



HAL
open science

ON ADAPTIVE WAVELET ESTIMATION OF A QUADRATIC FUNCTIONAL FROM A DECONVOLUTION PROBLEM

Christophe Chesneau

► **To cite this version:**

Christophe Chesneau. ON ADAPTIVE WAVELET ESTIMATION OF A QUADRATIC FUNCTIONAL FROM A DECONVOLUTION PROBLEM. 2007. hal-00174239v3

HAL Id: hal-00174239

<https://hal.science/hal-00174239v3>

Preprint submitted on 14 Oct 2007 (v3), last revised 15 Oct 2008 (v4)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

ON ADAPTIVE WAVELET ESTIMATION OF A QUADRATIC FUNCTIONAL FROM A DECONVOLUTION PROBLEM

Christophe Chesneau

*Laboratoire de Mathématiques Nicolas Oresme,
Université de Caen Basse-Normandie,
Campus II, Science 3,
14032 Caen, France.*

<http://www.chesneau-stat.com> chesneau@math.unicaen.fr

Abstract

We consider the following situation: an unknown function f is observed in Gaussian white noise after convolution with a known function g . We wish to estimate the quadratic functional $\int f^2(t)dt$ from the observations. To reach this goal, we propose an adaptive estimator based on wavelet thresholding. We prove that it achieves near optimal rates of convergence under the mean squared error over a range of smoothness classes.

Key words: Deconvolution, Quadratic functionals, Adaptive curve estimation, Wavelets, Global thresholding.

1991 MSC: Primary: 62G07, Secondary: 62N01.

1 MOTIVATION

We consider the stochastic process $\{Y(t); t \in [0, 1]\}$ defined by

$$dY(t) = \left(\int_0^1 f(t-u)g(u)du \right) dt + n^{-1/2}dW(t), \quad (1.1)$$

where $\{W(t); t \in [0, 1]\}$ is a standard Brownian motion, f an unknown one-periodic function such that $\int_0^1 f^2(t)dt < \infty$, and g a known one-periodic function such that $\int_0^1 g^2(t)dt < \infty$. We suppose that the Fourier coefficients of g decay in a polynomial fashion. The goal is to estimate f , or a quantity depending on f , from $\{Y(t); t \in [0, 1]\}$. The equation (1.1) modelizes the important

problem of recovery of noisy signals in linear motion blur. Further details can be found in Bertero and Boccacci (1998).

In the literature, many authors have examined the estimation of f . See, for instance, Cavalier and Tsybakov (2002), Johnstone et al. (2004) and Chesneau (2007). In this paper, we focus our attention on a different problem: the estimation of the quadratic functional $Q(f)$ defined by

$$Q(f) = \|f\|_2^2 = \int_0^1 f^2(t)dt.$$

The problem of estimating quadratic functionals is closely connected to the construction of confidence balls in nonparametric function estimation. It has already been investigated for a wide variety of models (density, regression, Gaussian model in white noise, density derivatives, ...). See, for instance, Bickel and Ritov (1988), Donoho and Nussbaum (1990), Kerkycharian and Picard (1996), Efromovich and Low (1996), Gayraud and Tribouley (1999), Johnstone (2001a,b), Laurent (2005), Cai and Low (2005), Rivoirard and Tribouley (2007) and Butucea (2007). However, to our knowledge, the estimation of $Q(f) = \|f\|_2^2$ from the convolution model in Gaussian white noise (1.1) has never been studied.

To estimate $Q(f) = \|f\|_2^2$, we propose an adaptive wavelet estimator based on the global thresholding. The idea of this thresholding is the following: we decompose the quadratic function $Q(f)$ by using an appropriate wavelet basis, we estimate the associated coefficients via natural estimators, then, at each level of the wavelets, we keep all of these estimators if, and only if, the corresponding l_2 norm is greater than a fixed threshold. This technic has been initially developed by Gayraud and Tribouley (1999) for the quadratic functional estimation in the standard Gaussian white noise model.

Naturally, the construction of our estimator depends on the complexity of the convolution model (1.1) and, in particular, the function g . To evaluate his theoretical performances, we adopt the asymptotic minimax approach under the mean squared error over Besov balls. More precisely, if \widehat{Q}_n denotes our estimator and $B_{2,\infty}^s(M)$ denotes the Besov balls, we aim to determine the smallest rates of convergence φ_n such that

$$\lim_{n \rightarrow \infty} \varphi_n^{-1} \sup_{f \in B_{2,\infty}^s(M)} \mathbb{E} \left(\left(\widehat{Q}_n - \|f\|_2^2 \right)^2 \right) < \infty.$$

In this statistical framework, we prove that \widehat{Q}_n achieves near optimal of convergence. The proof is divided into two parts. The first part is devoted to the upper bounds. We use a suitable decomposition of the risk of \widehat{Q}_n and some concentration inequalities. The second part concerns the lower bounds. They are obtained via a specific theorem proved by Tsybakov (2004).

The paper is organized as follows. In Section 2, we present wavelets and Besov balls. Section 3 clarifies the assumption made on g and introduces some intermediate estimators. The main estimator of the study is defined in Section 4. Section 5 is devoted to the results. The proofs are postponed in Section 6.

2 WAVELETS AND BESOV BALLS

We consider an orthonormal wavelet basis generated by dilation and translation of a "father" Meyer-type wavelet ϕ and a "mother" Meyer-type wavelet ψ . The particularities of such wavelets are that the Fourier transforms of ϕ and ψ have bounded support. More precisely, we have $\text{supp}(F(\phi)) \subset [-4\pi 3^{-1}, 4\pi 3^{-1}]$ and $\text{supp}(F(\psi)) \subset [-8\pi 3^{-1}, -2\pi 3^{-1}] \cup [2\pi 3^{-1}, 8\pi 3^{-1}]$, where, for any function $h \in \mathbb{L}^1([0, 1])$, $F(h)$ denotes the Fourier transform of h defined by $F(h)(l) = \int_0^1 h(x)e^{-2i\pi lx} dx$. Moreover, for any $l \in [-2\pi, -\pi] \cup [\pi, 2\pi]$, there exists a constant $c > 0$ such that $|F(\psi)(l)| \geq c$. For further details about Meyer-type wavelets, see Walter (1994), and Zayed and Walter (1996).

For the purposes of this paper, we use the periodized wavelet bases on the unit interval. For any $x \in [0, 1]$, any integer j and any $k \in \{0, \dots, 2^j - 1\}$, let $\phi_{j,k}(x) = 2^{j/2}\phi(2^j x - k)$ and $\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k)$ be the elements of the wavelet basis, and

$$\phi_{j,k}^{per}(x) = \sum_{l \in \mathbb{Z}} \phi_{j,k}(x - l), \quad \psi_{j,k}^{per}(x) = \sum_{l \in \mathbb{Z}} \psi_{j,k}(x - l),$$

their periodized versions. There exists an integer τ such that the collection ζ defined by $\zeta = \{\phi_{\tau,k}^{per}, k = 0, \dots, 2^\tau - 1; \psi_{j,k}^{per}, j = \tau, \dots, \infty, k = 0, \dots, 2^j - 1\}$ constitutes an orthonormal basis of $\mathbb{L}_{per}^2([0, 1])$, the set of square integrable one-periodic functions on $[0, 1]$. In what follows, the superscript "per" will be suppressed from the notations for convenience.

