing the extended Gram matrix follows, whereby model order reduction very close to the optimal can be carried out without any optimising iterative procedure.

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