Two-scale numerical simulation of the weakly compressible 1D isentropic Euler equations

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1 Introduction

This paper enters a work programme concerning the development of two-scale numerical methods to solve PDEs with oscillatory singular perturbations linked to physical phenomena. Recently, these methods have been tested on simple problems. For instance, in Ailliot, Frénod and Monbet[1], such a method is used to manage the tide oscillation for long term drift forecast of objects in the ocean; in Frénod, Salvarani and Sonnendrücker[5], such a method is used to simulate a charged particle beam in a periodic focusing channel.

The ultimate goal of this work programme is to propose efficient numerical methods to simulate plasmas submitted to strong magnetic field. Of course, simulations of magnetic confinement fusion are part of this ultimate goal.

Before going further towards this ultimate goal, several questions concerning the behaviour of the concerned methods in front of non-linearities and non-smooth solutions need to be investigated. The field of weakly compressible 1D isentropic Euler equations offers a relatively confortable framework in order to tackle these questions.

Indeed, the mathematical context established in Grenier[9], Klainerman and Majda[10, 11], Fortenbach, Frénod, Munz and Sonnendrücker[3], Frénod, Raviart and Sonnendrücker[4], Frénod and Sonnendrücker[6, 7], Majda[14], Métivier and Schochet[16], Munz[17], Schochet[20, 21, 22, 23], offers a nice marked out way. This is the first motivation of the present paper.

A second motivation originates from the fact that robust numerical methods such as finite volumes set out in Godunov[8], LeVeque[12] and Roe[19] have been developed in order to solve compressible or incompressible Euler equations. Nevertheless, when they are used to simulate the flow in an experiment with conditions inducing a small Mach number, the CPU time cost of these methods is too high for operational applications because of the very small time step required in order to capture high speed travelling waves that appear in this case.

The precise aim of this paper is to develop a two-scale numerical method to solve

$$\partial_t u^\epsilon + \frac{1}{2}\, \partial_x\big((u^\epsilon)^2\big) + \partial_x\big(q^\epsilon(\rho^\epsilon)\big) + \frac{1}{\epsilon}\, \partial_x \rho^\epsilon = 0\,,$$

$$\partial_t \rho^\epsilon + \partial_x(\rho^\epsilon u^\epsilon) + \frac{1}{\epsilon}\, \partial_x u^\epsilon = 0\,,$$

$$u^\epsilon_{|t=0} = u_0\,,$$

$$\rho^\epsilon_{|t=0} = \rho_0\,, \tag{1.1}$$

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where $(x, t) \in \mathbb{T}^1 \times [0, T)$. In this model, ϵ is the Mach number, $u^\epsilon = u^\epsilon(x, t)$ is the dimensionless speed of the fluid, $\rho^\epsilon = \rho^\epsilon(x, t)$ is linked to the dimensionless density $\tilde{\rho}^\epsilon = \tilde{\rho}^\epsilon(x, t)$ by the relation

$$\rho^\epsilon = \frac{\tilde{\rho}^\epsilon - 1}{\epsilon}, \tag{1.2}$$

and

$$q^\epsilon(\rho^\epsilon) = \frac{\gamma - 2}{2} (\rho^\epsilon)^2 + \epsilon \, q_1^\epsilon(\rho^\epsilon), \tag{1.3}$$

where γ is the adiabatic coefficient of the fluid and $q_1^\epsilon(\rho^\epsilon)$ is regular. This form of the isentropic Euler equations is set out in Grenier[9] (p. 494) and Métivier and Schochet[16]. Equation (1.1) is obtained from the classical form of the 1D isentropic Euler equations

$$\begin{aligned}
\partial_t u^\epsilon + u^\epsilon \, \partial_x u^\epsilon + \frac{1}{\epsilon^2 \, \tilde{\rho}^\epsilon} \, \partial_x \big(p(\tilde{\rho}^\epsilon)\big) &= 0, \\
\partial_t \tilde{\rho}^\epsilon + \partial_x \big(\tilde{\rho}^\epsilon u^\epsilon\big) &= 0, \\
u^\epsilon(x, 0) &= u_0(x), \\
\tilde{\rho}^\epsilon(x, 0) &= 1 + \epsilon \, \rho_0(x),
\end{aligned} \tag{1.4}$$

where usually, pressure function is given by $p(\tilde{\rho}^\epsilon) = \frac{(\tilde{\rho}^\epsilon)^\gamma}{\gamma}$. Using the relation (1.2) between ρ^ϵ and $\tilde{\rho}^\epsilon$, we obtain

$$\frac{1}{\epsilon^2 \, \tilde{\rho}^\epsilon} \, \partial_x \big(p(\tilde{\rho}^\epsilon)\big) = \frac{1}{\epsilon^2 \, (1 + \epsilon \, \rho^\epsilon)} \, \partial_x \Big(\frac{(1 + \epsilon \, \rho^\epsilon)^\gamma}{\gamma}\Big) = \frac{(1 + \epsilon \, \rho^\epsilon)^{\gamma - 2}}{\epsilon} \, \partial_x \rho^\epsilon. \tag{1.5}$$

Making and expansion of $(1 + \epsilon \, \rho^\epsilon)^{\gamma - 2}$, we obtain

$$\frac{1}{\epsilon^2 \, \tilde{\rho}^\epsilon} \, \partial_x \big(p(\tilde{\rho}^\epsilon)\big) = \frac{1}{\epsilon} \, (\partial_x \rho^\epsilon) \big(1 + (\gamma - 2) \, \epsilon \, \rho^\epsilon + \epsilon^2 \, q_1^\epsilon(\rho^\epsilon)\big), \tag{1.6}$$

where $q_1^\epsilon(\rho^\epsilon)$ is regular, and, by introducing this result in the first equation of (1.4), we deduce the first equation of (1.1). In order to obtain the second equation of (1.1), we introduce the relation (1.2) in the second equation of (1.4).

In order to achieve this precise aim, inspired by two-scale convergence theory developed in Nguetseng[18] and Allaire[2], we establish a homogenized model describing the mean behaviour of $(u^\epsilon, \rho^\epsilon)$. This homogenized model neither contains nor generates high frequency oscillations but only their mean action. We also establish a way to reconstruct the oscillating solution $(u^\epsilon, \rho^\epsilon)$ from obtained mean behaviour. The homogenized model and the reconstruction procedure involve an additional variable which is a copy of the time variable and which allows the transfer of informations between the mean behaviour and the high frequency oscillations.

2   Results

In this section, we claim the mathematical results leading to the construction of our two-scale numerical method.

After recalling existence and properties of solutions of (1.1), we give a first theorem giving the homogenized model and convergence properties. Then, we set out a numerical approximation for this homogenized model based on Roe's finite volume method which

is the two-scale numerical method that allows to solve (1.1). Finally, we give a theorem giving some properties about the convergence of the considered numerical scheme.

It is well known (see Grenier[9], Klainerman and Majda[10], Majda[14], Metivier and Schochet[16], or Schochet[20, 22, 23]) that, with initial data satisfying

$$u_0, \rho_0 \in H^s(\mathbb{T}^1) \quad \text{with} \quad s > \frac{3}{2}, \tag{2.1}$$

there exists an existence time $T > 0$, independent of ϵ, such that the system (1.1) admits a unique solution $(u^\epsilon, \rho^\epsilon)$ in $\big(\mathcal{C}\big(0, T; H^s(\mathbb{T}^1)\big)\big)^2 \cap \big(\mathcal{C}^1\big(0, T; H^{s-1}(\mathbb{T}^1)\big)\big)^2$ for any $\epsilon > 0$. Furthermore, the sequence $(u^\epsilon, \rho^\epsilon)_{\epsilon > 0}$ is bounded in $\big(\mathcal{C}\big(0, T; H^s(\mathbb{T}^1)\big)\big)^2 \cap \big(\mathcal{C}^1\big(0, T; H^{s-1}(\mathbb{T}^1)\big)\big)^2$ for the usual norm.

Having this existence result at hand, we can claim the following theorem giving the homogenized model.

Theorem 1. *If we fix $\epsilon_0 > 0$ and $s \geq 2$, there exists a constant $K > 0$ such that, for any $\epsilon \in]0, \epsilon_0]$, we have*

$$\left(\int_0^T \left\| u^\epsilon(\cdot, t) - U\big(\cdot, \frac{t}{\epsilon}, t\big) \right\|_{L^2(\mathbb{T}^1)}^2 dt \right)^{\frac{1}{2}} \leq K\epsilon,$$

$$\left(\int_0^T \left\| \rho^\epsilon(\cdot, t) - R\big(\cdot, \frac{t}{\epsilon}, t\big) \right\|_{L^2(\mathbb{T}^1)}^2 dt \right)^{\frac{1}{2}} \leq K\epsilon, \tag{2.2}$$

where U and R are defined by

$$U(x, \tau, t) = F(x - \tau, t) + B(x + \tau, t) + \frac{\overline{u}}{2\pi},$$

$$R(x, \tau, t) = F(x - \tau, t) - B(x + \tau, t) + \frac{\overline{\rho}}{2\pi}, \tag{2.3}$$

with $\overline{u} = \int_{\mathbb{T}^1} u_0(x)\, dx$, $\overline{\rho} = \int_{\mathbb{T}^1} \rho_0(x)\, dx$, and $F, B \in \mathcal{C}\big(0, T; H^s(\mathbb{T}^1)\big)$ satisfying

$$\int_{T^1} F(x, t)\, dx = \int_{\mathbb{T}^1} B(x, t)\, dx = 0 \qquad \forall\, t \in [0, T), \tag{2.4}$$

and

$$\partial_t F + \partial_x \Big(\frac{\gamma + 1}{4} F^2 + \frac{2\overline{u} + (\gamma - 1)\overline{\rho}}{4\pi} F \Big) = 0,$$

$$\partial_t B + \partial_x \Big(\frac{\gamma + 1}{4} B^2 + \frac{2\overline{u} - (\gamma - 1)\overline{\rho}}{4\pi} B \Big) = 0,$$

$$F_{|t=0} = \frac{1}{2} \Big(u_0 + \rho_0 - \frac{\overline{u} + \overline{\rho}}{2\pi} \Big),$$

$$B_{|t=0} = \frac{1}{2} \Big(u_0 - \rho_0 - \frac{\overline{u} - \overline{\rho}}{2\pi} \Big). \tag{2.5}$$

A way to interpret this theorem is that if we can compute some good approximations F_h and B_h of the solutions F and B of (2.5), we are able to reconstruct some functions U_h and R_h in the same way as we reconstruct U and R in (2.3). Then the obtained functions $(x,t) \mapsto U_h(x, \frac{t}{\epsilon}, t)$ and $(x,t) \mapsto R_h(x, \frac{t}{\epsilon}, t)$ are good approximations of $(x,t) \mapsto u^\epsilon(x,t)$ and $(x,t) \mapsto \rho^\epsilon(x,t)$ respectively.

Based on this idea, we build our two-scale numerical method: firstly, we consider a uniform mesh on $\mathbb{T}^1 \times [0,T]$ with space step h and time step k, and we use the notations $x_i = ih$ and $t_n = nk$. Then we build F_h and B_h such that

$$F_h(x,t) = F_i^n \text{ and } B_h(x,t) = B_i^n \quad \forall (x,t) \in [x_{i-1/2}, x_{i+1/2}[\times[t_n, t_{n+1}[, \qquad (2.6)$$

where F_i^n and B_i^n are computed with Roe's finite volume approximation of (2.5):

$$\begin{aligned}
F_i^{n+1} = F_i^n - \frac{k}{2h}\Bigg(&\frac{\gamma+1}{4}\big((F_{i+1}^n)^2 - (F_{i-1}^n)^2\big) + \frac{2\overline{u} + (\gamma-1)\overline{\rho}}{4\pi}(F_{i+1}^n - F_{i-1}^n) \\
&- \left|\frac{\gamma+1}{4}(F_{i+1}^n + F_i^n) + \frac{2\overline{u} + (\gamma-1)\overline{\rho}}{4\pi}\right|(F_{i+1}^n - F_i^n) \\
&+ \left|\frac{\gamma+1}{4}(F_i^n + F_{i-1}^n) + \frac{2\overline{u} + (\gamma-1)\overline{\rho}}{4\pi}\right|(F_i^n - F_{i-1}^n) \Bigg),
\end{aligned} \qquad (2.7)$$

$$\begin{aligned}
B_i^{n+1} = B_i^n - \frac{k}{2h}\Bigg(&\frac{\gamma+1}{4}\big((B_{i+1}^n)^2 - (B_{i-1}^n)^2\big) + \frac{2\overline{u} - (\gamma-1)\overline{\rho}}{4\pi}(B_{i+1}^n - B_{i-1}^n) \\
&- \left|\frac{\gamma+1}{4}(B_{i+1}^n + B_i^n) + \frac{2\overline{u} - (\gamma-1)\overline{\rho}}{4\pi}\right|(B_{i+1}^n - B_i^n) \\
&+ \left|\frac{\gamma+1}{4}(B_i^n + B_{i-1}^n) + \frac{2\overline{u} - (\gamma-1)\overline{\rho}}{4\pi}\right|(B_i^n - B_{i-1}^n) \Bigg).
\end{aligned} \qquad (2.8)$$

Finally, we compute U_h and R_h by

$$\begin{aligned}
U_h(x, \tau, t) &= F_h(x - \tau, t) + B_h(x + \tau, t) + \frac{\overline{u}}{2\pi}, \\
R_h(x, \tau, t) &= F_h(x - \tau, t) - B_h(x + \tau, t) + \frac{\overline{\rho}}{2\pi},
\end{aligned} \qquad (2.9)$$

and we have the following convergence result.

