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# ANALYTICITY AND REGULARITY FOR A CLASS OF SECOND ORDER EVOLUTION EQUATIONS

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**Résumé.** On étudie la conservation de la régularité et l'effet régularisant pour l'équation  $u'' + Au + cA^\alpha u' = 0$  où  $A$  est un opérateur auto-adjoint positif sur un espace de Hilbert réel  $H$  et  $\alpha \in (0, 1]$ ;  $c > 0$ . Si  $\alpha \geq \frac{1}{2}$  l'équation engendre un semi-groupe analytique sur  $D(A^{1/2}) \times H$ , et si  $\alpha \in (0, \frac{1}{2})$  une propriété de régularisation plus faible mais optimale est prouvée. Enfin des propriétés de conservation de la régularité associée à d'autres normes sont obtenues, avec comme exemple typique d'application l'équation des ondes  $u_{tt} - \Delta u - c\Delta u_t = 0$  avec condition de Dirichlet homogène dans un domaine borné pour laquelle la régularité  $C_0(\Omega) \times C_0(\Omega)$  est conservée for  $t > 0$ , un saisissant contraste avec le cas conservatif  $u_{tt} - \Delta u = 0$  dans lequel la régularité  $C_0(\Omega)$  peut-être perdue même pour un état initial  $(u_0, 0)$  avec  $u_0 \in C_0(\Omega) \cap C^1(\bar{\Omega})$ .

**Abstract.** The regularity conservation as well as the smoothing effect are studied for the equation  $u'' + Au + cA^\alpha u' = 0$  where  $A$  is a positive selfadjoint operator on a real Hilbert space  $H$  and  $\alpha \in (0, 1]$ ;  $c > 0$ . When  $\alpha \geq \frac{1}{2}$  the equation generates an analytic semigroup on  $D(A^{1/2}) \times H$ , and if  $\alpha \in (0, \frac{1}{2})$  a weaker optimal smoothing property is established. Some conservation properties in other norms are established, a typical example being the strongly dissipative wave equation  $u_{tt} - \Delta u - c\Delta u_t = 0$  with Dirichlet boundary conditions in a bounded domain for which the space  $C_0(\Omega) \times C_0(\Omega)$  is conserved for  $t > 0$ , in sharp contrast with the conservative case  $u_{tt} - \Delta u = 0$  for which  $C_0(\Omega)$ -regularity can be lost even starting from an initial state  $(u_0, 0)$  with  $u_0 \in C_0(\Omega) \cap C^1(\bar{\Omega})$ .

**Keywords:** Regularity, analytic semi-group, smoothing effect

**AMS classification numbers:** 35B65, 35D10, 35L05, 35L90

## 1. INTRODUCTION AND NOTATION

This paper is mainly devoted to a detailed study of the regularity conservation as well as the smoothing effect for the equation

$$(1.1) \quad u'' + Au + cA^\alpha u' = 0$$

where  $A$  is a positive selfadjoint operator on a real Hilbert space  $H$  and  $\alpha \geq 0$ ,  $c > 0$ . Throughout the text we assume

$$A \geq \eta I, \quad \eta > 0.$$

In particular the operator  $A^s$  is well defined and bounded  $H \rightarrow H$  for all  $s \leq 0$ . We identify  $H$  with its topological dual and we therefore have

$$D(A) \subset V = D(A^{1/2}) \subset H = H' \subset V' \subset D(A)'.$$

More generally we define a monotone nonincreasing one-parameter family of Hilbert spaces by the formula

$$H^s = \begin{cases} D(A^{s/2}) & \text{if } s \geq 0 \\ (D(A^{-s/2}))' & \text{if } s < 0. \end{cases}$$

We are interested in the smoothing effect and the conservation of regularity for the evolution equation (1.1). The plan of the paper is as follows: Section 2 is devoted to well-posedness of (1.1) considered as a first order system in  $V \times H$ . Section 3 deals with compactness properties of the resolvent and the semi-group associated to (1.1). Sections 4 and 5 are devoted to the study of time and spatial smoothing properties and especially to a simple direct proof of analyticity for  $\alpha \geq 1/2$ . A more general result, motivated by a conjecture of Chen & Russell [2] can be found in [3] but their proof is quite involved, relying on a stationary type of argument involving complex resolvent. Here we give a pure real and dynamical argument based only on elementary tools such as inner products and integration with respect to  $t$ . In Section 6 we investigate some regularity conservation properties which are specific to (1.1) with  $c > 0$  since the conservative problem corresponding to  $c = 0$  does not satisfy those properties anymore, cf. eg. [5]. Finally Section 5 contains the basic examples of application.

