N-cyclic functions and multiple subharmonic solutions of Duffing's equation

Sana Gasmi, Alain Haraux

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Abstract. We introduce, in the abstract framework of finite isometry groups on a Hilbert space, a generalization of antiperiodicity called N-cyclicity. The non-existence of N-cyclic solutions of a certain type for the autonomous ODE $x'' + g(x) = 0$ implies the existence of N different subharmonic solutions for some forced equations of the type $x'' + g(x) + cx' = \varepsilon f(t)$ where c and ε are some positive constants and f is, for instance, a sinusoidal function.

Résumé. On introduit, dans le cadre général des groupes finis d'isométries sur un espace de Hilbert réel, une généralisation de l'anti-périodicité appelée N-cyclicité. L'inexistence de solutions N-cycliques d'un certain type pour l'équation autonome $x'' + g(x) = 0$ permet de déduire l'existence of N solutions sous-harmoniques pour l'équation de Duffing forcée avec dissipation $x'' + g(x) + cx' = \varepsilon f(t)$ où c et ε sont des constantes positives et f est par exemple, une fonction sinusoidale.

Keywords: second order equation, periodic solution, subharmonic

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1. Introduction.

The problem of existence as well as multiplicity of periodic solutions of the forced Duffing's equation

$$x'' + g(x) + cx' = f(t) \tag{1.1}$$

has been the object of many works in both undamped (case c = 0) and damped case. A considerable number of papers have been devoted to this subject in the past, cf. for instance the well known contributions of K.O. Friedrichs & J.J. Stoker [9], M.L. Cartwright & J.E. Littlewood [5], M.E. Levenson [13], W. S. Loud [15 - 17], etc. In 1976, G. Morris [19] obtained the rather surprising result that all solutions of the undamped equation are bounded when g is a cubic nonlinearity, together with the existence of infinitely many subharmonic solutions of any order. This result was generalized by T. Ding [7] and complemented in 1984 by A. Bahri & H. Berestycki [1] who showed, in a wide context, the existence of infinitely many periodic trajectories for the superlinear undamped forced equation. In view of the finiteness result in the case of first order scalar equations, cf. V. A. Pliss [21] and L. Brull & J. Mawhin [4], it is quite natural to wonder whether the dissipative equation with forcing can produce an infinite number of periodic solutions. This question received a partial answer in 1959 in the paper [16] of W.S. Loud who gave a method to construct an arbitrary large number of those solutions by small perturbation from periodic orbits of the autonomous problem, relying on a small parameter approach using standard and modified implicit function arguments. The construction requires a smallness assumption on both forcing term f, of the form εf_0 where ε is small and damping coefficient c, required to be smaller than a certain function of ε. However, [16] does not give a very explicit way to build a large number of solutions. A more precise picture is given in the paper by J.K. Hale & P.Z. Taboas [11] in which the implicit function theorem is used via bifurcation theory.

When g is odd and for instance, convex and increasing for positive values of the argument, all solutions of the autonomous equation

$$u'' + g(u) = 0 \tag{1.2}$$

are well known to be anti-periodic, which means that for some number $\tau > 0$ (called an anti-period of u we have

$$\forall t \in \mathbb{R}, \quad u(t+\tau) = -u(t)$$

An immediate calculation shows that any anti-periodic function u with anti-period τ is 2τ-periodic. Moreover if $u \not\equiv 0$, the set of all-antiperiods for u is made of all odd multiples of a positive number, the smallest anti-period of u. Nonlinear differential equations with odd nonlinearity and an anti-periodic forcing term tend to have, as a general rule, at least one anti-periodic solution, which means that anti-periodicity of the exterior force in presence of odd nonlinearities prevents the resonance phenomenon classically observed in non-coercive situations. This was noticed first by H. Okochi [20] for monotone systems and generalized by the second author in [12]. In particular equation (1.1) with g odd and f anti-periodic always has at least one anti-periodic solution. Subsequently, A. Beaulieu [2] and P. Souplet [22] obtained uniqueness results for this solution when f is small in the uniform norm compared to the damping coefficient and the first counterexample to uniqueness was obtained by P. Souplet [22] who exhibited two different anti-periodic solutions of (1.1) for a suitable C^∞ forcing term.

A natural question is then whether the method of W.S. Loud [17] can be used in the anti-periodic framework, especially to construct multiple anti-periodic trajectories or anti-periodic subharmonic solutions with large anti-periods. This program was partially carried out by the first author in [10] and specifically, existence of 4 different anti-periodic solutions was obtained when $g(s) = \alpha s + \beta s^3$. The purpose of the present work is to produce existence results for subharmonic solutions with arbitrarily large anti-periods. These solutions automatically provide many anti-periodic solutions, the translates of one of them by multiples of the period of the driving force f. In order to do that it will be convenient to define some generalizations of periodic and anti-periodic functions, called below cyclic and anti-cyclic functions.