Let m be an integer such that $m \geq \tau$. A function $f \in \mathbb{L}_{per}^2([0, 1])$ can be expanded into a wavelet series as

$$f(x) = \sum_{k=0}^{2^m-1} \alpha_{m,k} \phi_{m,k}(x) + \sum_{j=m}^{\infty} \sum_{k=0}^{2^j-1} \beta_{j,k} \psi_{j,k}(x),$$

where $\alpha_{m,k} = \int_0^1 f(t)\phi_{m,k}(t)dt$ and $\beta_{j,k} = \int_0^1 f(t)\psi_{j,k}(t)dt$. For further details about wavelet bases on the unit interval, we refer to Cohen et al. (1993).

Now, let us present the Besov balls $B_{2,\infty}^s(M)$. Let $M \in (0, \infty)$ and $s \in (0, \infty)$. We say that a one-periodic function f belongs to the Besov balls $B_{2,\infty}^s(M)$ if, and only if, there exists a constant $M_* > 0$ such that the associated wavelet

coefficients satisfy

$$\sup_{j \geq \tau} 2^{2js} \sum_{k=0}^{2^j-1} \beta_{j,k}^2 \leq M_*.$$

These sets contain all the Besov balls $B_{p,\infty}^s(M)$ with $p \geq 2$. See, for instance, Meyer (1992).

3 PRELIMINARY STUDY

In this section, we clarify the assumption made on g and we introduce some intermediate estimators.

Assumption on g . As mentioned in Section 1, we suppose that the Fourier coefficients of g decay in a polynomial fashion. In other words, there exist three constants, $c > 0$, $C > 0$ and $\delta > 2^{-1}$, such that, for $|l|$ large enough, g satisfies

$$c|l|^{-\delta} \leq |F(g)(l)| \leq C|l|^{-\delta}. \quad (3.1)$$

The factor δ will play a determinant role in our study.

The assumption (3.1) is satisfied for a wide variety of functions. An immediate example is the square integrable one-periodic function g defined by $g(x) = \sum_{l \in \mathbb{Z}} e^{-|x+l|}$. Since, for any $l \in \mathbb{Z}$, we have $F(g)(l) = 2(1 + 4\pi^2 l^2)^{-1}$, g satisfies the inequality (3.1) with $\delta = 2$.

Preliminary to the estimation of $\|f\|_2^2$ from $\{Y(t); t \in [0, 1]\}$. Thanks to the orthonormality of the wavelet basis, for any integer $m \geq \tau$, the unknown quadratic functional $Q(f) = \|f\|_2^2$ can be decomposed as

$$\|f\|_2^2 = \sum_{k=0}^{2^m-1} \alpha_{m,k}^2 + \sum_{j=m}^{\infty} \sum_{k=0}^{2^j-1} \beta_{j,k}^2.$$

Thus, the first step to estimate $\|f\|_2^2$ consists in estimating the unknown coefficients $(\alpha_{m,k}^2)_k$ and $(\beta_{j,k}^2)_{j,k}$.

In what follows, we investigate the estimation of $(\beta_{j,k}^2)_{j,k}$. For any $l \in \mathbb{Z}$, we have $F\left(\int_0^1 f(t-u)g(u)du\right)(l) = F((f \star g)(t))(l) = F(f)(l)F(g)(l)$, where \star denotes the standard convolution product on the unit interval. Thus, if we set

$$y_l = \int_0^1 e^{-2\pi i l t} dY(t), \quad f_l = F(f)(l), \quad g_l = F(g)(l), \quad e_l = \int_0^1 e^{-2\pi i l t} dW(t),$$

then we have

$$y_l = f_l g_l + n^{-1/2} e_l.$$

The Parseval theorem and the translation theorem of the Fourier transform give

$$\begin{aligned}
\sum_{l \in \mathbb{Z}} y_l g_l^{-1} F(\psi_{j,k})(l) &= \sum_{l \in \mathbb{Z}} f_l F(\psi_{j,k})(l) + n^{-1/2} \sum_{l \in \mathbb{Z}} g_l^{-1} F(\psi_{j,k})(l) e_l \\
&= \beta_{j,k} + n^{-1/2} \sum_{l \in \mathbb{Z}} g_l^{-1} F(\psi_{j,k})(l) e_l \\
&= \beta_{j,k} + n^{-1/2} \sum_{l \in \mathbb{Z}} g_l^{-1} e^{2i\pi lk/(2^j)} F(\psi_{j,0})(l) e_l.
\end{aligned}$$

Since $n^{-1/2} \sum_{l \in \mathbb{Z}} g_l^{-1} e^{2i\pi lk/(2^j)} F(\psi_{j,0})(l) e_l \sim \mathcal{N}(0, n^{-1} \sum_{l \in \mathbb{Z}} |g_l|^{-2} |F(\psi_{j,0})(l)|^2)$, we have

$$\widehat{\beta}_{j,k} = \sum_{l \in \mathbb{Z}} y_l g_l^{-1} F(\psi_{j,k})(l) \sim \mathcal{N}\left(\beta_{j,k}, n^{-1} \sum_{l \in \mathbb{Z}} |g_l|^{-2} |F(\psi_{j,0})(l)|^2\right).$$

Therefore, the random variable $\widehat{\theta}_{j,k}$ defined by

$$\widehat{\theta}_{j,k} = \widehat{\beta}_{j,k}^2 - n^{-1} \sum_{l \in \mathbb{Z}} |g_l|^{-2} |F(\psi_{j,0})(l)|^2$$

is an unbiased (and a natural) estimator of $\beta_{j,k}^2$.

We are now in the position to describe the main estimator of the study.

