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ON ADAPTIVE WAVELET ESTIMATION OF A QUADRATIC FUNCTIONAL FROM A DECONVOLUTION PROBLEM

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Abstract

We consider the following situation: an unknown function f is observed in Gaussian white noise after convolution with a known function g . We wish to estimate the quadratic functional $\int f^2(t)dt$ from the observations. To reach this goal, we propose an adaptive estimator based on wavelet thresholding. We prove that it achieves near optimal rates of convergence under the mean squared error over a range of smoothness classes.

Key words: Deconvolution, Quadratic functionals, Adaptive curve estimation, Wavelets, Global thresholding.

1991 MSC: Primary: 62G07, Secondary: 62N01.

1 MOTIVATION

We consider the stochastic process $\{Y(t); t \in [0, 1]\}$ defined by

$$dY(t) = \left(\int_0^1 f(t-u)g(u)du \right) dt + n^{-1/2}dW(t), \quad (1.1)$$

where $\{W(t); t \in [0, 1]\}$ is a standard Brownian motion, f an unknown one-periodic function such that $\int_0^1 f^2(t)dt < \infty$, and g a known one-periodic function such that $\int_0^1 g^2(t)dt < \infty$. We suppose that the Fourier coefficients of g decay in a polynomial fashion. The goal is to estimate f , or a quantity depending on f , from $\{Y(t); t \in [0, 1]\}$. The equation (1.1) modelizes the important

problem of recovery of noisy signals in linear motion blur. Further details can be found in Bertero and Boccacci (1998).

In the literature, many authors have examined the estimation of f . See, for instance, Cavalier and Tsybakov (2002), Johnstone et al. (2004) and Chesneau (2007). In this paper, we focus our attention on a different problem: the estimation of the quadratic functional $Q(f)$ defined by

$$Q(f) = \|f\|_2^2 = \int_0^1 f^2(t)dt.$$

The problem of estimating quadratic functionals is closely connected to the construction of confidence balls in nonparametric function estimation. It has already been investigated for a wide variety of models (density, regression, Gaussian model in white noise, density derivatives, ...). See, for instance, Bickel and Ritov (1988), Donoho and Nussbaum (1990), Kerkycharian and Picard (1996), Efromovich and Low (1996), Gayraud and Tribouley (1999), Johnstone (2001a,b), Laurent (2005), Cai and Low (2005) and Butucea (2007). However, to our knowledge, the estimation of $Q(f) = \|f\|_2^2$ from the convolution model in Gaussian white noise (1.1) has never been studied.

To estimate $Q(f) = \|f\|_2^2$, we propose an adaptive wavelet estimator based on the global thresholding. The idea of this thresholding is the following: we decompose the quadratic function $Q(f)$ by using an appropriate wavelet basis, we estimate the associated coefficients via natural estimators, then, at each level of the wavelets, we keep all of these estimators if, and only if, the corresponding l_2 norm is greater than a fixed threshold. This technic has been initially developed by Gayraud and Tribouley (1999) for the quadratic functional estimation in the standard Gaussian white noise model.

Naturally, the construction of our estimator depends on the complexity of the convolution model (1.1) and, in particular, the function g . To evaluate his theoretical performances, we adopt the (asymptotic) minimax approach under the mean squared error over Besov balls. More precisely, if \widehat{Q}_n denotes our estimator and $B_{2,\infty}^s(M)$ denotes the Besov balls, we aim to determine the smallest rates of convergence φ_n such that

$$\lim_{n \rightarrow \infty} \varphi_n^{-1} \sup_{f \in B_{2,\infty}^s(M)} \mathbb{E} \left(\left(\widehat{Q}_n - \|f\|_2^2 \right)^2 \right) < \infty.$$

In this statistical framework, we prove that \widehat{Q}_n achieves near optimal of convergence. The proof is divided into two parts. The first part is devoted to the upper bounds. A key tool is an inequality for weighted chi-squared established by Johnstone (2001b). The second part concerns the lower bounds. They are obtained via a specific theorem proved by Tsybakov (2004).

The paper is organized as follows. In Section 2, we present wavelets and Besov

balls. Section 3 clarifies the assumption made on g and introduces some intermediate estimators. The main estimator of the study is defined in Section 4. Section 5 is devoted to the results. The proofs are postponed in Section 6.

2 WAVELETS AND BESOV BALLS

We consider an orthonormal wavelet basis generated by dilation and translation of a "father" Meyer-type wavelet ϕ and a "mother" Meyer-type wavelet ψ . The particularities of such wavelets are that the Fourier transforms of ϕ and ψ have bounded support. More precisely, we have $\text{supp}(F(\phi)) \subset [-4\pi 3^{-1}, 4\pi 3^{-1}]$ and $\text{supp}(F(\psi)) \subset [-8\pi 3^{-1}, -2\pi 3^{-1}] \cup [2\pi 3^{-1}, 8\pi 3^{-1}]$, where, for any function $h \in \mathbb{L}^1([0, 1])$, $F(h)$ denotes the Fourier transform of h defined by $F(h)(l) = \int_0^1 h(x)e^{-2i\pi lx} dx$. For further details about Meyer-type wavelets, see Walter (1994), and Zayed and Walter (1996).

For the purposes of this paper, we use the periodized wavelet bases on the unit interval. For any $x \in [0, 1]$, any integer j and any $k \in \{0, \dots, 2^j - 1\}$, let

$$\phi_{j,k}(x) = 2^{j/2}\phi(2^j x - k), \quad \psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k)$$

be the elements of the wavelet basis and

$$\phi_{j,k}^{per}(x) = \sum_{l \in \mathbb{Z}} \phi_{j,k}(x - l), \quad \psi_{j,k}^{per}(x) = \sum_{l \in \mathbb{Z}} \psi_{j,k}(x - l),$$

their periodized versions. There exists an integer τ such that the collection ζ defined by $\zeta = \{\phi_{\tau,k}^{per}, k = 0, \dots, 2^\tau - 1; \psi_{j,k}^{per}, j = \tau, \dots, \infty, k = 0, \dots, 2^j - 1\}$ constitutes an orthonormal basis of $\mathbb{L}^2([0, 1])$. In what follows, the superscript "per" will be suppressed from the notations for convenience. A one-periodic function $f \in \mathbb{L}^2([0, 1])$ can be expanded into a wavelet series as

$$f(x) = \sum_{k=0}^{2^\tau-1} \alpha_{\tau,k} \phi_{\tau,k}(x) + \sum_{j=\tau}^{\infty} \sum_{k=0}^{2^j-1} \beta_{j,k} \psi_{j,k}(x),$$

where $\alpha_{\tau,k} = \int_0^1 f(t)\phi_{\tau,k}(t)dt$ and $\beta_{j,k} = \int_0^1 f(t)\psi_{j,k}(t)dt$. For further details about wavelet bases on the unit interval, we refer to Cohen et al. (1993).

