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On the p -adic Leopoldt transform of a power series

Bruno Anglès¹

Let p be an odd prime number. Let X be the projective limit for the norm maps of the p -Sylow subgroups of the ideal class groups of $\mathbb{Q}(\zeta_{p^{n+1}})$, $n \geq 0$. Let $\Delta = \text{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$ and let θ be an even and non-trivial character of Δ . Then X is a $\mathbb{Z}_p[[T]]$ -module and the characteristic ideal of the isotypic component $X(\omega\theta^{-1})$ is generated by a power series $f(T, \theta) \in \mathbb{Z}_p[[T]]$ such that (see for example [2]):

$$\forall n \geq 1, n \equiv 0 \pmod{p-1}, f((1+p)^{1-n} - 1, \theta) = L(1-n, \theta),$$

where $L(s, \theta)$ is the usual Dirichlet L -series. Therefore, it is natural and interesting to study the properties of the power series $f(T, \theta)$.

We denote by $\overline{f(T, \theta)} \in \mathbb{F}_p[[T]]$ the reduction of $f(T, \theta)$ modulo p . Then B. Ferrero and L. Washington have proved ([3]):

$$\overline{f(T, \theta)} \neq 0.$$

Note that, in fact, we have ([1]):

$$\overline{f(T, \theta)} \notin \mathbb{F}_p[[T^p]].$$

W. Sinnott has proved the following ([8]):

$$\overline{f(T, \theta)} \notin \mathbb{F}_p(T).$$

But, note that $\forall a \in \mathbb{Z}_p^*, \mathbb{F}_p[[T]] = \mathbb{F}_p[[(1+T)^a - 1]]$. Therefore it is natural to introduce the notion of a pseudo-polynomial which is an element $F(T)$ in $\mathbb{F}_p[[T]]$ such that there exist an integer $r \geq 1$, $c_1, \dots, c_r \in \mathbb{F}_p$, $a_1, \dots, a_r \in \mathbb{Z}_p$, such that $F(T) = \sum_{i=1}^r c_i (1+T)^{a_i}$. An element of $\mathbb{F}_p[[T]]$ will be called a pseudo-rational function if it is the quotient of two pseudo-polynomials. In this paper, we prove that $\overline{f(T, \theta)}$ is not a pseudo-rational function (part 1) of Theorem 4.5). This latter result suggests the following question: is $\overline{f(T, \theta)}$ algebraic over $\mathbb{F}_p(T)$? We suspect that this is not the case but we

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have no evidence for it. Note that, by the result of Ferrero and Washington, we can write:

$$\overline{f(T, \theta)} = T^{\lambda(\theta)} U(T),$$

where $\lambda(\theta) \in \mathbb{N}$ and $U(T) \in \mathbb{F}_p[[T]]^*$. S. Rosenberg has proved that ([6]):

$$\lambda(\theta) \leq (4p(p-1))^{\phi(p-1)},$$

where ϕ is Euler's totient function. In this paper, we improve Rosenberg's bound (part 2) of Theorem 4.5):

$$\lambda(\theta) < \left(\frac{p-1}{2}\right)^{\phi(p-1)}.$$

This implies that the lambda invariant of the field $\mathbb{Q}(\zeta_p)$ is less than $2\left(\frac{p-1}{2}\right)^{\phi(p-1)+1}$ (see Corollary 4.6 for the precise statement for an abelian number field). Note that this bound is certainly far from the truth, because according to a heuristic argument due to Ferrero and Washington (see [5]) and to Grennberg's conjecture:

$$\lambda(\mathbb{Q}(\zeta_p)) = \sum_{\theta \in \widehat{\Delta}, \theta \neq 1 \text{ and even}} \lambda(\theta) \leq \frac{\text{Log}(p)}{\text{Log}(\text{Log}(p))}.$$

The author is indebted to Warren Sinnott for communicating some of his unpublished works (note that Lemma 4.2 is due to Warren Sinnott). The author also thanks Filippo Nuccio for pointing out the work of J. Kraft and L. Washington ([4]).

1 Notations

Let p be an odd prime number and let K be a finite extension of \mathbb{Q}_p . Let O_K be the valuation ring of K and let π be a prime of K . We set $\mathbb{F}_q = O_K/\pi O_K$, it is a finite field having q elements and its characteristic is p . Let T be an indeterminate over K , we set $\Lambda = O_K[[T]]$. Observe that $\Lambda/\pi\Lambda \simeq \mathbb{F}_q[[T]]$. Let $F(T) \in \Lambda \setminus \{0\}$, then we can write in an unique way ([9], Theorem 7.3):

$$F(T) = \pi^{\mu(F)} P(T) U(T),$$

where $U(T)$ is a unit of Λ , $\mu(F) \in \mathbb{N}$, $P(T) \in O_K[[T]]$ is a monic polynomial such that $P(T) \equiv T^{\lambda(F)} \pmod{\pi}$ for some integer $\lambda(F) \in \mathbb{N}$. If $F(T) = 0$, we set $\mu(F) = \lambda(F) = \infty$. An element $F(T) \in \Lambda$ is called a pseudo-polynomial (see also [6], Definition 2) if there exist some integer $r \geq 1$, $c_1, \dots, c_r \in O_K$, $a_1, \dots, a_r \in \mathbb{Z}_p$, such that:

$$F(T) = \sum_{i=1}^r c_i (1+T)^{a_i}.$$

We denote the ring of pseudo-polynomials in Λ by A . Let $\delta \in \mathbb{Z}/(p-1)\mathbb{Z}$ and $F(T) \in \Lambda$, we set:

$$\gamma_\delta(F(T)) = \frac{1}{p-1} \sum_{\eta \in \mu_{p-1}} \eta^\delta F((1+T)^\eta - 1).$$

Then $\gamma_\delta : \Lambda \rightarrow \Lambda$ is a O_K -linear map and:

- for $\delta, \delta' \in \mathbb{Z}/(p-1)\mathbb{Z}$, $\gamma_\delta \gamma_{\delta'} = 0$ if $\delta \neq \delta'$ and $\gamma_\delta^2 = \gamma_\delta$,
- $\sum_{\delta \in \mathbb{Z}/(p-1)\mathbb{Z}} \gamma_\delta = \text{Id}_\Lambda$.

For $F(T) \in \Lambda$, we set:

$$D(F(T)) = (1+T) \frac{d}{dT} F(T),$$

$$U(F(T)) = F(T) - \frac{1}{p} \sum_{\zeta \in \mu_p} F(\zeta(1+T) - 1) \in \Lambda.$$

Then $D, U : \Lambda \rightarrow \Lambda$ are O_K -linear maps. Observe that:

- $U^2 = U$,
- $DU = UD$,
- $\forall \delta \in \mathbb{Z}/(p-1)\mathbb{Z}$, $\gamma_\delta U = U \gamma_\delta$,
- $\forall \delta \in \mathbb{Z}/(p-1)\mathbb{Z}$, $D \gamma_\delta = \gamma_{\delta+1} D$.

