



HAL
open science

GloptiPoly 3: moments, optimization and semidefinite programming

Didier Henrion, Jean-Bernard Lasserre, Johan Lofberg

► **To cite this version:**

Didier Henrion, Jean-Bernard Lasserre, Johan Lofberg. GloptiPoly 3: moments, optimization and semidefinite programming. Optimization Methods and Software, 2009, 24 (4-5), pp. 761-779. hal-00172442

HAL Id: hal-00172442

<https://hal.science/hal-00172442>

Submitted on 17 Sep 2007

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

GloptiPoly 3: moments, optimization and semidefinite programming

Didier Henrion^{1,2}, Jean-Bernard Lasserre^{1,3}, Johan Löfberg⁴

Version 3.0 of September 17, 2007

Abstract

We describe a major update of our Matlab freeware GloptiPoly for parsing generalized problems of moments and solving them numerically with semidefinite programming.

1 What is GloptiPoly ?

Gloptipoly 3 is intended to solve, or at least approximate, the Generalized Problem of Moments (GPM), an infinite-dimensional optimization problem which can be viewed as an extension of the classical problem of moments [8]. From a theoretical viewpoint, the GPM has developments and impact in various areas of mathematics such as algebra, Fourier analysis, functional analysis, operator theory, probability and statistics, to cite a few. In addition, and despite a rather simple and short formulation, the GPM has a large number of important applications in various fields such as optimization, probability, finance, control, signal processing, chemistry, cristallography, tomography, etc. For an account of various methodologies as well as some of potential applications, the interested reader is referred to [1, 2] and the nice collection of papers [5].

The present version of GloptiPoly 3 can handle moment problems with polynomial data. Many important applications in e.g. optimization, probability, financial economics and

¹LAAS-CNRS, University of Toulouse, France

²Faculty of Electrical Engineering, Czech Technical University in Prague, Czech Republic

³Institute of Mathematics, University of Toulouse, France

⁴Department of Electrical Engineering, Linköping University, Sweden

optimal control, can be viewed as particular instances of the GPM, and (possibly after some transformation) of the GPM with polynomial data.

The approach is similar to that used in the former version 2 of GloptiPoly [3]. The software allows to build up a hierarchy of semidefinite programming (SDP), or linear matrix inequality (LMI) relaxations of the GPM, whose associated monotone sequence of optimal values converges to the global optimum. For more details on the approach, the interested reader is referred to [8].

2 Installation

GloptiPoly 3 is a freeware subject to the General Public Licence (GPL) policy. It can be downloaded at

www.laas.fr/~henrion/software/gloptipoly3

The package, available as a compressed archive, consists of several m-files and subdirectories, and it contains no binaries. Extracted files are placed in a `gloptipoly3` directory that should be declared in the Matlab working path, using e.g. Matlab's command

```
>> addpath gloptipoly3
```

GloptiPoly 3 uses by default the semidefinite programming solver SeDuMi [11], so this package should be properly installed. Other semidefinite solvers can also be used provided they are installed and interfaced through YALMIP [10].

3 Getting started

Please type the command

```
>> gloptipolydemo
```

to run interactively the basic example that follows.

Consider the classical problem of minimizing globally the two-dimensional six-hump camel back function [3]

$$\min_{x \in \mathbb{R}^2} g_0(x) = 4x_1^2 + x_1x_2 - 4x_2^2 - 2.1x_1^4 + 4x_2^4 + \frac{1}{3}x_1^6.$$

The function has six local minima, two of them being global minima.

Using GloptiPoly 3, this optimization problem can be modeled as a moment problem as follows:

```
>> mpol x1 x2
>> g0 = 4*x1^2+x1*x2-4*x2^2-2.1*x1^4+4*x2^4+x1^6/3
Scalar polynomial
4x1^2+x1x2-4x2^2-2.1x1^4+4x2^4+0.33333x1^6
>> P = msdp(min(g0));
GloptiPoly 3.0
Define moment SDP problem
...
(GloptiPoly output suppressed)
...
Generate moment SDP problem
>> P
Moment SDP problem
Measure label          = 1
Relaxation order       = 3
Decision variables     = 27
Semidefinite inequalities = 10x10
```

Once the moment problem is modeled, a semidefinite solver can be used to solve it numerically. Here we use SeDuMi [11] which is assumed to be installed and accessible from the Matlab working path:

```

>> [status,obj] = msol(P)
GloptiPoly 3.0
Solve moment SDP problem
*****
Calling SeDuMi
SeDuMi 1.1R3 by AdvOL, 2006 and Jos F. Sturm, 1998-2003.
...
(SeDuMi output suppressed)
...
2 globally optimal solutions extracted
>> status
status =
     1
>> obj
obj =
    -1.0316
>> x = double([x1 x2]);
x(:,:,1) =
    0.0898   -0.7127
x(:,:,2) =
   -0.0898    0.7127

```

The flag `status = 1` means that the moment problem is solved successfully and that GloptiPoly can extract two globally optimal solutions reaching the objective function `obj = -1.0316`.