Theorem 2. *If $s \geq 2$, then the approximations $U_h(\cdot, \tau, \cdot)$ and $R_h(\cdot, \tau, \cdot)$ converge to $U(\cdot, \tau, \cdot)$ and $R(\cdot, \tau, \cdot)$ in $L^1([0,T] \times \mathbb{T}^1)$ for any $\tau \in \mathbb{T}^1$. Furthermore, if $s \geq 3$, the local truncation errors of the numerical scheme (2.6)-(2.7)-(2.8) are first order accurate.*

Remark 3. *The proof of the convergence result (2.2) and of Theorem 2 do not work if $s \in]\frac{3}{2}, 2[$. However, in this case, the regularity is enough to get*

$$\begin{aligned}
u^\epsilon - U(\cdot, \frac{\cdot}{\epsilon}, \cdot) &\rightharpoonup 0 \quad weakly - *, \\
\rho^\epsilon - R(\cdot, \frac{\cdot}{\epsilon}, \cdot) &\rightharpoonup 0 \quad weakly - *,
\end{aligned} \qquad (2.10)$$

in $L^\infty(0, T; H^s(\mathbb{T}^1))$ when $\epsilon \to 0$, with U and R solution of (2.3)-(2.5).

3 Construction of the two-scale numerical method

3.1 Homogenized model

We present here the construction of the model (2.5). We recall that we start from the weakly compressible 1D isentropic Euler equations (1.1)-(1.3). Introducing the functions f^ϵ, b^ϵ defined by

$$f^\epsilon(x,t) = \frac{1}{2}\left(u^\epsilon\left(x + \frac{t}{\epsilon}, t\right) + \rho^\epsilon\left(x + \frac{t}{\epsilon}, t\right) - \frac{\overline{u} + \overline{\rho}}{2\pi}\right),$$

$$b^\epsilon(x,t) = \frac{1}{2}\left(u^\epsilon\left(x + \frac{t}{\epsilon}, t\right) - \rho^\epsilon\left(x + \frac{t}{\epsilon}, t\right) - \frac{\overline{u} - \overline{\rho}}{2\pi}\right),$$

(3.1)

we can rewrite the system (1.1):

$$\partial_t f^\epsilon(x,t) + \frac{\epsilon}{2}\,\partial_x\left(q_1^\epsilon\left(f^\epsilon(x,t) - b^\epsilon\left(x + \frac{2t}{\epsilon}, t\right) + \frac{\overline{\rho}}{2\pi}\right)\right)$$

$$+ \partial_x\left(\frac{\gamma + 1}{4}\left(f^\epsilon(x,t)\right)^2 + \frac{\gamma - 3}{4}\left(b^\epsilon\left(x + \frac{2t}{\epsilon}, t\right)\right)^2 + \frac{2\overline{u} + (\gamma - 1)\overline{\rho}}{4\pi}\,f^\epsilon(x,t)\right.$$

$$\left. + \frac{(3 - \gamma)\overline{\rho}}{4\pi}\,b^\epsilon\left(x + \frac{2t}{\epsilon}, t\right) + \frac{3 - \gamma}{2}\,f^\epsilon(x,t)b^\epsilon\left(x + \frac{2t}{\epsilon}, t\right)\right) = 0\,,$$

(3.2)

$$\partial_t b^\epsilon(x,t) + \frac{\epsilon}{2}\,\partial_x\left(q_1^\epsilon\left(f^\epsilon\left(x - \frac{2t}{\epsilon}, t\right) - b^\epsilon(x,t) + \frac{\overline{\rho}}{2\pi}\right)\right)$$

$$+ \partial_x\left(\frac{\gamma + 1}{4}\left(b^\epsilon(x,t)\right)^2 + \frac{\gamma - 3}{4}\left(f^\epsilon\left(x - \frac{2t}{\epsilon}, t\right)\right)^2 + \frac{2\overline{u} - (\gamma - 1)\overline{\rho}}{4\pi}\,b^\epsilon(x,t)\right.$$

$$\left. - \frac{(3 - \gamma)\overline{\rho}}{4\pi}\,f^\epsilon\left(x - \frac{2t}{\epsilon}, t\right) + \frac{3 - \gamma}{2}\,f^\epsilon\left(x - \frac{2t}{\epsilon}, t\right)b^\epsilon(x,t)\right) = 0\,,$$

(3.3)

equipped with

$$f^\epsilon_{|t=0} = f_0 = \frac{1}{2}\left(u_0 + \rho_0 - \frac{\overline{u} + \overline{\rho}}{2\pi}\right),$$

(3.4)

$$b^\epsilon_{|t=0} = b_0 = \frac{1}{2}\left(u_0 - \rho_0 - \frac{\overline{u} - \overline{\rho}}{2\pi}\right).$$

(3.5)

Since the sequences $(f^\epsilon)_{\epsilon>0}$ and $(b^\epsilon)_{\epsilon>0}$ are bounded in $L^\infty\left(0, T; H^s(\mathbb{T}^1)\right)$, there exist two functions F and B in $L^\infty\left(0, T; H^s(\mathbb{T}^1)\right)$ such that for some subsequences always denoted $(f^\epsilon)_{\epsilon>0}$ and $(b^\epsilon)_{\epsilon>0}$, we have

$$f^\epsilon \rightharpoonup F \quad \text{weakly-* in } L^\infty\left(0, T; H^s(\mathbb{T}^1)\right),$$

(3.6)

$$b^\epsilon \rightharpoonup B \quad \text{weakly-* in } L^\infty\left(0, T; H^s(\mathbb{T}^1)\right),$$

(3.7)

for $\epsilon \to 0$.

A first property of F and B is that their averages are equal to 0: to show this, we integrate (3.2)-(3.3) with respect to x to obtain

$$\partial_t \left(\int_{\mathbb{T}^1} f^\epsilon(x,t)\,dx \right) = \partial_t \left(\int_{\mathbb{T}^1} b^\epsilon(x,t)\,dx \right) = 0\,.$$

Then we deduce that, for all $t \in [0,T)$, we have

$$\int_{\mathbb{T}^1} f^\epsilon\,dx = \int_{\mathbb{T}^1} f_0\,dx = 0\,, \quad \int_{\mathbb{T}^1} b^\epsilon\,dx = \int_{\mathbb{T}^1} b_0\,dx = 0\,. \tag{3.8}$$

These results, combined with the fact that f^ϵ and b^ϵ weakly-* converge to F and B respectively, lead to the results (2.4).

Furthermore, usual compactness results (for example Lions[13]) yield that the functional space

$$\mathcal{U} = \left\{ g \in L^\infty\big(0,T; H^s(\mathbb{T}^1)\big) \,:\, \partial_t g \in L^\infty\big(0,T; H^{s-1}(\mathbb{T}^1)\big) \right\}, \tag{3.9}$$

provided with the usual product norm, is compactly embedded in $L^\infty\big(0,T; H^{s-1}(\mathbb{T}^1)\big)$. As a consequence, since $(f^\epsilon)_{\epsilon>0}$ and $(b^\epsilon)_{\epsilon>0}$ are bounded in $\mathcal{U}$, we have

$$f^\epsilon \xrightarrow{\epsilon \to 0} F \quad \text{strongly in } L^\infty\big(0,T; H^{s-1}(\mathbb{T}^1)\big)\,, \tag{3.10}$$

$$b^\epsilon \xrightarrow{\epsilon \to 0} B \quad \text{strongly in } L^\infty\big(0,T; H^{s-1}(\mathbb{T}^1)\big)\,. \tag{3.11}$$

Since f^ϵ and b^ϵ are continuous in t, we can remark that $F, B \in \mathcal{C}\big(0,T; H^{s-1}(\mathbb{T}^1)\big)$, so the convergence result (3.10)-(3.11) is also true in $\mathcal{C}\big(0,T; H^{s-1}(\mathbb{T}^1)\big)$. Having now this convergence result at hand, we can look for the constraint equations on F and B. For that, we multiply the equations (3.2)-(3.3) by a regular function φ with compact support on $\mathbb{T}^1 \times [0,T)$ and we integrate with respect to x and t on $\mathbb{T}^1 \times [0,T)$. We obtain

$$
\begin{aligned}
-\int_0^T &\int_{\mathbb{T}^1} f^\epsilon(x,t)\,\partial_t\varphi(x,t)\,dx\,dt - \int_{\mathbb{T}^1} f_0(x)\,\varphi(x,0)\,dx \\
&- \int_0^T \int_{\mathbb{T}^1} \frac{\gamma+1}{4} \big(f^\epsilon(x,t)\big)^2\, \partial_x\varphi(x,t)\,dx\,dt \\
&- \int_0^T \int_{\mathbb{T}^1} \frac{2\overline{u}+(\gamma-1)\overline{\rho}}{4\pi}\, f^\epsilon(x,t)\, \partial_x\varphi(x,t)\,dx\,dt \\
&- \int_0^T \int_{\mathbb{T}^1} \frac{\gamma-3}{4} \left(b^\epsilon\big(x+\tfrac{2t}{\epsilon},t\big)\right)^2 \partial_x\varphi(x,t)\,dx\,dt \\
&- \int_0^T \int_{\mathbb{T}^1} \frac{(3-\gamma)\overline{\rho}}{4\pi}\, b^\epsilon\big(x+\tfrac{2t}{\epsilon},t\big)\, \partial_x\varphi(x,t)\,dx\,dt \\
&- \int_0^T \int_{\mathbb{T}^1} \frac{3-\gamma}{2}\, f^\epsilon(x,t)\, b^\epsilon\big(x+\tfrac{2t}{\epsilon},t\big)\, \partial_x\varphi(x,t)\,dx\,dt \\
&- \frac{\epsilon}{2} \int_0^T \int_{\mathbb{T}^1} q_1^\epsilon \left(f^\epsilon(x,t) - b^\epsilon\big(x+\tfrac{2t}{\epsilon},t\big) + \frac{\overline{\rho}}{2\pi} \right) \partial_x\varphi(x,t)\,dx\,dt = 0\,,
\end{aligned}
\tag{3.12}
$$

and

$$-\int_0^T \int_{\mathbb{T}^1} b^\epsilon(x,t)\,\partial_t\varphi(x,t)\,dx\,dt - \int_{\mathbb{T}^1} b_0(x)\,\varphi(x,0)\,dx$$

$$-\int_0^T \int_{\mathbb{T}^1} \frac{\gamma+1}{4}\big(b^\epsilon(x,t)\big)^2\,\partial_x\varphi(x,t)\,dx\,dt$$

$$-\int_0^T \int_{\mathbb{T}^1} \frac{2\overline{u}-(\gamma-1)\overline{\rho}}{4\pi}\,b^\epsilon(x,t)\,\partial_x\varphi(x,t)\,dx\,dt$$

$$-\int_0^T \int_{\mathbb{T}^1} \frac{\gamma-3}{4}\Big(f^\epsilon\big(x-\frac{2t}{\epsilon},t\big)\Big)^2\,\partial_x\varphi(x,t)\,dx\,dt \qquad (3.13)$$

$$+\int_0^T \int_{\mathbb{T}^1} \frac{(3-\gamma)\overline{\rho}}{4\pi}\,f^\epsilon\big(x-\frac{2t}{\epsilon},t\big)\,\partial_x\varphi(x,t)\,dx\,dt$$

$$-\int_0^T \int_{\mathbb{T}^1} \frac{3-\gamma}{2}\,f^\epsilon\big(x-\frac{2t}{\epsilon},t\big)\,b^\epsilon(x,t)\,\partial_x\varphi(x,t)\,dx\,dt$$

$$-\frac{\epsilon}{2}\int_0^T \int_{\mathbb{T}^1} q_1^\epsilon\Big(f^\epsilon\big(x-\frac{2t}{\epsilon},t\big)-b^\epsilon(x,t)+\frac{\overline{\rho}}{2\pi}\Big)\,\partial_x\varphi(x,t)\,dx\,dt = 0\,.$$

Because of (3.10) and (3.11), passing to the limit in the four first terms of (3.12) and (3.13) is straightforward. Because of the factor $\frac{\epsilon}{2}$ in front of the last term of (3.12) and (3.13), we deduce that these terms converge to 0.