Now, let us present the Besov balls $B_{2,\infty}^s(M)$. Let $M \in (0, \infty)$ and $s \in (0, \infty)$. We say that a one-periodic function f belongs to the Besov balls $B_{2,\infty}^s(M)$ if, and only if, there exists a constant $M^* > 0$ such that the associated wavelet coefficients satisfy

$$2^{2\tau} \sum_{k=0}^{2^\tau-1} \alpha_{\tau,k}^2 + \sup_{j \geq \tau} 2^{2js} \sum_{k=0}^{2^j-1} \beta_{j,k}^2 \leq M^*.$$

These sets contain all the Besov balls $B_{p,\infty}^s(M)$ with $p \geq 2$. See, for instance, Meyer (1992)

3 PRELEMINARIES

In this section, we clarify the assumption made on g and we introduce some intermediate estimators.

Assumption on g . As mentioned in Section 1, we suppose that the Fourier coefficients of g decay in a polynomial fashion. In other words, there exist three constants, $c > 0$, $C > 0$ and $\delta \geq 0$, such that, for $|l|$ large enough, g satisfies

$$c|l|^{-\delta} \leq |F(g)(l)| \leq C|l|^{-\delta}. \quad (3.1)$$

The factor δ will play a determinant role in our study.

The assumption (3.1) is satisfied for a wide variety of functions. An immediate example is the one-periodic function g defined by $g(x) = \sum_{l \in \mathbb{Z}} e^{-|x+l|}$. It satisfies the inequality (3.1) with $\delta = 2$.

Preliminaries to the estimation of $\|f\|_2^2$ from $\{Y(t); t \in [0, 1]\}$. Thanks to the orthonormality of the wavelet basis, the unknown quadratic functional $Q(f) = \|f\|_2^2$ can be decomposed as

$$\|f\|_2^2 = \sum_{k=0}^{2^\tau-1} \alpha_{\tau,k}^2 + \sum_{j=\tau}^{\infty} \sum_{k=0}^{2^j-1} \beta_{j,k}^2.$$

Thus, the first step to estimate $\|f\|_2^2$ consists in estimating the unknown coefficients $(\alpha_{\tau,k}^2)_k$ and $(\beta_{j,k}^2)_{j,k}$. In what follows, we investigate the estimation of $(\beta_{j,k}^2)_{j,k}$. Since, for any $l \in \mathbb{Z}$, $F\left(\int_0^1 f(t-u)g(u)du\right)(l) = F(f)(l)F(g)(l)$, if we set

$$y_l = \int_0^1 e^{-2\pi i l t} dY(t), \quad f_l = F(f)(l), \quad g_l = F(g)(l) \quad \text{and} \quad e_l = \int_0^1 e^{-2\pi i l t} dB(t),$$

then we have

$$y_l = f_l g_l + e_l.$$

The Parseval theorem yields

$$\begin{aligned} \sum_{l \in \mathbb{Z}} y_l g_l^{-1} F(\psi_{j,k})(l) &= \sum_{l \in \mathbb{Z}} f_l F(\psi_{j,k})(l) + n^{-1/2} \sum_{l \in \mathbb{Z}} g_l^{-1} F(\psi_{j,k})(l) e_l \\ &= \beta_{j,k} + n^{-1/2} \sum_{l \in \mathbb{Z}} g_l^{-1} F(\psi_{j,k})(l) e_l. \end{aligned}$$

Since, $n^{-1/2} \sum_{l \in \mathbb{Z}} g_l^{-1} F(\psi_{j,k})(l) e_l \sim \mathcal{N} \left(0, n^{-1} \sum_{l \in \mathbb{Z}} g_l^{-2} F^2(\psi_{j,k})(l) \right)$, we have

$$\widehat{\beta}_{j,k} = \sum_{l \in \mathbb{Z}} y_l g_l^{-1} F(\psi_{j,k})(l) \sim \mathcal{N} \left(\beta_{j,k}, n^{-1} \sum_{l \in \mathbb{Z}} g_l^{-2} F^2(\psi_{j,k})(l) \right).$$

Therefore, the random variable $\widehat{\theta}_{j,k}$ defined by

$$\widehat{\theta}_{j,k} = \widehat{\beta}_{j,k}^2 - n^{-1} \sum_{l \in \mathbb{Z}} g_l^{-2} F^2(\psi_{j,k})(l)$$

is an unbiased (and a natural) estimator of $\beta_{j,k}^2$.

We are now in the position to describe the main estimator of the study.

4 ESTIMATOR

Set $(a)_+ = \max(a, 0)$. Let j_1 be an integer such that

$$2^{-1} n^{1/(2\delta+1/2)} < 2^{j_1} \leq n^{1/(2\delta+1/2)}.$$

We define the thresholding estimator \widehat{Q}_n by

$$\widehat{Q}_n = \sum_{k=0}^{2^\tau-1} \left(\widehat{\alpha}_{\tau,k}^2 - n^{-1} \eta_{\tau,k}^2 \right) + \sum_{j=\tau}^{j_1} \left(\sum_{k=0}^{2^j-1} \left(\widehat{\beta}_{j,k}^2 - n^{-1} \sigma_{j,k}^2 \right) - n^{-1} \sigma_{j,\infty}^2 \lambda_n \right)_+, \quad (4.1)$$

where

$$\widehat{\alpha}_{\tau,k} = \sum_{l \in \mathbb{Z}} y_l g_l^{-1} F(\phi_{\tau,k})(l), \quad \widehat{\beta}_{j,k} = \sum_{l \in \mathbb{Z}} y_l g_l^{-1} F(\psi_{j,k})(l),$$

$$\eta_{\tau,k}^2 = \sum_{l \in \mathbb{Z}} g_l^{-2} F^2(\phi_{\tau,k})(l), \quad \sigma_{j,k}^2 = \sum_{l \in \mathbb{Z}} g_l^{-2} F^2(\psi_{j,k})(l),$$

$$\sigma_{j,\infty}^2 = \sup_{k=0, \dots, 2^j-1} \sigma_{j,k}^2 \text{ and } \lambda_n = \lambda_{n,j} = 2(\log 2)^{1/2} (1 + 4\delta)^{1/2} (j2^j)^{1/2}.$$

As mentioned in Section 1, the construction of \widehat{Q}_n is based on the global thresholding technic initially developed by Gayraud and Tribouley (1999) for the quadratic functional estimation in the standard Gaussian white noise model.

Notice that the definition of \widehat{Q}_n does not require any a priori knowledge on f . It is adaptive.