## 2. THE INITIAL VALUE PROBLEM

In this section we consider the equation

$$(2.1) \quad u'' + Au + Bu' = 0$$

where  $B = B^* \geq 0$  on  $H$  and in addition for some constants  $C > c > 0$

$$cA^\alpha \leq B \leq CA^\alpha.$$

**Theorem 2.1.** *Let  $(u^0, u^1) \in V \times H$  be given. For any  $T > 0$  there is a unique  $u \in C([0, T], V) \cap C^1([0, T], H)$  such that, setting  $\beta = \max\{1, \alpha\}$  we have*

$$\begin{cases} u' \in L^2(0, T; H^\alpha) & u'' \in L^2(0, T; H^{-\beta}) \\ u'' = -Au - Bu' & \text{in } L^2(0, T; H^{-\beta}) \\ u(0) = u^0 & u'(0) = u^1. \end{cases}$$

Moreover the function

$$E(t) := \frac{1}{2} \{ |u(t)|_V^2 + |u'(t)|_H^2 \}$$

is absolutely continuous with

$$E'(t) = - \langle Bu', u' \rangle_{H^{-\alpha}, H^\alpha}$$

almost everywhere on  $(0, T)$ .

*Proof.* For the existence part we introduce  $J_\lambda = (I + \lambda A)^{-1}$  and  $K_\lambda = J_\lambda^n$  for some integer  $n \geq \max\{1, \alpha\}$ . Then we solve

$$u_\lambda'' + Au_\lambda + K_\lambda BK_\lambda u_\lambda' = 0; \quad u_\lambda(0) = u^0, \quad u_\lambda'(0) = u^1.$$

The identity

$$\int_0^T (BK_\lambda u_\lambda', K_\lambda u_\lambda') dt + \frac{1}{2} \{ \|u_\lambda'(T)\|_H^2 + \|u_\lambda(T)\|_V^2 \} = \frac{1}{2} \{ \|u^1\|_H^2 + \|u^0\|_V^2 \}$$

allows to pass to the limit as  $\lambda \rightarrow 0$  along a suitable subsequence.

For the uniqueness part as well as the energy identity we start with a solution  $u \in L^2(0, T; V) \cap H^1(0, T; H)$  of

$$u'' + Au = f \in L^2(0, T; H^{-\beta})$$

We show that  $v_\lambda = K_\lambda u$  satisfies

$$\frac{d}{dt} \frac{1}{2} \{ \|v_\lambda'(t)\|_H^2 + \|v_\lambda(t)\|_V^2 \} = (K_\lambda f, K_\lambda u') = \langle f, K_\lambda^2 u' \rangle$$

Then we integrate and let  $\lambda \rightarrow 0$ . Finally we choose  $f = -Bu'$ . Uniqueness follows then by linearity from the energy identity applied with  $u^0 = u^1 = 0$ .  $\square$

### 3. COMPACTNESS OF THE RESOLVENT

In this section we assume  $B = cA^\alpha$ ,  $c > 0$ . Setting  $v = u'$ ,  $U = (u, v) \in V \times H = \mathcal{H}$ , and denoting by  $\tilde{B}$  the unique extension to  $\mathcal{L}(V, H^{1-2\alpha})$  of  $B \in \mathcal{L}(D(B), H)$ , we find that the equation (2.1) becomes

$$U' + \mathcal{A}U = 0$$

where

$$D(\mathcal{A}) = \{(u, v) \in V \times V, \quad Au + \tilde{B}v \in H\}$$

and

$$\mathcal{A}(u, v) = (-v, Au + \tilde{B}v)$$

so that

$$\|\mathcal{A}(u, v)\|_{\mathcal{H}} \sim \|v\|_V + \|Au + \tilde{B}v\|_H$$

**Theorem 3.1.** *a) If  $\alpha < 1$ , then  $D(\mathcal{A}) \subset H^\gamma \times V$  with  $\gamma = \min\{2, 3 - 2\alpha\}$ . As a consequence if the imbedding  $V \rightarrow H$  is compact, then so is  $(I + \mathcal{A})^{-1} : \mathcal{H} \rightarrow \mathcal{H}$ .*

*b) If  $\alpha \geq 1$ , then we have*

$$\forall z \in V, \quad U(z) =: (cz, -A^{1-\alpha}z) \in D(\mathcal{A})$$

with

$$\|U(z)\|_{\mathcal{H}} + \|\mathcal{A}U(z)\|_{\mathcal{H}} \leq K\|z\|_V.$$

In particular if  $\dim H = \infty$ ,  $(I + \mathcal{A})^{-1} : \mathcal{H} \rightarrow \mathcal{H}$  is not compact. Hence in this case the semi-group  $\mathcal{S}(t)$  generated on  $\mathcal{H}$  by  $\mathcal{A}$  is never compact.