If $F(T) \in \Lambda$, we denote its reduction modulo π by $\overline{F(T)} \in \mathbb{F}_q[[T]]$. If $f : \Lambda \rightarrow \Lambda$ is a O_K -linear map, we denote its reduction modulo π by $\overline{f} : \mathbb{F}_q[[T]] \rightarrow \mathbb{F}_q[[T]]$. For all $n \geq 0$, we set $\omega_n(T) = (1+T)^{p^n} - 1$.

Let B be a commutative and unitary ring. We denote the set of invertible elements of B by B^* .

We fix κ a topological generator of $1+p\mathbb{Z}_p$. Let $x \in \mathbb{Z}_p$ and let $n \geq 1$, we denote the unique integer $k \in \{0, \dots, p^n - 1\}$ such that $x \equiv k \pmod{p^n}$ by $[x]_n$. Let $\omega : \mathbb{Z}_p^* \rightarrow \mu_{p-1}$ be the Teichmüller character, i.e. $\forall a \in \mathbb{Z}_p^*$, $\omega(a) \equiv a \pmod{p}$. Let $x, y \in \mathbb{Z}_p$, we write:

- $x \sim y$ if there exists $\eta \in \mu_{p-1}$ such that $y = \eta x$,
- $x \equiv y \pmod{\mathbb{Q}^*}$ if there exists $z \in \mathbb{Q}^*$ such that $y = zx$.

The function Log_p will denote the usual p -adic logarithm. v_p will denote the usual p -adic valuation on \mathbb{C}_p such that $v_p(p) = 1$.

Let ρ be a Dirichlet character of conductor f_ρ . Recall that the Bernoulli numbers $B_{n,\rho}$ are defined by the following identity:

$$\sum_{a=1}^{f_\rho} \frac{\rho(a)e^{aZ}}{e^{f_\rho Z} - 1} = \sum_{n \geq 0} \frac{B_{n,\rho}}{n!} Z^{n-1},$$

where $e^Z = \sum_{n \geq 0} Z^n/n!$. If $\rho = 1$, for $n \geq 2$, $B_{n,1}$ is the n th Bernoulli number.

Let $x \in \mathbb{R}$. We denote the biggest integer less than or equal to x by $[x]$. The function Log will denote the usual logarithm.

2 Preliminaries

Let $\delta \in \mathbb{Z}/(p-1)\mathbb{Z}$. In this section, we will recall the construction of the p -adic Leopoldt transform Γ_δ (see [5], Theorem 6.2) which is a O_K -linear map from Λ to Λ .

First, observe that $(\pi^n, \omega_n(T)) = \pi^n \Lambda + \omega_n(T) \Lambda$, $n \geq 1$, is a basis of neighbourhood of zero in Λ :

Lemma 2.1

- 1) $\forall n \geq 1$, $(\pi, T)^{2n} \subset (\pi^n, T^n) \subset (\pi, T)^n$.
- 2) $\forall n \geq 1$, $\omega_n(T) \in (p^{[n/2]}, T^{p^{[n/2]+1}})$.
- 3) Let $N \geq 1$, set $n = [\text{Log}(N)/\text{Log}(p)]$. We have:

$$T^N \in (p^{[n/2]}, \omega_{[n/2]+1}(T)).$$

Proof Note that assertion 1) is obvious. Assertion 2) comes from the fact:

$$\forall k \in \{1, \dots, p^n\}, v_p\left(\frac{p^n!}{k!(p^n - k)!}\right) = n - v_p(k).$$

To prove assertion 3), it is enough to prove the following:

$\forall n \geq 0$, there exist $\delta_0^{(n)}(T), \dots, \delta_n^{(n)}(T) \in \mathbb{Z}[T]$ such that:

$$T^{p^n} = \sum_{i+j=n} \omega_i(T) p^j \delta_j^{(n)}(T).$$

Let's prove this latter fact by recurrence on n . Note that the result is clear if $n = 0$. Let's assume that it is true for n and let's prove the assertion for $n + 1$. Let $r(T) \in \mathbb{Z}[T]$ such that:

$$\frac{\omega_{n+1}(T)}{\omega_n(T)} + pr(T) = T^{p^n(p-1)}.$$

Then:

$$T^{p^{n+1}} = T^{p^n} \frac{\omega_{n+1}(T)}{\omega_n(T)} + pr(T) T^{p^n}.$$

Note that there exists $q(T) \in \mathbb{Z}[T]$ such that:

$$\frac{\omega_{n+1}(T)}{\omega_n(T)} = \omega_n(T)^{p-1} + pq(T).$$

Thus:

$$T^{p^{n+1}} = \omega_{n+1}(T) \delta_0^{(n)}(T) + \sum_{i+j=n, j \geq 1} (\omega_n(T)^{p-1} + pq(T)) \omega_i(T) p^j \delta_j^{(n)}(T) + \sum_{i+j=n} \omega_i(T) p^{j+1} \delta_j^{(n)}(T) r(T)$$

Thus, there exist $\delta_0^{(n+1)}(T), \dots, \delta_{n+1}^{(n+1)}(T) \in \mathbb{Z}[T]$ such that:

$$T^{p^{n+1}} = \sum_{i+j=n+1} \omega_i(T) p^j \delta_j^{(n+1)}(T). \diamond$$

The following Lemma will be useful in the sequel (for a similar result see [6], Lemma 5):

Lemma 2.2 *Let $F(T) \in A$. Write $F(T) = \sum_{i=1}^r \beta_i (1+T)^{\alpha_i}$, $\beta_1, \dots, \beta_r \in O_K$, $\alpha_1, \dots, \alpha_r \in \mathbb{Z}_p$, and $\alpha_i \neq \alpha_j$ for $i \neq j$. Let $N = \text{Max}\{v_p(\alpha_i - \alpha_j), i \neq j\}$. Let $n \geq 1$ be an integer. Then:*

$$F(T) \equiv 0 \pmod{(\pi^n, \omega_{N+1}(T))} \Leftrightarrow \forall i = 1, \dots, r, \beta_i \equiv 0 \pmod{\pi^n}.$$

Proof We have:

$$F(T) \equiv \sum_{i=1}^r \beta_i (1+T)^{[\alpha_i]_{N+1}} \pmod{\omega_{N+1}(T)}.$$

Therefore $F(T) \equiv 0 \pmod{(\pi^n, \omega_{N+1}(T))}$ if and only if we have:

$$\sum_{i=1}^r \beta_i (1+T)^{[\alpha_i]_{N+1}} \equiv 0 \pmod{\pi^n}.$$

But for $i \neq j$, $[\alpha_i]_{N+1} \neq [\alpha_j]_{N+1}$. Therefore $\sum_{i=1}^r \beta_i (1+T)^{[\alpha_i]_{N+1}} \equiv 0 \pmod{\pi^n}$ if and only if:

$$\forall i = 1, \dots, r, \beta_i \equiv 0 \pmod{\pi^n}. \diamond$$

Observe that U, D, γ_δ are continuous O_K -linear maps by Lemma 2.1 and the following Lemma:

Lemma 2.3 *Let $F(T) \in \Lambda$ and let $n \geq 0$.*

- 1) $F(T) \equiv 0 \pmod{\omega_n(T)} \Rightarrow \gamma_\delta(F(T)) \equiv 0 \pmod{\omega_n(T)}$.
- 2) $F(T) \equiv 0 \pmod{\omega_n(T)} \Rightarrow D(F(T)) \equiv 0 \pmod{(p^n, \omega_n(T))}$.
- 3) If $n \geq 1$, $F(T) \equiv 0 \pmod{\omega_n(T)} \Rightarrow U(F(T)) \equiv 0 \pmod{\omega_n(T)}$.