4 From version 2 to version 3

The major changes incorporated into GloptiPoly when passing from version 2 to 3 can be summarized as follows:

- Use of native polynomial objects and object-oriented programming with specific classes for multivariate polynomials, measures, moments, and corresponding overloaded operators. In contrast with version 2, the Symbolic Toolbox for Matlab (gateway to the Maple kernel) is not required anymore to process polynomial data.
- Generalized problems of moments featuring several measures with semialgebraic support constraints and linear moment constraints can be processed and solved. Version 2 was limited to moment problems on a unique measure without moment constraints.
- Explicit moment substitutions are carried out to reduce the number of variables and constraints.
- The moment problems can be solved numerically with any semidefinite solver, provided it is interfaced through YALMIP. In contrast, version 2 used only the solver SeDuMi.

5 Solving generalized problems of moments

GloptiPoly 3 uses advanced Matlab features for object-oriented programming and overloaded operators. The user should be familiar with the following basic objects.

5.1 Multivariate polynomials (mpol)

A multivariate polynomial is an affine combination of monomials, each monomial depending on a set of variables. Variables can be declared in the Matlab working space as follows:

```
>> clear
>> mpol x
>> x
Scalar polynomial
```

```

x
>> mpol y 2
>> y
2-by-1 polynomial vector
(1,1):y(1)
(2,1):y(2)
>> mpol z 3 2
>> z
3-by-2 polynomial matrix
(1,1):z(1,1)
(2,1):z(2,1)
(3,1):z(3,1)
(1,2):z(1,2)
(2,2):z(2,2)
(3,2):z(3,2)

```

Variables, monomials and polynomials are defined as objects of class `mpol`.

All standard Matlab operators have been overloaded for `mpol` objects:

```

>> y*y'-z'*z+x^3
2-by-2 polynomial matrix
(1,1):y(1)^2-z(1,1)^2-z(2,1)^2-z(3,1)^2+x^3
(2,1):y(1)y(2)-z(1,1)z(1,2)-z(2,1)z(2,2)-z(3,1)z(3,2)+x^3
(1,2):y(1)y(2)-z(1,1)z(1,2)-z(2,1)z(2,2)-z(3,1)z(3,2)+x^3
(2,2):y(2)^2-z(1,2)^2-z(2,2)^2-z(3,2)^2+x^3

```

Use the instruction

```
>> mset clear
```

to delete all existing GloptiPoly variables from the Matlab working space.

5.2 Measures (meas)

Variables can be associated with real-valued measures, and one variable is associated with only one measure. For GloptiPoly, measures are identified with a label, a positive integer. When starting a GloptiPoly session, the default measure has label 1. By default, all created variables are associated with the current measure. Measures can be handled with the class `meas` as follows:

```
>> mset clear
>> mpol x
>> mpol y 2
>> meas
Measure 1 on 3 variables: x,y(1),y(2)
>> meas(y) % create new measure
Measure 2 on 2 variables: y(1),y(2)
>> m = meas
1-by-2 vector of measures
1:Measure 1 on 1 variable: x
2:Measure 2 on 2 variables: y(1),y(2)
>> m(1)
Measure number 1 on 1 variable: x
```

The above script creates a measure $d\mu_1(x)$ on \mathbb{R} and a measure $d\mu_2(y)$ on \mathbb{R}^2 .

Use the instruction

```
>> mset clearmeas
```

to delete all existing GloptiPoly measures from the working space. Note that this does not delete existing GloptiPoly variables.

5.3 Moments (mom)

Linear combinations of moments of a given measure can be manipulated with the `mom` class as follows:

```
>> mom(1+2*x+3*x^2)
Scalar moment
I[1+2x+3x^2]d[1]
>> mom(y*y')
2-by-2 moment matrix
(1,1):I[y(1)^2]d[2]
(2,1):I[y(1)y(2)]d[2]
(1,2):I[y(1)y(2)]d[2]
(2,2):I[y(2)^2]d[2]
```

The notation $I[p]d[k]$ stands for $\int p d\mu_k$ where p is a polynomial of the variables associated with measure $d\mu_k$, and k is the measure label.