*Proof.* a) If  $U \in D(\mathcal{A})$ , then  $v \in V$  and  $Au + \tilde{B}v \in H$ , hence

$$u \in D(A) + A^{-1}(\tilde{B}V) = H^2 + A^{\alpha-1}(H^1) = H^\gamma.$$

If  $V \rightarrow H$  is compact and  $\alpha < 1$ , then  $H^\gamma \rightarrow V$  is compact, therefore since

$$(I + \mathcal{A})^{-1} \in \mathcal{L}(\mathcal{H}, H^\gamma \times \mathcal{V})$$

the result is now obvious.

b) Assume now  $\alpha \geq 1$ . Then clearly  $\forall z \in V$ ,  $U(z) =: (cz, -A^{1-\alpha}z) \in D(\mathcal{A})$  because both  $cz$  and  $-A^{1-\alpha}z$  are in  $V$  and in addition

$$A(cz) + cA^\alpha(-A^{1-\alpha}z) = 0 \in H.$$

Moreover we have

$$\|U(z)\|_{\mathcal{H}} \leq \|cz\|_V + \|A^{1-\alpha}z\|_H \leq c\|z\|_V + C'\|z\|_H \leq C''\|z\|_V$$

and

$$\|\mathcal{A}U(z)\|_{\mathcal{H}} \leq \|A^{1-\alpha}z\|_V + \|A(cz) + cA^\alpha(-A^{1-\alpha}z)\|_H = \|A^{1-\alpha}z\|_V \leq C'''\|z\|_V.$$

Hence

$$\|U(z)\|_{\mathcal{H}} + \|\mathcal{A}U(z)\|_{\mathcal{H}} \leq K\|z\|_V$$

with  $K := C'' + C'''$ . Finally when  $z \in V$  varies in the unit ball

$$\{\|z\|_V \leq 1\}$$

the first projection of  $U(z)$  covers the entire ball of radius  $c$  in  $V$ , therefore  $(I + \mathcal{A})^{-1} : \mathcal{H} \rightarrow \mathcal{H}$  is not compact  $\square$

#### 4. ANALYTIC TYPE TIME SMOOTHING EFFECT.

In this section we give a new and short proof of a result previously obtained by Chen & Triggiani [3]. However our proof seems to be limited to the case  $B = cA^\alpha$ ,  $c > 0$  or at least to require that  $B$  commutes with  $A$ .

**Theorem 4.1.** *For any  $\alpha \geq 1/2$ , the semi-group  $\mathcal{S}(t)$  generated on  $\mathcal{H}$  by  $\mathcal{A}$  is analytic, more precisely*

$$\forall t > 0, \quad S(t)U^0 \in D(\mathcal{A})$$

and

$$(4.1) \quad \forall t > 0, \quad \|\mathcal{A}S(t)U^0\|_{\mathcal{H}} \leq \frac{C}{t}\|U^0\|_{\mathcal{H}}.$$

*Proof.* We set

$$E := \frac{1}{2} \left( |u'(0)|^2 + |A^{1/2}u(0)|^2 \right) = \frac{1}{2} \|U^0\|_{\mathcal{H}}^2.$$

Multiplying the equation by  $u'$  we have immediately

$$(4.2) \quad \forall t \geq 0, \quad \int_0^t (Bu', u') ds \leq E.$$

In particular, with  $C = 1/c$

$$(4.3) \quad \forall t \geq 0, \quad \int_0^t |A^{\frac{\alpha}{2}}u'(s)|^2 ds \leq CE.$$

Taking the inner product in  $H$  of (2.1) by  $A^{1-\alpha}u(s)$  and integrating, we find

$$\int_0^t |A^{1-\frac{\alpha}{2}}u(s)|^2 ds = - \int_0^t (u'' + Bu', A^{1-\alpha}u) ds.$$

Now we have

$$\begin{aligned} \int_0^t (Bu', A^{1-\alpha}u) ds &= c \int_0^t (A^\alpha u', A^{1-\alpha}u) ds = c \int_0^t (Au, u') ds \\ &= \frac{c}{2} (|A^{1/2}u(t)|^2 - |A^{1/2}u(0)|^2) \geq -cE. \end{aligned}$$

Next, integrating by parts we find

$$\int_0^t (u'', A^{1-\alpha}u) ds = [(u', A^{1-\alpha}u)]_0^t - \int_0^t (u', A^{1-\alpha}u') ds.$$

Since  $\alpha \geq 1/2$ , we have  $1 - \alpha \leq 1/2$  and  $1 - \alpha \leq \alpha$ , hence

$$- \int_0^t (u'', A^{1-\alpha}u) ds \leq C_1 E$$

therefore

$$(4.4) \quad \int_0^t |A^{1-\frac{\alpha}{2}}u(s)|^2 ds \leq (c + C_1)E = C_2 E$$

hence

$$\int_0^t (|A^{1-\frac{\alpha}{2}}u(s)|^2 + |A^{\frac{\alpha}{2}}u'(s)|^2) ds \leq C_3 E.$$

