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# Sharp estimates of bounded solutions to some semilinear second order dissipative equations

Cyrine Fitouri & Alain Haraux

**Abstract.** Let  $H, V$  be two real Hilbert spaces such that  $V \subset H$  with continuous and dense imbedding, and let  $F \in C^1(V)$  be convex. By using differential inequalities, a close-to-optimal ultimate bound of the energy is obtained for solutions in  $C^1(\mathbb{R}^+, V) \cap W_{loc}^{2,\infty}(\mathbb{R}^+, V')$  to  $u'' + cu' + bu + \nabla F(u) = f(t)$  whenever  $f \in L^\infty(\mathbb{R}, H)$ .

**Résumé.** Soient  $H, V$  deux espaces de Hilbert réels tels que  $V \subset H$  avec injection continue et dense, et soit  $F \in C^1(V)$  convexe. Au moyen d'inéquations différentielles, une borne proche de l'optimalité est établie pour l'énergie des solutions dans  $C^1(\mathbb{R}^+, V) \cap W_{loc}^{2,\infty}(\mathbb{R}^+, V')$  de  $u'' + cu' + bu + \nabla F(u) = f(t)$  pour tout  $f \in L^\infty(\mathbb{R}, H)$ .

**Keywords:** second order equation, bounded solution

**AMS classification numbers:** 34C15, 34C25, 34C27, 34D05, 34D30

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# 1. Introduction

Let  $H$  be a real Hilbert space. In the sequel we denote by  $(u, v)$  the inner product of two vectors  $u, v$  in  $H$  and by  $|u|$  the H-norm of  $u$ . We consider a second Hilbert space  $V \subset H$  with continuous and dense imbedding and we denote by  $\|u\|$  the V-norm of  $u$ . The duality pairing between  $\varphi \in V'$  and  $u \in V$  is denoted by  $\langle \varphi, u \rangle$ . We identify  $H$  with its dual which implies  $H \subset V'$  and the identity

$$\forall u \in H, \forall v \in V, \quad \langle u, v \rangle = (u, v)$$

Let  $b, c$  be two positive constants and  $F \in C^1(V)$  be **convex, nonnegative**. The equation

$$u'' + cu' + bu + \nabla F(u) = f(t) \tag{1.1}$$

is a natural vector generalization of the scalar ODE

$$u'' + cu' + g(u) = f(t) \tag{1.2}$$

considered, after [1], in [8] (cf. also [6] for a pure differential inequality treatment) under the hypothesis

$$g \in C^1, \quad g' \geq b > 0 \tag{1.3}$$

In addition when  $F$  is a nonnegative quadratic form on  $V$ , equation (1.1) becomes

$$u'' + cu' + Au = f(t) \tag{1.4}$$

where  $A = bI + \nabla^2 F$  is a linear self-adjoint operator and  $A \geq bI$ . The results of this paper extend some results from both [8] and [7] on the ultimate bound of solutions to (1.2) and (1.4) respectively. In addition the result of [7] is improved for  $c$  large. This comes from a different proof based on a new energy functional which allows the extension to the case of a nonlinear strongly monotone conservative term.

The plan of the paper is as follows. In Section 2 we give the statement of our general results. Sections 3 and 4 are devoted to the proof of this result for  $c \leq 2\sqrt{b}$  and  $c \geq 2\sqrt{b}$ , respectively. In Section 5 we specify the improvement of the previous results for equations (1.2) and (1.4) and we give an application of the general result to a sharp estimate of the size of the attractor of a semilinear dissipative wave equation in a bounded domain.

## 2- Main results.

The following general result will be established in Sections 3 and 4.

**Theorem 2.1.** *Let  $b, c$  be two positive constants and  $F \in C^1(V)$  be **nonnegative and convex**. Then for any solution  $u \in C^1(\mathbb{R}^+, V) \cap W_{loc}^{2,\infty}(\mathbb{R}^+, V')$  of (1.1),  $u$  is bounded with values in  $H$  with*

$$\overline{\lim}_{t \rightarrow +\infty} |u(t)| \leq \max\left\{\frac{1}{b}, \frac{2}{c\sqrt{b}}\right\} \overline{\lim}_{t \rightarrow +\infty} |f(t)| \quad (2.1)$$

and moreover, introducing for each  $u \in V$

$$G(u) = \frac{b}{2}|u|^2 + F(u)$$

$G(u(t))$  is bounded on  $\mathbb{R}^+$  with the estimate

$$2 \overline{\lim}_{t \rightarrow +\infty} G(u(t)) \leq \left(\frac{4}{c^2} + \frac{1}{b}\right) \overline{\lim}_{t \rightarrow +\infty} |f(t)|^2 \quad (2.2)$$