5 MAIN RESULTS

Theorem 5.1 below determines the rates of convergence achieved by the estimator (4.1) under the mean squared error over the Besov balls $B_{2,\infty}^s(M)$.

Theorem 5.1 *Let us consider the convolution model defined by (1.1). Assume that g satisfies the inequality (3.1). Let \hat{Q}_n be the estimator defined by (4.1). Then, there exists a constant $C > 0$ such that, for n large enough, we have*

$$\sup_{f \in B_{2,\infty}^s(M)} \mathbb{E} \left(\left(\hat{Q}_n - \|f\|_2^2 \right)^2 \right) \leq C\varphi_n,$$

where

$$\varphi_n = \begin{cases} n^{-1}, & \text{when } s > \delta + 4^{-1}, \\ \left(n(\log n)^{-1/2} \right)^{-4s/(2s+2\delta+1/2)}, & \text{when } s \in (0, \delta + 4^{-1}]. \end{cases}$$

One part of the proof of Theorem 5.1 uses an inequality for weighted chi-squared established by Johnstone (2001b).

Notice that, the rates of convergence φ_n presented in Theorem 5.1 are equal to $\max \left(n^{-1}, \left(n(\log n)^{-1/2} \right)^{-4s/(2s+2\delta+1/2)} \right)$.

These rates of convergence are similar to those obtained by Butucea (2007) (see Theorem 4) for a different model (the density convolution), with a different estimator (the deconvolution kernel estimator), and under a different risk ($R(h, f) = \mathbb{E}(|h - \|f\|_2^2|)$).

The result of Theorem 5.1 raises the following question: are the rates of convergence φ_n the optimal? Theorem 5.2 below provides the answer by determining the minimax lower bounds.

Theorem 5.2 *Let us consider the convolution model defined by (1.1). Assume that g satisfies the inequality (3.1). Then, there exists a constant $c > 0$ such that, for n large enough, we have*

$$\inf_{\tilde{Q}_n} \sup_{f \in B_{2,\infty}^s(M)} \mathbb{E} \left(\left(\tilde{Q}_n - \|f\|_2^2 \right)^2 \right) \geq c\varphi_n^*,$$

where $\inf_{\tilde{Q}_n}$ denotes the infimum over all the possible estimators of $\|f\|_2^2$, and

$$\varphi_n^* = \begin{cases} n^{-1}, & \text{when } s > \delta + 4^{-1}, \\ n^{-4s/(2s+2\delta+1/2)}, & \text{when } s \in (0, \delta + 4^{-1}]. \end{cases}$$

The proof of Theorem 5.2 is based on a general theorem established by Tsybakov (2004).

The results of Theorems 5.1 and 5.2 show that the estimator \widehat{Q}_n is near optimal under the mean squared error over the Besov balls $B_{2,\infty}^s(M)$. "Near" is due to the case $s \in (0, \delta + 4^{-1}]$ where there is an extra logarithmic term.

6 PROOFS

In this section, c and C denote positive constants which can take different values for each mathematical term. They are independent of f and n .

PROOF OF THEOREM 5.1. The orthonormality of the wavelet basis yields

$$\|f\|_2^2 = \sum_{k=0}^{2^\tau-1} \alpha_{\tau,k}^2 + \sum_{j=\tau}^{j_1} \sum_{k=0}^{2^j-1} \beta_{j,k}^2 + \sum_{j=j_1+1}^{\infty} \sum_{k=0}^{2^j-1} \beta_{j,k}^2.$$

The definition of \widehat{Q}_n (see (4.1)) and an elementary of convexity give

$$\mathbb{E} \left(\left(\widehat{Q}_n - \|f\|_2^2 \right)^2 \right) \leq 3(R + S + T), \quad (6.1)$$

where

$$R = \mathbb{E} \left(\left(\sum_{k=0}^{2^\tau-1} \left(\widehat{\alpha}_{\tau,k}^2 - n^{-1} \eta_{\tau,k}^2 - \alpha_{\tau,k}^2 \right) \right)^2 \right),$$

$$S = \mathbb{E} \left(\left(\sum_{j=\tau}^{j_1} \left(\sum_{k=0}^{2^j-1} \left(\widehat{\beta}_{j,k}^2 - n^{-1} \sigma_{j,k}^2 \right) - n^{-1} \sigma_{j,\infty}^2 \lambda_n \right) - \sum_{k=0}^{2^j-1} \beta_{j,k}^2 \right)^2 \right)$$

and

$$T = \left(\sum_{j=j_1+1}^{\infty} \sum_{k=0}^{2^j-1} \beta_{j,k}^2 \right)^2.$$

Let us investigate the upper bounds for R , S and T in turn.

The upper bound for R . Using the Cauchy-Schwarz inequality, we have

$$R \leq 2^\tau \sum_{k=0}^{2^\tau-1} \mathbb{E} \left(\left(\widehat{\alpha}_{\tau,k}^2 - n^{-1} \eta_{\tau,k}^2 - \alpha_{\tau,k}^2 \right)^2 \right). \quad (6.2)$$

For the sake of simplicity, set $\sum_{l \in \mathbb{Z}} g_l^{-1} F(\phi_{\tau,k})(l) e_l = \eta_{\tau,k} z$ where $z \sim \mathcal{N}(0, 1)$. We have

$$\begin{aligned}\widehat{\alpha}_{\tau,k}^2 - n^{-1}\eta_{\tau,k}^2 - \alpha_{\tau,k}^2 &= \left(\alpha_{\tau,k} + n^{-1/2}\eta_{\tau,k}z\right)^2 - n^{-1}\eta_{\tau,k}^2 - \alpha_{\tau,k}^2 \\ &= 2n^{-1/2}\eta_{\tau,k}z\alpha_{\tau,k} + n^{-1}\eta_{\tau,k}^2(z^2 - 1).\end{aligned}$$

Using the elementary inequality $(x + y)^2 \leq 2(|x|^2 + |y|^2)$, $x, y \in \mathbb{R}$, and the fact that $\mathbb{E}(z^2) = 1$ and $\mathbb{E}((z^2 - 1)^2) = \mathbb{V}(z^2) = 2$, we obtain

$$\begin{aligned}\mathbb{E}\left(\left(\widehat{\alpha}_{\tau,k}^2 - n^{-1}\eta_{\tau,k}^2 - \alpha_{\tau,k}^2\right)^2\right) &\leq C\mathbb{E}\left(n^{-1}\eta_{\tau,k}^2z^2\alpha_{\tau,k}^2 + n^{-2}\eta_{\tau,k}^4(z^2 - 1)^2\right) \\ &\leq C\left(n^{-1}\eta_{\tau,k}^2\alpha_{\tau,k}^2 + n^{-2}\eta_{\tau,k}^4\right).\end{aligned}$$