Since  $\frac{1-\alpha}{2} \leq \frac{\alpha}{2}$ , we deduce

$$(4.5) \quad \int_0^t (|A^{1-\frac{\alpha}{2}}u(s)|^2 + |A^{\frac{1-\alpha}{2}}u'(s)|^2) ds \leq C_4 E.$$

We introduce now

$$w(t) = A^{\frac{1-\alpha}{2}}u(t).$$

The basic estimate (4.2) applied to  $w$  instead of  $u$  gives

$$\int_s^t |A^{\frac{1}{2}}u'(\sigma)|^2 d\sigma = \int_s^t |A^{\frac{\alpha}{2}}w'(\sigma)|^2 d\sigma \leq C_5 (|A^{1-\frac{\alpha}{2}}u(s)|^2 + |A^{\frac{1-\alpha}{2}}u'(s)|^2).$$

By integrating on  $(0, t)$  and using (4.5) we obtain

$$\int_0^t \int_s^t |A^{\frac{1}{2}}u'(\sigma)|^2 d\sigma ds \leq C_5 C_4 E$$

which by Fubini's theorem reduces to

$$\int_0^t \sigma |A^{\frac{1}{2}}u'(\sigma)|^2 d\sigma \leq C_5 C_4 E.$$

Hence for some  $K > 0$

$$(4.6) \quad \forall t > 0, \quad \int_0^t s |A^{\frac{1}{2}}u'(s)|^2 ds \leq K E.$$

We shall in fact establish for some  $M > 0$

$$(4.7) \quad \forall t > 0, \quad \int_0^t s (|A^{\frac{1}{2}}u'(s)|^2 + |u''(s)|^2) ds \leq M E.$$

To this end, first we observe that as a consequence of (4.6)

$$\inf_{t \leq s \leq 2t} s^2 |A^{\frac{1}{2}} u'(s)|^2 \leq 2t \inf_{t \leq s \leq 2t} s |A^{\frac{1}{2}} u'(s)|^2 \leq 2KE.$$

Now we choose  $\tau \in (t, 2t)$  for which  $u'(\tau) \in D(A^{1/2})$  with

$$\tau^2 |A^{\frac{1}{2}} u'(\tau)|^2 \leq 2KE$$

and we integrate on  $(0, \tau)$  after taking the inner product by  $su''(s)$ . We find

$$\int_0^\tau s |u''|^2 ds = - \int_0^\tau [s(Au, u'') + s(Bu', u'')] ds.$$

First we have

$$- \int_0^\tau s(Bu', u'') ds = - \frac{1}{2} \int_0^\tau s(Bu', u')' ds = - \left[ \frac{s}{2} (Bu', u') \right]_0^\tau + \int_0^\tau \frac{1}{2} (Bu', u') ds \leq \frac{1}{2} E.$$

Then

$$\begin{aligned} - \int_0^\tau s(Au, u'') ds &= - \int_0^\tau (sAu, (u')') ds = - (sAu, u')_0^\tau + \int_0^\tau ((sAu)', u') ds \\ &\leq \tau |A^{\frac{1}{2}} u(\tau)| |A^{\frac{1}{2}} u'(\tau)| + \int_0^\tau (Au, u') ds + \int_0^\tau s |A^{\frac{1}{2}} u'|^2 ds \\ &\leq \tau |A^{\frac{1}{2}} u(\tau)| |A^{\frac{1}{2}} u'(\tau)| + \frac{1}{2} |A^{\frac{1}{2}} u(\tau)|^2 + \int_0^\tau s |A^{\frac{1}{2}} u'|^2 ds \leq K'E. \end{aligned}$$

By adding we find

$$\int_0^\tau s |u''|^2 ds \leq K'' E$$

and together with (4.6) this provides (4.7) with  $t$  replaced by  $\tau$ . Since  $t \leq \tau$  we obtain (4.7). Finally since the function

$$t \rightarrow (|A^{\frac{1}{2}} u'(t)|^2 + |u''(t)|^2)$$

is nonincreasing, we have

$$(4.8) \quad \forall t > 0, \quad 2 \int_0^t s (|A^{\frac{1}{2}} u'(s)|^2 + |u''(s)|^2) ds \geq t^2 (|A^{\frac{1}{2}} u'(t)|^2 + |u''(t)|^2).$$

By combining (4.7) and (4.8), we obtain (4.1)  $\square$

**Theorem 4.2.** *For any  $\alpha < 1/2$ , the semi-group  $\mathcal{S}(t)$  generated on  $\mathcal{H}$  by  $\mathcal{A}$  satisfies*

$$\forall t > 0, \quad \mathcal{S}(t)U^0 \in D(\mathcal{A})$$

and

$$(4.9) \quad \forall t > 0, \quad \|\mathcal{A}\mathcal{S}(t)U^0\|_{\mathcal{H}} \leq \frac{C}{t^\beta} \|U^0\|_{\mathcal{H}}$$

with  $\beta = \frac{1}{2\alpha}$ . In addition if  $\mathcal{A}$  is unbounded with  $\mathcal{A}^{-1}$  compact, (4.9) is not satisfied for any  $\beta > \frac{1}{2\alpha}$ . In particular in this case the semi-group  $\mathcal{S}(t)$  is not analytic.