Note that it makes no sense to define moments over several measures, or nonlinear moment expressions:

```
>> mom(x*y(1))
??? Error using ==> mom.mom
Invalid partitioning of measures in moments
>> mom(x)*mom(y(1))
??? Error using ==> mom.times
Invalid moment product
```

Note also the distinction between a constant term and the mass of a measure:

```
>> 1+mom(x)
Scalar moment
1+I[x]d[1]
```

```
>> mom(1+x)
Scalar moment
I[1+x]d[1]
>> mass(x)
Scalar moment
I[1]d[1]
```

Finally, let us mention three equivalent notations to refer to the mass of a measure:

```
>> mass(meas(y))
Scalar moment
I[1]d[2]
>> mass(y)
Scalar moment
I[1]d[2]
>> mass(2)
Scalar moment
I[1]d[2]
```

The first command refers explicitly to the measure, the second command is a handy shortcut to refer to a measure via its variables, and the third command refers to GloptiPoly's labeling of measures.

5.4 Support constraints (supcon)

By default, a measure on n variables is defined on the whole \mathbb{R}^n . We can restrict the support of a measure to a given semialgebraic set as follows:

```
>> 2*x^2+x^3 == 2+x
Scalar measure support equality
2x^2+x^3 == 2+x
>> disk = (y'*y <= 1)
```

Scalar measure support inequality

$$y(1)^2 + y(2)^2 \leq 1$$

Support constraints are modeled by objects of class `supcon`. The first command means that variable x must satisfy $x^3 + 2x^2 - x - 2 = (x - 1)(x + 1)(x + 2) = 0$, i.e. measure $d\mu_1(x)$ must be discrete, a linear combination of three Dirac at 1, -1 and -2 . The second command restricts measure $d\mu_2(y)$ within the unit disk.

Note that it makes no sense to define a support constraint on several measures:

```
>> x+y(1) <= 1
?? Error using ==> supcon.supcon
Invalid reference to several measures
```

5.5 Moment constraints (`momcon`)

We can constrain linearly the moments of several measures:

```
>> mom(x^2+2) == 1+mom(y(1)^3*y(2))
Scalar moment equality constraint
I[2+x^2]d[1] == 1+I[y(1)^3*y(2)]d[2]
>> mass(x)+mass(y) <= 2
Scalar moment inequality constraint
I[1]d[1]+I[1]d[2] <= 2
```

Moment constraints are modeled by objects of class `momcon`.

For GloptiPoly an objective function to be minimized or maximized is considered as a particular moment constraint:

```
>> min(mom(x^2+2))
Scalar moment objective function
min I[2+x^2]d[1]
```

```
>> max(x^2+2)
```

Scalar moment objective function

```
max I [2+x^2]d[1]
```

The latter syntax is a handy short-cut which directly converts an `mpol` object into an `momcon` object.

5.6 Floating point numbers (double)

Variables in a measure can be assigned numerical values:

```
>> m1 = assign(x,2)
```

Measure 1 on 1 variable: x

supported on 1 point

which is equivalent to enforcing a discrete support for the measure. Here $d\mu_1$ is set to the Dirac at the point 2.

The `double` operator converts a measure or its variables into a floating point number:

```
>> double(x)
```

ans =

2

```
>> double(m1)
```

ans =

2

Polynomials can be evaluated similarly:

```
>>double(1-2*x+3*x^2)
```

ans =

9

Discrete measure supports consisting of several points can be specified in an array:

```
>> m2 = assign(y,[-1 2 0;1/3 1/4 -2])
Measure 2 on 2 variables: y(1),y(2)
supported on 3 points
>> double(m2)
ans(:,:,1) =
    -1.0000
     0.3333
ans(:,:,2) =
     2.0000
     0.2500
ans(:,:,3) =
     0
    -2
```

5.7 Moment SDP problems (msdp)

GloptiPoly 3 can manipulate and solve Generalized Problems of Moments (GPM) as defined in [8]:

$$\begin{aligned} \min_{d\mu} \text{ (or max)} \quad & \sum_k \int_{\mathbb{K}_k} g_{0k}(x) d\mu_k(x) \\ \text{s.t.} \quad & \sum_k \int_{\mathbb{K}_k} h_{jk}(x) d\mu_k(x) \geq \text{ (or =) } b_j, \quad j = 0, 1, \dots \end{aligned}$$

where measures $d\mu_k$ are supported on basic semialgebraic sets

$$\mathbb{K}_k = \{x \in \mathbb{R}^{n_k} : g_{ik}(x) \geq 0, \quad i = 1, 2, \dots\}.$$

In the above notations, $g_{ik}(x)$, $h_{jk}(x)$ are given real polynomials and b_j are given real constants. The decision variables in the GPM are measures $d\mu_k(x)$, and GloptiPoly 3 allows to optimize over them through their moments

$$y_{\alpha_k} = \int_{\mathbb{K}_k} x^{\alpha_k} d\mu_k(x), \quad \alpha_k \in \mathbb{N}^{n_k}$$

where the α_k are multi-indices.