In order to find the limit of $\int_0^T \int_{\mathbb{T}^1} b^\epsilon\big(x+\frac{2t}{\epsilon},t\big)\,\partial_x\varphi(x,t)\,dx\,dt$ we firstly make a change of variables

$$\int_0^T \int_{\mathbb{T}^1} b^\epsilon\big(x+\frac{2t}{\epsilon},t\big)\,\partial_x\varphi(x,t)\,dx\,dt = \int_0^T \int_{\mathbb{T}^1} b^\epsilon(x,t)\,\partial_x\varphi\big(x-\frac{2t}{\epsilon},t\big)\,dx\,dt\,,$$

then, we write

$$\int_0^T \int_{\mathbb{T}^1} b^\epsilon\big(x+\frac{2t}{\epsilon},t\big)\,\partial_x\varphi(x,t)\,dx\,dt$$
$$= \int_0^T \int_{\mathbb{T}^1} \big(b^\epsilon(x,t)-B(x,t)\big)\,\partial_x\varphi\big(x-\frac{2t}{\epsilon},t\big)\,dx\,dt + \int_0^T \int_{\mathbb{T}^1} B(x,t)\,\partial_x\varphi\big(x-\frac{2t}{\epsilon},t\big)\,dx\,dt\,.$$

Then, using the Cauchy-Schwarz inequality, we show that

$$\lim_{\epsilon\to 0}\int_0^T \int_{\mathbb{T}^1} \big(b^\epsilon(x,t)-B(x,t)\big)\,\partial_x\varphi\big(x-\frac{2t}{\epsilon},t\big)\,dx\,dt = 0\,. \qquad (3.14)$$

If we define $\varphi^\epsilon(x,t) = \tilde{\varphi}(x,\frac{t}{\epsilon},t)$ where $\tilde{\varphi}(x,\tau,t) = \varphi(x-2\tau,t)$, the sequence $(\varphi^\epsilon)_{\epsilon>0}$ is bounded in $L^\infty\big(0,T;H^s(\mathbb{T}^1)\big)$. Then, there exists a function Φ in $L^\infty\big(0,T;L^2(\mathbb{T}^1;H^s(\mathbb{T}^1))\big)$ such that

$$\lim_{\epsilon\to 0}\int_0^T \int_{\mathbb{T}^1} \psi\big(x,\frac{t}{\epsilon},t\big)\,\varphi^\epsilon(x,t)\,dx\,dt = \int_0^T \int_{\mathbb{T}^1} \int_{\mathbb{T}^1} \psi(x,\tau,t)\,\Phi(x,\tau,t)\,d\tau\,dx\,dt \qquad (3.15)$$

for any regular function ψ defined on $\mathbb{T}^1 \times \mathbb{T}^1 \times [0,T)$. As described in Allaire[2], Φ is called the two-scale limit of φ^ϵ. Furthermore, following Marusic-Paloka and Piatnitski[15] and Allaire[2], since $\tilde{\varphi}$ is regular and 2π-periodic in τ, it is an easy game to prove that

$\Phi = \tilde{\varphi}$ in $L^\infty\big(0, T; L^2(\mathbb{T}^1; H^s(\mathbb{T}^1))\big)$. As a consequence, we have $\Phi(x, \tau, t) = \varphi(x - 2\tau, t)$ and

$$\varphi^\epsilon \rightharpoonup \int_{\mathbb{T}^1} \varphi(x - 2\tau, t)\, d\tau \quad \text{weakly-* in } L^\infty\big(0, T; H^s(\mathbb{T}^1)\big). \tag{3.16}$$

Using these results, we obtain

$$\lim_{\epsilon \to 0} \int_0^T \int_{\mathbb{T}^1} B(x, t) \partial_x \varphi\big(x - \frac{2t}{\epsilon}, t\big) \, dx\, dt$$

$$= -\int_0^T \int_{\mathbb{T}^1} \int_{\mathbb{T}^1} \big(\partial_x B(x, t)\big)\, \varphi(x - 2\tau, t)\, d\tau\, dx\, dt \tag{3.17}$$

$$= -\frac{1}{2} \int_0^T \int_{\mathbb{T}^1} B(x, t) \left(\int_{\mathbb{T}^1} \partial_\tau \big(\varphi(x - 2\tau, t)\big)\, d\tau \right) dx\, dt = 0\,,$$

and finally

$$\lim_{\epsilon \to 0} \int_0^T \int_{\mathbb{T}^1} b^\epsilon\big(x + \frac{2t}{\epsilon}, t\big)\, \partial_x \varphi(x, t)\, dx\, dt = 0\,. \tag{3.18}$$

In the same way, we obtain

$$\lim_{\epsilon \to 0} \int_0^T \int_{\mathbb{T}^1} f^\epsilon\big(x - \frac{2t}{\epsilon}, t\big)\, \partial_x \varphi(x, t)\, dx\, dt = 0\,, \tag{3.19}$$

$$\lim_{\epsilon \to 0} \int_0^T \int_{\mathbb{T}^1} \big(b^\epsilon\big(x + \frac{2t}{\epsilon}, t\big)\big)^2 \partial_x \varphi(x, t)\, dx\, dt = 0\,, \tag{3.20}$$

$$\lim_{\epsilon \to 0} \int_0^T \int_{\mathbb{T}^1} \big(f^\epsilon\big(x - \frac{2t}{\epsilon}, t\big)\big)^2 \partial_x \varphi(x, t)\, dx\, dt = 0\,. \tag{3.21}$$

To find the limit of $\int_0^T \int_{\mathbb{T}^1} f^\epsilon(x, t)\, b^\epsilon\big(x + \frac{2t}{\epsilon}, t\big)\, \partial_x \varphi(x, t)\, dx\, dt$, we notice that

$$\int_0^T \int_{\mathbb{T}^1} f^\epsilon(x, t)\, b^\epsilon\big(x + \frac{2t}{\epsilon}, t\big)\, \partial_x \varphi(x, t)\, dx\, dt$$

$$= \int_0^T \int_{\mathbb{T}^1} \big(f^\epsilon(x, t) - F(x, t)\big)\, B\big(x + \frac{2t}{\epsilon}, t\big)\, \partial_x \varphi(x, t)\, dx\, dt$$

$$+ \int_0^T \int_{\mathbb{T}^1} F(x, t) \big(b^\epsilon\big(x + \frac{2t}{\epsilon}, t\big) - B\big(x + \frac{2t}{\epsilon}, t\big)\big)\, \partial_x \varphi(x, t)\, dx\, dt$$

$$+ \int_0^T \int_{\mathbb{T}^1} \big(f^\epsilon(x, t) - F(x, t)\big) \big(b^\epsilon\big(x + \frac{2t}{\epsilon}, t\big) - B\big(x + \frac{2t}{\epsilon}, t\big)\big)\, \partial_x \varphi(x, t)\, dx\, dt$$

$$+ \int_0^T \int_{\mathbb{T}^1} F(x, t)\, B\big(x + \frac{2t}{\epsilon}, t\big)\, \partial_x \varphi(x, t)\, dx\, dt\,.$$

With Cauchy-Schwarz and Hölder's inequalities, we easily show that

$$\int_0^T \int_{\mathbb{T}^1} \big(f^\epsilon(x, t) - F(x, t)\big)\, B\big(x + \frac{2t}{\epsilon}, t\big)\, \partial_x \varphi(x, t)\, dx\, dt \to 0\,, \quad (3.22)$$

$$\int_0^T \int_{\mathbb{T}^1} F(x, t) \big(b^\epsilon\big(x + \frac{2t}{\epsilon}, t\big) - B\big(x + \frac{2t}{\epsilon}, t\big)\big)\, \partial_x \varphi(x, t)\, dx\, dt \to 0\,, \quad (3.23)$$

$$\int_0^T \int_{\mathbb{T}^1} \big(f^\epsilon(x, t) - F(x, t)\big) \big(b^\epsilon\big(x + \frac{2t}{\epsilon}, t\big) - B\big(x + \frac{2t}{\epsilon}, t\big)\big)\, \partial_x \varphi(x, t)\, dx\, dt \to 0\,, \quad (3.24)$$

when $\epsilon \to 0$. If we define the functions $B^\epsilon(x,t) = B\left(x + \frac{2t}{\epsilon}, t\right)$, we remark that the sequence $(B^\epsilon)_{\epsilon>0}$ is bounded in $L^\infty\left(0, T; H^{s-1}(\mathbb{T}^1)\right)$ and then admits a two-scale limit $\mathfrak{B}$ in $L^\infty\left(0, T; L^\infty(\mathbb{T}^1; H^{s-1}(\mathbb{T}^1))\right)$ satisfying

$$\mathfrak{B}(x, \tau, t) = B(x + 2\tau, t) \qquad \forall\, (x, \tau, t) \in \mathbb{T}^1 \times \mathbb{T}^1 \times [0, T)\,, \tag{3.25}$$

$$B^\epsilon \rightharpoonup \int_{\mathbb{T}^1} B(x + 2\tau, t)\, d\tau = 0 \quad \text{weakly-* in } L^\infty\left(0, T; H^s(\mathbb{T}^1)\right)\,. \tag{3.26}$$

This means that, for any function $\psi \in \mathcal{C}\left(0, T; H^{s-1}(\mathbb{T}^1)\right)$, we have

$$\lim_{\epsilon \to 0} \int_0^T \int_{T^1} B^\epsilon(x, t)\, \psi(x, t)\, dx\, dt = 0\,. \tag{3.27}$$

Setting then $\psi = F\, \partial_x \varphi$ in (3.27), we obtain

$$\lim_{\epsilon \to 0} \int_0^T \int_{\mathbb{T}^1} F(x, t)\, B\left(x + \frac{2t}{\epsilon}, t\right) \partial_x \varphi(x, t)\, dx\, dt = 0\,. \tag{3.28}$$

Hence

$$\lim_{\epsilon \to 0} \int_0^T \int_{\mathbb{T}^1} f^\epsilon(x, t)\, b^\epsilon\left(x + \frac{2t}{\epsilon}, t\right) \partial_x \varphi(x, t)\, dx\, dt = 0\,. \tag{3.29}$$

The same method also gives

$$\lim_{\epsilon \to 0} \int_0^T \int_{\mathbb{T}^1} b^\epsilon(x, t)\, f^\epsilon\left(x - \frac{2t}{\epsilon}, t\right) \partial_x \varphi(x, t)\, dx\, dt = 0\,. \tag{3.30}$$

With these results, we can pass to the limit in (3.12)-(3.13) and get

$$-\int_0^T \int_{\mathbb{T}^1} F(x, t)\, \partial_t \varphi(x, t)\, dx\, dt - \int_{\mathbb{T}^1} f_0(x)\, \varphi(x, 0)\, dx$$
$$-\int_0^T \int_{\mathbb{T}^1} \left(\frac{\gamma + 1}{4}\left(F(x, t)\right)^2 + \frac{2\overline{u} + (\gamma - 1)\overline{\rho}}{4\pi}\, F(x, t)\right) \partial_x \varphi(x, t)\, dx\, dt = 0\,, \tag{3.31}$$

and

$$-\int_0^T \int_{\mathbb{T}^1} B(x, t)\, \partial_t \varphi(x, t)\, dx\, dt - \int_{\mathbb{T}^1} b_0(x)\, \varphi(x, 0)\, dx$$
$$-\int_0^T \int_{\mathbb{T}^1} \left(\frac{\gamma + 1}{4}\left(B(x, t)\right)^2 + \frac{2\overline{u} - (\gamma - 1)\overline{\rho}}{4\pi}\, B(x, t)\right) \partial_x \varphi(x, t)\, dx\, dt = 0\,. \tag{3.32}$$

Remembering the definition (3.4)-(3.5) of f_0 and b_0, we recognize here the weak formulation of (2.5).

Under the hypothesis of Theorem 1 about the initial data in (2.5), we deduce that the solution (F, B) belongs to $\left(\mathcal{C}\left(0, T; H^s(\mathbb{T}^1)\right)\right)^2$ and is unique in this space (we can apply the Theorem 3.6.1 of Serre[24] for example). Finally, from uniqueness, we have all the convergence results above for the whole sequences $(f^\epsilon)_{\epsilon>0}$, $(b^\epsilon)_{\epsilon>0}$ and not only for some subsequences.

3.2 Convergence of the homogenized model: proof of Theorem 1

In this section, we will prove inequalities (2.2). Before going further, we remark that convergence (3.6)-(3.7) is already proved and that, thanks to (2.3) and (3.1), these inequalities are equivalent to the theorem below.

Theorem 4. *For $\epsilon_0 > 0$ and $s \geq 2$ fixed, there exists a constant $K > 0$ such that*

$$\left(\int_0^T \left\| f^\epsilon(\cdot, t) - F(\cdot, t) \right\|_{L^2(\mathbb{T}^1)}^2 dt \right)^{\frac{1}{2}} \leq K\epsilon, \tag{3.33}$$

$$\left(\int_0^T \left\| b^\epsilon(\cdot, t) - B(\cdot, t) \right\|_{L^2(\mathbb{T}^1)}^2 dt \right)^{\frac{1}{2}} \leq K\epsilon, \tag{3.34}$$

for any $\epsilon \in \left]0, \epsilon_0\right]$.

Proof of theorem 4: to prove this theorem, in the same spirit of Frénod, Raviart and Sonnendrücker[4], we introduce the functions γ^ϵ and δ^ϵ defined by

$$\gamma^\epsilon(x, t) = \frac{1}{\epsilon} \left(f^\epsilon(x, t) - F(x, t) \right) - W\left(x, \frac{t}{\epsilon}, t\right),$$
$$\delta^\epsilon(x, t) = \frac{1}{\epsilon} \left(B(x, t) - b^\epsilon(x, t) \right) - V\left(x, \frac{t}{\epsilon}, t\right), \tag{3.35}$$

with W and V defined on $\mathbb{T}^1 \times \mathbb{T}^1 \times [0, T)$ by

$$W(x, \tau, t) = -\frac{\gamma - 3}{4} \int_0^\tau \partial_x \Big(\left(B(x + 2\theta, t)\right)^2 - 2F(x, t)B(x + 2\theta, t)$$
$$- \frac{\overline{\rho}}{\pi} B(x + 2\theta, t) \Big) d\theta,$$
$$V(x, \tau, t) = \frac{\gamma - 3}{4} \int_0^\tau \partial_x \Big(\left(F(x - 2\theta, t)\right)^2 - 2F(x - 2\theta, t)B(x, t)$$
$$+ \frac{\overline{\rho}}{\pi} F(x - 2\theta, t) \Big) d\theta. \tag{3.36}$$

Concerning sequences $(\gamma^\epsilon)_{\epsilon \in \left]0, \epsilon_0\right]}$ and $(\delta^\epsilon)_{\epsilon \in \left]0, \epsilon_0\right]}$, we have the property below.