*Proof.* The beginning of proof of Theorem 4.1 applies until formula (4.3). Then taking the inner product in  $H$  of (2.1) by  $A^\alpha u(s)$  and integrating we find

$$\int_0^t |A^{\frac{1+\alpha}{2}} u(s)|^2 ds = - \int_0^t (u'' + Bu', A^\alpha u) ds.$$

Now we have since  $\alpha \leq 1/2$

$$\int_0^t (Bu', A^\alpha u) ds = c \int_0^t (A^\alpha u', A^\alpha u) ds = \frac{c}{2} (|A^{\alpha/2} u(t)|^2 - |A^{\alpha/2} u(0)|^2) \geq -C_1 E.$$

Next, integrating by parts we find

$$\int_0^t (u'', A^\alpha u) ds = [(u', A^\alpha u)]_0^t - \int_0^t (u', A^\alpha u') ds$$

hence since  $\alpha \leq 1/2$  and by using (4.3)

$$- \int_0^t (u'', A^\alpha u) ds \leq C_2 E$$

therefore

$$(4.10) \quad \int_0^t |A^{\frac{1+\alpha}{2}} u(s)|^2 ds \leq (C_1 + C_2) = C_3 E.$$

Combining (4.10) with (4.3) we deduce

$$(4.11) \quad \int_0^t (|A^{\frac{1}{2}+\frac{\alpha}{2}} u(s)|^2 + |A^{\frac{\alpha}{2}} u'(s)|^2) ds \leq C_4 E$$

from which we deduce

$$(4.12) \quad |A^{\frac{1}{2}+\frac{\alpha}{2}} u(t)| + |A^{\frac{\alpha}{2}} u'(t)| \leq \frac{C}{\sqrt{t}} (|A^{\frac{1}{2}} u(0)| + |A^0 u'(0)|).$$

Since time-translation and multiplication by  $A^\alpha$  commutes with the equation, replacing  $t$  by  $\frac{t}{n}$  and iterating (4.12)  $n$  times we deduce easily

$$(4.13) \quad |A^{\frac{1}{2}+\frac{n\alpha}{2}} u(t)| + |A^{\frac{n\alpha}{2}} u'(t)| \leq \frac{C(n)}{(\sqrt{t})^n} (|A^{\frac{1}{2}} u(0)| + |A^0 u'(0)|).$$

Now (4.13) is valid for integer values of  $n$  and by interpolation, we extend it easily for all real  $n > 0$ . Finally choosing  $n = \frac{1}{\alpha}$  we obtain (4.9).

In order to prove the optimality result, we set  $p = \frac{1}{\alpha}$ , we select  $\gamma > 0$  and for any  $\lambda > 0$  with  $\gamma\lambda^p - \lambda^2 > 0$  we set

$$\omega := \sqrt{\gamma\lambda^p - \lambda^2}; \quad y(t) = y_\lambda(t) = e^{-\lambda t} \cos \omega t.$$

Then

$$\begin{aligned} y'(t) &= -\lambda e^{-\lambda t} \cos \omega t - \omega e^{-\lambda t} \sin \omega t \\ y''(t) &= \lambda^2 e^{-\lambda t} \cos \omega t + 2\omega\lambda e^{-\lambda t} \sin \omega t - \omega^2 e^{-\lambda t} \cos \omega t \end{aligned}$$

and

$$y''(t) + 2\lambda y'(t) = -(\lambda^2 + \omega^2) e^{-\lambda t} \cos \omega t$$

so that  $y$  is a solution of

$$(4.14) \quad y'' + \gamma\lambda^p y + 2\lambda y' = 0$$

which satisfies

$$\lambda^p y^2(0) + y'^2(0) = \lambda^p + \lambda^2.$$



We are interested in the behavior of the energy for *large* values of  $\lambda$  and *small* values of  $t$  when  $\gamma \geq \gamma_0 > 0$ . We select

$$t := t(\lambda) = \pi \frac{[\frac{\omega}{\lambda}] + 1/2}{\omega}$$

where  $[\frac{\omega}{\lambda}]$  denotes the integer part of  $\frac{\omega}{\lambda}$ . As  $\lambda \rightarrow \infty$  we have

$$t(\lambda) \sim \frac{\pi}{\lambda}$$

and it follows

$$\begin{aligned} \lambda^p y'^2(t(\lambda)) &\sim \lambda^p \omega^2 e^{-2\pi} \\ t^p(\lambda) \lambda^p y'^2(t(\lambda)) &\sim \pi^p \omega^2 e^{-2\pi} \sim \pi^p e^{-2\pi} \gamma \lambda^p \end{aligned}$$