This paper is divided in 9 sections. In section 2, 3 and 4 we introduce an abstract notion of cyclicity in the framework of finite isometry groups on a Hilbert space, and we apply it to define N- cyclic and anticyclic functions. Section 5 contains non-existence results of N- cyclic and anticyclic solutions of (1.2) under suitable assumptions on g, satisfied in particular by a variety of polynomial nonlinearities. Sections 6, 7 and 8 are devoted to the existence of many subharmonic solutions in both periodic and anti-periodic frameworks. Section 9 concludes the paper with a list of 3 questions.

2. A simple algebraic property.

Let H be a real Hilbert space. In the sequel we denote by (u, v) the inner product of two vectors u, v in H and by $|u|$ the H-norm of u. We consider a finite group G of (linear) isometries on H, which means

$$\forall g \in G, \forall u \in H, \quad |g(u)| = |u|$$

We define the (linear) isotropy space Γ of G as the following subset of H

$$u \in \Gamma \iff \forall g \in G,\ g(u) = u$$

Theorem 2.1. *We have the following equivalence*

$$u \in \Gamma^{\perp} [\iff \forall v \in \Gamma,\ (u, v) = 0] \iff \sum_{g \in G} g(u) = 0$$

Proof. First we note that for any $y \in H$ we have $\sum_{g \in G} g(y) \in \Gamma$ indeed

$$\forall h \in G, \quad h(\sum_{g \in G} g(y)) = \sum_{g \in G} h(g(y)) = \sum_{g \in G} (hg)(y) = \sum_{g \in G} g(y)$$

since

$$\{hg | g \in G\} = G$$

In particular if $u \in \Gamma^{\perp}$ we have

$$\forall y \in H, \quad (\sum_{g \in G} g(u), y) = (\sum_{g \in G} g^{-1}(u), y) = (\sum_{g \in G} g^{*}(u), y) = (\sum_{g \in G} g(y), u) = 0$$

where we used the isometry property through $\forall g \in G,\ g^{-1} = g^{*}$. Choosing $y = \sum_{g \in G} g(u)$ we find $\sum_{g \in G} g(u) = 0$.

Conversely if $\sum_{g \in G} g(u) = 0$, we notice that

$$\forall v \in \Gamma, \quad 0 = (\sum_{g \in G} g(u), v) = (\sum_{g \in G} g^{-1}(u), v) = (\sum_{g \in G} g^{*}(u), v) = (\sum_{g \in G} g(v), u) = 0$$

Since

$$\sum_{g \in G} g(v) = (card\, G)v$$

this yields

$$\forall v \in \Gamma, \quad (v, u) = 0$$

Definition 2.2. An element function $u \in H$ is called G -cyclic if we have

$$\sum_{g \in G} g(u) = 0$$

Remark 2.3. Theorem 2.1 means that the orthogonal of the vector space of G-cyclic elements of H is equal to the set of common fixed points of all transformations $g \in G$, that we called the isotropy space of G.

3- N-cyclic functions

Definition 3.1. A function $f \in C(\mathbb{R})$ is said N-cyclic if there exists $\tau > 0$ for which

$$\forall t \in \mathbb{R}, \quad \sum_{k=0}^{N-1} f(t + k\tau) = 0 \tag{3.1}$$

In addition when (3.1) is fulfilled we say that f is τ, N-cyclic.

Proposition 3.2. *Any τ, N-cyclic function is $N\tau$-periodic.*

Proof. As a consequence of (3.1) we have

$$\forall t \in \mathbb{R}, \quad f(t) + \sum_{k=1}^{N-1} f(t + k\tau) = 0$$

Applying (3.1) to the translate function $f(. + \tau)$ we have also

$$\forall t \in \mathbb{R}, \quad \sum_{k=1}^{N-1} f(t + k\tau) + f(t + N\tau) = 0$$

The result follows immediately.

Remark 3.3. When $N = 2$, an N-cyclic function is just an antiperiodic function. In addition all anti-periodic functions are N-cyclic functions for all even values of N.

The following property is an easy consequence of Theorem 2.1.