In addition for  $c \leq 2\sqrt{b}$

$$\overline{\lim}_{t \rightarrow +\infty} |u'(t)| \leq \left(\frac{2}{c} + \frac{1}{\sqrt{b}}\right) \overline{\lim}_{t \rightarrow +\infty} |f(t)| \leq \frac{4}{c} \overline{\lim}_{t \rightarrow +\infty} |f(t)| \quad (2.3)$$

and for  $c > 2\sqrt{b}$

$$\overline{\lim}_{t \rightarrow +\infty} |u'(t)| \leq \frac{c\sqrt{2}}{\sqrt{b}(c^2 - 2b)} \overline{\lim}_{t \rightarrow +\infty} |f(t)| \leq \frac{2}{\sqrt{b}} \overline{\lim}_{t \rightarrow +\infty} |f(t)| \quad (2.4)$$

**Remark 2.2.** In the limiting case  $c = 2\sqrt{b}$ , the four constants in (2.3) and (2.4) are equal:

$$\frac{2}{c} + \frac{1}{\sqrt{b}} = \frac{4}{c} = \frac{c\sqrt{2}}{\sqrt{b}(c^2 - 2b)} = \frac{2}{\sqrt{b}}$$

On the other hand when  $\frac{c}{\sqrt{b}} \rightarrow 0$  the left constant is equivalent to  $\frac{2}{c}$  and when  $\frac{c}{\sqrt{b}} \rightarrow +\infty$  the constant  $\frac{c\sqrt{2}}{\sqrt{b}(c^2 - 2b)}$  is equivalent to  $\frac{\sqrt{2}}{\sqrt{b}}$ . However (2.4) is weak compared to the estimate given in [8] who found  $\frac{4}{c}$  in all cases, for the scalar equation (1.2). We do not know whether the same result is true for the general equation (1.1).

In the applications it is sometimes useful to consider the slightly different situation of solutions defined and bounded on the whole real line. This is the object of our second result.

**Theorem 2.3.** *Let  $b, c$  and  $F, G$  be as in the statement of Theorem 2.1. Then for any solution  $u \in C_b(\mathbb{R}, V) \cap C_b^1(\mathbb{R}, H) \cap W_{loc}^{2,\infty}(\mathbb{R}, V')$  of (1.1), the following estimates are valid*

$$\forall t \in \mathbb{R}, \quad |u(t)| \leq \max\left\{\frac{1}{b}, \frac{2}{c\sqrt{b}}\right\} \|f\|_{L^\infty(\mathbb{R}, H)} \quad (2.5)$$

$$\forall t \in \mathbb{R}, \quad 2G(u(t)) \leq \left(\frac{4}{c^2} + \frac{1}{b}\right) \|f\|_{L^\infty(\mathbb{R}, H)} \quad (2.6)$$

In addition for  $c \leq 2\sqrt{b}$

$$\forall t \in \mathbb{R}, \quad |u'(t)| \leq \left(\frac{2}{c} + \frac{1}{\sqrt{b}}\right) \|f\|_{L^\infty(\mathbb{R}, H)} \leq \frac{4}{c} \|f\|_{L^\infty(\mathbb{R}, H)} \quad (2.7)$$

and for  $c > 2\sqrt{b}$

$$\forall t \in \mathbb{R}, \quad |u'(t)| \leq \frac{c\sqrt{2}}{\sqrt{b(c^2 - 2b)}} \|f\|_{L^\infty(\mathbb{R}, H)} \leq \frac{2}{\sqrt{b}} \|f\|_{L^\infty(\mathbb{R}, H)} \quad (2.8)$$

### 3- Proof in the case of a small damping.

This section is devoted to the proof of Theorems 2.1 and 2.3 under the hypothesis

$$c \leq 2\sqrt{b} \quad (3.1)$$

In this case we can use the following energy functional :

$$\Phi(t) = |u'(t)|^2 + 2G(u(t)) + c(u(t), u'(t)) \quad (3.2)$$

Here we have, setting

$$g(u) := bu + \nabla F(u) = \nabla G(u)$$

$$\Phi' = \langle u'' + g(u), 2u' \rangle + c|u'|^2 + c\langle u'', u \rangle = -c|u'|^2 + c\langle f - g(u) - cu', u \rangle + 2\langle f, u' \rangle$$

$$\Phi' = -c(u'^2 + \langle g(u), u \rangle + c\langle u, u' \rangle) + \langle f, 2u' + cu \rangle \quad (3.3)$$