In particular for any  $\varepsilon > 0$  we have

$$\lim_{\lambda \rightarrow \infty} (t(\lambda))^{p-\varepsilon} \frac{\lambda^p y'_\lambda{}^2(t(\lambda)) + y''_\lambda{}^2(t(\lambda))}{\lambda^p y^2(0) + y'^2(0)} = \infty$$

uniformly for  $\gamma \geq \gamma_0 > 0$ .

Finally let  $\mu$  be a large eigenvalue of  $A$  with associated eigenfunction  $\phi_\mu$  and set

$$\gamma = \mu(c/2)^{-p}; \quad \lambda = \left(\frac{\mu}{\gamma}\right)^{1/p} \iff \gamma \lambda^p = \mu.$$

Since  $y = y(\lambda)$  is a solution of

$$y'' + \mu y + 2\gamma^{-1/p} \mu^{1/p} y' = y'' + \mu y + c\mu^\alpha y' = 0.$$

It is now easy to see that  $u(t) := y(t)\phi_\mu$  is a solution of (2.1) which does not satisfy (4.4) for any  $\beta > \frac{1}{2\alpha}$ .  $\square$

Theorems 4.1 and 4.2 imply a stronger time-smoothing effect property. More precisely we have

**Theorem 4.3.** *For any  $\alpha > 0$ , and for any solution  $u$  of*

$$u'' + Au + cA^\alpha u' = 0$$

*we have*

$$\forall \delta > 0, u \in C^\infty([\delta, \infty), V)$$

*In addition the operator*

$$(u(0), u'(0)) \in V \times H \rightarrow u^{(k)} \in L^\infty[\delta, \infty), V)$$

*is bounded for each fixed value of  $k$ .*

## 5. EXISTENCE OF A SPATIAL SMOOTHING EFFECT

Combining the result of Theorem 4.3 and the inclusion  $D(\mathcal{A}) \subset H^\gamma \times V$  obtained in Theorem 3.1 for  $0 < \alpha < 1$ , by an easy induction argument we obtain

**Theorem 5.1.** *Assuming  $\alpha < 1$ , for any solution  $u$  of*

$$u'' + Au + cA^\alpha u' = 0$$

*we have*

$$\forall n \in \mathbb{N}, \quad \forall \delta > 0, u \in C^\infty([\delta, \infty), D(A^n))$$

In addition the operator

$$(u(0), u'(0)) \in V \times H \rightarrow u^{(k)} \in L^\infty([\delta, \infty), D(A^n))$$

is bounded for each fixed value of  $k$  and  $n$ .

**Remark.** If on the other hand  $\alpha \geq 1$  there is no spatial smoothing effect anymore. For instance if we consider the special case

$$\Omega = (0, \pi) \quad H = L^2(\Omega) \quad V = H_0^1(\Omega) \quad A = -\Delta, \quad B = 2A$$

we have special solutions of the form

$$u(t, x) = \sum a_n e^{(-n^2 + \sqrt{n^2(n^2-1)})t} \sin nx$$

with

$$u(0) = \sum a_n \sin nx$$

and

$$\begin{aligned} \|u(0)\|_{D(A^s)}^2 &= \sum n^{4s} a_n^2 \\ \|u(t)\|_{D(A^s)}^2 &= \sum n^{4s} a_n^2 e^{(-n^2 + \sqrt{n^2(n^2-1)})2t}. \end{aligned}$$

Since

$$-n^2 + \sqrt{n^2(n^2-1)} = \frac{-n^2}{n^2 + \sqrt{n^2(n^2-1)}} \geq -1$$

we find

$$\|u(0)\|_{D(A^s)}^2 \leq \|u(t)\|_{D(A^s)}^2 e^{2t}$$

This formal estimate can be easily worked out to show that if  $u(t) \in D(A^s)$  for some  $t > 0$ , we must have  $u(0) \in D(A^s)$ . Hence there is no spatial smoothing effect whatsoever in such a situation.

## 6. REGULARITY: CONSERVATION PROPERTIES

Let  $L = L^* \geq 0$  on  $H$  and assume that there is a second Banach space  $X$  such that

$$(6.1) \quad \bigcap_{n \geq 1} D(L^n) \subset X \subset H$$

with dense imbeddings. The norm in  $X$  is denoted by  $\|\cdot\|$  and we assume that  $\exp(-tL)$  is a  $C^0$  semigroup of bounded operators on  $X$ .