Let \mathcal{D}_τ be the support of $\phi_{\tau,k}$. The assumption made on g (see (3.1)) and the fact that $\sup_{l \in \mathbb{Z}} F^2(\phi_{\tau,k})(l) = \sup_{l \in \mathcal{D}_\tau} F^2(\phi_{\tau,k})(l) \leq C2^{-\tau}$ imply that

$$\begin{aligned}\eta_{\tau,k} &= \left(\sum_{l \in \mathbb{Z}} g_l^{-2} F^2(\phi_{\tau,k})(l)\right)^{1/2} \leq C \left(2^{-\tau} \sum_{l \in \mathcal{D}_\tau} g_l^{-2}\right)^{1/2} \leq C \left(\sup_{l \in \mathcal{D}_\tau} g_l^{-2}\right)^{1/2} \\ &\leq C2^{\delta\tau} \leq C.\end{aligned}\tag{6.3}$$

Moreover, since $f \in B_{2,\infty}^s(M)$,

$$\sum_{k=0}^{2^\tau-1} \alpha_{\tau,k}^2 \leq M_* 2^{-2\tau} \leq C.\tag{6.4}$$

Putting (6.2), (6.3) and (6.4) together, we have

$$\begin{aligned}R &\leq C \left(n^{-1} \sum_{k=0}^{2^\tau-1} \eta_{\tau,k}^2 \alpha_{\tau,k}^2 + n^{-2} \sum_{k=0}^{2^\tau-1} \eta_{\tau,k}^4\right) \leq C \left(n^{-1} \sum_{k=0}^{2^\tau-1} \alpha_{\tau,k}^2 + n^{-2}\right) \\ &\leq Cn^{-1} \leq C \max\left(n^{-1}, \left(n(\log n)^{-1/2}\right)^{-4s/(2s+2\delta+1/2)}\right) = C\varphi_n.\end{aligned}\tag{6.5}$$

We obtain the desired upper bound for R .

The upper bound for S . We consider the following lemma.

Lemma 6.1 *Let X_1, \dots, X_n be n random variables. Then*

$$\mathbb{E}\left(\left(\sum_{i=1}^n X_i\right)^2\right) \leq \left(\sum_{i=1}^n \left(\mathbb{E}(X_i^2)\right)^{1/2}\right)^2.$$

Applying Lemma 6.1 above, we obtain

$$S \leq \left(\sum_{j=\tau}^{j_1} \left[\mathbb{E} \left(\left(\left(\sum_{k=0}^{2^j-1} (\widehat{\beta}_{j,k}^2 - n^{-1}\sigma_{j,k}^2) - n^{-1}\sigma_{j,\infty}^2 \lambda_n \right)_+ - \sum_{k=0}^{2^j-1} \beta_{j,k}^2 \right)^2 \right) \right]^{1/2} \right)^2. \quad (6.6)$$

Now, let us consider Lemma 6.2 below. It is a consequence of a result proved by Johnstone (2001b). Further details are given in the proof of Lemma 6.2.

Lemma 6.2 *We adopt the notations of the previous sections. There exist two constants c and γ such that, for any $j \in \{\tau, \dots, j_1\}$, we have*

$$\begin{aligned} & \mathbb{E} \left(\left(\left(\sum_{k=0}^{2^j-1} (\widehat{\beta}_{j,k}^2 - n^{-1}\sigma_{j,k}^2) - n^{-1}\sigma_{j,\infty}^2 \lambda_n \right)_+ - \sum_{k=0}^{2^j-1} \beta_{j,k}^2 \right)^2 \right) \\ & \leq \gamma \sigma_{j,\infty}^4 \left(cn^{-2}2^{-4\delta j} + \min \left(2 \left(\sum_{k=0}^{2^j-1} \beta_{j,k}^2 \sigma_{j,k}^{-2} \right)^2, v_j \right) \right), \end{aligned}$$

where

$$v_j = 2n^{-2}2^j + 4n^{-1} \sum_{k=0}^{2^j-1} \beta_{j,k}^2 \sigma_{j,k}^{-2} + 4(\log 2)(1 + 4\delta) (n^{-2}j2^j).$$

Applying Lemma 6.2 above and an elementary inequality of convexity, we have

$$\begin{aligned} & \left[\mathbb{E} \left(\left(\left(\sum_{k=0}^{2^j-1} (\widehat{\beta}_{j,k}^2 - n^{-1}\sigma_{j,k}^2) - \sigma_{j,\infty}^2 n^{-1} \lambda_n \right)_+ - \sum_{k=0}^{2^j-1} \beta_{j,k}^2 \right)^2 \right) \right]^{1/2} \\ & \leq C \sigma_{j,\infty}^2 \left(n^{-1}2^{-2\delta j} + \min \left(\sum_{k=0}^{2^j-1} \beta_{j,k}^2 \sigma_{j,k}^{-2}, v_j^{1/2} \right) \right). \quad (6.7) \end{aligned}$$

Let \mathcal{C}_j be the support of $\psi_{j,k}$. Since $\sup_{l \in \mathbb{Z}} F^2(\psi_{j,k})(l) = \sup_{l \in \mathcal{C}_j} F^2(\psi_{j,k})(l) \leq C2^{-j}$, we have

$$\sigma_{j,\infty}^2 = \sup_{k=0, \dots, 2^j-1} \sigma_{j,k}^2 \leq C2^{-j} \sum_{l \in \mathcal{C}_j} g_l^{-2} \leq C \sup_{l \in \mathcal{C}_j} g_l^{-2} \leq C2^{2\delta j}. \quad (6.8)$$

It follows from (6.6), (6.7) and (6.8) that

$$S \leq C(I + J)^2, \quad (6.9)$$

where

$$I = n^{-1} \sum_{j=\tau}^{j_1} 2^{2\delta j} 2^{-2\delta j}$$

and

$$J = \sum_{j=\tau}^{j_1} 2^{2\delta j} \min \left(\sum_{k=0}^{2^j-1} \beta_{j,k}^2 \sigma_{j,k}^{-2}, v_j^{1/2} \right).$$

Let us study the upper bounds for I and J in turn.

The upper bound for I . We have

$$\begin{aligned} I &= n^{-1}(j_1 - \tau + 1) \leq Cn^{-1} \log n \\ &\leq C \max \left(n^{-1/2}, \left(n(\log n)^{-1/2} \right)^{-2s/(2s+2\delta+1/2)} \right) = C\varphi_n^{1/2}. \end{aligned} \quad (6.10)$$

The upper bound for J . We distinguish the case $s > \delta + 4^{-1}$ and the case $s \in (0, \delta + 4^{-1}]$.