By convexity of  $F$  we have on the other hand

$$\forall u \in V, \quad \langle g(u), u \rangle = b|u|^2 + \langle \nabla F(u), u \rangle \geq b|u|^2 + F(u) = \frac{b}{2}|u|^2 + G(u)$$

Hence

$$(u'^2 + \langle g(u), u \rangle + c(u, u')) \geq \frac{1}{2}(u'^2 + 2G(u) + c(u, u')) + \frac{1}{2}(u'^2 + b|u|^2 + c(u, u'))$$

Therefore (3.3) implies

$$\Phi'(t) \leq -\frac{c}{2}\Phi(t) - \frac{c}{2}(u'^2 + b|u|^2 + c(u, u')) + f(2u' + cu) \quad (3.4)$$

On the other hand since  $2G(u) \geq b|u|^2$  we have by (3.1)

$$|2u' + cu|^2 = 4|u'|^2 + 4c(u, u') + c^2|u|^2 \leq 4|u'|^2 + 4c(u, u') + 4b|u|^2 \leq 4\Phi$$

hence, using

$$(f, 2u' + cu) \leq \frac{2}{c}|f|^2 + \frac{c}{8}|2u' + cu|^2 \leq \frac{2}{c}|f|^2 + \frac{c}{2}\Phi$$

we deduce from (3.4) the inequality

$$\Phi' \leq -\frac{c}{2}\Phi + \frac{2}{c}|f|^2 \quad (3.5)$$

In particular we find that  $\Phi$  is bounded with

$$\overline{\lim}_{t \rightarrow +\infty} \Phi(t) \leq \frac{4}{c^2} \overline{\lim}_{t \rightarrow +\infty} |f(t)|^2$$

Fix any number

$$A > \frac{4}{c^2} \overline{\lim}_{t \rightarrow +\infty} |f(t)|^2$$

Then for  $t$  large enough we have

$$|u'(t)|^2 + 2G(u(t)) + c(u(t), u'(t)) \leq A \quad (3.6)$$

In particular for  $t$  large enough

$$b|u(t)|^2 + c(u(t), u'(t)) \leq A$$

and this means

$$\frac{c}{2}(|u(t)|^2)' + b|u(t)|^2 \leq A$$

In particular

$$b \overline{\lim}_{t \rightarrow +\infty} |u(t)|^2 \leq A$$

and by minimizing  $A$  we deduce

$$b \overline{\lim}_{t \rightarrow +\infty} |u(t)|^2 \leq \frac{4}{c^2} \overline{\lim}_{t \rightarrow +\infty} |f(t)|^2 \quad (3.7)$$

Finally from (3.5) we deduce for any  $A$  as above and all  $t$  large enough

$$2G(u(t)) \leq A - |u'(t)|^2 - c(u(t), u'(t)) \leq A + \frac{c^2}{4}|u(t)|^2$$

and then (2.2) follows from (3.7). To check (2.3) we start from (3.6) and (3.7) which give

$$|u'(t)|^2 + b|u(t)|^2 + c(u(t), u'(t)) \leq A$$

valid for  $t$  large enough, in particular for  $t$  large:

$$|u'(t) + \frac{c}{2}u(t)|^2 = |u'(t)|^2 + \frac{c^2}{4}|u(t)|^2 + c(u(t), u'(t)) \leq A$$

hence

$$|u'(t)| \leq |u'(t) + \frac{c}{2}u(t)| + \frac{c}{2}|u(t)| \leq A^{\frac{1}{2}} + \frac{c}{2\sqrt{b}}A^{\frac{1}{2}}$$

from which (2.3) follows at once by letting

$$A \longrightarrow \frac{4}{c^2} \overline{\lim}_{t \rightarrow +\infty} |f(t)|^2$$

The proof of Theorem 2.3 follows the same steps but at each stage the inequalities are valid for all  $t \in \mathbb{R}$  and the upper limits are replaced by uniform bounds.