4 ESTIMATOR

Let j_0 and j_1 be two integers such that

$$2^{-1} \left(n(\log n)^{-1}\right)^{1/(4\delta+1)} < 2^{j_0} \leq \left(n(\log n)^{-1}\right)^{1/(4\delta+1)}$$

and

$$2^{-1} n^{1/(2\delta+1/2)} < 2^{j_1} \leq n^{1/(2\delta+1/2)}.$$

Set $(a)_+ = \max(a, 0)$. Let κ be a positive real number. We define the thresholding estimator \widehat{Q}_n by

$$\widehat{Q}_n = \sum_{k=0}^{2^{j_0}-1} \left(\widehat{\alpha}_{j_0,k}^2 - n^{-1} \eta_{j_0}^2\right) + \sum_{j=j_0}^{j_1} \left(\sum_{k=0}^{2^j-1} (\widehat{\beta}_{j,k}^2 - n^{-1} \sigma_j^2) - \kappa n^{-1} \sigma_j^2 (j2^j)^{1/2}\right)_+, \tag{4.1}$$

where

$$\begin{aligned}\widehat{\alpha}_{j_0,k} &= \sum_{l \in \mathcal{D}_{j_0}} y_l g_l^{-1} F(\phi_{j_0,k})(l), & \widehat{\beta}_{j,k} &= \sum_{l \in \mathcal{C}_j} y_l g_l^{-1} F(\psi_{j,k})(l), \\ \eta_{j_0}^2 &= \sum_{l \in \mathcal{D}_{j_0}} |g_l|^{-2} F^2(\phi_{j_0,0})(l), & \sigma_j^2 &= \sum_{l \in \mathcal{C}_j} |g_l|^{-2} |F(\psi_{j,0})(l)|^2.\end{aligned}\quad (4.2)$$

Here, for any $k \in \{0, \dots, 2^j - 1\}$, \mathcal{D}_{j_0} denotes the support of $F(\phi_{j_0,k})(l)$ and \mathcal{C}_j the support of $F(\psi_{j,k})(l)$.

As mentioned in Section 1, \widehat{Q}_n is based on the global thresholding technic initially developed by Gayraud and Tribouley (1999) for the quadratic functional estimation in the standard Gaussian white noise model.

Notice that the definition of \widehat{Q}_n does not require any a priori knowledge on f . It is adaptive.

5 MAIN RESULTS

Theorem 5.1 below determines the rates of convergence achieved by the estimator (4.1) under the mean squared error over the Besov balls $B_{2,\infty}^s(M)$.

Theorem 5.1 *Let us consider the convolution model defined by (1.1). Assume that g satisfies the inequality (3.1). Let \widehat{Q}_n be the estimator defined by (4.1) with a large enough κ . Then, there exists a constant $C > 0$ such that, for n large enough, we have*

$$\sup_{f \in B_{2,\infty}^s(M)} \mathbb{E} \left(\left(\widehat{Q}_n - \|f\|_2^2 \right)^2 \right) \leq C \varphi_n,$$

where

$$\varphi_n = \begin{cases} n^{-1}, & \text{when } s > \delta + 4^{-1}, \\ \left(n(\log n)^{-1/2} \right)^{-4s/(2s+2\delta+1/2)}, & \text{when } s \in (0, \delta + 4^{-1}]. \end{cases}$$

The proof of Theorem 5.1 uses a decomposition of the risk similar to Proposition 3 of Rivoirard and Tribouley (2007) and two concentration inequalities of the estimators $(\widehat{\beta}_{j,k})_{k=0,\dots,2^j-1}$.

Notice that, the rates of convergence φ_n presented in Theorem 5.1 are equal to $\max \left(n^{-1}, \left(n(\log n)^{-1/2} \right)^{-4s/(2s+2\delta+1/2)} \right)$. These rates of convergence are similar to those obtained by Butucea (2007) (see Theorem 4) for a different model

(the density convolution), with a different estimator (the deconvolution kernel estimator), and under a different risk ($R(h, f) = \mathbb{E}(|h - \|f\|_2^2|)$).

The result of Theorem 5.1 raises the following question: are the rates of convergence φ_n the optimal? Theorem 5.2 below provides the answer by determining the minimax lower bounds.

Theorem 5.2 *Let us consider the convolution model defined by (1.1). Assume that g satisfies the inequality (3.1). Then, there exists a constant $c > 0$ such that, for n large enough, we have*

$$\inf_{\tilde{Q}_n} \sup_{f \in B_{2,\infty}^s(M)} \mathbb{E} \left(\left(\tilde{Q}_n - \|f\|_2^2 \right)^2 \right) \geq c\varphi_n^*,$$

where $\inf_{\tilde{Q}_n}$ denotes the infimum over all the possible estimators of $\|f\|_2^2$, and

$$\varphi_n^* = \begin{cases} n^{-1}, & \text{when } s > \delta + 4^{-1}, \\ n^{-4s/(2s+2\delta+1/2)}, & \text{when } s \in (0, \delta + 4^{-1}]. \end{cases}$$

The proof of Theorem 5.2 is based on a general theorem established by Tsybakov (2004).

The results of Theorems 5.1 and 5.2 show that the estimator \hat{Q}_n is near optimal under the mean squared error over the Besov balls $B_{2,\infty}^s(M)$. "Near" is due to the case $s \in (0, \delta + 4^{-1}]$ where there is an extra logarithmic term.

6 PROOFS

In this section, c and C denote positive constants which can take different values for each mathematical term. They are independent of f and n .

PROOF OF THEOREM 5.1. Lemma 6.1 below presents a "functional" upper bound for the mean squared risk of \hat{Q}_n without any assumption on the smoothness of f .

Lemma 6.1 *Let \hat{Q}_n be the estimator defined by (4.1) with a large enough κ . There exist two constants $C_1 > 0$ and $C_2 > 0$ such that*

$$\mathbb{E} \left(\left(\hat{Q}_n - \|f\|_2^2 \right)^2 \right) \leq C_1(\mathbf{A} + \mathbf{B} + \mathbf{C} + \mathbf{D}),$$

where

$$\mathbf{A} = n^{-2}2^{(1+4\delta)j_0}, \quad \mathbf{B} = n^{-1}\|f\|_2^2, \quad \mathbf{C} = \left(\sum_{j=j_1+1}^{\infty} \sum_{k=0}^{2^j-1} \beta_{j,k}^2 \right)^2$$

and

$$\mathbf{D} = \left(\sum_{j=j_0}^{j_1} \min \left(\sum_{k=0}^{2^j-1} \beta_{j,k}^2, C_2 \kappa 2^{2\delta j} n^{-1} (j2^j)^{1/2} \right) \right)^2.$$

Therefore, to prove Theorem 5.1, it is enough to investigate the upper bounds for the terms \mathbf{A} , \mathbf{B} , \mathbf{C} and \mathbf{D} defined in Lemma 6.1 above.

The upper bound for \mathbf{A} . Thanks to the definition of j_0 , we have

$$\mathbf{A} = n^{-2}2^{(1+4\delta)j_0} \leq n^{-1} \leq \max \left(n^{-1}, \left(n(\log n)^{-1/2} \right)^{-4s/(2s+2\delta+1/2)} \right) = \varphi_n.$$

The upper bound for \mathbf{B} . We have

$$\mathbf{B} = n^{-1}\|f\|_2^2 \leq C \max \left(n^{-1}, \left(n(\log n)^{-1/2} \right)^{-4s/(2s+2\delta+1/2)} \right) = C\varphi_n.$$

The upper bound for \mathbf{C} . Since $f \in B_{2,\infty}^s(M)$, we have

$$\begin{aligned} \mathbf{C} &= \left(\sum_{j=j_1+1}^{\infty} \sum_{k=0}^{2^j-1} \beta_{j,k}^2 \right)^2 \leq C \left(\sum_{j=j_1+1}^{\infty} 2^{-2js} \right)^2 \leq C 2^{-4j_1 s} \leq C n^{-4s/(2\delta+1/2)} \\ &\leq C \max \left(n^{-1}, \left(n(\log n)^{-1/2} \right)^{-4s/(2s+2\delta+1/2)} \right) = C\varphi_n. \end{aligned}$$

The upper bound for \mathbf{D} . We distinguish the case $s > \delta + 4^{-1}$ and the case $s \in (0, \delta + 4^{-1}]$.