Proof The assertions 1) and 2) are obvious. It remains to prove 3). Observe that, by [9], Proposition 7.2, we have:

$$\forall G(T) \in \Lambda, G(T) \equiv 0 \pmod{\omega_n(T)} \Leftrightarrow \forall \zeta \in \mu_{p^n}, G(\zeta - 1) = 0.$$

Now, let $F(T) \in \Lambda$, $F(T) \equiv 0 \pmod{\omega_n(T)}$. Since the map: $\mu_{p^n} \rightarrow \mu_{p^n}$, $x \mapsto \zeta x$, is a bijection for all $\zeta \in \mu_{p^n}$, we get:

$$\forall \zeta \in \mu_{p^n}, U(F)(\zeta - 1) = 0.$$

Therefore:

$$U(F(T)) \equiv 0 \pmod{\omega_n(T)}. \diamond$$

Let $s \in \mathbb{Z}_p$. For $n \geq 0$, set:

$$k_n(s, \delta) = [s]_{n+1} + \delta_n p^{n+1} \in \mathbb{N} \setminus \{0\},$$

where $\delta_n \in \{1, \dots, p-1\}$ is such that $[s]_{n+1} + \delta_n \equiv \delta \pmod{p-1}$. Observe that:

- $\forall n \geq 0, k_n(s, \delta) \equiv \delta \pmod{p-1}$ and $k_n(s, \delta) \equiv s \pmod{p^{n+1}}$,
- $\forall n \geq 0, k_{n+1}(s, \delta) > k_n(s, \delta)$,
- $s = \lim_n k_n(s, \delta)$.

In particular:

$$\forall a \in \mathbb{Z}_p, \forall n \geq 0, a^{k_{n+1}(s, \delta)} \equiv a^{k_n(s, \delta)} \pmod{p^{n+1}}.$$

Now, let $F(T) \in A$. Write $F(T) = \sum_{i=1}^r \beta_i (1+T)^{\alpha_i}$, $\beta_1, \dots, \beta_r \in O_K$, $\alpha_1, \dots, \alpha_r \in \mathbb{Z}_p$. We set:

$$\Gamma_\delta(F(T)) = \sum_{\alpha_i \in \mathbb{Z}_p^*} \beta_i \omega^\delta(\alpha_i) (1+T)^{\frac{\text{Log}_p(\alpha_i)}{\text{Log}_p(\kappa)}}.$$

Thus, we have a surjective O_K -linear map: $\Gamma_\delta : A \rightarrow A$. Note that:

Lemma 2.4 *Let $F(T) \in A$.*

1) *Let $s \in \mathbb{Z}_p$. Then:*

$$\forall n \geq 0, \Gamma_\delta(F)(\kappa^s - 1) \equiv D^{k_n(s, \delta)}(F)(0) \pmod{p^{n+1}}.$$

2) *Let $n \geq 1$. Assume that $F(T) \equiv 0 \pmod{\omega_n(T)}$. Then $\Gamma_\delta(F(T)) \equiv 0 \pmod{\omega_{n-1}(T)}$.*

Proof For $a \in \mathbb{Z}_p^*$, write $a = \omega(a) \langle a \rangle$, where $\langle a \rangle \in 1 + p\mathbb{Z}_p$. Let's write:

$$F(T) = \sum_{i=1}^r \beta_i (1+T)^{\alpha_i},$$

$\beta_1, \dots, \beta_r \in O_K$, $\alpha_1, \dots, \alpha_r \in \mathbb{Z}_p$. We have:

$$D^{k_n(s, \delta)}(F(T)) = \sum_{i=1}^r \beta_i \alpha_i^{k_n(s, \delta)} (1+T)^{\alpha_i}.$$

Thus:

$$D^{k_n(s, \delta)}(F(T)) \equiv \sum_{\alpha_i \in \mathbb{Z}_p^*} \beta_i \omega^\delta(\alpha_i) \langle \alpha_i \rangle^s (1+T)^{\alpha_i} \pmod{p^{n+1}}.$$

But recall that:

$$\Gamma_\delta(F)(\kappa^s - 1) = \sum_{\alpha_i \in \mathbb{Z}_p^*} \beta_i \omega^\delta(\alpha_i) \langle \alpha_i \rangle^s.$$

Assertion 1) follows easily. Now, let's suppose that $F(T) \equiv 0 \pmod{\omega_n(T)}$ for some $n \geq 1$. Then:

$$\forall a \in \{0, \dots, p^n - 1\}, \quad \sum_{\alpha_i \equiv a \pmod{p^n}} \beta_i = 0.$$

This implies that:

$$\forall a \in \{0, \dots, p^{n-1} - 1\}, \quad \sum_{\alpha_i \in \mathbb{Z}_p^*, \text{Log}_p(\alpha_i)/\text{Log}_p(\kappa) \equiv a \pmod{p^{n-1}}} \omega^\delta(\alpha_i) \beta_i = 0.$$

But recall that:

$$\Gamma_\delta(F(T)) = \sum_{\alpha_i \in \mathbb{Z}_p^*} \beta_i \omega^\delta(\alpha_i) (1 + T)^{\frac{\text{Log}_p(\alpha_i)}{\text{Log}_p(\kappa)}}.$$

Thus $\Gamma_\delta(F(T)) \equiv 0 \pmod{\omega_{n-1}(T)}$. \diamond

Proposition 2.5 *Let $F(T) \in \Lambda$. There exists an unique power series $\Gamma_\delta(F(T)) \in \Lambda$ such that:*

$$\forall s \in \mathbb{Z}_p \forall n \geq 0, \Gamma_\delta(F)(\kappa^s - 1) \equiv D^{k_n(s, \delta)}(F)(0) \pmod{p^{n+1}}.$$

Proof Let $(F_N(T))_{N \geq 0}$ be a sequence of elements in A such that:

$$\forall N \geq 0, F(T) \equiv F_N(T) \pmod{\omega_N(T)}.$$

Fix $N \geq 1$. Then:

$$\forall m \geq N, F_m(T) \equiv F_N(T) \pmod{\omega_N(T)}.$$

Therefore, by Lemma 2.4, we have:

$$\forall m \geq N, \Gamma_\delta(F_m(T)) \equiv \Gamma_\delta(F_N(T)) \pmod{\omega_{N-1}(T)}.$$