5.8 Solving moment problems `msol`

Once a moment problem is defined, it can be solved numerically with the instruction `msol`. In the sequel we give several examples of GPMs handled with GloptiPoly 3.

5.8.1 Unconstrained minimization

Following [6], given a multivariate polynomial $g_0(x)$, the unconstrained optimization problem

$$\min_{x \in \mathbb{R}^n} g_0(x)$$

can be formulated as a linear moment optimization problem

$$\begin{aligned} \min_{d\mu} \quad & \int g_0(x) d\mu(x) \\ \text{s.t.} \quad & \int d\mu(x) = 1 \end{aligned}$$

where measure $d\mu$ lives in the space \mathbb{B}^n of finite Borel signed measures on \mathbb{R}^n . The equality constraint indicates that the mass of $d\mu$ is equal to one, or equivalently, that $d\mu$ is a probability measure.

In general, this linear (hence convex) reformulation of a (typically nonconvex) polynomial problem is not helpful because there is no computationally efficient way to represent measures and their underlying Borel spaces. The approach proposed in [6] consists in using convex semidefinite representations of the space \mathbb{B}^n truncated to finite degree moments. GloptiPoly 3 allows to input such moment optimization problems in an user-friendly way, and to solve them using existing software for semidefinite programming (SDP).

In Section 3 we already encountered an example of an unconstrained polynomial optimization solved with GloptiPoly 3. Let us revisit this example:

```
>> mset clear
>> mpol x1 x2
>> g0 = 4*x1^2+x1*x2-4*x2^2-2.1*x1^4+4*x2^4+x1^6/3
Scalar polynomial
4x1^2+x1x2-4x2^2-2.1x1^4+4x2^4+0.33333x1^6
```

```

>> P = msdp(min(g0));
...
>> msol(P)
...
2 globally optimal solutions extracted
Global optimality certified numerically

```

This indicates that the global minimum is attained with a discrete measure supported on two points. The measure can be constructed from the knowledge of its first moments of degree up to 6:

```

>> meas
Measure 1 on 2 variables: x1,x2
  with moments of degree up to 6, supported on 2 points
>> double(meas)
ans(:,:,1) =
    0.0898
   -0.7127
ans(:,:,2) =
   -0.0898
    0.7127
>> double(g0)
ans(:,:,1) =
   -1.0316
ans(:,:,2) =
   -1.0316

```

When converting to floating point numbers with the operator `double`, it is essential to make the distinction between `mpol` and `mom` objects:

```

>> v = mmon([x1 x2],2)
1-by-6 polynomial vector

```

```

(1,1):1
(1,2):x1
(1,3):x2
(1,4):x1^2
(1,5):x1x2
(1,6):x2^2
>> double(v)
ans(:,:,1) =
    1.0000    0.0898   -0.7127    0.0081   -0.0640    0.5079
ans(:,:,2) =
    1.0000   -0.0898    0.7127    0.0081   -0.0640    0.5079
>> double(mom(v))
ans =
    1.0000    0.0000   -0.0000    0.0081   -0.0640    0.5079

```

The first instruction `mmon` generates a vector of monomials `v` of class `mpol`, so the command `double(v)` calls the convertor `@mpol/double` which evaluates a polynomial expression on the discrete support of a measure (here two points). The last command `double(mom(v))` calls the convertor `@mom/double` which returns the value of the moments obtained after solving the moment problem.

Note that when inputting moment problems on a unique measure whose mass is not constrained, GloptiPoly assumes by default that the measure has mass one, i.e. that we are seeking a probability measure. Therefore, if `g0` is the polynomial defined previously, the two instructions

```
>> P = msdp(min(g0));
```

and

```
>> P = msdp(min(g0), mass(meas(g0))==1);
```

are equivalent. See also Section 5.3 for handling masses of measures and Section 5.8.2 for

more information on mass constraints.