- *The case $s > \delta + 4^{-1}$.* Since $f \in B_{2,\infty}^s(M)$, the definition of j_0 yields

$$\begin{aligned} \mathbf{D} &= \left(\sum_{j=j_0}^{j_1} \min \left(\sum_{k=0}^{2^j-1} \beta_{j,k}^2, C_2 \kappa 2^{2\delta j} n^{-1} (j2^j)^{1/2} \right) \right)^2 \leq C \left(\sum_{j=j_0}^{j_1} \sum_{k=0}^{2^j-1} \beta_{j,k}^2 \right)^2 \\ &\leq C \left(\sum_{j=j_0}^{j_1} 2^{-2js} \right)^2 \leq C 2^{-4j_0 s} \leq C \left(n(\log n)^{-1} \right)^{-4s/(4\delta+1)} \leq C n^{-1}. \end{aligned}$$

- *The case $s \in (0, \delta + 4^{-1}]$.* Let j_2 be an integer such that

$$2^{-1} \left(n(\log n)^{-1/2} \right)^{1/(2s+2\delta+1/2)} < 2^{j_2} \leq \left(n(\log n)^{-1/2} \right)^{1/(2s+2\delta+1/2)}.$$

Notice that $j_0 \leq j_2 \leq j_1$. We have the following decomposition

$$\mathbf{D} = (\mathbf{L} + \mathbf{M})^2,$$

where

$$\mathbf{L} = \sum_{j=j_0}^{j_2} \min \left(\sum_{k=0}^{2^j-1} \beta_{j,k}^2, C_2 \kappa 2^{2\delta j} n^{-1} (j 2^j)^{1/2} \right)$$

and

$$\mathbf{M} = \sum_{j=j_2+1}^{j_1} \min \left(\sum_{k=0}^{2^j-1} \beta_{j,k}^2, C_2 \kappa 2^{2\delta j} n^{-1} (j 2^j)^{1/2} \right).$$

Let us study the upper bounds for \mathbf{L} and \mathbf{M} in turn.

It follows from the definition of j_2 that

$$\begin{aligned} \mathbf{L} &\leq C n^{-1} \sum_{j=j_0}^{j_2} j^{1/2} 2^{(2\delta+1/2)j} \leq C n^{-1} (\log n)^{1/2} 2^{(2\delta+1/2)j_2} \\ &\leq C \left(n(\log n)^{-1/2} \right)^{-2s/(2s+2\delta+1/2)}. \end{aligned}$$

Since $f \in B_{2,\infty}^s(M)$, the definition of j_2 yields

$$\begin{aligned} \mathbf{M} &\leq \sum_{j=j_2+1}^{j_1} \sum_{k=0}^{2^j-1} \beta_{j,k}^2 \leq C \sum_{j=j_2+1}^{j_1} 2^{-2js} \leq C 2^{-2j_2 s} \\ &\leq C \left(n(\log n)^{-1/2} \right)^{-2s/(2s+2\delta+1/2)}. \end{aligned}$$

Putting these two inequalities together, we obtain

$$\mathbf{D} = (\mathbf{L} + \mathbf{M})^2 \leq C \left(n(\log n)^{-1/2} \right)^{-4s/(2s+2\delta+1/2)}.$$

Combining the obtained upper bounds for \mathbf{A} , \mathbf{B} , \mathbf{C} and \mathbf{D} , for any $s > 0$, we obtain

$$\mathbb{E} \left(\left(\widehat{Q}_n - \|f\|_2^2 \right)^2 \right) \leq C_1 (\mathbf{A} + \mathbf{B} + \mathbf{C} + \mathbf{D}) \leq C \varphi_n.$$

Theorem 5.1 is proved.

PROOF OF LEMMA 6.1. The proof of Lemma 6.1 is similar to Proposition 3 of Rivoirard and Tribouley (2007). If we analyze the proof of this proposition, Lemma 6.1 follows from Lemmas 6.2 and 6.3 below.

Lemma 6.2 *There exist two constants $c_1 > 0$ and $C_1 > 0$ such that, for any integer $j \geq \tau$, we have*

$$c_1 2^{2\delta j} \leq \sigma_j^2 \leq C_1 2^{2\delta j}.$$

Moreover, we have $\eta_{j_0}^2 \leq C_1 2^{2\delta j_0}$.

(Proof of Lemma 6.2.) Using the inequalities $\sup_{l \in \mathcal{C}_j} |F(\psi_{j,0})(l)|^2 \leq C 2^{-j}$, $\text{Card}(\mathcal{C}_j) \leq C 2^j$ and the assumption made on g (see (3.1)), we have

$$\sigma_j^2 = \sum_{l \in \mathcal{C}_j} |g_l|^{-2} |F(\psi_{j,0})(l)|^2 \leq C 2^{-j} \sum_{l \in \mathcal{C}_j} |g_l|^{-2} \leq C \sup_{l \in \mathcal{C}_j} |g_l|^{-2} \leq C 2^{2\delta j}.$$

In the same way, we prove that $\eta_{j_0}^2 \leq C_1 2^{2\delta j_0}$.

Recall that, for any $l \in [-2\pi, -\pi] \cup [\pi, 2\pi]$, there exists a constant $c > 0$ such that $|F(\psi)(l)| \geq c$. Therefore, there exists a set \mathcal{H}_j such that $\inf_{l \in \mathcal{H}_j} |F(\psi_{j,0})(l)|^2 \geq c 2^{-j}$, $\text{Card}(\mathcal{H}_j) \geq c 2^j$ and, thanks to the assumption made on g (see (3.1)), $\inf_{l \in \mathcal{H}_j} |g_l|^{-2} \geq c 2^{2\delta j}$. Hence,

$$c 2^{2\delta j} \leq c \inf_{l \in \mathcal{H}_j} |g_l|^{-2} \leq c 2^{-j} \sum_{l \in \mathcal{H}_j} |g_l|^{-2} \leq \sum_{l \in \mathcal{C}_j} |g_l|^{-2} |F(\psi_{j,0})(l)|^2 = \sigma_j^2.$$

This ends the proof of Lemma 6.2.

Lemma 6.3 below presents two concentration inequalities satisfied by the estimator $\widehat{\beta}_{j,k}$.