This implies that the sequence $(\Gamma_\delta(F_N(T)))_{N \geq 1}$ converges in Λ to some power series $G(T) \in \Lambda$. Observe that, since Λ is compact, we have:

$$\forall N \geq 1, G(T) \equiv \Gamma_\delta(F_N(T)) \pmod{\omega_{N-1}(T)}.$$

In particular:

$$\forall N \geq 1, G(\kappa^s - 1) \equiv \Gamma_\delta(F_N)(\kappa^s - 1) \pmod{p^N}.$$

Thus, applying Lemma 2.4, we get:

$$\forall N \geq 1, G(\kappa^s - 1) \equiv D^{k_{N-1}(s,\delta)}(F_N)(0) \pmod{p^N}.$$

But:

$$\forall N \geq 1, D^{k_{N-1}(s,\delta)}(F(T)) \equiv D^{k_{N-1}(s,\delta)}(F_N(T)) \pmod{(p^N, \omega_N(T))}.$$

Therefore:

$$\forall N \geq 1, G(\kappa^s - 1) \equiv D^{k_{N-1}(s,\delta)}(F)(0) \pmod{p^N}.$$

Now, set $\Gamma_\delta(F(T)) = G(T)$. The Proposition follows easily. \diamond

3 Some properties of the p -adic Leopoldt transform

We need the following fundamental result:

Proposition 3.1 *Let $\delta \in \mathbb{Z}/(p-1)\mathbb{Z}$ and let $F(T) \in \Lambda$. Let $m, n \in \mathbb{N} \setminus \{0\}$. then:*

$$\Gamma_\delta(F(T)) \equiv 0 \pmod{(\pi^n, \omega_{m-1}(T))} \Leftrightarrow \gamma_{-\delta}U(F(T)) \equiv 0 \pmod{(\pi^n, \omega_m(T))}.$$

Proof A similar result has been obtained by S. Rosenberg ([6], Lemma 8).

We begin by proving that Γ_δ is a continuous O_K -linear map. By Lemma 2.1, this comes from the following fact:

Let $F(T) \in \Lambda$. Let $n \geq 1$ and assume that $F(T) \equiv 0 \pmod{\omega_n(T)}$, then $\Gamma_\delta(F(T)) \equiv 0 \pmod{\omega_{n-1}(T)}$.

Indeed, let $(F_N(T))_{N \geq 0}$ be a sequence of elements in A such that:

$$\forall N \geq 0, F(T) \equiv F_N(T) \pmod{\omega_N(T)}.$$

By the proof of Proposition 2.5:

$$\forall N \geq 1, \Gamma_\delta(F(T)) \equiv \Gamma_\delta(F_N(T)) \pmod{\omega_{N-1}(T)}.$$

Now, by Lemma 2.4:

$$\Gamma_\delta(F_n(T)) \equiv 0 \pmod{\omega_{n-1}(T)}.$$

The assertion follows.

Now, since $\Gamma_\delta, \gamma_{-\delta}, U$ are continuous O_K -linear maps, it suffices to prove the Proposition in the case where $F(T) \in A$. Write $F(T) = \sum_{i=1}^r \beta_i (1+T)^{\alpha_i}$, $\beta_1, \dots, \beta_r \in O_K$, $\alpha_1, \dots, \alpha_r \in \mathbb{Z}_p$. Let $I \subset \{\alpha_1, \dots, \alpha_r\}$ be a set of representatives of the classes of $\alpha_1, \dots, \alpha_r$ for the relation \sim . For $x \in I$, $x \not\equiv 0 \pmod{p}$, set:

$$\beta_x = \sum_{\alpha_i \sim x} \beta_i \frac{\alpha_i}{x}.$$

We get:

$$(p-1)\gamma_{-\delta}U(F(T)) = \sum_{\eta \in \mu_{p-1}} \sum_{x \in I, x \in \mathbb{Z}_p^*} \eta^{-\delta} \beta_x (1+T)^{\eta x}.$$

Now observe that:

$$\Gamma_\delta(F(T)) = \Gamma_\delta \gamma_{-\delta} U(F(T)) = \sum_{x \in I, x \in \mathbb{Z}_p^*} \beta_x \omega^\delta(x) (1+T)^{\text{Log}_p(x)/\text{Log}_p(\kappa)}.$$

Therefore $\Gamma_\delta(F(T)) \equiv 0 \pmod{(\pi^n, \omega_{m-1}(T))}$ if and only if:

$$\forall a \in \{0, \dots, p^{m-1}-1\}, \quad \sum_{x \in I, x \in \mathbb{Z}_p^*, \text{Log}_p(x)/\text{Log}_p(\kappa) \equiv a \pmod{p^{m-1}}} \beta_x \omega^\delta(x) \equiv 0 \pmod{\pi^n}.$$

Now, observe that for $a \in \{0, \dots, p^m-1\}$, there exists at most one $\eta \in \mu_{p-1}$ such that $[\eta x]_m = a$, and if such a η exists it is equal to $\omega(a)\omega^{-1}(x)$. Therefore $\Gamma_\delta(F(T)) \equiv 0 \pmod{(\pi^n, \omega_{m-1}(T))}$ if and only if:

$$\forall a \in \{0, \dots, p^m-1\}, \quad \sum_{x \in I, x \in \mathbb{Z}_p^*, \exists \eta x \in \mu_{p-1}, [\eta x]_m = a} \beta_x \eta_x^{-\delta} \equiv 0 \pmod{\pi^n}.$$

This latter property is equivalent to $\gamma_{-\delta}U(F(T)) \equiv 0 \pmod{(\pi^n, \omega_m(T))}$. \diamond

Now, we can list the basic properties of Γ_δ :

Proposition 3.2 *Let $\delta \in \mathbb{Z}/(p-1)\mathbb{Z}$.*

- 1) $\Gamma_\delta : \Lambda \rightarrow \Lambda$ is a surjective and continuous O_K -linear map.
- 2) $\forall F(T) \in \Lambda$, $\Gamma_\delta(F(T)) = \Gamma_\delta \gamma_{-\delta} U(F(T))$.
- 3) $\forall a \in \mathbb{Z}_p^*$, $\Gamma_\delta(F((1+T)^a - 1)) = \omega^\delta(a) (1+T)^{\text{Log}_p(a)/\text{Log}_p(\kappa)} \Gamma_\delta(F(T))$.
- 4) Let κ' be another topological generator of $1+p\mathbb{Z}_p$ and let Γ'_δ be the p -adic Leopoldt transform associated to κ' and δ . Then:

$$\forall F(T) \in \Lambda, \Gamma'_\delta(F(T)) = \Gamma_\delta(F) ((1+T)^{\text{Log}_p(\kappa)/\text{Log}_p(\kappa')} - 1).$$

5) Let $F(T) \in \Lambda$. Then $\mu(\Gamma_\delta(F(T))) = \mu(\gamma_{-\delta}U(F(T)))$ and:

$$\forall N \geq 1, \lambda(\Gamma_\delta(F(T))) \geq p^{N-1} \Leftrightarrow \lambda(\gamma_{-\delta}U(F(T))) \geq p^N.$$