First we consider the problem

$$(6.2) \quad u'' + L^2 u + cLu' = 0$$

Then we have

**Theorem 6.1.** *Let  $c \geq 2$ , and let  $L$  be coercive on  $X$ . Introducing*

$$D_X(L) = \{x \in D(L), Lx \in X\}$$

*assume that the set  $\{x \in D_X(L), \|Lx\| \leq 1\}$  is closed in  $X$  and that  $\exp(-tL)$  is an analytic semigroup of bounded operators on  $X$ . Let  $u$  be the unique solution of (6.2) satisfying*

$$u(0) = u^0 \in D_X(L), \quad u'(0) = u^1 \in X$$

whose existence is insured by Theorem 2.1 with  $A = L^2$  and  $B = cL$ . Then we have

$$\forall t > 0, \quad u(t) \in D_X(L), \quad u'(t) \in X$$

with

$$(6.3) \quad \sup_{t>0} \{\|u(t)\| + \|Lu(t)\| + \|u'(t)\|\} \leq C(\|u^0\| + \|Lu^0\| + \|u^1\|)$$

*Proof.* We start with

$$(u^0, u^1) \in \left(\bigcap_{n \geq 1} D(L^n)\right)^2$$

The main idea is to look for  $\alpha, \beta > 0$  such that

$$u'' + L^2u + cLu' = (u' + \alpha Lu)' + \beta L(u' + \alpha Lu).$$

This identity reduces to the system

$$\alpha + \beta = c; \quad \alpha\beta = 1$$

which has real solutions  $(\alpha, \beta)$  as a consequence of the assumption  $c \geq 2$ . Then since  $v = u' + \alpha Lu$  is a solution of

$$v' + \beta Lv = 0$$

we have (cf. e.g. [6]) with  $K = \|u^1\| + \alpha\|Lu^0\|$

$$\forall t > 0, \quad \|v(t)\|_{D_X(L^{1/2})} \leq C_1 t^{-1/2} \|u^1 + \alpha Lu^0\| \leq C_1 t^{-1/2} K.$$

Then we have

$$u(t) = e^{-\alpha L t} u^0 + \int_0^t e^{-\alpha L(t-s)} v(s) ds$$

hence by [6]

$$\forall t > 0, \quad \|u(t)\|_{D_X(L)} \leq C_2 \|u^0\|_{D_X(L)} + C_3 \int_0^t C_1 (t-s)^{-1/2} s^{-1/2} K ds.$$

This provides the bound on  $\|u(t)\| + \|Lu(t)\|$ . Finally since

$$\forall t > 0, \quad \|v(t)\| \leq K$$

and

$$u'(t) = v(t) - \alpha Lu(t)$$

the estimate on  $\|u'(t)\|$  follows easily. Then the general case

$$(u^0, u^1) \in D_X(L) \times X$$

follows by a density argument. □

We next consider the problem

$$(6.4) \quad u'' + Lu + cLu' = 0.$$

We assume that  $\exp(-tL)$  is a  $C^0$  semigroup of bounded operators on  $X$ . Then we have

**Theorem 6.2.** *Under the condition (6.1), Let  $u$  be the unique solution of (6.4) satisfying*

$$u(0) = u^0 \in X, \quad u'(0) = u^1 \in X$$

*whose existence is insured by Theorem 2.1 with  $A = L$  and  $B = cL$ . Then we have*

$$\forall t > 0, \quad u(t) \in X \quad u'(t) \in X$$

*If in addition we assume*

$$(6.5) \quad \|\exp(-tL)\|_{\mathcal{L}(X)} \leq Me^{-\lambda t}$$

*for some  $M > 1, \lambda > 0$ . such that*

$$(6.6) \quad c^2\lambda > M + 1$$

*then  $(u, u')$  is bounded for  $t \geq 0$  and*

$$(6.7) \quad \sup_{t>0} \{\|u(t)\| + \|u'(t)\|\} \leq C(c, M, \lambda)(\|u^0\| + \|u^1\|)$$

.

*Proof.* As before we consider first

$$(u^0, u^1) \in \left(\bigcap_{n \geq 1} D(L^n)\right)^2$$

The main idea is to introduce

$$v := u' + \frac{1}{c}u$$

so that

$$v' + cLv = u'' + \frac{1}{c}u' + cLu' + Lu = \frac{1}{c}u' = \frac{1}{c}(v - \frac{1}{c}u)$$