Lemma 6.3 *Let $\widehat{\beta}_{j,k}$ be the estimator defined by (4.2). For λ large enough, there exists a constant $C > 0$ such that, for any integer $j \geq \tau$, we have*

$$\mathbb{P} \left(\left| \sum_{k=0}^{2^j-1} \beta_{j,k} (\widehat{\beta}_{j,k} - \beta_{j,k}) \right| \geq \lambda n^{-1/2} j^{1/2} 2^{\delta j} \left(\sum_{k=0}^{2^j-1} \beta_{j,k}^2 \right)^{1/2} \right) \leq 2 \exp(-C \lambda^2 j) \quad (6.1)$$

and

$$\mathbb{P} \left(\left(\sum_{k=0}^{2^j-1} (\widehat{\beta}_{j,k} - \beta_{j,k})^2 \right)^{1/2} \geq \lambda n^{-1/2} 2^{j/2} 2^{\delta j} \right) \leq 2 \exp(-C \lambda^2 2^j). \quad (6.2)$$

(Proof of Lemma 6.3.) The first part of the proof is devoted to the bound (6.1) and the second part, to the bound (6.2).

- *Proof of the bound (6.1).* Since the random variable $(e_l)_l$ are i.i.d. with

$e_l \sim \mathcal{N}(0, 1)$, we have

$$\sum_{k=0}^{2^j-1} \beta_{j,k} (\widehat{\beta}_{j,k} - \beta_{j,k}) = n^{-1/2} \sum_{l \in C_j} e_l g_l^{-1} \sum_{k=0}^{2^j-1} \beta_{j,k} F(\psi_{j,k})(l) \sim \mathcal{N}(0, \theta_j^2),$$

where

$$\theta_j^2 = n^{-1} \sum_{l \in C_j} |g_l|^{-2} \left(\sum_{k=0}^{2^j-1} \beta_{j,k} F(\psi_{j,k})(l) \right)^2.$$

The condition made on g (see (3.1)) yields $\sup_{l \in C_j} |g_l|^{-2} \leq C2^{2\delta j}$. It follows from this, the Plancherel theorem and the orthonormality of the wavelet basis that

$$\begin{aligned} \theta_j^2 &= n^{-1} \sum_{l \in C_j} |g_l|^{-2} \left(F \left(\sum_{k=0}^{2^j-1} \beta_{j,k} \psi_{j,k} \right) (l) \right)^2 \\ &\leq Cn^{-1} 2^{2\delta j} \sum_{l \in C_j} \left(F \left(\sum_{k=0}^{2^j-1} \beta_{j,k} \psi_{j,k} \right) (l) \right)^2 \\ &= Cn^{-1} 2^{2\delta j} \int_0^1 \left(\sum_{k=0}^{2^j-1} \beta_{j,k} \psi_{j,k}(x) \right)^2 dx = Cn^{-1} 2^{2\delta j} \sum_{k=0}^{2^j-1} \beta_{j,k}^2. \end{aligned}$$

Applying a standard Gaussian concentration inequality, we obtain

$$\begin{aligned} &\mathbb{P} \left(\left| \sum_{k=0}^{2^j-1} \beta_{j,k} (\widehat{\beta}_{j,k} - \beta_{j,k}) \right| \geq \lambda n^{-1/2} j^{1/2} 2^{\delta j} \left(\sum_{k=0}^{2^j-1} \beta_{j,k}^2 \right)^{1/2} \right) \\ &\leq 2 \exp \left(- \left(\lambda n^{-1/2} j^{1/2} 2^{\delta j} \left(\sum_{k=0}^{2^j-1} \beta_{j,k}^2 \right)^{1/2} \right)^2 / (2\theta_j^2) \right) \leq 2 \exp(-C\lambda^2 j). \end{aligned}$$

This ends the proof of the bound (6.1).

- *Proof of the bound (6.2).* To obtain the bound (6.2), we need to use the Cirelson inequality. It is presented in Lemma 6.4 below.

Lemma 6.4 (Cirelson et al. (1976)) *Let D be a subset of \mathbb{R} . Let $(\eta_t)_{t \in D}$ be a centered Gaussian process. If $\mathbb{E}(\sup_{t \in D} \eta_t) \leq N$ and $\sup_{t \in D} \mathbb{V}(\eta_t) \leq V$ then, for any $x > 0$, we have*

$$\mathbb{P} \left(\sup_{t \in D} \eta_t \geq x + N \right) \leq \exp(-x^2/(2V)).$$

We have $\widehat{\beta}_{j,k} - \beta_{j,k} = n^{-1/2} \sum_{l \in C_j} e_l g_l^{-1} F(\psi_{j,k})(l) \sim \mathcal{N}(0, n^{-1} \sigma_j^2)$, where $\sigma_j^2 = \sum_{l \in C_j} |g_l|^{-2} |F(\psi_{j,0})(l)|^2$. Consider the set Ω defined by $\Omega = \{a = (a_k); \sum_{k=0}^{2^j-1} |a_k|^2 \leq 1\}$ and the centered Gaussian process $Z(a)$ defined by

$$Z(a) = \sum_{k=0}^{2^j-1} a_k (\widehat{\beta}_{j,k} - \beta_{j,k}) = n^{-1/2} \sum_{l \in C_j} e_l g_l^{-1} \sum_{k=0}^{2^j-1} a_k F(\psi_{j,k})(l).$$

By an argument of duality, we have $\sup_{a \in \Omega} Z(a) = \left(\sum_{k=0}^{2^j-1} (\widehat{\beta}_{j,k} - \beta_{j,k})^2 \right)^{1/2}$. Now, let us determine the values of N and V which appeared in the Cirelson inequality (see Lemma 6.4).

– *Value of N .* The Cauchy-Schwarz inequality and Proposition 6.2 imply that

$$\begin{aligned} \mathbb{E} \left(\sup_{a \in \Omega} Z(a) \right) &= \mathbb{E} \left(\left(\sum_{k=0}^{2^j-1} (\widehat{\beta}_{j,k} - \beta_{j,k})^2 \right)^{1/2} \right) \leq \left(\sum_{k=0}^{2^j-1} \mathbb{E} \left((\widehat{\beta}_{j,k} - \beta_{j,k})^2 \right) \right)^{1/2} \\ &\leq C \left(n^{-1} \sum_{k=0}^{2^j-1} \sigma_j^2 \right)^{1/2} \leq C n^{-1/2} 2^{j/2} 2^{\delta j}. \end{aligned} \quad (6.3)$$

Hence $N = C n^{-1/2} 2^{j/2} 2^{\delta j}$.