Proof The assertions 1),2),3),4) come from the fact that $\Gamma_\delta, \gamma_{-\delta}, U$ are continuous and that these assertions are true for pseudo-polynomials. The assertion 5) is a direct application of Proposition 3.1 . \diamond

Let's recall the following remarkable result due to W. Sinnott:

Proposition 3.3 Let $r_1(T), \dots, r_s(T) \in \mathbb{F}_q(T) \cap \mathbb{F}_q[[T]]$. Let $c_1, \dots, c_s \in \mathbb{Z}_p \setminus \{0\}$ and suppose that:

$$\sum_{i=1}^s r_i((1+T)^{c_i} - 1) = 0.$$

Then:

$$\forall a \in \mathbb{Z}_p, \sum_{c_i \equiv a \pmod{\mathbb{Q}^*}} r_i((1+T)^{c_i} - 1) \in \mathbb{F}_q.$$

Proof See [8], Proposition 1. \diamond

Let's give a first application of this latter result:

Proposition 3.4 Let $\delta \in \mathbb{Z}/(p-1)\mathbb{Z}$ and let $F(T) \in K(T) \cap \Lambda$.

1) If δ is odd or if $\delta = 0$, then:

$$\mu(\Gamma_\delta(F(T))) = \mu(U(F(T)) + (-1)^\delta U(F((1+T)^{-1} - 1))).$$

2) If δ is even and $\delta \neq 0$, then:

$$\mu(\Gamma_\delta(F(T))) = \mu(U(F(T)) + U(F((1+T)^{-1} - 1)) - 2U(F)(0)).$$

Proof The case $\delta = 0$ has already been obtained by Sinnott ([7], Theorem 1). We prove 1), the proof of 2) is quite similar. Now, observe that 1) is a consequence of Proposition 3.2 and the following fact:

Let $F(T) \in K(T) \cap \Lambda$, then $\mu(\gamma_{-\delta}(F(T))) = \mu(F(T) + (-1)^\delta F((1+T)^{-1} - 1))$.

Let's prove this fact. Let $r(T) \in \Lambda$, observe that:

$$\gamma_{-\delta}(r(T)) = (-1)^\delta \gamma_{-\delta}(r((1+T)^{-1} - 1)).$$

We can assume that $F(T) + (-1)^\delta F((1+T)^{-1} - 1) \neq 0$. Write:

$$F(T) + (-1)^\delta F((1+T)^{-1} - 1) = \pi^m G(T),$$

where $m \in \mathbb{N}$, and $G(T) \in \Lambda \setminus \pi\Lambda$. Note that $G(T) \in K(T)$. We must prove that $\gamma_{-\delta}(G(T)) \not\equiv 0 \pmod{\pi}$. Suppose that it is not the case, i.e. $\gamma_{-\delta}(G(T)) \equiv 0 \pmod{\pi}$. Then:

$$G(0) \equiv 0 \pmod{\pi}.$$

Furthermore, by Proposition 3.3, there exists $c \in O_K$ such that:

$$G(T) + (-1)^\delta G((1+T)^{-1} - 1) \equiv c \pmod{\pi}.$$

But, we must have $c \equiv 0 \pmod{\pi}$. Observe that:

$$G(T) = (-1)^\delta G((1+T)^{-1} - 1).$$

Therefore we get $G(T) \equiv 0 \pmod{\pi}$ which is a contradiction. \diamond

Lemma 3.5 *Let $F(T) \in \mathbb{F}_q(T) \cap \mathbb{F}_q[[T]]$. Then $F(T)$ is a pseudo-polynomial if and only if there exists some integer $n \geq 0$ such that $(1+T)^n F(T) \in \mathbb{F}_q[T]$.*

Proof Assume that $F(T)$ is a pseudo-polynomial. We can suppose that $F(T) \neq 0$. Write:

$$F(T) = \sum_{i=1}^r c_i (1+T)^{a_i},$$

where $c_1, \dots, c_r \in \mathbb{F}_q^*$, $a_1, \dots, a_r \in \mathbb{Z}_p$ and $a_i \neq a_j$ for $i \neq j$. Since $F(T) \in \mathbb{F}_q(T)$ there exist $m, n \in \mathbb{N} \setminus \{0\}$, $m > \text{Max}\{v_p(a_i - a_j), i \neq j\}$, such that:

$$(T^{q^n} - T)^{q^m} F(T) \in \mathbb{F}_q[T].$$

Thus:

$$\sum_{i=1}^r c_i (1+T)^{a_i + q^{n+m}} - \sum_{i=1}^r c_i (1+T)^{a_i + q^m} \in \mathbb{F}_q[T].$$

Observe that:

- $\forall i, j \in \{1, \dots, r\}$, $a_i + q^{n+m} \neq a_j + q^m$,
- $a_i + q^m = a_j + q^m \Leftrightarrow i = j$.

Thus, by Lemma 2.2, we get:

$$\forall i \in \{1, \dots, r\}, a_i + q^m \in \mathbb{N}.$$

Therefore $(1+T)^{q^m} F(T) \in \mathbb{F}_q[T]$. The Lemma follows. \diamond

Let's give a second application of Proposition 3.3:

Proposition 3.6 *Let $\delta \in \mathbb{Z}/(p-1)\mathbb{Z}$ and let $F(T) \in \mathbb{F}_q(T) \cap \mathbb{F}_q[[T]]$. Suppose that there exist an integer $r \in \{0, \dots, (p-3)/2\}$, $c_1, \dots, c_r \in \mathbb{Z}_p \setminus \{0\}$, $G_1(T), \dots, G_r(T) \in \mathbb{F}_q(T) \cap \mathbb{F}_q[[T]]$ and a pseudo-polynomial $R(T) \in \mathbb{F}_q[[T]]$ such that:*

$$\overline{\gamma}_\delta(F(T)) = R(T) + \sum_{i=1}^r G_i((1+T)^{c_i} - 1).$$

Then, there exists an integer $n \geq 0$ such that:

$$(1+T)^n(F(T) + (-1)^\delta F((1+T)^{-1} - 1)) \in \mathbb{F}_q[T].$$

Proof Note that if $\eta, \eta' \in \mu_{p-1} : \eta \equiv \eta' \pmod{\mathbb{Q}^*} \Leftrightarrow \eta = \eta' \text{ or } \eta = -\eta'$. Since $r < (p-1)/2$, by Proposition 3.3, there exists $\eta \in \mu_{p-1}$ such that:

$$\overline{\eta}^\delta F((1+T)^\eta - 1) + \overline{-\eta}^\delta F((1+T)^{-\eta} - 1) \text{ is a pseudo - polynomial.}$$