The case $s > \delta + 4^{-1}$. Let j_2 be an integer such that

$$2^{-1} \left(n(\log n)^{-1} \right)^{1/(4\delta+1)} < 2^{j_2} \leq \left(n(\log n)^{-1} \right)^{1/(4\delta+1)}.$$

Notice that $j_2 \leq j_1$. We have the following decomposition

$$J = G + H, \quad (6.11)$$

where

$$G = \sum_{j=\tau}^{j_2} 2^{2\delta j} \min \left(\sum_{k=0}^{2^j-1} \beta_{j,k}^2 \sigma_{j,k}^{-2}, v_j^{1/2} \right)$$

and

$$H = \sum_{j=j_2+1}^{j_1} 2^{2\delta j} \min \left(\sum_{k=0}^{2^j-1} \beta_{j,k}^2 \sigma_{j,k}^{-2}, v_j^{1/2} \right).$$

Let us investigate the upper bounds for G and H in turn.

The upper bound for G . Since $\inf_{l \in \mathbb{Z}} F^2(\psi_{j,k})(l) = \inf_{l \in \mathcal{C}_j} F^2(\psi_{j,k})(l) \geq C2^{-j}$, we have

$$\sigma_{j,k}^{-2} \leq C \left(2^{-j} \sum_{l \in \mathcal{C}_j} g_l^{-2} \right)^{-1} \leq C \left(\inf_{l \in \mathcal{C}_j} g_l^{-2} \right)^{-1} \leq C2^{-2\delta j}. \quad (6.12)$$

Using an elementary inequality of convexity, the inequality (6.12), and the fact that $f \in B_{2,\infty}^s(M)$ (i.e. $\sum_{k=0}^{2^j-1} \beta_{j,k}^2 \leq M^* 2^{-2js}$), we obtain

$$\begin{aligned}
v_j^{1/2} &\leq C \left(n^{-1/2} \left(\sum_{k=0}^{2^j-1} \beta_{j,k}^2 \sigma_{j,k}^{-2} \right)^{1/2} + n^{-1} j^{1/2} 2^{j/2} \right) \\
&\leq C \left(n^{-1/2} 2^{-(s+\delta)j} + n^{-1} j^{1/2} 2^{j/2} \right).
\end{aligned} \tag{6.13}$$

It follows from the inequality (6.13) and the definition of j_2 that

$$\begin{aligned}
G &\leq \sum_{j=\tau}^{j_2} 2^{2\delta j} v_j^{1/2} \leq C \left(n^{-1/2} \sum_{j=\tau}^{j_2} 2^{-(s-\delta)j} + n^{-1} \sum_{j=\tau}^{j_2} j^{1/2} 2^{(2\delta+1/2)j} \right) \\
&\leq C \left(n^{-1/2} + n^{-1} j_2^{1/2} 2^{(2\delta+1/2)j_2} \right) \leq C n^{-1/2}.
\end{aligned} \tag{6.14}$$

The upper bound for H . It follows from the inequality $\sigma_{j,k}^{-2} \leq C 2^{-2\delta j}$ (see (6.12)), the fact that $f \in B_{2,\infty}^s(M)$ and the definition of j_2 that

$$\begin{aligned}
H &\leq \sum_{j=j_2+1}^{j_1} \sum_{k=0}^{2^j-1} \beta_{j,k}^2 (2^{2\delta j} \sigma_{j,k}^{-2}) \leq C \sum_{j=j_2+1}^{j_1} \sum_{k=0}^{2^j-1} \beta_{j,k}^2 \leq C \sum_{j=j_2+1}^{j_1} 2^{-2js} \\
&\leq C 2^{-2j_2 s} \leq C \left(n(\log n)^{-1} \right)^{-2s/(4\delta+1)} \leq C n^{-1/2}.
\end{aligned} \tag{6.15}$$

Combining (6.11), (6.14) and (6.15), we obtain

$$J = G + H \leq C n^{-1/2}. \tag{6.16}$$

By putting the inequalities (6.9), (6.10) and (6.16) together, we have

$$S \leq C (I + J)^2 \leq C n^{-1}. \tag{6.17}$$

The case $s \in (0, \delta + 4^{-1}]$. Let j_3 be an integer such that

$$2^{-1} \left(n(\log n)^{-1/2} \right)^{1/(2s+2\delta+1/2)} < 2^{j_3} \leq \left(n(\log n)^{-1/2} \right)^{1/(2s+2\delta+1/2)}.$$

Notice that $j_3 \leq j_1$. We have the following decomposition

$$J = L + M, \tag{6.18}$$

where

$$L = \sum_{j=\tau}^{j_3} 2^{2\delta j} \min \left(\sum_{k=0}^{2^j-1} \beta_{j,k}^2 \sigma_{j,k}^{-2}, v_j^{1/2} \right)$$

and

$$M = \sum_{j=j_3+1}^{j_1} 2^{2\delta j} \min \left(\sum_{k=0}^{2^j-1} \beta_{j,k}^2 \sigma_{j,k}^{-2}, v_j^{1/2} \right).$$

Let us study the upper bounds for L and M in turn.

The upper bound for L . Set $(a)_+ = \max(a, 0)$. Using the inequality (6.13), we have

$$\begin{aligned}
L &\leq \sum_{j=\tau}^{j_3} 2^{2\delta j} v_j^{1/2} \leq C \left(n^{-1/2} \sum_{j=\tau}^{j_3} 2^{-(s-\delta)j} + n^{-1} \sum_{j=\tau}^{j_3} j^{1/2} 2^{(2\delta+1/2)j} \right) \\
&\leq C \left(n^{-1/2} 2^{j_3(\delta-s)_+} + n^{-1} (\log n)^{1/2} 2^{(2\delta+1/2)j_3} \right) \\
&\leq C \left(n^{-((\delta-s)_+ + s + \delta + 1/4)/(2s+2\delta+1/2)} + \left(n(\log n)^{-1/2} \right)^{-2s/(2s+2\delta+1/2)} \right) \\
&\leq C \left(n(\log n)^{-1/2} \right)^{-2s/(2s+2\delta+1/2)}. \tag{6.19}
\end{aligned}$$

The upper bound for M . By the inequality $\sigma_{j,k}^{-2} \leq C 2^{-2\delta j}$ (see (6.12)), the fact that $f \in B_{2,\infty}^s(M)$ and the definition of j_3 , we have

$$\begin{aligned}
M &\leq \sum_{j=j_3+1}^{j_1} \sum_{k=0}^{2^j-1} \beta_{j,k}^2 (2^{2\delta j} \sigma_{j,k}^{-2}) \leq C \sum_{j=j_3+1}^{j_1} \sum_{k=0}^{2^j-1} \beta_{j,k}^2 \leq C \sum_{j=j_3+1}^{j_1} 2^{-2js} \\
&\leq C 2^{-2j_3 s} \leq C \left(n(\log n)^{-1/2} \right)^{-2s/(2s+2\delta+1/2)}. \tag{6.20}
\end{aligned}$$