Let  $J$  be any closed subinterval of  $[0, +\infty)$ . By setting

$$\|u\|_{J, \infty} := \sup_{t \in J} \|u(t)\|$$

and similarly

$$\|v\|_{J, \infty} := \sup_{t \in J} \|v(t)\|$$

from

$$v' + cLv - \frac{1}{c}v = -\frac{1}{c^2}u$$

we obtain

$$\forall t \in J, \quad \|v(t)\| \leq M \exp\left(\frac{1}{c}t\right)\|v^0\| + \frac{M}{c^2} \exp\left(\frac{1}{c}t\right)|J|\|u\|_{J, \infty} \leq C(\|v^0\| + |J|\|u\|_{J, \infty})$$

for  $|J| \leq 1$  and

$$\forall t \in J, \quad \|u(t)\| \leq \|u^0\| + c\|v\|_{J, \infty}$$

The result easily follows locally by selecting  $|J|$  small enough, for instance  $Cc|J| \leq \frac{1}{2}$  and using a density argument. Then a simple induction argument concludes the proof, since the condition on  $|J|$  is independent of the initial data and the equation is autonomous.

When (6.5)-(6.6) are assumed, the same calculation for an arbitray  $J$  now gives

$$\forall t \in J, \quad \|v(t)\| \leq M \exp[-(\lambda c - \frac{1}{c})t] \|v_0\| + \frac{M}{\lambda c^3 - c} \|u\|_\infty$$

and

$$\forall t \in J, \quad \|u(t)\| \leq \|u^0\| + c\|v\|_\infty$$

Hence if  $\lambda c^2 > 1$  we find

$$\|v\|_\infty \leq M\|v_0\| + \frac{M}{\lambda c^3 - c} \|u\|_\infty \leq M\|v_0\| + \frac{M}{\lambda c^3 - c} (\|u^0\| + c\|v\|_\infty)$$

which gives the desired bound for  $v$  if

$$\frac{M}{\lambda c^2 - 1} < 1$$

a condition equivalent to (6.6). Then the bound on  $u$  follows automatically. This bound is valid on each interval  $J = [0, T]$  and the result (6.7) follows. The general case is obtained by density.  $\square$

## 7. MAIN EXAMPLES

**Example 7.1.** Let  $\Omega$  be a bounded open subset of  $\mathbb{R}^N$ . We consider the problem

$$(7.1) \quad \begin{cases} u_{tt} - \Delta u - c\Delta u_t = 0 & \text{on } \mathbb{R}^+ \times \Omega \\ u = \Delta u = 0 & \text{on } \mathbb{R}^+ \times \partial\Omega \\ u(0) = u^0 \quad u'(0) = u^1 \end{cases}$$

We set

$$H = L^2(\Omega) \quad ; \quad V = H_0^1(\Omega) \\ X = C_0(\Omega) = \{u \in C(\bar{\Omega}), \quad u = 0 \text{ on } \partial\Omega\}$$

For any  $(u^0, u^1) \in V \times H$ , the unique mild solution  $u$  of (7.1) belongs to  $C^\infty((0, \infty), V)$ . If in addition we assume  $(u^0, u^1) \in X \times X$ , then for all  $t \geq 0$   $(u(t, \cdot), u_t(t, \cdot)) \in X \times X$ . For  $c$  large enough we have the stronger property  $u \in W^{1, \infty}((0, \infty), X)$ . The same result is also valid if  $X$  is replaced by  $X_p = L^p(\Omega)$  for any  $p \in [2, \infty)$ .

**Example 7.2.** Let  $\Omega$  be a bounded open subset of  $\mathbb{R}^N$ . We consider the problem

$$(7.2) \quad \begin{cases} u_{tt} + \Delta^2 u - c\Delta u_t = 0 & \text{on } \mathbb{R}^+ \times \Omega \\ u = \Delta u = 0 & \text{on } \mathbb{R}^+ \times \partial\Omega \\ u(0) = u^0 \quad u'(0) = u^1 \end{cases}$$

For any  $(u^0, u^1) \in V \times H$  we set

$$H = L^2(\Omega) \quad ; \quad V = \{u \in H_0^1(\Omega), \Delta u \in H\} \\ X = C_0(\Omega) = \{u \in C(\bar{\Omega}), \quad u = 0 \text{ on } \partial\Omega\}$$

For any  $(u^0, u^1) \in V \times H$ , the unique mild solution  $u$  of (7.1) belongs to  $C^\infty((0, \infty), V) \cap C^\infty((0, \infty) \times \Omega)$ . If in addition we assume  $c \geq 2$  and  $(u^0, u^1) \in X \times X$ , then  $u \in L^\infty((0, \infty), X)$ . Finally if  $c \geq 2$ ,  $(u^0, u^1) \in X \times X$

and  $\Delta^2 u^0 \in X$ , then  $\Delta^2 u \in L^\infty((0, \infty), X)$ ;  $u_t \in L^\infty((0, \infty), X)$ . The same result is also valid if  $X$  is replaced by  $X_p = L^p(\Omega)$  for any  $p \in [2, \infty)$ .

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