Lemma 5. *Under the hypothesis of theorem 4, there exists a contant $M > 0$, independent of $\epsilon \in \left]0, \epsilon_0\right]$ and $t \in [0, T)$, such that*

$$\partial_t \Big(\left\| \gamma^\epsilon(\cdot, t) \right\|_{L^2(\mathbb{T}^1)}^2 + \left\| \delta^\epsilon(\cdot, t) \right\|_{L^2(\mathbb{T}^1)}^2 \Big)$$
$$\leq M \left(\frac{3}{2} + \sqrt{2} \right) \Big(\left\| \gamma^\epsilon(\cdot, t) \right\|_{L^2(\mathbb{T}^1)}^2 + \left\| \delta^\epsilon(\cdot, t) \right\|_{L^2(\mathbb{T}^1)}^2 \Big) + \sqrt{2}M, \tag{3.37}$$

for any $\epsilon \in \left]0, \epsilon_0\right]$ and $t \in [0, T)$.

Having this inequality at hand, we can apply Gronwall's lemma to find a constant $L > 0$, independent of t and ϵ, such that

$$\left\| \gamma^\epsilon(\cdot, t) \right\|_{L^2(\mathbb{T}^1)}^2 + \left\| \delta^\epsilon(\cdot, t) \right\|_{L^2(\mathbb{T}^1)}^2 \leq L, \qquad \forall (t, \epsilon) \in [0, T) \times \left]0, \epsilon_0\right]. \tag{3.38}$$

Finally, we define the constant K by

$$K = \sqrt{T}\left[\sqrt{L} + \max\left(\sup_{\substack{t \in [0,T) \\ \tau \in \mathbb{T}^1}} \left\|W(\cdot,\tau,t)\right\|_{H^s(\mathbb{T}^1)}, \sup_{\substack{t \in [0,T) \\ \tau \in \mathbb{T}^1}} \left\|V(\cdot,\tau,t)\right\|_{H^s(\mathbb{T}^1)}\right) \right], \qquad (3.39)$$

and we obtain the inequalities (3.33)-(3.34), giving the theorem.

Now, to finish the proof of Theorem 4, we have to prove Lemma 5. Firstly, we notice that $\partial_t \gamma^\epsilon$ and $\partial_t \delta^\epsilon$ read

$$
\begin{aligned}
\partial_t \gamma^\epsilon(x,t) = &\left(-\alpha\big(f^\epsilon(x,t) + F(x,t)\big) + \zeta B\big(x + \tfrac{2t}{\epsilon},t\big) - \beta_+ \right) \partial_x \gamma^\epsilon(x,t) \\
&+ \left(-\alpha \partial_x\big(f^\epsilon(x,t) + F(x,t)\big) + \zeta \partial_x B\big(x + \tfrac{2t}{\epsilon},t\big) \right) \gamma^\epsilon(x,t) \\
&+ \frac{\zeta}{2}\left(B\big(x + \tfrac{2t}{\epsilon},t\big) + b^\epsilon\big(x + \tfrac{2t}{\epsilon},t\big) - 2f^\epsilon(x,t) - \frac{\overline{\rho}}{\pi}\right) \partial_x \delta^\epsilon\big(x + \tfrac{2t}{\epsilon},t\big) \quad (3.40) \\
&+ \frac{\zeta}{2}\,\partial_x\left(B\big(x + \tfrac{2t}{\epsilon},t\big) + b^\epsilon\big(x + \tfrac{2t}{\epsilon},t\big) - 2f^\epsilon(x,t)\right) \delta^\epsilon\big(x + \tfrac{2t}{\epsilon},t\big) \\
&+ \Gamma^\epsilon(x,t)\,,
\end{aligned}
$$

$$
\begin{aligned}
\partial_t \delta^\epsilon(x,t) = &\left(-\alpha\big(B(x,t) + b^\epsilon(x,t)\big) + \zeta F\big(x - \tfrac{2t}{\epsilon},t\big) - \beta_- \right) \partial_x \delta^\epsilon(x,t) \\
&+ \left(-\alpha \partial_x\big(B(x,t) + b^\epsilon(x,t)\big) + \zeta \partial_x F\big(x - \tfrac{2t}{\epsilon},t\big) \right) \delta^\epsilon(x,t) \\
&+ \frac{\zeta}{2}\left(f^\epsilon\big(x - \tfrac{2t}{\epsilon},t\big) + F\big(x - \tfrac{2t}{\epsilon},t\big) - 2b^\epsilon(x,t) + \frac{\overline{\rho}}{\pi}\right) \partial_x \gamma^\epsilon\big(x - \tfrac{2t}{\epsilon},t\big) \quad (3.41) \\
&+ \frac{\zeta}{2}\,\partial_x\left(f^\epsilon\big(x - \tfrac{2t}{\epsilon},t\big) + F\big(x - \tfrac{2t}{\epsilon},t\big) - 2b^\epsilon(x,t)\right) \gamma^\epsilon\big(x - \tfrac{2t}{\epsilon},t\big) \\
&+ \Delta^\epsilon(x,t)\,,
\end{aligned}
$$

with $\alpha, \beta_+, \beta_-, \zeta, \Gamma^\epsilon(x,t)$ and $\Delta^\epsilon(x,t)$ defined by

$$\alpha = \frac{\gamma + 1}{4}\,, \quad \beta_+ = \frac{2\overline{u} + (\gamma - 1)\overline{\rho}}{4\pi}\,, \quad \beta_- = \frac{2\overline{u} - (\gamma - 1)\overline{\rho}}{4\pi}\,, \quad \zeta = \frac{\gamma - 3}{2}\,, \qquad (3.42)$$

$$
\begin{aligned}
\Gamma^\epsilon(x,t) = &-\alpha \partial_x\left(W\big(x, \tfrac{t}{\epsilon},t\big)\big(f^\epsilon(x,t) + F(x,t)\big)\right) - \beta_+ \partial_x W\big(x, \tfrac{t}{\epsilon},t\big) \\
&- \zeta \partial_x\left(f^\epsilon(x,t) V\big(x + \tfrac{2t}{\epsilon}, \tfrac{t}{\epsilon},t\big) - B\big(x + \tfrac{2t}{\epsilon},t\big) W\big(x, \tfrac{t}{\epsilon},t\big)\right) \\
&+ \frac{\zeta}{2}\,\partial_x\left(V\big(x + \tfrac{2t}{\epsilon}, \tfrac{t}{\epsilon},t\big)\big(B\big(x + \tfrac{2t}{\epsilon},t\big) + b^\epsilon\big(x + \tfrac{2t}{\epsilon},t\big)\big)\right) \quad (3.43) \\
&- \frac{\zeta\overline{\rho}}{2\pi}\,\partial_x V\big(x + \tfrac{2t}{\epsilon}, \tfrac{t}{\epsilon},t\big) - \frac{1}{2}\,\partial_x\left(q_1^\epsilon\big(\rho^\epsilon\big(x + \tfrac{t}{\epsilon},t\big)\big)\right) - (\partial_t W)\big(x, \tfrac{t}{\epsilon},t\big)
\end{aligned}
$$

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and

$$\Delta^\epsilon(x,t) = -\alpha\partial_x\Big(V\big(x,\tfrac{t}{\epsilon},t\big)\big(B(x,t)+b^\epsilon(x,t)\big)\Big) - \beta_-\partial_x V\big(x,\tfrac{t}{\epsilon},t\big)$$
$$- \zeta\partial_x\Big(W\big(x-\tfrac{2t}{\epsilon},\tfrac{t}{\epsilon},t\big)b^\epsilon(x,t) - F\big(x-\tfrac{2t}{\epsilon},t\big)V\big(x,\tfrac{t}{\epsilon},t\big)\Big)$$
$$+ \frac{\zeta}{2}\,\partial_x\bigg(W\big(x-\tfrac{2t}{\epsilon},\tfrac{t}{\epsilon},t\big)\Big(f^\epsilon\big(x-\tfrac{2t}{\epsilon},t\big)+F\big(x-\tfrac{2t}{\epsilon},t\big)\Big)\bigg)$$
$$+ \frac{\zeta\overline{\rho}}{2\pi}\,\partial_x W\big(x-\tfrac{2t}{\epsilon},\tfrac{t}{\epsilon},t\big) + \frac{1}{2}\,\partial_x\bigg(q_1^\epsilon\Big(\rho^\epsilon\big(x-\tfrac{t}{\epsilon},t\big)\Big)\bigg) - (\partial_t V)\big(x,\tfrac{t}{\epsilon},t\big).$$

(3.44)

Since the sequences $(f^\epsilon)_{\epsilon\in]0,\epsilon_0]}$ and $(b^\epsilon)_{\epsilon\in]0,\epsilon_0]}$ are bounded in $\mathcal{C}\big(0,T;H^s(\mathbb{T}^1)\big)$ with $s\geq 2$, F and B are in $\mathcal{C}\big(0,T;H^s(\mathbb{T}^1)\big)$, and q_1^ϵ is regular, we can find some constants $C_i > 0$ $(i=1,\ldots,8)$, independent of ϵ and t, such that

$$2\int_{\mathbb{T}^1}\Big(-\alpha\big(f^\epsilon(x,t)+F(x,t)\big)-\beta_+ + \zeta B\big(x+\tfrac{2t}{\epsilon},t\big)\Big)\gamma^\epsilon(x,t)\,\partial_x\gamma^\epsilon(x,t)\,dx$$
$$\leq C_1\big\|\gamma^\epsilon(\cdot,t)\big\|^2_{L^2(\mathbb{T}^1)}\,,$$

(3.45)

$$2\int_{\mathbb{T}^1}\Big(-\alpha\big(b^\epsilon(x,t)+B(x,t)\big)-\beta_- + \zeta F\big(x-\tfrac{2t}{\epsilon},t\big)\Big)\delta^\epsilon(x,t)\,\partial_x\delta^\epsilon(x,t)\,dx$$
$$\leq C_2\big\|\delta^\epsilon(\cdot,t)\big\|^2_{L^2(\mathbb{T}^1)}\,,$$

(3.46)

$$2\int_{\mathbb{T}^1}\Big(-\alpha\partial_x\big(f^\epsilon(x,t)+F(x,t)\big)+\zeta\partial_x B\big(x+\tfrac{2t}{\epsilon},t\big)\Big)\big(\gamma^\epsilon(x,t)\big)^2\,dx$$
$$\leq C_3\big\|\gamma^\epsilon(\cdot,t)\big\|^2_{L^2(\mathbb{T}^1)}\,,$$

(3.47)

$$2\int_{\mathbb{T}^1}\Big(-\alpha\partial_x\big(b^\epsilon(x,t)+B(x,t)\big)+\zeta\partial_x F\big(x-\tfrac{2t}{\epsilon},t\big)\Big)\big(\delta^\epsilon(x,t)\big)^2\,dx$$
$$\leq C_4\big\|\delta^\epsilon(\cdot,t)\big\|^2_{L^2(\mathbb{T}^1)}\,,$$

(3.48)

$$2\int_{\mathbb{T}^1}\frac{\zeta}{2}\,\partial_x\Big(B\big(x+\tfrac{2t}{\epsilon},t\big)+b^\epsilon\big(x+\tfrac{2t}{\epsilon},t\big)-2f^\epsilon(x,t)\Big)\delta^\epsilon\big(x+\tfrac{2t}{\epsilon},t\big)\gamma^\epsilon(x,t)\,dx$$
$$\leq C_5\big\|\gamma^\epsilon(\cdot,t)\big\|_{L^2(\mathbb{T}^1)}\big\|\delta^\epsilon(\cdot,t)\big\|_{L^2(\mathbb{T}^1)}\,,$$

(3.49)

$$2\int_{\mathbb{T}^1}\frac{\zeta}{2}\,\partial_x\Big(F\big(x-\tfrac{2t}{\epsilon},t\big)+f^\epsilon\big(x-\tfrac{2t}{\epsilon},t\big)-2b^\epsilon(x,t)\Big)\gamma^\epsilon\big(x-\tfrac{2t}{\epsilon},t\big)\delta^\epsilon(x,t)\,dx$$
$$\leq C_6\big\|\gamma^\epsilon(\cdot,t)\big\|_{L^2(\mathbb{T}^1)}\big\|\delta^\epsilon(\cdot,t)\big\|_{L^2(\mathbb{T}^1)}\,,$$

(3.50)

$$2\int_{\mathbb{T}^1}\Gamma^\epsilon(x,t)\,\gamma^\epsilon(x,t)\,dx \leq C_7\big\|\gamma^\epsilon(\cdot,t)\big\|_{L^2(\mathbb{T}^1)}\,,$$