5.8.2 Constrained minimization

Consider now the constrained polynomial optimization problem

$$\min_{x \in \mathbb{K}} g_0(x)$$

where

$$\mathbb{K} = \{x \in \mathbb{R}^n : g_i(x) \geq 0, i = 1, 2, \dots\}$$

is a basic semialgebraic set described by given polynomials $g_i(x)$. Following [6], this (non-convex polynomial) problem can be formulated as the (convex linear) moment problem

$$\begin{aligned} \min_{d\mu} \quad & \int_{\mathbb{K}} g_0(x) d\mu(x) \\ \text{s.t.} \quad & \int_{\mathbb{K}} d\mu(x) = 1 \end{aligned}$$

where the indeterminate is a probability measure $d\mu$ of \mathbb{B}^n which is now supported on set \mathbb{K} . In other words

$$\int_{\mathbb{R}^n / \mathbb{K}} d\mu(x) = 0.$$

As an example, consider the non-convex quadratic problem of Section 4.4 in [3]:

$$\begin{aligned} \min \quad & -2x_1 + x_2 - x_3 \\ \text{s.t.} \quad & 24 - 20x_1 + 9x_2 - 13x_3 + 4x_1^2 - 4x_1x_2 + 4x_1x_3 + 2x_2^2 - 2x_2x_3 + 2x_3^2 \geq 0 \\ & x_1 + x_2 + x_3 \leq 4, \quad 3x_2 + x_3 \leq 6 \\ & 0 \leq x_1 \leq 2, \quad 0 \leq x_2, \quad 0 \leq x_3 \leq 3 \end{aligned}$$

Each constraint in this problem is interpreted by GloptiPoly 3 as a support constraint on the measure associated with variable x , see Section 5.4:

```
>> mpol x 3
>> x(1)+x(2)+x(3) <= 4
Scalar measure support inequality
x(1)+x(2)+x(3) <= 4
```

The whole problem can be entered as follows:

```

>> mpol x 3
>> g0 = -2*x(1)+x(2)-x(3);
>> K = [24-20*x(1)+9*x(2)-13*x(3)+4*x(1)^2-4*x(1)*x(2) ...
+4*x(1)*x(3)+2*x(2)^2-2*x(2)*x(3)+2*x(3)^2 >= 0, ...
x(1)+x(2)+x(3) <= 4, 3*x(2)+x(3) <= 6, ...
0 <= x(1), x(1) <= 2, 0 <= x(2), 0 <= x(3), x(3) <= 3];
>> P = msdp(min(g0), K)
...
Moment SDP problem
Measure label          = 1
Relaxation order      = 1
Decision variables    = 9
Linear inequalities    = 8
Semidefinite inequalities = 4x4

```

The moment problem can then be solved:

```

>> [status,obj] = msol(P)
GloptiPoly 3.0
Solve moment SDP problem
...
Global optimality cannot be ensured
status =
    0
obj =
   -6.0000

```

Since `status=0` the moment SDP problem can be solved but it is impossible to detect global optimality. The value `obj=-6.0000` is then a lower bound on the global minimum of the quadratic problem.

The measure associated with the problem variables can be retrieved as follows:

```
>> mu = meas
Measure 1 on 3 variables: x(1),x(2),x(3)
with moments of degree up to 2
```

Its vector of moments can be built as follows:

```
>> mv = mvec(mu)
10-by-1 moment vector
(1,1):I[1]d[1]
(2,1):I[x(1)]d[1]
(3,1):I[x(2)]d[1]
(4,1):I[x(3)]d[1]
(5,1):I[x(1)^2]d[1]
(6,1):I[x(1)x(2)]d[1]
(7,1):I[x(1)x(3)]d[1]
(8,1):I[x(2)^2]d[1]
(9,1):I[x(2)x(3)]d[1]
(10,1):I[x(3)^2]d[1]
```

These moments are the decision variables of the SDP problem solved with the above `msol` command. Their numerical values can be retrieved as follows:

```
>> double(mv)
ans =
    1.0000
    2.0000
   -0.0000
    2.0000
    7.6106
    1.4671
    2.3363
    4.8335
```

```
0.5008
8.7247
```

The numerical moment matrix can be obtained using the following commands:

```
>> double(mmat(mu))
ans =
    1.0000    2.0000   -0.0000    2.0000
    2.0000    7.6106    1.4671    2.3363
   -0.0000    1.4671    4.8335    0.5008
    2.0000    2.3363    0.5008    8.7247
```