– *Value of V .* Using the fact that, for any $l, l' \in \mathbb{Z}$, we have $\mathbb{E}(e_l \overline{e_{l'}}) = \int_0^1 e^{-2i\pi(l-l')t} dt = 1_{\{l=l'\}}$, the assumption made on g (see (3.1)), the Plancherel inequality and the orthonormality of the wavelet basis, we obtain

$$\begin{aligned} \sup_{a \in \Omega} \mathbb{V}(Z(a)) &= \sup_{a \in \Omega} \left(\mathbb{E} \left(\sum_{k=0}^{2^j-1} \sum_{k'=0}^{2^j-1} a_k (\widehat{\beta}_{j,k} - \beta_{j,k}) \overline{a_{k'} (\widehat{\beta}_{j,k'} - \beta_{j,k'})} \right) \right) \\ &= n^{-1} \sup_{a \in \Omega} \left(\sum_{k=0}^{2^j-1} \sum_{k'=0}^{2^j-1} a_k \overline{a_{k'}} \sum_{l \in C_j} \sum_{l' \in C_j} g_l^{-1} F(\psi_{j,k})(l) (\overline{g_{l'}})^{-1} \overline{F(\psi_{j,k'})(l')} \mathbb{E}(e_l \overline{e_{l'}}) \right) \\ &= n^{-1} \sup_{a \in \Omega} \left(\sum_{k=0}^{2^j-1} \sum_{k'=0}^{2^j-1} a_k \overline{a_{k'}} \sum_{l \in C_j} |g_l|^{-2} F(\psi_{j,k})(l) \overline{F(\psi_{j,k'})(l)} \right) \\ &\leq C n^{-1} 2^{2\delta j} \sup_{a \in \Omega} \left(\sum_{k=0}^{2^j-1} \sum_{k'=0}^{2^j-1} a_k \overline{a_{k'}} \sum_{l \in C_j} F(\psi_{j,k})(l) \overline{F(\psi_{j,k'})(l)} \right) \\ &= C n^{-1} 2^{2\delta j} \sup_{a \in \Omega} \left(\sum_{k=0}^{2^j-1} \sum_{k'=0}^{2^j-1} a_k \overline{a_{k'}} \int_0^1 \psi_{j,k}(x) \overline{\psi_{j,k'}(x)} dx \right) \\ &= C n^{-1} 2^{2\delta j} \sup_{a \in \Omega} \left(\sum_{k=0}^{2^j-1} |a_k|^2 \right) \leq C 2^{2\delta j} n^{-1}. \end{aligned}$$

Hence $V = C2^{2\delta j}n^{-1}$. By taking λ large enough and $x = 2^{-1}\lambda n^{-1/2}2^{j/2}2^{\delta j}$, the Cirelson inequality described in Lemma 6.4 yields

$$\begin{aligned} & \mathbb{P} \left(\left(\sum_{k=0}^{2^j-1} (\widehat{\beta}_{j,k} - \beta_{j,k})^2 \right)^{1/2} \geq \lambda n^{-1/2} 2^{j/2} 2^{\delta j} \right) \\ & \leq \mathbb{P} \left(\sup_{a \in \Omega} Z(a) \geq x + N \right) \leq \exp(-x^2/(2V)) \leq 2 \exp(-C\lambda^2 2^j). \end{aligned}$$

The bound (6.2) is proved.

This ends Lemma 6.1.

PROOF OF THEOREM 5.2. We distinguish the case $s > \delta + 4^{-1}$ and the case $s \in (0, \delta + 4^{-1}]$.

The case $s > \delta + 4^{-1}$. Let us consider the two following functions

$$f_0(x) = 1, \quad f_1(x) = 1 + n^{-1/2}.$$

Clearly, f_0 and f_1 belong to $B_{2,\infty}^s(M)$. Moreover, we have

$$\left| \|f_0\|_2^2 - \|f_1\|_2^2 \right| = 2n^{-1/2} + n^{-1} \geq n^{-1/2} = \kappa_n.$$

Now, in order to apply Theorem 2.12 (iii) of Tsybakov (2004), we aim to bound (by a constant C) the chi-square divergence $\chi^2(\mathbb{P}_{f_1}, \mathbb{P}_{f_0})$ defined by

$$\chi^2(\mathbb{P}_{f_1}, \mathbb{P}_{f_0}) = \int \left(\frac{d\mathbb{P}_{f_1}}{d\mathbb{P}_{f_0}} \right)^2 d\mathbb{P}_{f_0} - 1, \quad (6.4)$$

where \mathbb{P}_h denotes the probability distribution of $\{Y(t); t \in [0, 1]\}$ indexed by the function h .

Let \star be the standard convolution product on the unit interval. The Girsanov theorem yields

$$\begin{aligned} \frac{d\mathbb{P}_{f_1}}{d\mathbb{P}_{f_0}} &= \exp \left(n \int_0^1 ((f_1 \star g)(t) - (f_0 \star g)(t)) dY(t) - \right. \\ & \quad \left. 2^{-1}n \int_0^1 ((f_1 \star g)^2(t) - (f_0 \star g)^2(t)) dt \right) \\ &= \exp \left(n^{1/2} \int_0^1 g(u) du \int_0^1 dY(t) - 2^{-1}n \left(\int_0^1 g(u) du \right)^2 (2n^{-1/2} + n^{-1}) \right). \end{aligned}$$

So, under \mathbb{P}_{f_0} , we have

$$\frac{d\mathbb{P}_{f_1}}{d\mathbb{P}_{f_0}} = \exp \left(\int_0^1 g(u) du \int_0^1 dW(t) - 2^{-1} \left(\int_0^1 g(u) du \right)^2 (2n^{1/2} + 1) \right).$$

Using the equality $\int \exp \left(2 \int_0^1 g(u) du \int_0^1 dW(t) \right) d\mathbb{P}_{f_0} = \exp \left(2 \left(\int_0^1 g(u) du \right)^2 \right)$, the Cauchy-Schwarz inequality and the fact that $\int_0^1 g^2(t) dt < \infty$, we have

$$\begin{aligned} & \int \left(\frac{d\mathbb{P}_{f_1}}{d\mathbb{P}_{f_0}} \right)^2 d\mathbb{P}_{f_0} \\ &= \exp \left(- \left(\int_0^1 g(u) du \right)^2 (2n^{1/2} + 1) \right) \int \exp \left(2 \int_0^1 g(u) du \int_0^1 dW(t) \right) d\mathbb{P}_{f_0} \\ &= \exp \left(- \left(\int_0^1 g(u) du \right)^2 (2n^{1/2} + 1) \right) \exp \left(2 \left(\int_0^1 g(u) du \right)^2 \right) \\ &\leq \exp \left(\left(\int_0^1 g(u) du \right)^2 \right) \leq \exp \left(\int_0^1 g^2(u) du \right) \leq C. \end{aligned}$$

Therefore, there exists a constant $C > 0$ such that

$$\chi^2(\mathbb{P}_{f_1}, \mathbb{P}_{f_0}) = \int \left(\frac{d\mathbb{P}_{f_1}}{d\mathbb{P}_{f_0}} \right)^2 d\mathbb{P}_{f_0} - 1 \leq C < \infty.$$

It follows from Theorem 2.12 (iii) of Tsybakov (2004) that

$$\inf_{\tilde{Q}_n} \sup_{f \in B_{2,\infty}^s(M)} \mathbb{E} \left(\left(\tilde{Q}_n - \|f\|_2^2 \right)^2 \right) \geq c\kappa_n^2 = cn^{-1} = c\varphi_n^*. \quad (6.5)$$

The case $s \in (0, \delta + 4^{-1}]$. Let us consider the two following functions

$$f_0(x) = 0, \quad f_1(x) = \gamma_{j_*} \sum_{k_*=0}^{2^{j_*}} w_{k_*} \psi_{j_*,k_*}(x), \quad (6.6)$$

where j_* is an integer to be chosen below, γ_{j_*} is a quantity to be chosen below, and $w_0, \dots, w_{2^{j_*}-1}$ are i.i.d. Rademacher random variables (i.e., for any $u \in \{0, \dots, 2^{j_*} - 1\}$, $\mathbb{P}(w_u = 1) = \mathbb{P}(w_u = -1) = 2^{-1}$).