(3.51)

$$2\int_{\mathbb{T}^1}\Delta^\epsilon(x,t)\,\delta^\epsilon(x,t)\,dx \leq C_8\big\|\delta^\epsilon(\cdot,t)\big\|_{L^2(\mathbb{T}^1)}\,.$$

(3.52)

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On the other hand, making in the first integral the change of variable $x \mapsto x + \tau$ and in the second one $x \mapsto x - \tau$, rearranging the terms and using the definitions of γ^ϵ and δ^ϵ, we have

$$\zeta \int_{\mathbb{T}^1} \left(b^\epsilon(x + 2\tau, t) + B(x + 2\tau, t) - 2f^\epsilon(x, t) - \frac{\overline{\rho}}{\pi} \right) \partial_x \delta^\epsilon(x + 2\tau, t)\, \gamma^\epsilon(x, t)\, dx$$

$$+ \zeta \int_{\mathbb{T}^1} \left(f^\epsilon(x - 2\tau, t) + F(x - 2\tau, t) - 2b^\epsilon(x, t) + \frac{\overline{\rho}}{\pi} \right) \partial_x \gamma^\epsilon(x - 2\tau, t)\, \delta^\epsilon(x, t)\, dx$$

$$= \zeta \int_{\mathbb{T}^1} \left(b^\epsilon(x + \tau, t) - B(x + \tau, t) \right) \gamma^\epsilon(x - \tau, t)\, \partial_x \delta^\epsilon(x + \tau, t)\, dx$$

$$+ 2\zeta \int_{\mathbb{T}^1} \left(F(x - \tau, t) - f^\epsilon(x - \tau, t) \right) \gamma^\epsilon(x - \tau, t)\, \partial_x \delta^\epsilon(x + \tau, t)\, dx$$

$$- 2\zeta \int_{\mathbb{T}^1} \left(F(x - \tau, t) - B(x + \tau, t) + \frac{\overline{\rho}}{2\pi} \right) \gamma^\epsilon(x - \tau, t)\, \partial_x \delta^\epsilon(x + \tau, t)\, dx$$

$$+ \zeta \int_{\mathbb{T}^1} \left(f^\epsilon(x - \tau, t) - F(x - \tau, t) \right) \delta^\epsilon(x + \tau, t)\, \partial_x \gamma^\epsilon(x - \tau, t)\, dx$$

$$+ 2\zeta \int_{\mathbb{T}^1} \left(B(x + \tau, t) - b^\epsilon(x + \tau, t) \right) \delta^\epsilon(x + \tau, t)\, \partial_x \gamma^\epsilon(x - \tau, t)\, dx$$

$$+ 2\zeta \int_{\mathbb{T}^1} \left(F(x - \tau, t) - B(x + \tau, t) + \frac{\overline{\rho}}{2\pi} \right) \delta^\epsilon(x + \tau, t)\, \partial_x \gamma^\epsilon(x - \tau, t)\, dx$$

$$= \zeta \int_{\mathbb{T}^1} \epsilon \left(\partial_x \gamma^\epsilon(x - \tau, t) - \partial_x \delta^\epsilon(x + \tau, t) \right) \gamma^\epsilon(x - \tau, t)\, \delta^\epsilon(x + \tau, t)\, dx$$

$$+ \zeta \int_{\mathbb{T}^1} \epsilon \partial_x \gamma^\epsilon(x - \tau, t) \left(W(x - \tau, \tau, t) + 2V(x + \tau, \tau, t) \right) \delta^\epsilon(x + \tau, t)\, dx$$

$$- \zeta \int_{\mathbb{T}^1} \epsilon \partial_x \delta^\epsilon(x + \tau, t) \left(V(x + \tau, \tau, t) + 2W(x - \tau, \tau, t) \right) \gamma^\epsilon(x - \tau, t)\, dx$$

$$- 2\zeta \int_{\mathbb{T}^1} \epsilon \partial_x \delta^\epsilon(x + \tau, t) \left(\gamma^\epsilon(x - \tau, t) \right)^2 dx$$

$$+ 2\zeta \int_{\mathbb{T}^1} \epsilon \partial_x \gamma^\epsilon(x - \tau, t) \left(\delta^\epsilon(x + \tau, t) \right)^2 dx$$

$$+ 2\zeta \int_{\mathbb{T}^1} R(x, \tau, t) \left(\delta^\epsilon(x + \tau, t) \partial_x \gamma^\epsilon(x - \tau, t) - \gamma^\epsilon(x - \tau, t) \partial_x \delta^\epsilon(x + \tau, t) \right) dx \,.$$

The sequences $(\epsilon \gamma^\epsilon)_{\epsilon \in]0, \epsilon_0]}$ and $(\epsilon \delta^\epsilon)_{\epsilon \in]0, \epsilon_0]}$ are bounded in $\mathcal{C}\left(0, T; H^s(\mathbb{T}^1)\right)$, so we can find some constants $C_i > 0$ $(i = 9, \ldots, 13)$, independent of ϵ and t, such that

$$\zeta \int_{\mathbb{T}^1} \epsilon \left(\partial_x \gamma^\epsilon(x - \tau, t) - \partial_x \delta^\epsilon(x + \tau, t) \right) \gamma^\epsilon(x - \tau, t)\, \delta^\epsilon(x + \tau, t)\, dx$$
$$\leq C_9 \left\| \gamma^\epsilon(\cdot, t) \right\|_{L^2(\mathbb{T}^1)} \left\| \gamma^\epsilon(\cdot, t) \right\|_{L^2(\mathbb{T}^1)}, \quad (3.53)$$

$$\zeta \int_{\mathbb{T}^1} \epsilon \partial_x \gamma^\epsilon(x - \tau, t) \left(W(x - \tau, \tau, t) + 2V(x + \tau, \tau, t) \right) \delta^\epsilon(x + \tau, t)\, dx$$
$$\leq C_{10} \left\| \delta^\epsilon(\cdot, t) \right\|_{L^2(\mathbb{T}^1)}, \quad (3.54)$$

$$\zeta \int_{\mathbb{T}^1} \epsilon \partial_x \delta^\epsilon(x + \tau, t) \left(V(x + \tau, \tau, t) + 2W(x - \tau, \tau, t) \right) \gamma^\epsilon(x - \tau, t)\, dx$$
$$\leq C_{11} \left\| \gamma^\epsilon(\cdot, t) \right\|_{L^2(\mathbb{T}^1)}, \quad (3.55)$$

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$$2\zeta \int_{\mathbb{T}^1} \epsilon \partial_x \delta^\epsilon(x+\tau,t) \big(\gamma^\epsilon(x-\tau,t)\big)^2 \, dx \le C_{12} \big\|\gamma^\epsilon(\cdot,t)\big\|^2_{L^2(\mathbb{T}^1)}, \tag{3.56}$$

$$2\zeta \int_{\mathbb{T}^1} \epsilon \partial_x \gamma^\epsilon(x-\tau,t) \big(\delta^\epsilon(x+\tau,t)\big)^2 \, dx \le C_{13} \big\|\delta^\epsilon(\cdot,t)\big\|^2_{L^2(\mathbb{T}^1)}. \tag{3.57}$$

For the last integral, we simply compute:

$$
\begin{aligned}
2\zeta \int_{\mathbb{T}^1} &R(x,\tau,t)\big(\delta^\epsilon(x+\tau,t)\partial_x\gamma^\epsilon(x-\tau,t) - \gamma^\epsilon(x-\tau,t)\partial_x\delta^\epsilon(x+\tau,t)\big) \, dx \\
&= 2\zeta \int_{\mathbb{T}^1} \partial_\tau\big(R(x,\tau,t)\big)\delta^\epsilon(x+\tau,t)\,\gamma^\epsilon(x-\tau,t) \, dx \\
&\quad - 2\zeta\partial_\tau\left(\int_{\mathbb{T}^1} R(x,\tau,t)\,\delta^\epsilon(x+\tau,t)\,\gamma^\epsilon(x-\tau,t) \, dx \right) \\
&= 2\zeta \int_{\mathbb{T}^1} \partial_\tau\big(R(x,\tau,t)\big)\delta^\epsilon(x+\tau,t)\,\gamma^\epsilon(x-\tau,t) \, dx \\
&= -2\zeta \int_{\mathbb{T}^1} \delta^\epsilon(x+\tau,t)\,\gamma^\epsilon(x-\tau,t)\,\partial_x U(x,\tau,t) \, dx \\
&\le C_{14}\big\|\gamma^\epsilon(\cdot,t)\big\|_{L^2(\mathbb{T}^1)}\big\|\delta^\epsilon(\cdot,t)\big\|_{L^2(\mathbb{T}^1)}
\end{aligned}
\tag{3.58}
$$

for a constant $C_{14} > 0$ independent of ϵ and t.

Having (3.45)-(3.58) at hand, we multiply (3.40) by $\gamma^\epsilon(x,t)$ and (3.41) by $\delta^\epsilon(x,t)$ to deduce

$$
\begin{aligned}
\partial_t\Big(\big\|\gamma^\epsilon(\cdot,t)\big\|^2_{L^2(\mathbb{T}^1)} + \big\|\delta^\epsilon(\cdot,t)\big\|^2_{L^2(\mathbb{T}^1)}\Big) \le M\Big(&\big\|\gamma^\epsilon(\cdot,t)\big\|^2_{L^2(\mathbb{T}^1)} + \big\|\delta^\epsilon(\cdot,t)\big\|^2_{L^2(\mathbb{T}^1)} \\
&+ \big\|\gamma^\epsilon(\cdot,t)\big\|_{L^2(\mathbb{T}^1)} + \big\|\delta^\epsilon(\cdot,t)\big\|_{L^2(\mathbb{T}^1)} \\
&+ \big\|\gamma^\epsilon(\cdot,t)\big\|_{L^2(\mathbb{T}^1)}\big\|\delta^\epsilon(\cdot,t)\big\|_{L^2(\mathbb{T}^1)}\Big),
\end{aligned}
\tag{3.59}
$$

with $M = \max(C_1 + C_3 + C_{12}, C_2 + C_4 + C_{13}, C_5 + C_6 + C_9 + C_{14}, C_7 + C_{11}, C_8 + C_{10})$. Inequality (3.37) is directly obtained from (3.59), ending the proof of lemma 5 and then the proof of theorem 4. $\square$

3.3 Properties of the finite volume scheme: proof of the Theorem 2

Thanks to (2.3) and (2.9), convergence of $\big(U_h(\cdot,\tau,\cdot), R_h(\cdot,\tau,t)\big)$ to $\big(U(\cdot,\tau,\cdot), R(\cdot,\tau,t)\big)$ for any $\tau \in \mathbb{T}^1$ is equivalent to the convergence of (F_h, B_h) to (F, B).

Furthermore, the equations satisfied by F and B are of the form

$$\partial_t q + \partial_x\big(f(q)\big) = 0, \quad q(x,0) = q_0(x), \quad \int_{\mathbb{T}^1} q(x,t) \, dx = 0, \quad \forall t, \tag{3.60}$$

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with $f(q) = \alpha q^2 + \beta q$, α and $\beta = \beta_{\pm}$ defined in (3.42). In the same way, the numerical method (2.6)-(2.7)-(2.8) is of the form

$$Q_h(x,t) = Q_i^n \qquad \forall\, [x_{i-1/2}, x_{i+1/2}[\times[t_n, t_{n+1}[\,,$$

$$Q_i^{n+1} = Q_i^n - \frac{k}{h}\big(\mathcal{F}(Q_{i+1}^n, Q_i^n) - \mathcal{F}(Q_i^n, Q_{i-1}^n)\big)\,, \tag{3.61}$$

$$\mathcal{F}(Q_i, Q_{i-1}) = \frac{1}{2}\big(f(Q_i) + f(Q_{i-1})\big) - \frac{1}{2}\big|\alpha(Q_i + Q_{i-1}) + \beta\big|(Q_i - Q_{i-1})\,,$$

with a space step $h = \dfrac{2\pi}{N_x + 1}$ and a time step k.

We notice that the CFL condition at the n-th time step is given by

$$\frac{k}{h}\max_{0 \le i \le N_x}\big|\alpha(Q_i^n + Q_{i-1}^n) + \beta\big| = \nu \le 1\,. \tag{3.62}$$

With these notations, proving Theorem 2 is equivalent to proving the theorem below.

Theorem 6. *If $s \ge 2$, the approximation Q_h converges to the solution q of (3.60) in $L^1\big([0,T] \times \mathbb{T}^1\big)$ norm. Furthermore, if $s \ge 3$, the local truncation error of the numerical scheme (3.61) is first order accurate.*

Proof of theorem 5: the *total variation* of a function $q \in L^1\big([0,T] \times \mathbb{T}^1\big)$ is defined by

$$TV_T(q) = \limsup_{\eta \to 0} \frac{1}{\eta} \int_0^T \int_{\mathbb{T}^1} \Big[\big|q(x+\eta, t) - q(x,t)\big| + \big|q(x, t+\eta) - q(x,t)\big|\Big]\, dx\, dt\,. \tag{3.63}$$

For an approximation Q_h computed with the numerical method (3.61), we have

$$TV_T(Q_h) = \sum_{n=0}^{T/k} \Big[k\, TV(Q^n) + \|Q^{n+1} - Q^n\|_1\Big]\,, \tag{3.64}$$

with $TV(Q)$ and $\|Q\|_1$ defined by

$$TV(Q) = \sum_{i=0}^{N_x} |Q_i - Q_{i-1}| \quad \text{and} \quad \|Q\|_1 = h\sum_{i=0}^{N_x} |Q_i|\,. \tag{3.65}$$

Introducing the sets $\mathcal{L}$ and $\mathcal{K}$ defined by

$$\mathcal{L} = \Big\{q \in L^1\big([0,T] \times \mathbb{T}^1\big) \,:\, \int_{\mathbb{T}^1} q\, dx = 0\, \forall t\Big\}, \quad \mathcal{K} = \big\{q \in \mathcal{L} \,:\, TV_T(q) \le R\big\}, \tag{3.66}$$

where R is a constant depending on q_0, it is well known that $\mathcal{K}$ is a compact subset of $\mathcal{L}$.