Therefore:

$$F(T) + (-1)^\delta F((1+T)^{-1} - 1) \text{ is a pseudo - polynomial.}$$

It remains to apply Lemma 3.5. \diamond

Let $F(T) \in \Lambda$. We say that $F(T)$ is a pseudo-rational function if $F(T)$ is the quotient of two pseudo-polynomials. For example, $\forall a \in \mathbb{Z}_p, \forall b \in \mathbb{Z}_p^*, \frac{(1+T)^a - 1}{(1+T)^b - 1}$ is a pseudo-rational function. We finish this section by giving a generalization of [8], Theorem1:

Theorem 3.7 *Let $\delta \in \mathbb{Z}/(p-1)\mathbb{Z}$ and let $F(T) \in \mathbb{F}_q(T) \cap \mathbb{F}_q[[T]]$. Then $\overline{\Gamma}_\delta(F(T))$ is a pseudo-rational function if and only if there exists some integer $n \geq 0$ such that:*

$$(1+T)^n(\overline{U}(F(T)) + (-1)^\delta \overline{U}(F((1+T)^{-1} - 1))) \in \mathbb{F}_q[T].$$

Proof Assume that $\overline{\Gamma}_\delta(F(T))$ is a pseudo-rational function. Then, by 3) of Proposition 3.2 and Proposition 3.1, there exist $c_1, \dots, c_r \in \mathbb{F}_q^*, a_1, \dots, a_r \in \mathbb{Z}_p, a_i \neq a_j$ for $i \neq j$, such that:

$$\overline{\Gamma}_\delta \overline{\gamma}_{-\delta} \overline{U} \left(\sum_{i=1}^r c_i F((1+T)^{a_i} - 1) \right) \text{ is a pseudo - polynomial.}$$

This implies, again by Proposition 3.1, that:

$$\overline{\gamma^{-\delta}}\overline{U}\left(\sum_{i=1}^r c_i F((1+T)^{\kappa^{a_i}} - 1)\right) \text{ is a pseudo - polynomial.}$$

Set:

$$G(T) = \overline{U}(F(T)) + (-1)^\delta \overline{U}(F((1+T)^{-1} - 1)) \in \mathbb{F}_q(T) \cap \mathbb{F}_q[[T]].$$

Now, by Proposition 3.3, there exist $d_1, \dots, d_\ell \in \mathbb{F}_q^*$, $b_1, \dots, b_\ell \in \mathbb{Z}_p$, $b_i \neq b_j$ for $i \neq j$, $\eta_1, \dots, \eta_\ell \in \mu_{p-1}$, with $\forall i, j \in \{1, \dots, \ell\}$, $\eta_i \kappa^{b_i} \equiv \eta_j \kappa^{b_j} \pmod{\mathbb{Q}^*}$, and $\eta_i \kappa^{b_i} \neq \eta_j \kappa^{b_j}$ for $i \neq j$, such that:

$$\sum_{i=1}^{\ell} d_i G((1+T)^{\eta_i \kappa^{b_i}} - 1) \text{ is a pseudo - polynomial.}$$

For $i = 1, \dots, \ell$, write:

$$\eta_i \kappa^{b_i} = \eta_1 \kappa^{b_1} x_i,$$

where $x_i \in \mathbb{Q}^* \cap \mathbb{Z}_p^*$, and $x_i \neq x_j$ for $i \neq j$. Since $G(T) = (-1)^\delta G((1+T)^{-1} - 1)$, we can assume that x_1, \dots, x_ℓ are positives. Now, we get:

$$\sum_{i=1}^{\ell} d_i G((1+T)^{x_i} - 1) \text{ is a pseudo - polynomial.}$$

Therefore, there exist $N_1, \dots, N_\ell \in \mathbb{N} \setminus \{0\}$, $N_i \neq N_j$ for $i \neq j$, such that:

$$\sum_{i=1}^{\ell} d_i G((1+T)^{N_i} - 1) \text{ is a pseudo - polynomial.}$$

Now, by Lemma 3.5, there exists some integer $N \geq 0$ such that:

$$(1+T)^N \left(\sum_{i=1}^{\ell} d_i G((1+T)^{N_i} - 1) \right) \in \mathbb{F}_q[T].$$

But, since $G(T) \in \mathbb{F}_q(T) \cap \mathbb{F}_q[[T]]$, $d_1, \dots, d_\ell \in \mathbb{F}_q^*$, $N_1, \dots, N_\ell \in \mathbb{N} \setminus \{0\}$ and $N_i \neq N_j$ for $i \neq j$, this implies that there exist some integer $n \geq 0$ such that $(1+T)^n G(T) \in \mathbb{F}_q[T]$. \diamond

4 Application to Kubota-Leopoldt p -adic L-functions

Let θ be a Dirichlet character of the first kind, $\theta \neq 1$ and θ even. We denote by $f(T, \theta)$ the Iwasawa power series attached to the p -adic L-function $L_p(s, \theta)$ (see [9], Theorem 7.10). Write:

$$\theta = \chi\omega^{\delta+1},$$

where χ is of conductor d , $d \geq 1$ and $d \not\equiv 0 \pmod{p}$, and $\delta \in \mathbb{Z}/(p-1)\mathbb{Z}$. Set $\kappa = 1 + pd$ and $K = \mathbb{Q}_p(\chi)$. We set:

$$F_\chi(T) = \frac{\sum_{a=1}^d \chi(a)(1+T)^a}{1 - (1+T)^d}.$$

Let's give the basic properties of $F_\chi(T)$:

Lemma 4.1

- 1) If $d \geq 2$, $F_\chi(T) \in \Lambda$.
- 2) If $d = 1$, $\forall \alpha \in \mathbb{Z}/(p-1)\mathbb{Z}$, $\alpha \neq 1$, $\gamma_\alpha(F_\chi(T)) \in \Lambda$.
- 3) $U(F_\chi(T)) = F_\chi(T) - \chi(p)F_\chi((1+T)^p - 1)$.
- 4) If $d \geq 2$, $F_\chi((1+T)^{-1} - 1) = \varepsilon F_\chi(T)$, where $\varepsilon = 1$ if χ is odd and $\varepsilon = -1$ if χ is even.
- 5) If $d = 1$, $F_\chi((1+T)^{-1} - 1) = -1 - F_\chi(T)$.

Proof 1), 4) and 5) are obvious.

2) For $d = 1$, we have:

$$F_\chi(T) = -1 + \frac{\sum_{a=0}^{p-1} (1+T)^a}{1 - (1+T)^p}.$$

Set:

$$G(T) = (1 - (1+T)^p)\gamma_\alpha(F_\chi(T)).$$

Note that:

$$\forall \eta \in \mu_{p-1}, \frac{1 - (1+T)^p}{1 - (1+T)^{\eta p}} \equiv \eta^{-1} \pmod{\omega_1(T)}.$$

Therefore:

$$(p-1)G(T) \equiv \sum_{\eta \in \mu_{p-1}} \eta^{\alpha-1} \sum_{a=0}^{p-1} (1+T)^{\eta a} \pmod{\omega_1(T)}.$$

Thus:

$$(p-1)G(T) \equiv \sum_{\eta \in \mu_{p-1}} \eta^{\alpha-1} \sum_{b=0}^{p-1} (1+T)^b \pmod{\omega_1(T)}.$$

Since $\alpha \neq 1$, we get:

$$G(T) \equiv 0 \pmod{\omega_1(T)}.$$

Therefore $\gamma_\alpha(F_\chi(T)) \in \Lambda$.