## 4 - Proof in the case of large damping.

This section is devoted to the proof of Theorems 2.1 and 2.3 under the hypothesis

$$c \geq 2\sqrt{b} \quad (4.1)$$

In this case we can use the following energy functional:

$$\Phi(t) := |u'(t)|^2 + 2G(u(t)) + \alpha(u(t), u'(t)) \quad (4.2)$$

where  $\alpha = c - \sqrt{c^2 - 4b}$ . We have

$$\Phi'(t) = \langle u'' + g(u), 2u' \rangle + \alpha|u'|^2 + \alpha \langle u'', u \rangle = (\alpha - 2c)|u'|^2 + \alpha \langle f - g(u) - cu', u \rangle + 2 \langle f, u' \rangle$$

$$\Phi'(t) = (\alpha - 2c)|u'|^2 + (f, 2u' + \alpha u) - \alpha \langle g(u), u \rangle - \alpha c(u, u')$$

Since  $\langle g(u), u \rangle \geq \frac{b}{2}|u|^2 + G(u)$  we obtain

$$\Phi'(t) \leq (\alpha - 2c)|u'|^2 - \alpha G(u) - \alpha \frac{b}{2}|u|^2 - c\alpha(u, u') + (f, 2u' + \alpha u)$$

Hence

$$\Phi'(t) \leq -\frac{\alpha}{2}\Phi(t) + \left(\frac{3\alpha}{2} - 2c\right)|u'|^2 - \alpha \frac{b}{2}|u|^2 + (-c\alpha + \frac{\alpha^2}{2})(u, u') + (f, 2u' + \alpha u)$$

On the other hand we have

$$(f, 2u' + \alpha u) \leq \frac{\alpha}{2b}|f|^2 + \frac{b}{2\alpha}(4|u'|^2 + \alpha^2|u|^2 + 4\alpha(u, u'))$$

therefore

$$\Phi'(t) \leq -\frac{\alpha}{2}\Phi(t) + \left(\frac{3\alpha}{2} - 2c + \frac{2b}{\alpha}\right)|u'|^2 + \left(-c\alpha + \frac{\alpha^2}{2} + 2b\right)(u, u') + \frac{\alpha}{2b}|f|^2$$

Since  $\alpha = c - \sqrt{c^2 - 4b}$  is a solution of the equation  $x^2 - 2cx + 4b = 0$  we have

$$-c\alpha + \frac{\alpha^2}{2} + 2b = 0$$

In addition

$$\frac{3\alpha}{2} - 2c + \frac{2b}{\alpha} = \alpha - c + \frac{\alpha}{2} - c + \frac{2b}{\alpha} = \alpha - c < 0$$

Hence

$$\Phi'(t) \leq -\frac{\alpha}{2}\Phi(t) + \frac{\alpha}{2b}|f|^2$$

In particular, we find that  $\Phi$  is bounded with

$$\overline{\lim}_{t \rightarrow \infty} \Phi(t) \leq \frac{1}{b} \overline{\lim}_{t \rightarrow \infty} |f|^2$$

Fix any number  $A$

$$A > \frac{1}{b} \overline{\lim}_{t \rightarrow \infty} |f|^2$$

Then for  $t$  large enough we have

$$|u'(t)|^2 + 2G(u(t)) + \alpha(u(t), u'(t)) \leq A \tag{4.3}$$

In particular for  $t$  large enough

$$b|u(t)|^2 + \alpha(u(t), u'(t)) \leq A$$



and this means

$$\frac{\alpha}{2}(|u(t)|^2)' + b|u(t)|^2 \leq A$$

In particular

$$b \overline{\lim}_{t \rightarrow \infty} |u(t)|^2 \leq A$$

and by minimizing A we deduce

$$b \overline{\lim}_{t \rightarrow \infty} |u(t)|^2 \leq \frac{1}{b} \overline{\lim}_{t \rightarrow \infty} |f|^2 \quad (4.4)$$

Finally from (4.3) and since  $\alpha \leq \frac{4b}{c}$  we deduce for any  $A$  as above and all  $t$  large enough

$$2G(u(t)) \leq A - |u'(t)|^2 - \alpha(u(t), u'(t)) \leq A + \frac{4b^2}{c^2}|u(t)|^2$$

and then (2.2) follows from (4.4). To check (2.4) we start from (4.3) and (4.4) which give

$$|u'(t)|^2 + 2b|u(t)|^2 + \alpha(u(t), u'(t)) \leq 2A$$

Valid for  $t$  large enough. Hence

$$|u'(t)|^2 \leq 2A - 2b|u(t)|^2 - \alpha(u(t), u'(t)) \leq 2A + \frac{\alpha^2}{8b}|u'(t)|^2$$