Since the numerical flux involved in (3.61) is continuous and satisfies

$$\mathcal{F}(q,q) = f(q) \quad \forall\, q\,, \tag{3.67}$$

the considered scheme is consistent with the conservation law (3.60). Hence, applying LeVeque[12], proving that Q_h converges to the solution q of (3.60) in $L^1\big([0,T] \times \mathbb{T}^1\big)$ norm for $h \to 0$ reduces to prove that the scheme is TV-stable. In other words, we need to prove that for $h \in [0, h_0]$, the approximation Q_h lies in some fixed set $\mathcal{K}$ where R only depends on the initial data q_0, the final time T and the function f.

The TV-stability is the consequence of two lemmas.

Lemma 7. *The numerical method (3.61) is TVD (Total Variation Diminishing), i.e.*

$$TV(Q^{n+1}) \le TV(Q^n) \quad \forall n. \tag{3.68}$$

Proof of the lemma 7: we develop $Q_{i+1}^{n+1} - Q_i^{n+1}$:

$$
\begin{aligned}
Q_{i+1}^{n+1} - Q_i^{n+1} = {} & Q_{i+1}^n - Q_i^n - \frac{k}{2h}\Big(f(Q_{i+2}^n) - f(Q_i^n) - f(Q_{i+1}^n) + f(Q_{i-1}^n) \\
& - \big|\alpha(Q_{i+2}^n + Q_{i+1}^n) + \beta\big|(Q_{i+2}^n - Q_{i+1}^n) \\
& + 2\big|\alpha(Q_{i+1}^n + Q_i^n) + \beta\big|(Q_{i+1}^n - Q_i^n) \\
& - \big|\alpha(Q_i^n + Q_{i-1}^n) + \beta\big|(Q_i^n - Q_{i-1}^n)\Big) \\
= {} & -\frac{k}{2h}\Big(1 - \mathrm{sg}\big(\alpha(Q_{i+2}^n + Q_{i+1}^n) + \beta\big)\Big)\big(\alpha(Q_{i+2}^n + Q_{i+1}^n) + \beta\big)(Q_{i+2}^n - Q_{i+1}^n) \\
& + \Big(1 - \frac{k}{h}\big|\alpha(Q_{i+1}^n + Q_i^n) + \beta\big|\Big)(Q_{i+1}^n - Q_i^n) \\
& + \frac{k}{2h}\Big(1 + \mathrm{sg}\big(\alpha(Q_i^n + Q_{i-1}^n) + \beta\big)\Big)\big(\alpha(Q_i^n + Q_{i-1}^n) + \beta\big)(Q_i^n - Q_{i-1}^n),
\end{aligned}
$$

where sg stands for the usual sign function. Using the CFL condition (3.62), we write:

$$
\begin{aligned}
\big|Q_{i+1}^{n+1} - Q_i^{n+1}\big| \le {} & \frac{k}{2h}\Big(1 - \mathrm{sg}\big(\alpha(Q_{i+2}^n + Q_{i+1}^n) + \beta\big)\Big)\big|\alpha(Q_{i+2}^n + Q_{i+1}^n) + \beta\big|\,|Q_{i+2}^n - Q_{i+1}^n| \\
& + \Big(1 - \frac{k}{h}\big|\alpha(Q_{i+1}^n + Q_i^n) + \beta\big|\Big)\,|Q_{i+1}^n - Q_i^n| \\
& + \frac{k}{2h}\Big(1 + \mathrm{sg}\big(\alpha(Q_i^n + Q_{i-1}^n) + \beta\big)\Big)\big|\alpha(Q_i^n + Q_{i-1}^n) + \beta\big|\,|Q_i^n - Q_{i-1}^n|.
\end{aligned}
$$

We deduce then

$$TV(Q^{n+1}) = \sum_{i=0}^{N_x} \big|Q_{i+1}^{n+1} - Q_i^{n+1}\big| \le \sum_{i=0}^{N_x} \big|Q_{i+1}^n - Q_i^n\big| = TV(Q^n) \tag{3.69}$$

giving the lemma. $\square$

As a consequence of this lemma, we deduce that there exists a constant $M_1 > 0$ which only depends on the initial data q_0 and such that

$$TV(Q^n) \le M_1 \quad \forall n. \tag{3.70}$$

Lemma 8. *For any n, we have $\big\|Q^{n+1} - Q^n\big\|_1 \le 2M_1 h$.*

Proof of lemma 8: we develop $Q_i^{n+1} - Q_i^n$:

$$
\begin{aligned}
Q_i^{n+1} - Q_i^n = {} & -\frac{k}{2h}\Big(f(Q_{i+1}^n) - f(Q_i^n) - \big|\alpha(Q_{i+1}^n + Q_i^n) + \beta\big|(Q_{i+1}^n - Q_i^n) \\
& - f(Q_i^n) + f(Q_{i-1}^n) + \big|\alpha(Q_i^n + Q_{i-1}^n) + \beta\big|(Q_i^n - Q_{i-1}^n)\Big) \\
= {} & -\frac{k}{2h}\Big(1 - \mathrm{sg}\big(\alpha(Q_{i+1}^n + Q_i^n) + \beta\big)\Big)\big(\alpha(Q_{i+1}^n + Q_i^n) + \beta\big)(Q_{i+1}^n - Q_i^n) \\
& + \frac{k}{2h}\Big(1 - \mathrm{sg}\big(\alpha(Q_i^n + Q_{i-1}^n) + \beta\big)\Big)\big(\alpha(Q_i^n + Q_{i-1}^n) + \beta\big)(Q_i^n - Q_{i-1}^n).
\end{aligned}
$$

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Then, using the CFL condition, we obtain

$$
\begin{aligned}
\left\| Q^{n+1} - Q^n \right\|_1 &= h \sum_{i=0}^{N_x} \left| Q_i^{n+1} - Q_i^n \right| \\
&\leq h \sum_{i=0}^{N_x} \left[\frac{k}{2h} \left| \alpha (Q_{i+1}^n + Q_i^n) + \beta \right| \left| Q_{i+1}^n - Q_i^n \right| \right] \\
&\quad + h \sum_{i=0}^{N_x} \left[\frac{k}{2h} \left| \alpha (Q_i^n + Q_{i-1}^n) + \beta \right| \left| Q_i^n - Q_{i-1}^n \right| \right] \\
&\leq 2h \sum_{i=0}^{N_x} \left| Q_i^n - Q_{i-1}^n \right|
\end{aligned}
$$

and we complete the proof by applying lemma 7. $\square$

End of the proof of theorem 5: combining all these results, we finally have the inequality

$$
TV_T(Q_h) \leq \left(1 + \frac{2h}{k} \right) T M_1 . \tag{3.71}
$$

Remarking that $\frac{h}{k}$ is bounded by a constant M_2 which only depends on the initial data q_0, we finally obtain that there exists a constant $C > 0$ which only depends on q_0 and T, and such that

$$
TV_T(Q_h) \leq C \tag{3.72}
$$

for any space step h. We conclude that the numerical method (3.61) is TV-stable, yielding the convergence of Q_h to q.

Now, we have to prove that, assuming that $s \geq 3$, the local truncation error is first order accurate. With the notation above, the expression of the local truncation error reads

$$
\begin{aligned}
e(x_i, t_n) = \frac{q_i^n - q_i^{n+1}}{k} - \frac{1}{2h} \Big(&f(q_{i+1}^n) - f(q_{i-1}^n) - \left| \alpha (q_{i+1}^n + q_i^n) + \beta \right| (q_{i+1}^n - q_i^n) \\
&+ \left| \alpha (q_i^n + q_{i-1}^n) + \beta \right| (q_i^n - q_{i-1}^n) \Big) ,
\end{aligned} \tag{3.73}
$$

where $q_i^n = q(x_i, t_n)$. According to the signs of $\alpha (q_i^n + q_{i-1}^n) + \beta$ and $\alpha (q_{i+1}^n + q_i^n) + \beta$, we distinguish four different cases.

- First case: we assume that $\alpha (q_{i+1}^n + q_i^n) + \beta \geq 0$ and $\alpha (q_i^n + q_{i-1}^n) + \beta \geq 0$. Then, the local truncation error becomes

$$
e(x_i, t_n) = \frac{q_i^n - q_i^{n+1}}{k} - \frac{f(q_i^n) - f(q_{i-1}^n)}{h} . \tag{3.74}
$$

Since $s \geq 3$, $q \in \mathcal{C}^2 \big([0, T) \times \mathbb{T}^1 \big)$, so we can write

$$
e(x_i, t_n) = -\partial_t q(x_i, t_n) - \partial_x \Big(f \big(q(x_i, t_n) \big) \Big) + \mathcal{O}(k) + \mathcal{O}(h) = \mathcal{O}(h + k) . \tag{3.75}
$$

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- Second case: we assume that $\alpha(q_{i+1}^n + q_i^n) + \beta \leq 0$ and $\alpha(q_i^n + q_{i-1}^n) + \beta \leq 0$. Then, the local truncation error becomes

$$e(x_i, t_n) = \frac{q_i^n - q_i^{n+1}}{k} - \frac{f(q_{i+1}^n) - f(q_i^n)}{h} \, . \tag{3.76}$$

Since $s \geq 3$, $q \in \mathcal{C}^2\big([0, T) \times \mathbb{T}^1\big)$, so we can write

$$e(x_i, t_n) = -\partial_t q(x_i, t_n) - \partial_x \Big(f\big(q(x_i, t_n)\big)\Big) + \mathcal{O}(k) + \mathcal{O}(h) = \mathcal{O}(h + k) \, . \tag{3.77}$$

- Third case: we assume that $\alpha(q_{i+1}^n + q_i^n) + \beta \geq 0$ and $\alpha(q_i^n + q_{i-1}^n) + \beta \leq 0$. Then, the function λ defined by

$$\lambda(x) = \alpha\big(q(x, t_n) + q(x - h, t_n)\big) + \beta \tag{3.78}$$

admits a zero denoted x_* in $[x_i, x_{i+1}]$. Since $s \geq 3$, λ is of class $\mathcal{C}^2$ and verifies

$$\begin{aligned} \lambda(x_i) &= \lambda(x_*) + \mathcal{O}(x_i - x_*) = \mathcal{O}(h) \, , \\ \lambda(x_{i+1}) &= \lambda(x_*) + \mathcal{O}(x_{i+1} - x_*) = \mathcal{O}(h) \, . \end{aligned} \tag{3.79}$$

In the same way, we have

$$f'(q_i^n) = 2\alpha q_i^n + \beta = \lambda(x_{i+1}) + \mathcal{O}(h) = \mathcal{O}(h) \, . \tag{3.80}$$

The local truncation error is of the form

$$e(x_i, t_n) = \frac{q_i^n - q_i^{n+1}}{k} = -\partial_t q(x_i, t_n) + \mathcal{O}(k) = f'\big(q(x_i, t_n)\big) \, \partial_x q(x_i, t_n) + \mathcal{O}(k) \, . \tag{3.81}$$

Then, using (3.80), we obtain

$$e(x_i, t_n) = \mathcal{O}(h + k) \, . \tag{3.82}$$

- Fourth case: we assume that $\alpha(q_{i+1}^n + q_i^n) + \beta \leq 0$ and $\alpha(q_i^n + q_{i-1}^n) + \beta \geq 0$. We proceed as we did in the precedent case, i.e. we deduce that the function λ defined by (3.78) admits a zero $x_* \in [x_i, x_{i+1}]$, and then the result (3.80) is true. Hence,

$$\begin{aligned} e(x_i, t_n) &= \frac{q_i^n - q_i^{n+1}}{k} - \frac{f(q_{i+1}^n) - f(q_{i-1}^n)}{h} \\ &= -\partial_t q(x_i, t_n) - 2\partial_x \Big(f\big(q(x_i, t_n)\big)\Big) + \mathcal{O}(h + k) \\ &= -f'(q_i^n)\partial_x q(x_i, t_n) + \mathcal{O}(h + k) \\ &= \mathcal{O}(h + k) \, . \end{aligned} \tag{3.83}$$

We conclude that, in any case, the local truncation error is $\mathcal{O}(h + k)$, i.e. first order accurate. $\square$

4 Numerical results

The first goal of this section is to numerically show that $u^\epsilon(x, t) - U_h(x, \frac{t}{\epsilon}, t) = \mathcal{O}(\epsilon)$ and $\rho^\epsilon(x, t) - R_h(x, \frac{t}{\epsilon}, t) = \mathcal{O}(\epsilon)$. Secondly, we will briefly analyze the gain in term of CPU time our two-scale numerical method brings when compared with a classical method consisting in solving directly (1.1). Lastly, by using our method, we will explore the simulation of experiments which were not accessible before because of too small Mach number.