Putting (6.18), (6.19) and (6.20) together, we obtain

$$J = L + M \leq C \left(n(\log n)^{-1/2} \right)^{-2s/(2s+2\delta+1/2)}. \tag{6.21}$$

Combining the inequalities (6.9), (6.10) and (6.21), we have

$$S \leq C (I + J)^2 \leq C \left(n(\log n)^{-1/2} \right)^{-4s/(2s+2\delta+1/2)}. \tag{6.22}$$

It follows from (6.17) and (6.22) that, for any $s > 0$, we have

$$S \leq C \max \left(n^{-1}, \left(n(\log n)^{-1/2} \right)^{-4s/(2s+2\delta+1/2)} \right) = C \varphi_n. \tag{6.23}$$

We obtain the desired upper bound for S .

The upper bound for T . Using the fact that $f \in B_{2,\infty}^s(M)$ and the definition of j_1 , we have

$$\begin{aligned}
T &= \left(\sum_{j=j_1+1}^{\infty} \sum_{k=0}^{2^j-1} \beta_{j,k}^2 \right)^2 \leq \left(\sum_{j=j_1+1}^{\infty} 2^{-2js} \right)^2 \leq C 2^{-4j_1 s} \leq C n^{-4s/(1/2+2\delta)} \\
&\leq C \max \left(n^{-1}, \left(n(\log n)^{-1/2} \right)^{-4s/(2s+2\delta+1/2)} \right) = C \varphi_n. \tag{6.24}
\end{aligned}$$

We obtain the desired upper bound for T .

It follows from (6.1), (6.5), (6.23) and (6.24) that

$$\sup_{f \in B_{2,\infty}^s(M)} \mathbb{E} \left(\left(\widehat{Q}_n - \|f\|_2^2 \right)^2 \right) \leq 3(R + S + T) \leq C\varphi_n,$$

where $\varphi_n = \max \left(n^{-1}, \left(n(\log n)^{-1/2} \right)^{-4s/(2s+2\delta+1/2)} \right)$.

Theorem 5.1 is proved.

PROOF OF LEMMA 6.2. First of all, notice that

$$\begin{aligned} W &= \left(\sum_{k=0}^{2^j-1} (\widehat{\beta}_{j,k}^2 - n^{-1}\sigma_{j,k}^2) - n^{-1}\sigma_{j,\infty}^2\lambda_n \right)_+ - \sum_{k=0}^{2^j-1} \beta_{j,k}^2 \\ &= \left(\sum_{k=0}^{2^j-1} \sigma_{j,k}^2 (\widehat{\beta}_{j,k}^2 \sigma_{j,k}^{-2} - n^{-1}) - n^{-1}\sigma_{j,\infty}^2\lambda_n \right)_+ - \sum_{k=0}^{2^j-1} \sigma_{j,k}^2 (\beta_{j,k}^2 \sigma_{j,k}^{-2}). \end{aligned}$$

Since $\widehat{\beta}_{j,k}\sigma_{j,k}^{-1} \sim \mathcal{N}(0, n^{-1})$, $\sigma_{j,k} \leq \sigma_{j,\infty}$ and $\lambda_n = (2d)^{1/2}(2\omega \log d)^{1/2}$ with $d = 2^j$ and $\omega = 1 + 4\delta$, an inequality for weighted chi-squared proved by Johnstone (2001b) (see Corollary 2.3 without the bound " $\rho \leq \rho_a$ " used in the corresponding proof) implies the existence of two constants $c > 0$ and $\gamma > 0$ such that

$$\mathbb{E} (W^2) \leq \gamma \sigma_{j,\infty}^4 \left(cn^{-2}d^{1-\beta} + \min \left(2 \left(\sum_{k=0}^{2^j-1} \beta_{j,k}^2 \sigma_{j,k}^{-2} \right)^2, v_j \right) \right),$$

where

$$v_j = \mathbb{V} \left(\sum_{k=0}^{2^j-1} (\widehat{\beta}_{j,k}^2 \sigma_{j,k}^{-2} - n^{-1}) \right) + n^{-2}\lambda_n^2.$$

After some computations, we obtain

$$\begin{aligned} &\mathbb{E} \left(\left(\left(\sum_{k=0}^{2^j-1} (\widehat{\beta}_{j,k}^2 - n^{-1}\sigma_{j,k}^2) - n^{-1}\sigma_{j,\infty}^2\lambda_n \right)_+ - \sum_{k=0}^{2^j-1} \beta_{j,k}^2 \right)^2 \right) \\ &\leq \gamma \sigma_{j,\infty}^4 \left(cn^{-2}2^{-4\delta j} + \min \left(2 \left(\sum_{k=0}^{2^j-1} \beta_{j,k}^2 \sigma_{j,k}^{-2} \right)^2, v_j \right) \right), \end{aligned}$$

where

$$v_j = 2n^{-2}2^j + 4n^{-1} \sum_{k=0}^{2^j-1} \beta_{j,k}^2 \sigma_{j,k}^{-2} + 4(\log 2)(1 + 4\delta) \left(n^{-2}j2^j\right).$$

Lemma 6.2 is proved.

PROOF OF THEOREM 5.2. We distinguish the case $s > \delta + 4^{-1}$ and the case $s \in (0, \delta + 4^{-1}]$.

The case $s > \delta + 4^{-1}$. Let us consider the two following functions

$$f_0(x) = 1, \quad f_1(x) = 1 + n^{-1/2}.$$

Clearly, f_0 and f_1 belong to $B_{2,\infty}^s(M)$. Moreover, we have

$$\left| \|f_0\|_2^2 - \|f_1\|_2^2 \right| = 2n^{-1/2} + n^{-1} \geq n^{-1/2} = \kappa_n.$$

Now, in order to apply Theorem 2.12 (iii) of Tsybakov (2004), we aim to bound (by a constant C) the chi-square divergence $\chi^2(\mathbb{P}_{f_1}, \mathbb{P}_{f_0})$ defined by

$$\chi^2(\mathbb{P}_{f_1}, \mathbb{P}_{f_0}) = \int \left(\frac{d\mathbb{P}_{f_1}}{d\mathbb{P}_{f_0}} \right)^2 d\mathbb{P}_{f_0} - 1, \quad (6.25)$$

where \mathbb{P}_h denotes the probability distribution of $\{Y(t); t \in [0, 1]\}$ indexed by the function h .