On the other hand we have

$$\alpha = c - \sqrt{c^2 - 4b} = \frac{4b}{c + \sqrt{c^2 - 4b}}$$

therefore

$$\frac{\alpha^2}{8b} = \frac{2b}{(c + \sqrt{c^2 - 4b})^2} \leq \frac{2b}{c^2}$$

so that we obtain

$$|u'(t)|^2 \leq 2A + \frac{2b}{c^2}|u'(t)|^2$$

from wich (2.4) follows at once by letting

$$A \longrightarrow \frac{1}{b} \overline{\lim}_{t \rightarrow +\infty} |f(t)|^2$$

The proof of Theorem 2.3 follows the same steps but at each stage the inequalities are valid for all  $t \in \mathbb{R}$  and the upper limits are replaced by uniform bounds.

## 5- Applications.

As mentioned in the introduction, Theorems 2.1 and 2.3 now enable us to improve several boundedness results which appeared previously in the Litterature.

### 5.1- Application to Duffing's equation.

When we apply Theorem 2.1 to Duffing's equation, we obtain immediately

**Corollary 5.1.1.** *Let  $b, c$  be two positive constants and let  $g$  satisfy (1.3). Then for any solution  $u \in C^1(\mathbb{R}^+) \cap W_{loc}^{2,\infty}(\mathbb{R}^+)$  of (1.2),  $u$  is bounded with*

$$\overline{\lim}_{t \rightarrow +\infty} |u(t)| \leq \max\left\{\frac{1}{b}, \frac{2}{c\sqrt{b}}\right\} \overline{\lim}_{t \rightarrow +\infty} |f(t)|$$

and moreover

$$2 \overline{\lim}_{t \rightarrow +\infty} G(u(t)) \leq \left(\frac{4}{c^2} + \frac{1}{b}\right) \overline{\lim}_{t \rightarrow +\infty} |f(t)|^2$$

where  $G$  is the primitive of  $g$  vanishing at 0. In addition for  $c \leq 2\sqrt{b}$

$$\overline{\lim}_{t \rightarrow +\infty} |u'(t)| \leq \left(\frac{2}{c} + \frac{1}{\sqrt{b}}\right) \overline{\lim}_{t \rightarrow +\infty} |f(t)| \leq \frac{4}{c} \overline{\lim}_{t \rightarrow +\infty} |f(t)|$$

and for  $c > 2\sqrt{b}$

$$\overline{\lim}_{t \rightarrow +\infty} |u'(t)| \leq \frac{c\sqrt{2}}{\sqrt{b}(c^2 - 2b)} \overline{\lim}_{t \rightarrow +\infty} |f(t)| \leq \frac{2}{\sqrt{b}} \overline{\lim}_{t \rightarrow +\infty} |f(t)|$$

**Remark 5.1.2.** This result improves on [8] but we do not recover the correct estimate on  $u'$  for  $c$  large.

### 5.2- The case of linear evolution equations.

Let  $H$  be a real Hilbert space. In the sequel we denote by  $(u, v)$  the inner product of two vectors  $u, v$  in  $H$  and by  $|u|$  the H-norm of  $u$ . Let  $A : D(A) \rightarrow H$  a possibly unbounded self-adjoint linear operator such that

$$\exists \lambda > 0, \forall u \in D(A), \quad (Au, u) \geq \lambda |u|^2$$

We consider the largest possible number satisfying the above inequality, in other terms

$$\lambda_1 = \inf_{u \in D(A), |u|=1} (Au, u)$$

We also introduce

$$V = D(A^{1/2})$$

endowed with the norm given by

$$\forall u \in V, \quad \|u\|^2 = |A^{1/2}u|^2$$

We recall that

$$\forall u \in D(A), \quad |A^{1/2}u|^2 = (Au, u)$$

It is clear that the norm just defined on  $V$  is equivalent to the graph norm of  $A^{1/2}$  as a consequence of our coerciveness assumption on  $A$ .