4.1 Convergence in ϵ

We consider here the initial data u_0 and ρ_0 defined by

$$u_0(x) = 1 + \frac{\cos(x)}{2} \quad \text{and} \quad \rho_0(x) = 1 + \frac{\sin(x)}{2}, \tag{4.1}$$

and we take a uniform mesh with $N_x + 1$ points denoted $x_0, \ldots, x_{N_x}$.

As illustrated in figures 1 and 2, the functions $U_h(x, \frac{t}{\epsilon}, t)$ and $R_h(x, \frac{t}{\epsilon}, t)$ are very close to u^ϵ and ρ^ϵ respectively. The numerical experiment showed in these figures is made with $\epsilon = 0.05$, $\gamma = 1$, and $N_x = 1023$, $T = 2.5$.
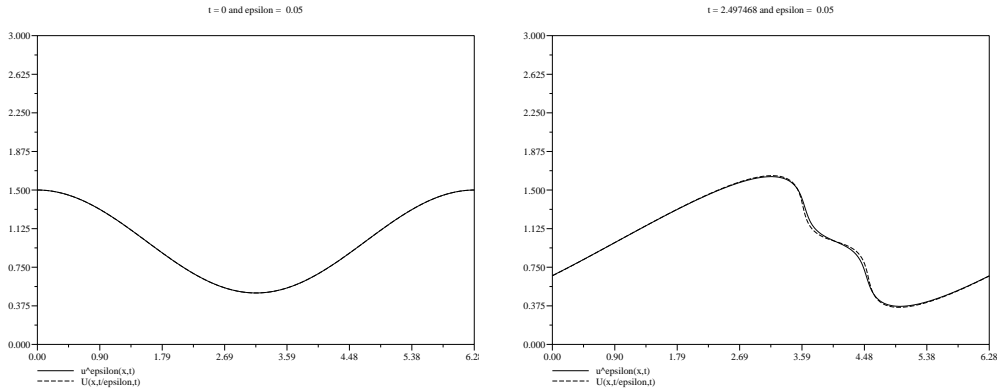
Figure 1: Comparison of $u^\epsilon(\cdot, t)$ and $U_h(\cdot, \frac{t}{\epsilon}, t)$ at times $t = 0$ and $t = 2.5$.
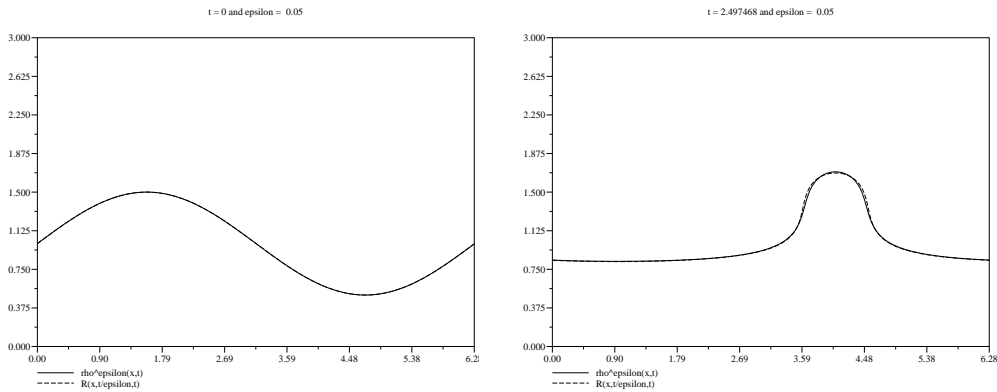
Figure 2: Comparison of $\rho^\epsilon(\cdot, t)$ and $R_h(\cdot, \frac{t}{\epsilon}, t)$ at times $t = 0$ and $t = 2.5$.

In order to quantify this good accuracy of $U_h(x, \frac{t}{\epsilon}, t)$ with u^ϵ and $R_h(x, \frac{t}{\epsilon}, t)$ with ρ^ϵ, we compute the errors $u^\epsilon(x, t) - U_h(x, \frac{t}{\epsilon}, t)$ and $\rho^\epsilon(x, t) - R_h(x, \frac{t}{\epsilon}, t)$ in $L^p\big([0, T) \times \mathbb{T}^1\big)$ norm ($p = 1, 2, \infty$), for several values of ϵ ranging in $\{0.01, 0.03, 0.05, 0.07, 0.1\}$, and we obtain the figures and the array below:
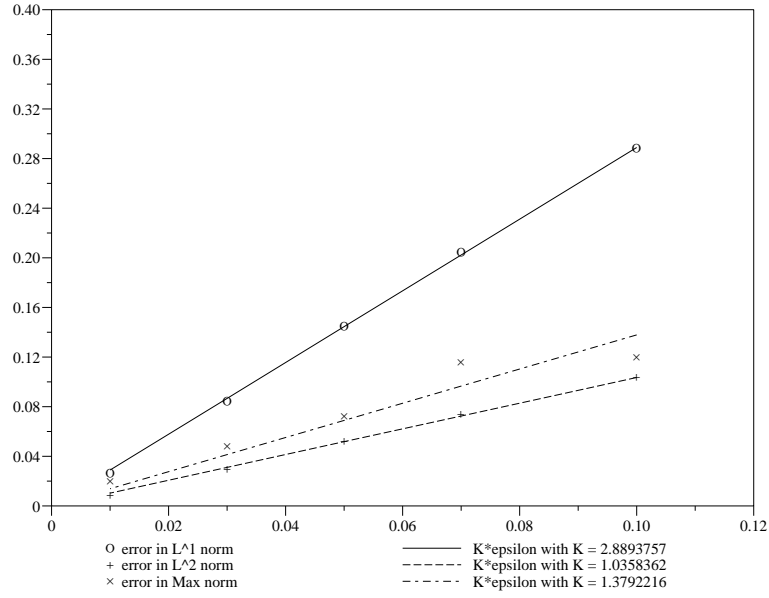
Figure 3: Error $u^\epsilon(x,t) - U_h(x, \frac{t}{\epsilon}, t)$ in L^1, L^2 and L^∞ norms.
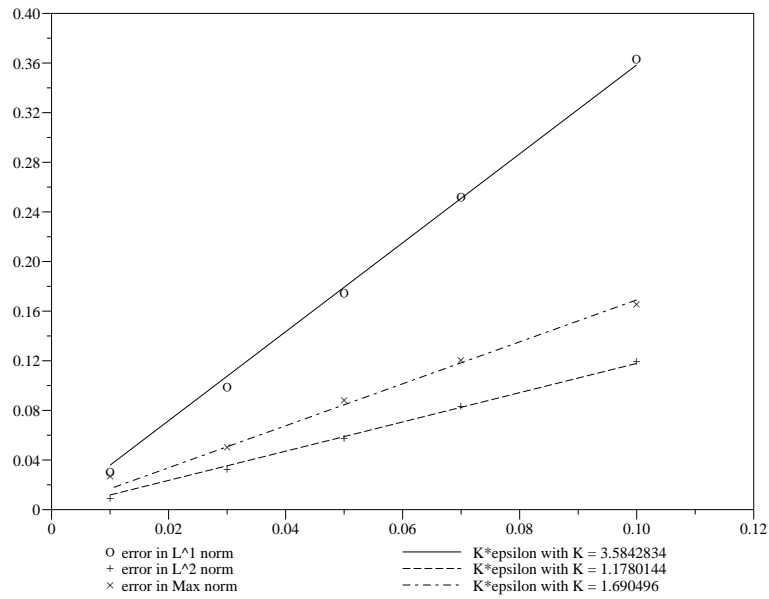
Figure 4: Error $\rho^\epsilon(x,t) - R_h(x, \frac{t}{\epsilon}, t)$ in L^1, L^2 and L^∞ norms.

Value	Error $u^\epsilon(x,t) - U_h(x, \frac{t}{\epsilon}, t)$			Error $\rho^\epsilon(x,t) - R_h(x, \frac{t}{\epsilon}, t)$		
of ϵ	L^1 norm	L^2 norm	L^∞ norm	L^1 norm	L^2 norm	L^∞ norm
0.1	0.2880016	0.1034517	0.1201132	0.3634635	0.1192177	0.1651374
0.07	0.2043288	0.0735498	0.1160378	0.2516643	0.0828920	0.1203342
0.05	0.1452756	0.0518309	0.0719549	0.1744198	0.0575395	0.0879887
0.03	0.0845621	0.0296311	0.0483544	0.0987818	0.0326775	0.0499608
0.01	0.0260695	0.0085251	0.0195340	0.0303524	0.0092954	0.0269734

Table 1: Errors $u^\epsilon(x,t) - U_h(x, \frac{t}{\epsilon}, t)$ and $\rho^\epsilon(x,t) - R_h(x, \frac{t}{\epsilon}, t)$.

The results above show an error $u^\epsilon(x,t) - U_h(x, \frac{t}{\epsilon}, t)$ in L^1 norm decreasing when $\epsilon \to 0$ as $K_1 \epsilon$ with $K_1 \approx 2.8893757$, as $K_2 \epsilon$ with $K_2 \approx 1.0358362$ in $L^2([0,T) \times \mathbb{T}^1)$ norm, and and as $K_\infty \epsilon$ with $K_\infty \approx 1.3792216$ in $L^\infty([0,T) \times \mathbb{T}^1)$ norm. Concerning the error $\rho^\epsilon(x,t) - R_h(x, \frac{t}{\epsilon}, t)$, we observe that it decreases when $\epsilon \to 0$ as $K_1' \epsilon$ with $K_1' \approx 3.5842834$ in $L^1([0,T) \times \mathbb{T}^1)$ norm, as $K_2' \epsilon$ with $K_2' \approx 1.1780144$ in $L^2([0,T) \times \mathbb{T}^1)$ norm, and and as $K_\infty' \epsilon$ with $K_\infty' \approx 1.690496$ in $L^\infty([0,T) \times \mathbb{T}^1)$ norm.

4.2 CPU time cost

One of the motivations of the present paper is the high CPU time cost of classical methods when used with small Mach number, as we explained it in the introduction. For example, if we apply Roe's finite volume scheme on the model (1.1), we must verify the CFL condition

$$\frac{k}{h} \max_{i,n} \left| \hat{u}_i^n \pm \frac{\sqrt{\overline{P}_i^n}}{\epsilon} \right| \leq 1 , \qquad (4.2)$$

where h is the space step, k is the time step, $\hat{u}_i^n$ and $\overline{P}_i^n$ are defined by

$$\hat{u}_i^n = \frac{u_i^n \sqrt{1 + \epsilon \rho_i^n} + u_{i-1}^n \sqrt{1 + \epsilon \rho_{i-1}^n}}{\sqrt{1 + \epsilon \rho_i^n} + \sqrt{1 + \epsilon \rho_{i-1}^n}} ,$$

$$\overline{P}_i^n = \begin{cases} \frac{(1+\epsilon\rho_i^n)^\gamma - (1+\epsilon\rho_{i-1}^n)^\gamma}{\gamma \epsilon (\rho_i^n - \rho_{i-1}^n)} & \text{if } \rho_i^n \neq \rho_{i-1}^n , \\ (1 + \epsilon \rho_i^n)^{\gamma - 1} & \text{else} . \end{cases} \qquad (4.3)$$

Hence, in order to garantee the stability of the scheme, the smaller ϵ is, the smaller k must be (and the higher the CPU time cost will be). One advantage of our two-scale numerical method is that its CFL condition (3.62) does not depend on ϵ, so k does not have to diminish with ϵ.

Value	Roe's method	Two-scale numerical method on (2.3)-(2.5)	
of ϵ	on (1.1)	Computation of (F_h, B_h)	Computation of (U_h, R_h)
0.1	24 m 34 s 36"	3 m 45 s 31"	42 s 79"
0.07	32 m 55 s 44"	3 m 45 s 22"	42 s 93"
0.05	44 m 7 s 21"	3 m 45 s 38"	42 s 74"
0.03	1 h 10 m 15 s	3 m 45 s 34"	42 s 80"
0.01	3 h 20 m 53 s 78"	3 m 45 s 12"	42 s 91"

Table 2: Comparison in terms of CPU time cost between Roe's method on (1.1) and the two-scale numerical method on (2.3)-(2.5) with the initial data (4.1), $N_x = 1023$, $\gamma = 1$.

The array above gives the comparison in terms of CPU time cost between Roe's method applied on the non-homogenized model (1.1), and our two-scale numerical method. These results have been obtained with the computer characteristics below:

- processor: Pentium© M 715,

- memory: 512 Mo DDR-RAM,

- operating system: SuSe© 9.1 Pro,

- compiler: gcc 3.3.3-41.

In particular, we distinguish the time cost for the computation of F_h and B_h on one hand, and the time cost for the reconstruction of $U_h(x, \frac{t}{\epsilon}, t)$ and $R_h(x, \frac{t}{\epsilon}, t)$ on the other hand.