Let \star be the standard convolution product on the unit interval. The Girsanov theorem yields

$$\begin{aligned} \frac{d\mathbb{P}_{f_1}}{d\mathbb{P}_{f_0}} &= \exp\left(n \int_0^1 ((f_1 \star g)(t) - (f_0 \star g)(t)) dY(t) - \right. \\ &\quad \left. 2^{-1}n \int_0^1 ((f_1 \star g)^2(t) - (f_0 \star g)^2(t)) dt\right) \\ &= \exp\left(n^{1/2} \int_0^1 g(u) du \int_0^1 dY(t) - 2^{-1}n \left(\int_0^1 g(u) du\right)^2 (2n^{-1/2} + n^{-1})\right). \end{aligned}$$

So, under \mathbb{P}_{f_0} , we have

$$\frac{d\mathbb{P}_{f_1}}{d\mathbb{P}_{f_0}} = \exp\left(\int_0^1 g(u) du \int_0^1 dB(t) - 2^{-1} \left(\int_0^1 g(u) du\right)^2 (2n^{1/2} + 1)\right).$$

Using the equality $\int \exp\left(2 \int_0^1 g(u) du \int_0^1 dB(t)\right) d\mathbb{P}_{f_0} = \exp\left(2 \left(\int_0^1 g(u) du\right)^2\right)$ and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
& \int \left(\frac{d\mathbb{P}_{f_1}}{d\mathbb{P}_{f_0}} \right)^2 d\mathbb{P}_{f_0} \\
&= \exp \left(- \left(\int_0^1 g(u) du \right)^2 (2n^{1/2} + 1) \right) \int \exp \left(2 \int_0^1 g(u) du \int_0^1 dB(t) \right) d\mathbb{P}_{f_0} \\
&= \exp \left(- \left(\int_0^1 g(u) du \right)^2 (2n^{1/2} + 1) \right) \exp \left(2 \left(\int_0^1 g(u) du \right)^2 \right) \\
&\leq \exp \left(\left(\int_0^1 g(u) du \right)^2 \right) \leq \exp \left(\int_0^1 g^2(u) du \right) \leq C.
\end{aligned}$$

Therefore, there exists a constant $C > 0$ such that

$$\chi^2(\mathbb{P}_{f_1}, \mathbb{P}_{f_0}) = \int \left(\frac{d\mathbb{P}_{f_1}}{d\mathbb{P}_{f_0}} \right)^2 d\mathbb{P}_{f_0} - 1 \leq C < \infty.$$

It follows from Theorem 2.12 (iii) of Tsybakov (2004) that

$$\inf_{\tilde{Q}_n} \sup_{f \in B_{2,\infty}^s(M)} \mathbb{E} \left(\left(\tilde{Q}_n - \|f\|_2^2 \right)^2 \right) \geq c\kappa_n^2 = cn^{-1} = c\varphi_n^*. \quad (6.26)$$

The case $s \in (0, \delta + 4^{-1}]$. Let us consider the two following functions

$$f_0(x) = 0, \quad f_1(x) = \gamma_{j_*} \sum_{k_*=0}^{2^{j_*}} w_{k_*} \psi_{j_*,k_*}(x), \quad (6.27)$$

where j_* is an integer to be chosen below, γ_{j_*} is a quantity to be chosen below, and $w_0, \dots, w_{2^{j_*}-1}$ are i.i.d. Rademacher random variables (i.e. $\mathbb{P}(w_u = 1) = \mathbb{P}(w_u = -1) = 2^{-1}$ for any $u \in \{0, \dots, 2^{j_*} - 1\}$).

Clearly, f_0 belong to $B_{2,\infty}^s(M)$. Thanks to the orthogonality of the wavelet basis, for any integer $j \geq \tau$ and any $k \in \{0, \dots, 2^j - 1\}$, we have $\beta_{j,k} = \int_0^1 f_1(x) \psi_{j,k}(x) dx = \gamma_{j_*} w_k$ for $j = j_*$ and 0 otherwise. Thus, if $\gamma_{j_*}^2 = M_* 2^{-j_*(2s+1)}$, then

$$2^{2j_*s} \sum_{k=0}^{2^{j_*}-1} \beta_{j_*,k}^2 = 2^{2j_*s} \gamma_{j_*}^2 \sum_{k=0}^{2^{j_*}-1} w_k^2 = \gamma_{j_*}^2 2^{j_*(2s+1)} = M_*.$$

Therefore, for such a choice of γ_{j_*} , we have $f_1 \in B_{2,\infty}^s(M)$.

Let j_* be an integer such that

$$2^{-1} n^{1/(2s+2\delta+1/2)} \leq 2^{j_*} \leq n^{1/(2s+2\delta+1/2)}. \quad (6.28)$$

With the previous value of γ_{j_*} , the orthonormality of the wavelet basis gives

$$\begin{aligned} \left| \|f_0\|_2^2 - \|f_1\|_2^2 \right| &= \|f_1\|_2^2 = \gamma_{j_*}^2 \sum_{k=0}^{2^{j_*}-1} w_k^2 = \gamma_{j_*}^2 2^{j_*} \geq C 2^{-2j_*s} \\ &\geq C n^{-2s/(2s+2\delta+1/2)} = \kappa_n. \end{aligned}$$

In what follows, we consider the function f_1 described by (6.27) with the integer j_* defined by (6.28) and the quantity $\gamma_{j_*}^2 = M_* 2^{-j_*(2s+1)}$. In order to apply Theorem 2.12 (iii) of Tsybakov (2004), we aim to bound (by a constant C) the chi-square divergence $\chi^2(\mathbb{P}_{f_1}, \mathbb{P}_{f_0})$ (see (6.25)).