Given  $f \in L^\infty(\mathbb{R}, H)$  the second order evolution equation

$$u'' + cu' + Au = f(t) \tag{1.4}$$

is well-known to have a unique bounded solution  $u \in C_b(\mathbb{R}, V) \cap C_b^1(\mathbb{R}, H)$ . Which attracts exponentially all solutions (and in particular all strong solutions) as  $t$  goes to infinity. As a consequence of Theorem 2.3 associated with a density argument for smooth forcing terms  $f$  we obtain

**Corollary 5.2.1.** *The unique bounded solution  $u$  of (1.4) satisfies*

$$\forall t \in \mathbb{R}, \quad |u(t)| \leq \max\left\{\frac{1}{\lambda_1}, \frac{2}{c\sqrt{\lambda_1}}\right\} \|f\|_{L^\infty(\mathbb{R}, H)}$$

$$\forall t \in \mathbb{R}, \quad \|u(t)\| \leq \sqrt{\frac{4}{c^2} + \frac{1}{\lambda_1}} \|f\|_{L^\infty(\mathbb{R}, H)}$$

In addition for  $c \leq 2\sqrt{\lambda_1}$  we have

$$\forall t \in \mathbb{R}, \quad |u'(t)| \leq \left(\frac{2}{c} + \frac{1}{\sqrt{\lambda_1}}\right) \|f\|_{L^\infty(\mathbb{R}, H)} \leq \frac{4}{c} \|f\|_{L^\infty(\mathbb{R}, H)}$$

and for  $c > 2\sqrt{\lambda_1}$

$$\forall t \in \mathbb{R}, \quad |u'(t)| \leq \frac{c\sqrt{2}}{\sqrt{\lambda_1}(c^2 - 2\lambda_1)} \|f\|_{L^\infty(\mathbb{R}, H)} \leq \frac{2}{\sqrt{\lambda_1}} \|f\|_{L^\infty(\mathbb{R}, H)}$$

**Remark 5.2.2.** Compared with the scalar case we lose here a factor  $\sqrt{2}$  for  $c$  small. This was already observed in the main result of [7]. On the other hand for  $c > 2\sqrt{\lambda_1}$  we improve the result of [7] by a factor  $\sqrt{2}$  and now all our estimates match in the limiting case, which was not the case in [7].

### 5.3- Attractors of semilinear hyperbolic problems.

Let  $\Omega$  be a bounded open domain in  $\mathbb{R}^N$  and  $b \geq 0, c > 0$ . We consider the problem

$$u'' - \Delta u + g(u) + cu' = a \sin u \quad (5.1)$$

with one of the boundary conditions

$$u = 0 \quad \text{on } \partial\Omega \quad (5.2)$$

or

$$\frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega \quad (5.3)$$

with  $g \in C^1$  such that for some nonnegative constants  $b, C, \gamma$  we have

$$\forall s \in \mathbb{R}, \quad b \leq g'(s) \leq C(1 + |s|)^\gamma \quad (5.4)$$

It is well known that under the growth condition (5.4) with

$$(N - 2)\gamma < 2 \quad (5.5)$$

problems (5.1)-(5.2) and (5.1)-(5.3) have unique solutions for given initial data in the energy space and these solutions can be approximated, cf e.g. [4], by solutions which satisfy the regularity conditions  $u \in C^1(\mathbb{R}^+, V) \cap W_{loc}^{2,\infty}(\mathbb{R}^+, V')$  where  $V = H_0^1(\Omega)$  in the first case and  $V = H^1(\Omega)$  in the second one. In addition the dynamical system generated by (5.1) is well known to have a compact attractor  $\mathcal{A}$  under the condition  $b > 0$  in the second case. The result of Theorem 2.1 now gives the following upper bound of the size of the u-projection of  $\mathcal{A}$ .

**Corollary 5.2.1.** *In the case of problem (5.1)-(5.2) we have*

$$\forall (u, v) \in \mathcal{A}, \quad \left\{ \int_{\Omega} (\|\nabla u\|^2 + bu^2) dx \right\}^{1/2} \leq a|\Omega|^{1/2} \sqrt{\frac{4}{c^2} + \frac{1}{\lambda_1(\Omega) + b}} \quad (5.6)$$

and for problem (5.1)-(5.3) we have

$$\forall (u, v) \in \mathcal{A}, \quad \left\{ \int_{\Omega} (\|\nabla u\|^2 + bu^2) dx \right\}^{1/2} \leq a|\Omega|^{1/2} \sqrt{\frac{4}{c^2} + \frac{1}{b}} \quad (5.7)$$

These estimates generalize a result from [7] and are, surprisingly enough, close to optimality even when  $g$  is linear, as was shown in [7]. Theorem 2.1 also provides the corresponding estimates on  $v = u'$  but they are less interesting and probably not quite optimal.

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