Proposition 3.4. *Let $f \in C(\mathbb{R})$ be $N\tau$-periodic. The following properties are equivalent*

i) f is τ, N-cyclic .

ii) For any τ-periodic function g we have $\int_0^{N\tau} f(s)g(s)ds = 0$

iii)

$$\forall k \in \mathbb{N}, \quad \int_0^{N\tau} f(s)\cos(2k\frac{\pi}{\tau}s)ds = \int_0^{N\tau} f(s)\sin(2k\frac{\pi}{\tau}s)ds = 0$$

Proof. We introduce the Hilbert space

$$H = \{f \in L^2_{loc}(\mathbb{R}), \quad f(t+N\tau) = f(t), \text{ a.e. on } \mathbb{R}\}$$

endowed with the inner product defined by

$$\langle f, g\rangle = \int_0^{N\tau} f(s)g(s)ds$$

We consider the finite multiplicative group G of isometries on H generated by the translation operator $\gamma : H \longrightarrow H$ defined by

$$\gamma f = f(. + \tau)$$

Then the equivalence of i) and ii) is a direct consequence of Theorem 2.1. The equivalence between ii) and iii) is a consequence of the fact that the family of functions $\cos(k\frac{\pi}{\tau}s + \phi)$ is total in $L^2(0, \tau)$.

4 - N-anticyclic functions

Definition 4.1. Let N be any odd integer. A function $f \in C(\mathbb{R})$ is said N-anticyclic if there exists $\tau > 0$ for which

$$\forall t \in \mathbb{R}, \quad \sum_{k=0}^{N-1} (-1)^k f(t + k\tau) = 0 \qquad (4.1)$$

In addition when (4.1) is fulfilled we say that f is τ, N-anticyclic.

Proposition 4.2. *Any τ, N-anticyclic function is $N\tau$-antiperiodic.*

Proof. As a consequence of (4.1) we have

$$\forall t \in \mathbb{R}, \quad f(t) + \sum_{k=1}^{N-1} (-1)^k f(t + k\tau) = 0$$

Applying (4.1) to the translate function $f(. + \tau)$ we have also

$$\forall t \in \mathbb{R}, \quad \sum_{k=1}^{N-1} (-1)^{k-1} f(t + k\tau) + f(t + N\tau) = 0$$

The result follows immediately.

The following property is an easy consequence of Theorem 2.1.

Proposition 4.3. *Let $f \in C(\mathbb{R})$ be $N\tau$-antiperiodic. The following properties are equivalent*

i) f is τ, N-anticyclic .

ii) For any τ-antiperiodic function g we have $\displaystyle\int_0^{N\tau} f(s)g(s)ds = 0$

iii)

$$\forall k \in \mathbb{N}, \quad \int_0^{N\tau} f(s)\cos((2k+1)\frac{\pi}{\tau}s)ds = \int_0^{N\tau} f(s)\sin((2k+1)\frac{\pi}{\tau}s)ds = 0$$

Proof. We introduce the Hilbert space

$$H = \{f \in L^2_{loc}(\mathbb{R}), \quad f(t + N\tau) = -f(t), \text{ a.e. on } \mathbb{R}\}$$

endowed with the inner product defined by

$$\langle f, g \rangle = \int_0^{N\tau} f(s)g(s)ds$$

We consider the finite multiplicative group G of isometries on H generated by the anti-translation operator $\gamma : H \longrightarrow H$ defined by

$$\gamma f = -f(. + \tau)$$

Then the equivalence of i) and ii) is a direct consequence of Theorem 2.1. The equivalence between ii) and iii) is a consequence of the fact that the family of functions $\cos((2k+1)\frac{\pi}{\tau}s+\phi)$ is total in the subspace of H made of all $\tau-$antiperiodic functions.

Remark 4.4. If N is even, (4.1) means that $g(t) = f(t+\tau)-f(t)$ defines an τ, $\frac{N}{2}$-cyclic function. Hence g is $\frac{N}{2}\tau$ -periodic. In this case f cannot be $\frac{N}{2}\tau$ antiperiodic unless g is 0. Then f must be $\tau - periodic$. Being both $\frac{N}{2}\tau$- antiperiodic and $\frac{N}{2}\tau$ -periodic, f must be 0. For the same reason in this case f cannot be $N\tau$ antiperiodic either.

Proposition 4.5. *Any τ, N-anticyclic function is 2τ, N-cyclic.*

Proof. This follows from the identity valid for all odd integers N:

$$\sum_{j=0}^{N-1}(\sum_{k=0}^{N-1}(-1)^k f(t+k\tau+j\tau)) = \sum_{k=0}^{N-1} f(t+2k\tau)$$

Proposition 4.6. *A 2τ, N-cyclic function is τ, N-anticyclic if and only if it is $N\tau$-antiperiodic.*

Proof. All we need to show is that a $N\tau$-antiperiodic function f which is 2τ, N-cyclic is actually τ, N-anticyclic. We use proposition 4.3, ii): for any τ-antiperiodic function g we have, since the product fg is an $N\tau$-periodic function

$$\int_0^{N\tau} f(s)g(s)ds = \frac{1}{2}\int_0^{2N\tau} f(s)g(s)ds$$

and since g is 2τ-periodic and f is 2τ, N-cyclic, by Proposition 3.3, ii) the integral in the RHS vanishes. Proposition 4.3, ii) then shows that f is τ, N-anticyclic.