As explained in [6], we can build a hierarchy of nested moment SDP problems, or relaxations, whose solutions converge monotonically and asymptotically to the global optimum, under mild technical assumptions. By default the command `msdp` builds the relaxation of lowest order, equal to half the degree of the highest degree monomial in the polynomial data. An additional input argument can be specified to build higher order relaxations:

```
>> P = msdp(min(g0), K, 2)
...
Moment SDP problem
  Measure label           = 1
  Relaxation order       = 2
  Decision variables     = 34
  Semidefinite inequalities = 10x10+8x(4x4)
>> [status,obj] = msol(P)
...
Global optimality cannot be ensured
status =
    0
obj =
   -5.6922
```

```

>> P = msdp(min(g0), K, 3)
...
Moment SDP problem
  Measure label           = 1
  Relaxation order        = 3
  Decision variables      = 83
  Semidefinite inequalities = 20x20+8x(10x10)
>> [status,obj] = msol(P)
...
Global optimality cannot be ensured
status =
    0
obj =
   -4.0684

```

We observe that the moment SDP problems feature an increasing number of variables and constraints. They generate a monotonically increasing sequence of lower bounds on the global optimum, which is eventually reached numerically at the fourth relaxation:

```

>> P = msdp(min(g0), K, 4)
...
Moment SDP problem
  Measure label           = 1
  Relaxation order        = 4
  Decision variables      = 164
  Semidefinite inequalities = 35x35+8x(20x20)
>> [status,obj] = msol(P)
...
2 globally optimal solutions extracted
Global optimality certified numerically
status =
    1

```

```

obj =
    -4.0000
>> double(x)
ans(:,:,1) =
    2.0000
    0.0000
    0.0000
ans(:,:,2) =
    0.5000
    0.0000
    3.0000
>> double(g0)
ans(:,:,1) =
    -4.0000
ans(:,:,2) =
    -4.0000

```

5.8.3 Rational minimization

Minimization of a rational function can also be formulated as a linear moment problem. Given two polynomials $g_0(x)$ and $h_0(x)$, consider the rational optimization problem

$$\min_{x \in \mathbb{K}} \frac{g_0(x)}{h_0(x)}$$

where

$$\mathbb{K} = \{x \in \mathbb{R}^n : g_i(x) \geq 0, i = 1, 2, \dots\}$$

is a basic semialgebraic set described by given polynomials $g_i(x)$. Following [4], the corresponding moment problem is given by

$$\begin{aligned} \min_{d\mu \in \mathbb{B}^n} \quad & \int_{\mathbb{K}} g_0(x) d\mu(x) \\ \text{s.t.} \quad & \int_{\mathbb{K}} h_0(x) d\mu(x) = 1. \end{aligned}$$

In contrast with the polynomial optimization problem of Section 5.8.2, the optimal measure $d\mu$ supported on \mathbb{K} is not necessarily a probability measure. Denoting $h_0(x) =$

$\sum_{\alpha} h_{0\alpha} x^{\alpha}$, the moments y_{α} of $d\mu$ must satisfy a linear constraint

$$\int_{\mathbb{K}} h_0(x) d\mu(x) = \sum_{\alpha} h_{0\alpha} y_{\alpha} = 1.$$

As an example, consider the one-variable rational minimization problem [4, Ex. 2]:

$$\min \frac{x^2 - x}{x^2 + 2x + 1}.$$

We can solve this problem with GloptiPoly 3 as follows:

```
>> mpol x
>> g0 = x^2-2*x; h0 = x^2+2*x+1;
>> P = msdp(min(g0), mom(h0) == 1);
>> [status,obj] = msol(P)
...
Global optimality certified numerically
status =
    1
obj =
   -0.3333
>> double(x)
ans =
    0.4999
```

5.8.4 Several measures

GloptiPoly 3 can handle several measures whose moments are linearly related.

For example, consider the GPM arising when solving polynomial optimal control problems as detailed in [7]. We are seeking two occupation measures $d\mu_1(x, u)$ and $d\mu_2(x)$ of a state vector $x(t)$ and input vector $u(t)$ whose time variation are governed by the differential equation

$$\frac{dx(t)}{dt} = f(x, u), \quad x(0) = x_0, \quad u(0) = u_0$$

with $f(x, u)$ a given polynomial mapping and x_0, u_0 given initial conditions. Measure $d\mu_1$ is supported on a given semialgebraic set \mathbb{K}_1 corresponding to constraints on x and u . Measure $d\mu_2$ is supported on a given semialgebraic set \mathbb{K}_2 corresponding to performance requirements. For example $\mathbb{K}_2 = 0$ indicates that state x must reach the origin.