Clearly, f_0 belong to $B_{2,\infty}^s(M)$. Thanks to the orthogonality of the wavelet basis, for any integer $j \geq \tau$ and any $k \in \{0, \dots, 2^j - 1\}$, we have $\beta_{j,k} = \int_0^1 f_1(x) \psi_{j,k}(x) dx = \gamma_{j_*} w_k$ for $j = j_*$, and 0 otherwise. Thus, if $\gamma_{j_*}^2 = M_* 2^{-j_*(2s+1)}$, then

$$2^{2j_*s} \sum_{k=0}^{2^{j_*}-1} \beta_{j_*,k}^2 = 2^{2j_*s} \gamma_{j_*}^2 \sum_{k=0}^{2^{j_*}-1} w_k^2 = \gamma_{j_*}^2 2^{j_*(2s+1)} = M_*.$$

Therefore, for such a choice of γ_{j_*} , we have $f_1 \in B_{2,\infty}^s(M)$.

Let j_* be an integer such that

$$2^{-1}n^{1/(2s+2\delta+1/2)} \leq 2^{j_*} \leq n^{1/(2s+2\delta+1/2)}. \quad (6.7)$$

With the previous value of γ_{j_*} , the orthonormality of the wavelet basis gives

$$\begin{aligned} \left| \|f_0\|_2^2 - \|f_1\|_2^2 \right| &= \|f_1\|_2^2 = \gamma_{j_*}^2 \sum_{k=0}^{2^{j_*}-1} w_k^2 = \gamma_{j_*}^2 2^{j_*} \geq C2^{-2j_*s} \\ &\geq Cn^{-2s/(2s+2\delta+1/2)} = \kappa_n. \end{aligned}$$

In what follows, we consider the function f_1 described by (6.6) with the integer j_* defined by (6.7) and the quantity $\gamma_{j_*} = M_*^{1/2}2^{-j_*(s+1/2)}$. In order to apply Theorem 2.12 (iii) of Tsybakov (2004), we aim to bound (by a constant C) the chi-square divergence $\chi^2(\mathbb{P}_{f_1}, \mathbb{P}_{f_0})$ (see (6.4)).

Let \star be the standard convolution product. Due to the definitions of f_0, f_1 , the variables $w_0, \dots, w_{2^{j_*}-1}$, and the Girsanov theorem, we have

$$\begin{aligned} \frac{d\mathbb{P}_{f_1}}{d\mathbb{P}_{f_0}} &= \prod_{k=0}^{2^{j_*}-1} [2^{-1} \exp \left(n\gamma_{j_*} \int_0^1 (\psi_{j_*,k} \star g)(t) dY(t) - 2^{-1}n\gamma_{j_*}^2 \int_0^1 (\psi_{j_*,k} \star g)^2(t) dt \right) \\ &\quad + 2^{-1} \exp \left(-n\gamma_{j_*} \int_0^1 (\psi_{j_*,k} \star g)(t) dY(t) - 2^{-1}n\gamma_{j_*}^2 \int_0^1 (\psi_{j_*,k} \star g)^2(t) dt \right)]. \end{aligned}$$

So, under \mathbb{P}_{f_0} , we have

$$\begin{aligned} &\frac{d\mathbb{P}_{f_1}}{d\mathbb{P}_{f_0}} \\ &= \prod_{k=0}^{2^{j_*}-1} [2^{-1} \exp \left(n^{1/2}\gamma_{j_*} \int_0^1 (\psi_{j_*,k} \star g)(t) dW(t) - 2^{-1}n\gamma_{j_*}^2 \int_0^1 (\psi_{j_*,k} \star g)^2(t) dt \right) \\ &\quad + 2^{-1} \exp \left(-n^{1/2}\gamma_{j_*} \int_0^1 (\psi_{j_*,k} \star g)(t) dW(t) - 2^{-1}n\gamma_{j_*}^2 \int_0^1 (\psi_{j_*,k} \star g)^2(t) dt \right)] \\ &= \prod_{k=0}^{2^{j_*}-1} 2^{-1} \exp \left(-2^{-1}n\gamma_{j_*}^2 \int_0^1 (\psi_{j_*,k} \star g)^2(t) dt \right) \times \\ &\quad \left[\exp \left(n^{1/2}\gamma_{j_*} \int_0^1 (\psi_{j_*,k} \star g)(t) dW(t) \right) + \exp \left(-n^{1/2}\gamma_{j_*} \int_0^1 (\psi_{j_*,k} \star g)(t) dW(t) \right) \right]. \end{aligned}$$

Hence,

$$\int \left(\frac{d\mathbb{P}_{f_1}}{d\mathbb{P}_{f_0}} \right)^2 d\mathbb{P}_{f_0} = \prod_{k=0}^{2^{j_*}-1} 2^{-2} \exp \left(-n\gamma_{j_*}^2 \int_0^1 (\psi_{j_*,k} \star g)^2(t) dt \right) \times \\ \int \left[\exp \left(n^{1/2} \gamma_{j_*} \int_0^1 (\psi_{j_*,k} \star g)(t) dW(t) \right) + \exp \left(-n^{1/2} \gamma_{j_*} \int_0^1 (\psi_{j_*,k} \star g)(t) dW(t) \right) \right]^2 d\mathbb{P}_{f_0}.$$

Since $\int \exp \left(2n^{1/2} \gamma_{j_*} \int_0^1 (\psi_{j_*,k} \star g)(t) dW(t) \right) d\mathbb{P}_{f_0} =$

$\int \exp \left(-2n^{1/2} \gamma_{j_*} \int_0^1 (\psi_{j_*,k} \star g)(t) dW(t) \right) d\mathbb{P}_{f_0} = \exp \left(2n\gamma_{j_*}^2 \int_0^1 (\psi_{j_*,k} \star g)^2(t) dt \right),$
we have

$$\int \left(\frac{d\mathbb{P}_{f_1}}{d\mathbb{P}_{f_0}} \right)^2 d\mathbb{P}_{f_0} \\ = \prod_{k=0}^{2^{j_*}-1} 2^{-1} \left(\exp \left(n\gamma_{j_*}^2 \int_0^1 (\psi_{j_*,k} \star g)^2(t) dt \right) + \exp \left(-n\gamma_{j_*}^2 \int_0^1 (\psi_{j_*,k} \star g)^2(t) dt \right) \right).$$