Let \star be the standard convolution product. Due to the definitions of f_0, f_1 , the variables $w_0, \dots, w_{2^{j_*}-1}$, and the Girsanov theorem, we have

$$\begin{aligned} \frac{d\mathbb{P}_{f_1}}{d\mathbb{P}_{f_0}} &= \prod_{k=0}^{2^{j_*}-1} \left[2^{-1} \exp \left(n \gamma_{j_*} \int_0^1 (\psi_{j_*,k} \star g)(t) dY(t) - 2^{-1} n \gamma_{j_*}^2 \int_0^1 (\psi_{j_*,k} \star g)^2(t) dt \right) \right. \\ &\quad \left. + 2^{-1} \exp \left(-n \gamma_{j_*} \int_0^1 (\psi_{j_*,k} \star g)(t) dY(t) - 2^{-1} n \gamma_{j_*}^2 \int_0^1 (\psi_{j_*,k} \star g)^2(t) dt \right) \right]. \end{aligned}$$

So, under \mathbb{P}_{f_0} , we have

$$\begin{aligned} \frac{d\mathbb{P}_{f_1}}{d\mathbb{P}_{f_0}} &= \prod_{k=0}^{2^{j_*}-1} \left[2^{-1} \exp \left(n^{1/2} \gamma_{j_*} \int_0^1 (\psi_{j_*,k} \star g)(t) dB(t) - 2^{-1} n \gamma_{j_*}^2 \int_0^1 (\psi_{j_*,k} \star g)^2(t) dt \right) \right. \\ &\quad \left. + 2^{-1} \exp \left(-n^{1/2} \gamma_{j_*} \int_0^1 (\psi_{j_*,k} \star g)(t) dB(t) - 2^{-1} n \gamma_{j_*}^2 \int_0^1 (\psi_{j_*,k} \star g)^2(t) dt \right) \right] \\ &= \prod_{k=0}^{2^{j_*}-1} 2^{-1} \exp \left(-2^{-1} n \gamma_{j_*}^2 \int_0^1 (\psi_{j_*,k} \star g)^2(t) dt \right) \times \\ &\quad \left[\exp \left(n^{1/2} \gamma_{j_*} \int_0^1 (\psi_{j_*,k} \star g)(t) dB(t) \right) + \exp \left(-n^{1/2} \gamma_{j_*} \int_0^1 (\psi_{j_*,k} \star g)(t) dB(t) \right) \right]. \end{aligned}$$

Hence,

$$\begin{aligned} \int \left(\frac{d\mathbb{P}_{f_1}}{d\mathbb{P}_{f_0}} \right)^2 d\mathbb{P}_{f_0} &= \prod_{k=0}^{2^{j_*}-1} 2^{-2} \exp \left(-n \gamma_{j_*}^2 \int_0^1 (\psi_{j_*,k} \star g)^2(t) dt \right) \times \\ &\quad \int \left[\exp \left(n^{1/2} \gamma_{j_*} \int_0^1 (\psi_{j_*,k} \star g)(t) dB(t) \right) + \exp \left(-n^{1/2} \gamma_{j_*} \int_0^1 (\psi_{j_*,k} \star g)(t) dB(t) \right) \right]^2 d\mathbb{P}_{f_0}. \end{aligned}$$

Since $\int \exp \left(2n^{1/2} \gamma_{j_*} \int_0^1 (\psi_{j_*,k} \star g)(t) dB(t) \right) d\mathbb{P}_{f_0} =$

$\int \exp \left(-2n^{1/2} \gamma_{j_*} \int_0^1 (\psi_{j_*,k} \star g)(t) dB(t) \right) d\mathbb{P}_{f_0} = \exp \left(2n \gamma_{j_*}^2 \int_0^1 (\psi_{j_*,k} \star g)^2(t) dt \right)$,
we have

$$\begin{aligned} & \int \left(\frac{d\mathbb{P}_{f_1}}{d\mathbb{P}_{f_0}} \right)^2 d\mathbb{P}_{f_0} \\ &= \prod_{k=0}^{2^{j_*}-1} 2^{-1} \left(\exp \left(n\gamma_{j_*}^2 \int_0^1 (\psi_{j_*,k} \star g)^2(t) dt \right) + \exp \left(-n\gamma_{j_*}^2 \int_0^1 (\psi_{j_*,k} \star g)^2(t) dt \right) \right). \end{aligned}$$

Using the inequality $2^{-1}(e^x + e^{-x}) \leq e^{x^2/2}$, $x \in \mathbb{R}$, we obtain

$$\int \left(\frac{d\mathbb{P}_{f_1}}{d\mathbb{P}_{f_0}} \right)^2 d\mathbb{P}_{f_0} \leq \prod_{k=0}^{2^{j_*}-1} \exp \left(2^{-1} n^2 \gamma_{j_*}^4 \left(\int_0^1 (\psi_{j_*,k} \star g)^2(t) dt \right)^2 \right).$$

Using the condition (3.1) and the Plancherel theorem, we obtain

$$\begin{aligned} \int_0^1 (\psi_{j_*,k} \star g)^2(t) dt &= \int_0^1 (F(\psi_{j_*,k} \star g))^2(t) dt = \int_0^1 F^2(\psi_{j_*,k})(t) F^2(g)(t) dt \\ &\leq C 2^{-2\delta j_*} \int_0^1 F^2(\psi_{j_*,k})(t) dt = C 2^{-2\delta j_*} \int_0^1 \psi_{j_*,k}^2(x) dx \\ &= C 2^{-2\delta j_*}. \end{aligned}$$

We deduce that

$$\int \left(\frac{d\mathbb{P}_{f_1}}{d\mathbb{P}_{f_0}} \right)^2 d\mathbb{P}_{f_0} \leq \prod_{k=0}^{2^{j_*}-1} \exp \left(C n^2 \gamma_{j_*}^4 2^{-4\delta j_*} \right) = \exp \left(C n^2 \gamma_{j_*}^4 2^{(1-4\delta)j_*} \right).$$

Thanks to the definitions of j_* and γ_{j_*} , we have

$$n^2 \gamma_{j_*}^4 2^{(1-4\delta)j_*} \leq C n^2 2^{-j_*(4s+2)} 2^{(1-4\delta)j_*} \leq C n^2 2^{-j_*(4s+1+4\delta)} \leq C.$$

It follows the existence of a constant $C > 0$ such that

$$\chi^2(\mathbb{P}_{f_1}, \mathbb{P}_{f_0}) = \int \left(\frac{d\mathbb{P}_{f_1}}{d\mathbb{P}_{f_0}} \right)^2 d\mathbb{P}_{f_0} - 1 \leq C < \infty.$$

Theorem 2.12 (iii) of Tsybakov (2004) yields

$$\inf_{\tilde{Q}_n} \sup_{f \in B_{2,\infty}^s(M)} \mathbb{E} \left(\left(\tilde{Q}_n - \|f\|_2^2 \right)^2 \right) \geq c \kappa_n^2 = c n^{-4s/(2s+2\delta+1)} = c \varphi_n^*. \quad (6.29)$$

Putting (6.26) and (6.29) together, we prove the existence of a constant $c > 0$ such that, for any $s > 0$, we have

$$\inf_{\tilde{Q}_n} \sup_{f \in B_{2,\infty}^s(M)} \mathbb{E} \left(\left(\tilde{Q}_n - \|f\|_2^2 \right)^2 \right) \geq c \varphi_n^*,$$

where $\varphi_n^* = \max \left(n^{-1}, n^{-4s/(2s+2\delta+1/2)} \right)$.

Theorem 5.2 is proved.

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