Given a polynomial test function $g(x)$ we can relax the dynamics constraint with the moment constraint

$$\int_{\mathbb{K}_2} g(x)d\mu_2(x) - g(x_0) = \int_{\mathbb{K}_1} \frac{dg(x)}{dx} f(x, u)d\mu_1(x, u)$$

linking linearly moments of $d\mu_1$ and $d\mu_2$. As explained in [7], a lower bound on the minimum time achievable by any feedback control law $u(x)$ is then obtained by minimizing the mass of $d\mu_1$ over all possible measures $d\mu_1, d\mu_2$ satisfying the support and moment constraints. The gap between the lower bound and the exact minimum time is narrowed by enlarging the class of test functions g .

In the following script we solve this moment problem in the case of a double integrator with state and input constraints:

```
% bounds on minimal achievable time for optimal control of
% double integrator with state and input constraints

x0 = [1; 1]; u0 = 0; % initial conditions
d = 6; % maximum degree of test function

% analytic minimum time
if x0(1) >= -(x0(2)^2-2)/2
    tmin = 1+x0(1)+x0(2)+x0(2)^2/2;
elseif x0(1) >= -x0(2)^2/2*sign(x0(2))
    tmin = 2*sqrt(x0(1)+x0(2)^2/2)+x0(2);
else
    tmin = 2*sqrt(-x0(1)+x0(2)^2/2)-x0(2);
end
```

```

% occupation measure for constraints
mpol x1 2
mpol u1
m1 = meas([x1;u1]);

% occupation measure for performance
mpol x2 2
m2 = meas(x2);

% dynamics
scaling = tmin; % time scaling
f = scaling*[x1(2);u1];

% test function
g1 = mmon(x1,d);
g2 = mmon(x2,d);

% initial condition
assign([x1;u1],[x0;u0]);
g0 = double(g1);

% moment problem
P = msdp(min(mass(m1)),...
    u1^2 <= 1,... % input constraint
    x1(2) >= -1,... % state constraint
    x2'*x2 <= 0,... % performance = reach the origin
    mom(g2) - g0 == mom(diff(g1,x1)*f)); % linear moment constraints

% solve
[status,obj] = msol(P);
obj = scaling*obj;

```

```
disp(['Minimum time = ' num2str(tmin)]);
disp(['LMI ' int2str(d) ' lower bound = ' num2str(obj)])
```

For the initial condition $x_0 = [1 \ 1]$ the exact minimum time is equal to 3.5. In Table 1 we report the monotonically increasing sequence of lower bounds obtained by solving moment problems with test functions of increasing degrees. We used the above script and the semidefinite solver SeDuMi 1.1R3.

degree	2	4	6	8	10	12	14	16
bound	1.0019	2.3700	2.5640	2.9941	3.3635	3.4813	3.4964	3.4991

Table 1: Minimum time optimal control for double integrator with state and input constraints: lower bounds on exact minimal time 3.5 achieved by solving moment problems with test functions of increasing degrees.

5.9 Using YALMIP

By default GloptiPoly 3 uses the semidefinite solver SeDuMi [11] for solving numerically SDP moment problems. It is however possible to use any solver interfaced through YALMIP [10] by setting a configuration flag with the `mset` command:

```
>> mset('yalmip',true)
```

Parameters for YALMIP, handled with the YALMIP command `sdpsettings`, can be forwarded to GloptiPoly 3 with the `mset` command. For example, the following command tells YALMIP to use the SDPT3 solver (instead of SeDuMi) when solving moment problems with GloptiPoly:

```
>> mset(sdpsettings('solver','sdpt3'));
```

5.10 SeDuMi parameters settings

The default parameters settings of SeDuMi [11] can be altered as follows:

```
>> pars.eps = 1e-10;  
>> mset(pars)
```

where `pars` is a structure of parameters consistent with SeDuMi's format.

5.11 Exporting moment SDP problems

A moment problem `P` of class `msdp` can be converted into SeDuMi's input format:

```
>> [A,b,c,K] = msedumi(P);
```

The SDP problem can then be solved with SeDuMi as follows:

```
>> [x,y,info] = sedumi(A,b,c,K);
```

See [11] for more information on SeDuMi's input data format.

Similarly, a moment SDP problem can be converted into YALMIP's input format:

```
>> [F,h,y] = myalmip(P);
```

where variable `F` contains the LMI constraints (YALMIP class `lmi`), `h` is the objective function (YALMIP class `sdpvar`) and `y` is the vector of moments (YALMIP class `sdpvar`). The SDP problem can then be solved with any semidefinite solver interfaced through YALMIP as follows:

```
>> solvesdp(F,h);  
>> ysol = double(y);
```

5.12 Moment substitutions

By performing explicit moment substitutions it is often possible to reduce significantly the number of variables and constraints in moment SDP problems. Version 2 of GloptiPoly implemented these substitutions for mixed-integer 0-1 problems only [3]. With version 3, these substitutions can be carried out in full generality.