Using the inequality $2^{-1}(e^x + e^{-x}) \leq e^{x^2/2}$, $x \in \mathbb{R}$, we obtain

$$\int \left(\frac{d\mathbb{P}_{f_1}}{d\mathbb{P}_{f_0}} \right)^2 d\mathbb{P}_{f_0} \leq \prod_{k=0}^{2^{j_*}-1} \exp \left(2^{-1} n^2 \gamma_{j_*}^4 \left(\int_0^1 (\psi_{j_*,k} \star g)^2(t) dt \right)^2 \right).$$

Using the condition made on g (see (3.1)) and the Plancherel theorem, we obtain

$$\int_0^1 (\psi_{j_*,k} \star g)^2(t) dt = \sum_{l \in \mathbb{Z}} (F(\psi_{j_*,k} \star g))^2(l) = \sum_{l \in \mathcal{C}_{j_*}} |F(\psi_{j_*,k})(l)|^2 |g_l|^2 \\ \leq C 2^{-2\delta j_*} \sum_{l \in \mathcal{C}_{j_*}} |F(\psi_{j_*,k})(l)|^2 = C 2^{-2\delta j_*} \int_0^1 \psi_{j_*,k}^2(x) dx \\ = C 2^{-2\delta j_*}.$$

We deduce that

$$\int \left(\frac{d\mathbb{P}_{f_1}}{d\mathbb{P}_{f_0}} \right)^2 d\mathbb{P}_{f_0} \leq \prod_{k=0}^{2^{j_*}-1} \exp \left(C n^2 \gamma_{j_*}^4 2^{-4\delta j_*} \right) = \exp \left(C n^2 \gamma_{j_*}^4 2^{(1-4\delta)j_*} \right).$$

Thanks to the definitions of j_* and γ_{j_*} , we have

$$n^2 \gamma_{j_*}^4 2^{(1-4\delta)j_*} \leq C n^2 2^{-j_*(4s+2)} 2^{(1-4\delta)j_*} \leq C n^2 2^{-j_*(4s+1+4\delta)} \leq C.$$

It follows the existence of a constant $C > 0$ such that

$$\chi^2(\mathbb{P}_{f_1}, \mathbb{P}_{f_0}) = \int \left(\frac{d\mathbb{P}_{f_1}}{d\mathbb{P}_{f_0}} \right)^2 d\mathbb{P}_{f_0} - 1 \leq C < \infty.$$

Theorem 2.12 (iii) of Tsybakov (2004) yields

$$\inf_{\tilde{Q}_n} \sup_{f \in B_{2,\infty}^s(M)} \mathbb{E} \left(\left(\tilde{Q}_n - \|f\|_2^2 \right)^2 \right) \geq c\kappa_n^2 = cn^{-4s/(2s+2\delta+1/2)} = c\varphi_n^*. \quad (6.8)$$

Putting (6.5) and (6.8) together, we prove the existence of a constant $c > 0$ such that, for any $s > 0$, we have

$$\inf_{\tilde{Q}_n} \sup_{f \in B_{2,\infty}^s(M)} \mathbb{E} \left(\left(\tilde{Q}_n - \|f\|_2^2 \right)^2 \right) \geq c\varphi_n^*,$$

where $\varphi_n^* = \max \left(n^{-1}, n^{-4s/(2s+2\delta+1/2)} \right)$.

Theorem 5.2 is proved.

References

- BERTERO, M. and BOCCACCI, P. (1998). *Introduction to Inverse Problems in Imaging*. Institute of physics, bristol and philadelphia ed.
- BICKEL, P. J. and RITOV, Y. (1988). Estimating integrated squared density derivatives: Sharp best order of convergence estimates. *Sankhya Ser. A*, 50:381–393.
- BUTUCEA, C. (2007). Goodness-of-fit testing and quadratic functional estimation from indirect observations. *Annals of Statistics (to appear)*, 66.
- CAI, T. and LOW, M. (2005). Non-quadratic estimators of a quadratic functional. *Ann. Statist.*, 33:2930–2956.
- CAVALIER, L. and TSYBAKOV, A. (2002). Sharp adaptation for inverse problems with random noise. *Probability Theory and Related Fields*, 123:323–354.
- CHESNEAU, C. (2007). Wavelet estimation via block thresholding : A minimax study under the lp risk. *Statistica Sinica (to appear)*.
- CIRELSON, B., IBRAGIMOV, I., and SUDAKOV, V. (1976). *Norm of Gaussian sample functions*. Berlin, springer verlag ed.
- COHEN, A., DAUBECHIES, I., JAWERTH, B., and VIAL, P. (1993). Wavelets on the interval and fast wavelet transforms. *Applied and Computational Harmonic Analysis*, 24(1):54–81.
- DONOHO, D. L. and NUSSBAUM, M. (1990). Minimax quadratic estimation of a quadratic functional. *J. Complexity*, 6:290–323.
- EFROMOVICH, S. Y. and LOW, M. (1996). On optimal adaptive estimation of a quadratic functional. *Ann. Statist.*, 24:1106–1125.
- GAYRAUD, G. and TRIBOULEY, K. (1999). Wavelet methods to estimate an integrated quadratic functional: Adaptivity and asymptotic law. *Statist. Probab. Lett.*, 44:109–122.

- JOHNSTONE, I., KERKYACHARIAN, G., PICARD, D., and RAIMONDO, M. (2004). Wavelet deconvolution in a periodic setting. *Journal of the Royal Statistical Society. Series B. Methodological*, 66:547–573.
- JOHNSTONE, I. M. (2001a). Chi-square oracle inequalities. *In State of the Art in Probability and Statistics. Festschrift for Willem R. van Zwet (M. de Gunst, C. Klaassen and A. van der Waart, eds.) IMS, Beachwood, OH.*, 24:399–418.
- JOHNSTONE, I. M. (2001b). Thresholding for weighted chi-squared. *Statistica Sinica*, 11:691–704.
- KERKYACHARIAN, G. and PICARD, D. (1996). Estimating nonquadratic functionals of a density using haar wavelets. *Annals of Statistics*, 24(2):485–507.
- LAURENT, B. (2005). Adaptive estimation of a quadratic functional of a density by model selection. *ESAIM P, S*, 9:1–18.
- MEYER, Y. (1992). *Wavelets and Operators*. Cambridge University Press, Cambridge.
- RIVOIRARD, V. and TRIBOULEY, K. (2007). The maxiset point of view for estimating integrated quadratic functionals. *Statistica Sinica (to appear)*.
- TSYBAKOV, A. (2004). *Introduction à l'estimation nonparamétrique*. Springer Verlag.
- WALTER, G. G. (1994). *Wavelets and other orthogonal systems in applications*. CRC Press, Boca Raton.
- ZAYED, A. and WALTER, G. (1996). Characterization of analytic functions in term of their wavelet coefficients. *Complex Variables*, 29:265–276.