4.3 Numerical experiments with small Mach number

In this paragraph, we present some numerical results obtained with our two-scale numerical method on simulations of experiments inducing a very small Mach number. These results are obtained with the initial data (4.1), $N_x + 1 = 1024$ points in x, and $\gamma = 1$.
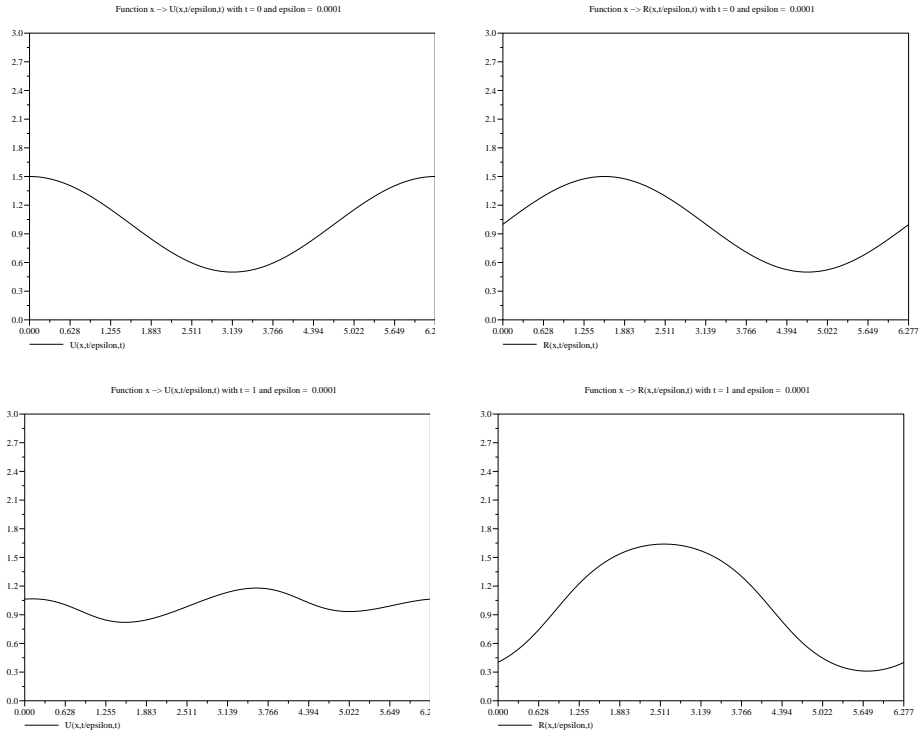

Figure 5: $U_h(x, \frac{t}{\epsilon}, t)$ (left) and $R_h(x, \frac{t}{\epsilon}, t)$ (right) with $\epsilon = 10^{-4}$ at times $t = 0$, $t = 1$.
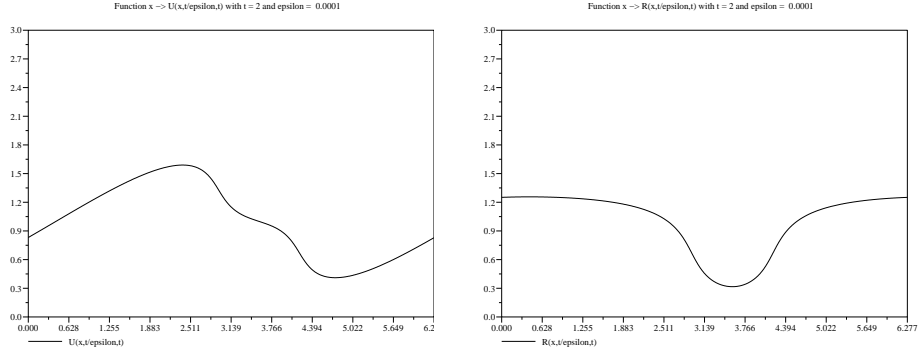
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Figure 6: $U_h(x, \frac{t}{\epsilon}, t)$ (left) and $R_h(x, \frac{t}{\epsilon}, t)$ (right) with $\epsilon = 10^{-4}$ at time $t = 2$.
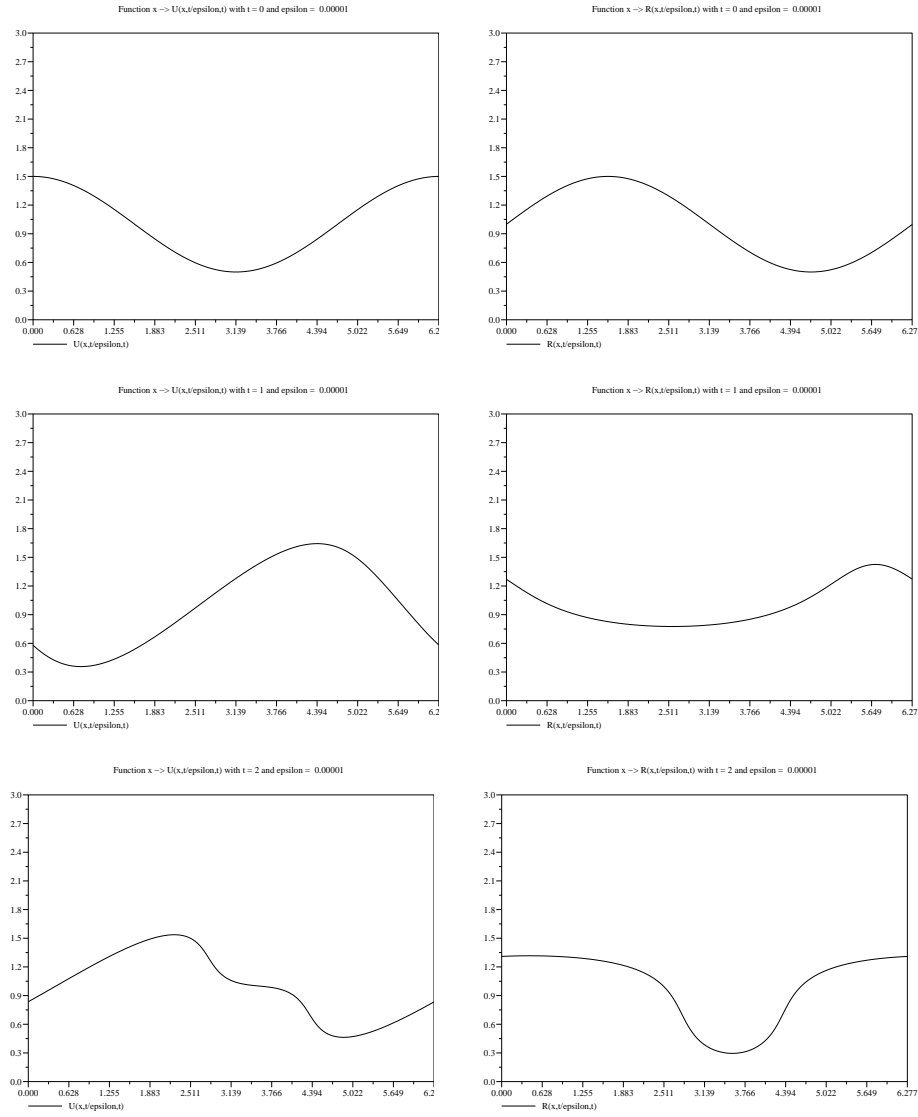

Figure 7: $U_h(x, \frac{t}{\epsilon}, t)$ (left) and $R_h(x, \frac{t}{\epsilon}, t)$ (right) with $\epsilon = 10^{-5}$ at times $t = 0$, $t = 1$, $t = 2$.
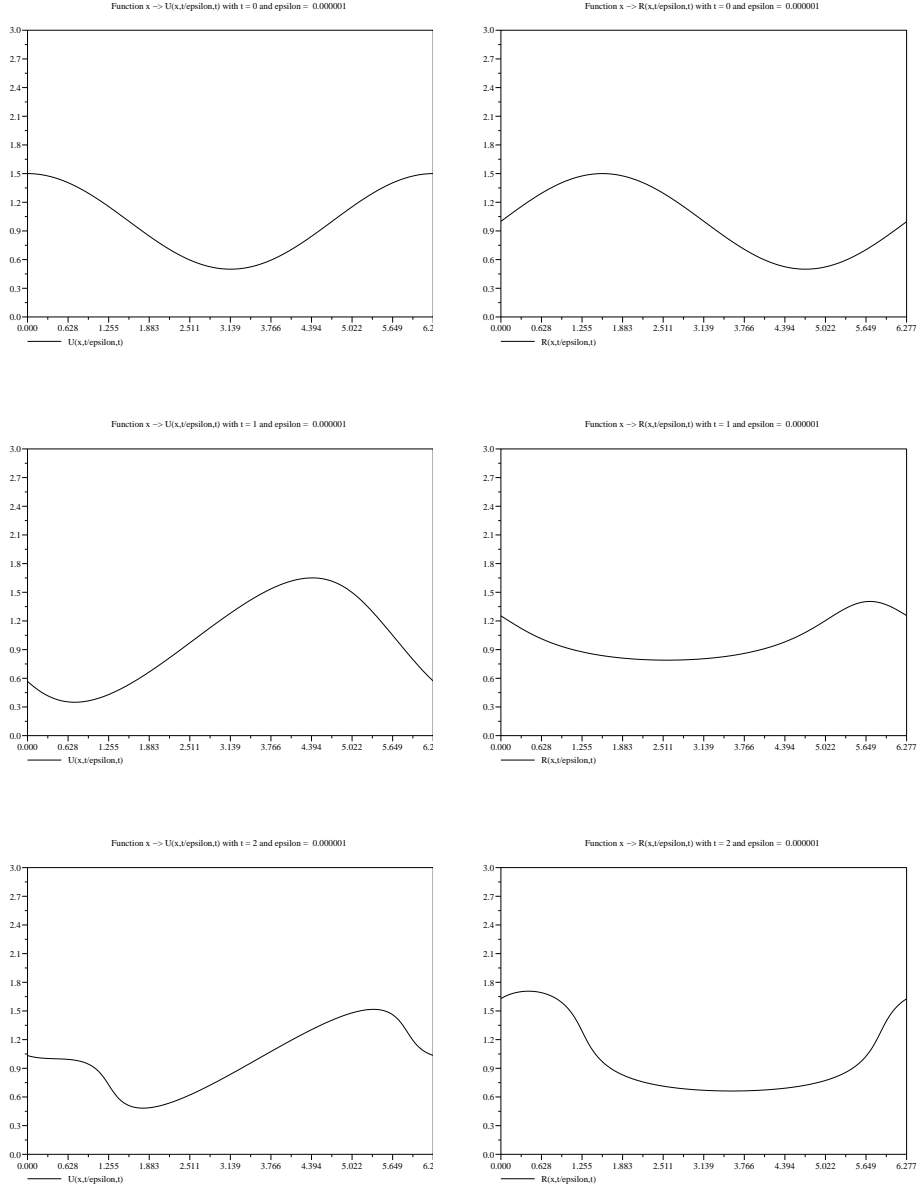
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Figure 8: $U_h(x, \frac{t}{\epsilon}, t)$ (left) and $R_h(x, \frac{t}{\epsilon}, t)$ (right) with $\epsilon = 10^{-6}$ at times $t = 0$, $t = 1$, $t = 2$.

We can notice that these results have been obtained after a few minutes of computation just as described in the table below:

Value of ϵ	Computation of (F_h, B_h)	Computation of (U_h, R_h)
10^{-4}	3 m 45 s 71"	42 s 84"
10^{-5}	3 m 45 s 95"	42 s 87"
10^{-6}	3 m 45 s 75"	42 s 98"

Furthermore, if we continue the simulation to final time $T \approx 3.2$ for ϵ ranging in $\{10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}, 10^{-6}\}$, we remark that discontinuities appear at time $t \approx$

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3.11 independently of ϵ and that, for ϵ ranging in $\{0.01, 0.03, 0.05, 0.07, 0.1\}$, these discontinuities also appear in the solution $(u^\epsilon, \rho^\epsilon)$ of (1.1) at the same time.

5 Conclusion

In this paper, we have developped a two-scale numerical method for the weakly compressible 1D Euler equations. By using two-scale convergence tools from Nguetseng[18] and Allaire[2], we have developped a new model independent of the Mach number ϵ. Then, using a finite volume scheme on this model, we have obtained numerical approximations that converge to the solutions of the weakly compressible 1D Euler equation when we refine the mesh on $\mathbb{T}^1 \times [0, T]$ and when $\epsilon \to 0$.

Furthermore, we have proved that this two-scale numerical method is first order accurate in ϵ. One the other hand, numerical results have confirmed this accuracy in ϵ and also have proved that, even with a very small Mach number, the two-scale numerical method allows us to simulate experiments with a reasonable CPU time cost.

Motived by the behaviour of the two-scale method, we are now ready to develop such a method on a model which describes a plasma submitted to a strong magnetic field such as the Vlasov-Poisson model established in Frénod and Sonnendrücker [7]. Furthermore, we only studied the behaviour of the two-scale numerical method in front of smooth initial data: it can be interesting to study the behaviour of the method on non-smooth initial data. Concerning the periodicity, we have built our numerical method in a periodic framework for mathematical reasons. However, it must be possible to extend this method to almost periodic framework in order to study problems like the noise generated by the blades of a turbine or a fan. If we go further, it can be interesting to improve the method in order to use it for studying the noise generated by multidimensional structures like cooling systems for computers or air conditioning, even if the links between time scales and space scales are not clear in this framework.

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