3) For $d = 1$, we have:

$$U(F_\chi(T)) = \frac{\sum_{a=1}^{p-1} (1+T)^a}{1 - (1+T)^p} = F_\chi(T) - F_\chi((1+T)^p - 1).$$

Now, let $d \geq 2$. Set $q_0 = \kappa = 1 + pd$. Note that:

$$F_\chi(T) = \frac{\sum_{a=1}^{q_0} \chi(a)(1+T)^a}{1 - (1+T)^{q_0}}.$$

Therefore:

$$U(F_\chi(T)) = \frac{\sum_{a=1, a \not\equiv 0 \pmod{p}}^{q_0} \chi(a)(1+T)^a}{1 - (1+T)^{q_0}}.$$

But:

$$F_\chi(T) - \chi(p)F_\chi((1+T)^p - 1) = \frac{\sum_{a=1}^{q_0} \chi(a)(1+T)^a}{1 - (1+T)^{q_0}} - \chi(p) \frac{\sum_{a=1}^d \chi(a)(1+T)^{pa}}{1 - (1+T)^{q_0}}.$$

The Lemma follows easily. \diamond

Lemma 4.2 *Assume that $d \geq 2$. The denominator of $F_\chi(T)$ is $\phi_d(1+T)$ where $\phi_d(X)$ is the d th cyclotomic polynomial and the same is true for $\overline{F_\chi(T)}$.*

Proof Let $\zeta \in \mu_d$. If ζ is not a primitive d th root of unity, then, by [9], Lemma 4.7, we have:

$$\sum_{a=1}^d \chi(a)\zeta^a = 0.$$

If ζ is a primitive d th root of unity, then by [9], Lemma 4.8, we have:

$$\sum_{a=1}^d \chi(a)\zeta^a \not\equiv 0 \pmod{\tilde{\pi}},$$

where $\tilde{\pi}$ is any prime of $K(\mu_d)$. \diamond

Lemma 4.3 *The derivative of $\gamma_{-\delta}(F_\chi(T))$ is not a pseudo-polynomial modulo π .*

Proof We first treat the case $d \geq 2$. By 3) and 4) of Lemma 4.1, Lemma 4.2 and Proposition 3.6, $\overline{\gamma_{-\delta}U(F_\chi(T))}$ is not a pseudo-polynomial. But observe that $\overline{U} = \overline{D^{p-1}}$. Thus $\overline{D\gamma_{-\delta}(F_\chi(T))}$ is not a pseudo-polynomial.

For the case $d = 1$. Set $F_\chi(T) = F_\chi(T) - 2F_\chi((1+T)^2 - 1) = 1 - \frac{1}{2+T}$. Observe that:

$$\begin{aligned} - F_\chi((1+T)^{-1} - 1) &= 1 - \widetilde{F_\chi(T)}, \\ - U(\widetilde{F_\chi(T)}) &= \widetilde{F_\chi(T)} - \widetilde{F_\chi((1+T)^p - 1)}. \end{aligned}$$

Therefore, as in the case $d \geq 2$, $\overline{\gamma_{-\delta}U(F_\chi(T))}$ is not a pseudo-polynomial. Thus $\overline{\gamma_{-\delta}U(F_\chi(T))}$ is not a pseudo-polynomial. And one can conclude as in the case $d \geq 2$. \diamond

Lemma 4.4

$$\Gamma_\delta \gamma_{-\delta}(F_\chi(T)) = f\left(\frac{1}{1+T} - 1, \theta\right).$$

Proof We treat the case $d = 1$, the case $d \geq 2$ is quite similar. Set $T = e^Z - 1$. We get:

$$\gamma_{-\delta}(F_\chi(T)) = \sum_{n \geq 0, n \equiv 1 + \delta \pmod{p-1}} \frac{B_n}{n!} Z^{n-1}.$$

Thus, by [9], Theorem 5.11, we get:

$$\forall k \in \mathbb{N}, k \equiv \delta \pmod{p-1}, D^k \gamma_{-\delta} U(F_\chi)(0) = L_p(-k, \theta).$$

But, by Proposition 2.5, we have for $s \in \mathbb{Z}_p$:

$$\Gamma_\delta \gamma_{-\delta} U(F_\chi)(\kappa^s - 1) = \lim_n D^{k_n(s, \delta)} \gamma_{-\delta} U(F_\chi)(0) = L_p(-s, \theta) = f(\kappa^{-s} - 1, \theta).$$

The Lemma follows. \diamond

We can now state and prove our main result:

Theorem 4.5

- 1) $f(T, \theta)$ is not a pseudo-rational function.
- 2) $\lambda(f(T, \theta)) < (\frac{p-1}{2}\phi(d))^{\phi(p-1)}$, where ϕ is Euler's totient function.

Proof

1) Assume the contrary, i.e. $\overline{f(T, \theta)}$ is a pseudo-rational function. Then $\overline{f(\frac{1}{1+T} - 1, \theta)}$ is also a pseudo-rational function. Thus $\overline{\Gamma_\delta \gamma_{-\delta} \overline{U(F_\chi(T))}}$ is a pseudo-rational function.

We first treat the case $d \geq 2$. By Theorem 3.7, there exists an integer $n \geq 0$ such that $(1+T)^n (\overline{U(F_\chi(T))}) + (-1)^\delta \overline{U(F_\chi((1+T)^{-1} - 1))} \in \mathbb{F}_q[T]$. This is a contradiction by 3) and 4) of Lemma 4.1 and Lemma 4.2.

For the case $d = 1$. We work with $\widetilde{F_\chi(T)} = F_\chi(T) - 2F_\chi((1+T)^2 - 1) = 1 - \frac{1}{2+T}$. Then, by Proposition 3.2, $\overline{\Gamma_\delta \gamma_{-\delta} \overline{U(\widetilde{F_\chi(T)})}}$ is a pseudo-rational function. We get a contradiction as in the case $d \geq 2$.