GloptiPoly 3 carries out moment substitutions as soon as the left hand-side of a support or moment equality constraint consists of an isolated monic monomial. Otherwise, no substitution is achieved and the equality constraint is preserved.

For example, consider the AW_2^9 Max-Cut problem studied in [3, §4.7], with variables x_i taking values -1 or $+1$ for $i = 1, \dots, 9$. These integer constraints can be expressed algebraically as $x_i^2 = 1$. The following piece of code builds up the third relaxation of this problem:

```
>> W = diag(ones(8,1),1)+diag(ones(7,1),2)+diag([1 1],7)+diag(1,8);
>> W = W+W'; n = size(W,1); e = ones(1,n); Q = (diag(e*W)-W)/4;
>> mset clear
>> mpol('x', n)
>> P = msdp(max(x'*Q*x), x.^2 == 1, 3)
```

GloptiPoly 3.0

Define moment SDP problem

Valid objective function

Number of support constraints = 9 including 9 substitutions

Number of moment constraints = 0

Measure #1

Maximum degree = 2

Number of variables = 9

Number of moments = 5005

Order of SDP relaxation = 3

Mass of measure 1 set to one

Total number of monomials = 5005

```
Perform moment substitutions
Perform support substitutions
Number of monomials after substitution = 465
Generate moment and support constraints
Generate moment SDP problem
```

```
Moment SDP problem
  Measure label          = 1
  Relaxation order      = 3
  Decision variables    = 465
  Semidefinite inequalities = 130x130
```

We see that out of the 5005 moments (corresponding to all the monomials of 9 variables of degree up to 6), only 465 linearly independent moments appear in a reduced moment matrix of dimension 130.

With the following syntax, moment substitutions are not carried out:

```
>> P = msdp(max(x'*Q*x), x.^2-1 == 0, 3)
...
Mass of measure 1 set to one
Total number of monomials = 5005
Perform moment substitutions
Number of monomials after substitution = 5004
Generate moment and support constraints
Generate moment SDP problem
```

```
Moment SDP problem
  Measure label          = 1
  Relaxation order      = 3
  Decision variables    = 5004
  Linear equalities     = 6435
```

Semidefinite inequalities = 220x220

Only the mass is substituted, and the remaining 5004 moments linked by 6435 linear equalities (many of which are redundant) now appear explicitly in a full-size moment matrix of dimension 220.

References

- [1] N. I. Akhiezer. The classical moment problem. Hafner, New York, 1965.
- [2] N. I. Akhiezer, M. G. Krein. Some questions in the theory of moments. American Mathematical Society Translations Vol. 2, 1962.
- [3] D. Henrion, J. B. Lasserre. GloptiPoly: global optimization over polynomials with Matlab and SeDuMi. ACM Transactions on Mathematical Software, Vol. 29, No. 2, pp. 165-194, 2003.
- [4] D. Jibetean, E. de Klerk. Global optimization of rational functions: a semidefinite programming approach. Mathematical Programming, Vol. 106, No. 1, pp. 93-109, 2006.
- [5] H. J. Landau. Moments in mathematics. In H. J. Landau (Editor), Proceedings of Symposium on Applied Mathematics, Vol. 37, American Mathematical Society, 1980.
- [6] J. B. Lasserre. Global optimization with polynomials and the problem of moments. SIAM Journal on Optimization, Vol. 11, No. 3, pp. 796–817, 2001.
- [7] J. B. Lasserre, C. Prieur, D. Henrion. Nonlinear optimal control: numerical approximation via moments and LMI relaxations. Proceedings of the joint IEEE Conference on Decision and Control and European Control Conference, Sevilla, Spain, December 2005.
- [8] J. B. Lasserre. A semidefinite programming approach to the generalized problem of moments. To appear in Mathematical Programming, Series B, 2007.

- [9] M. Laurent. Moment matrices and optimization over polynomials - A survey on selected topics. CWI Amsterdam, The Netherlands, September 2005.
- [10] J. Löfberg. YALMIP : a toolbox for modeling and optimization in Matlab. Proceedings of the IEEE Symposium on Computer-Aided Control System Design (CACSD), Taipei, Taiwan, 2004. See control.ee.ethz.ch/~joloef/yalmip.php
- [11] J. F. Sturm and the Advanced Optimization Laboratory at McMaster University, Canada. SeDuMi version 1.1R3, October 2006. See sedumi.mcmaster.ca