2) Our proof is inspired by a method introduced by S. Rosenberg ([6]). We first treat the case $d = 1$. Note that we can assume that $\lambda(f(T, \theta)) \geq 1$. Now, by Lemma 4.3:

$$\mu(\gamma_{-\delta}(F_\chi(T))) = 0.$$

Futhermore, we have:

$$\gamma_{-\delta}(F_\chi)(0) \equiv 0 \pmod{\pi}.$$

Therefore, by 3) of Lemma 4.1, we get:

$$\lambda(\gamma_{-\delta} U(F_\chi(T))) = \lambda(\gamma_{-\delta}(F_\chi(T))).$$

Therefore we have to evaluate $\lambda(\gamma_{-\delta}(F_\chi(T)))$. Set $F(T) = \frac{-1}{T}$. Since δ is odd, we have:

$$\gamma_{-\delta}(F_\chi(T)) = \gamma_{-\delta}(F(T)).$$

Observe that $F((1+T)^{-1} - 1) = 1 - F(T)$. Let $S \subset \mu_{p-1} \setminus \{1, -1\}$ be a set of representatives of $\mu_{p-1}/\{1, -1\}$. We have:

$$(p-1)\gamma_{-\delta}(F(T)) = 2 \sum_{\eta \in S} \eta^{-\delta} F((1+T)^\eta - 1) - \sum_{\eta \in S} \eta^{-\delta}.$$

Set:

$$G(T) = \left(\prod_{\eta \in S} ((1+T)^\eta - 1) \right) \gamma_{-\delta}(F(T)).$$

Then:

- $\mu(G(T)) = 0$,
- $\lambda(G(T)) = \frac{p-1}{2} + \lambda(\gamma_{-\delta}(F(T)))$.

For $S' \subset S$, write $t(S') = \sum_{x \in S'} x$. We can write:

$$G(T) = \sum_{S' \subset S} a_{S'} (1+T)^{t(S')},$$

where $a_{S'} \in O_K$. Set:

$$N = \text{Max}\{v_p(t(S') - t(S'')), S', S'' \subset S, t(S') \neq t(S'')\}.$$

It is clear that:

$$p^N < \left(\frac{p-1}{2}\right)^{\phi(p-1)}.$$

But, by Lemma 2.2, we have:

$$\lambda(G(T)) < p^{N+1}.$$

Thus, by Proposition 3.2, we get:

$$\lambda(f(T, \theta)) = \lambda\left(f\left(\frac{1}{1+T} - 1, \theta\right)\right) < p^N < \left(\frac{p-1}{2}\right)^{\phi(p-1)}.$$

Now, we treat the general case, i.e. $d \geq 2$. Again we can assume that $\lambda(f(T, \theta)) \geq 1$. Thus as in the case $d = 1$, we get:

$$\lambda(\gamma_{-\delta}U(F_\chi(T))) = \lambda(\gamma_{-\delta}(F_\chi(T))).$$

Now, by Lemma 4.2, we can write:

$$F_\chi(T) = \frac{\sum_{a=0}^{\phi(d)-1} r_a (1+T)^a}{\phi_d(1+T)},$$

where $r_a \in O_K$ for $a \in \{0, \dots, \phi(d) - 1\}$. Let again $S \subset \mu_{p-1}$ be a set of representatives of $\mu_{p-1}/\{1, -1\}$. By Lemma 4.1, we have:

$$(p-1)\gamma_{-\delta}(F_\chi(T)) = 2 \sum_{\eta \in S} \eta^{-\delta} F_\chi((1+T)^\eta - 1).$$

Set:

$$G(T) = \left(\prod_{\eta \in S} \phi_d((1+T)^\eta)\right) \gamma_{-\delta}(F_\chi(T)).$$

We have:

$$G(T) = \sum_{a=0}^{\phi(d)-1} \sum_{\eta \in S} \sum_{S' \subset S \setminus \{\eta\}} \sum_{\underline{d}=(d_{\eta'})_{\eta' \in S'}, d_{\eta'} \in \{0, \dots, \phi(d)\}} b_{S', \underline{d}} (1+T)^{a\eta + \sum_{\eta' \in S'} d_{\eta'} \eta'},$$

where $b_{S', \underline{d}} \in O_K$. Note that again $\mu(G(T)) = 0$ and that $\lambda(G(T)) = \lambda(\gamma_{-\delta}(F_\chi(T)))$. Now, for $a, b \in \{0, \dots, \phi(d) - 1\}$, $\eta_1, \eta_2 \in S$, $S_1 \in S \setminus \{\eta_1\}$, $S_2 \in S \setminus \{\eta_2\}$, set:

$$V = a\eta_1 + \sum_{\eta \in S_1} d_\eta \eta - b\eta_2 - \sum_{\eta \in S_2} d'_\eta \eta,$$

where $\forall \eta \in S_1$, $d_\eta \in \{0, \dots, \phi(d)\}$, and $\forall \eta \in S_2$, $d'_\eta \in \{0, \dots, \phi(d)\}$.
If $\eta_1 = \eta_2$ then we can write:

$$V = (a - b)\eta_1 + \sum_{\eta \in S'} u_\eta \eta,$$

where $|u_\eta| \in \{0, \dots, \phi(d)\}$ and $|S'| \leq \frac{p-3}{2}$.
If $\eta_1 \neq \eta_2$, we can write:

$$V = a'\eta_1 + b'\eta_2 + \sum_{\eta \in S'} u_\eta \eta,$$

where $|a'|, |b'|, |u_\eta| \in \{0, \dots, \phi(d)\}$, and $|S'| \leq \frac{p-5}{2}$. Therefore, if $V \neq 0$, we get:

$$p^{v_p(V)} < \left(\frac{p-1}{2}\phi(d)\right)^{\phi(p-1)}.$$

Now, we can conclude as in the case $d = 1$. \diamond

Let E be a number field and let E_∞/E be the cyclotomic \mathbb{Z}_p -extension of E . For $n \geq 0$, let A_n be the p th Sylow subgroup of the ideal class group of the n th layer in E_∞/E . Then, by [9], Theorem 13.13, there exist $\mu_p(E) \in \mathbb{N}$, $\lambda_p(E) \in \mathbb{N}$ and $\nu_p(E) \in \mathbb{Z}$, such that for all sufficiently large n :

$$|A_n| = p^{\mu_p(E)p^n + \lambda_p(E)n + \nu_p(E)}.$$

Recall that it is conjectured that $\mu_p(E) = 0$ and if E is an abelian number field it has been proved by B. Ferrero and L. Washington ([3]).

Corollary 4.6 *Let F be an abelian number field of conductor N . Write $N = p^m d$, where $m \in \mathbb{N}$ and $d \geq 1$, $d \not\equiv 0 \pmod{p}$. Then:*

$$\lambda_p(F) < 2\left(\frac{p-1}{2}\phi(d)\right)^{\phi(p-1)+1}.$$

Proof Set, for all $n \geq 0$, $q_n = p^{n+1}d$. Then $F \subset \mathbb{Q}(\mu_{q_m})$. It is not difficult to see that (see the arguments in the proof of Theorem 7.15 in [9]) :

$$\lambda_p(F) \leq \lambda_p(\mathbb{Q}(\mu_{q_m})).$$

But, note that:

$$\lambda_p(\mathbb{Q}(\mu_{q_m})) = \lambda_p(\mathbb{Q}(\mu_{q_0})).$$

Now, by [9] Proposition 13.32 and Theorem 7.13:

$$\lambda_p(\mathbb{Q}(\mu_{q_0})) \leq 2 \sum_{\theta \text{ even}, \theta \neq 1, f_\theta | q_0} \lambda(f(T, \theta)).$$

It remains to apply Theorem 4.5. \diamond

Note that the bound of this latter Corollary is certainly far from the truth even in the case $p = 3$ (see [4]).

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