5- Some results on N-cyclic solutions of x” + g(x) = 0.

Under some conditions on g it is well-known that all solutions of

$$x'' + g(x) = 0 \tag{5.1}$$

are antiperiodic, hence 2-cyclic. On the other hand except in the linear case, N-cyclic solutions for $N > 2$ (and N odd) are exceptional. The following 2 results are easy to prove and will allow us to construct multiple periodic and anti-periodic solutions for the forced dissipative equation.

Theorem 5.1. *Assume that g is odd and absolutely monotone on $\mathbb{R}$. Then assuming*

$$\exists n_0 \geq 1, \quad g^{(2n_0+1)}(0) > 0 \tag{5.2}$$

there is no nontrivial 3-cyclic solution and no nontrivial 3-anticyclic solution of (5.1).

Theorem 5.2. *Assume g is an odd polynomial function of degree > 1 on $\mathbb{R}$. Then if u is a non-trivial τ - antiperiodic solution of (5.1), there are arbitrarily large odd integers N for which u is neither $\frac{\tau}{N}$, N - anticyclic nor $\frac{2\tau}{N}$, N - cyclic .*

Proof of Theorem 5.1. It is sufficient to prove that if u, v, w are 3 solutions of (5.1) such that

$$u + v + w = 0$$

the product uvw is identically 0. This was established in [10] in the special case of a cubic polynomial. In general it is well known [3] that any odd absolutely monotone function on $\mathbb{R}$ is analytic with

$$g(s) = \sum_{n=0}^{\infty} a_n s^{2n+1}$$

where

$$\forall n \in \mathbb{N}, \quad n! a_n = g^{(2n+1)}(0) \geq 0$$

In particular $a_{n_0} > 0$.

Lemma 5.3. *For any $n \in \mathbb{N}^*$ we have*

$$\forall (x, y) \in \mathbb{R}^2, \quad xy(x + y) \neq 0 \Longrightarrow \frac{(x + y)^{2n+1} - x^{2n+1} - y^{2n+1}}{xy(x + y)} > 0$$

Proof. By homogeneity it is sufficient to prove the result when $x = 1$ and $y = t$, with $t \neq 0, t \neq -1$. The function

$$F(t) = (1 + t)^{2n+1} - (1 + t^{2n+1})$$

is a polynomial which vanishes for t = 0 and t = -1 and therefore is the product of $t(t + 1)$ by a polynomial of degree $2n - 1$. Using

$$F'(t) = (2n + 1)[(1 + t)^{2n} - t^{2n}]$$

it is immediate to check that $F(t)$ is positive on $(-\infty,-1)\cup(0,\infty)$ and negative on $(-1,0)$. The result follows easily.

End of proof of Theorem 5.1. If u,v,w are 3 solutions of (5.1) such that

$$u+v+w=0$$

a trivial calculation shows that

$$\forall t\in\mathbb{R},\quad g(u(t)+v(t))=g(u(t))+g(v(t))$$

Hence

$$\forall t\in\mathbb{R},\quad \sum_{n=0}^{\infty}a_n[(u(t)+v(t))^{2n+1}-u(t)^{2n+1}-v(t)^{2n+1}]=0$$

If for $t=a$ we assume

$$u(a)v(a)(u(a)+v(a))\neq 0$$

by using Lemma 4.3 we obtain after division a sum of nonnegative terms equal to 0. In particular this gives

$$a_{n_0}\frac{(x+y)^{2n_0+1}-x^{2n_0+1}-y^{2n_0+1}}{xy(x+y)}=0$$

with $x=u(a)$, $y=v(a)$, which contradicts Lemma 5.3 since by hypothesis we have $a_{n_0}>0$ and $xy(x+y)\neq 0$. We conclude that uvw is identically 0. To obtain the result is is now sufficient to apply this property with $v(t)=\pm u(t+\tau),\quad w(t)=u(t+2\tau)$. If u is nontrivial, the 3 functions u, v, w have only a finite number of zeroes by the Cauchy-Lipschitz property and consequently the product uvw also. This completes the proof of Theorem 5.1.

Proof of Theorem 5.2. Actually a simple algebraic argument gives the result: if u is $\frac{2\tau}{N}$, N - cyclic for all odd integers greater than N_0, say, it means that for all those odd integers N we have

$$\forall k\in\mathbb{N},\quad \int_0^{2\tau}u(s)\cos(Nk\frac{\pi}{\tau}s)ds=\int_0^{2\tau}u(s)\sin(Nk\frac{\pi}{\tau}s)ds=0$$

In particular, letting $k=1$ or $k=2$ we see that $v(t)=u(\frac{\pi}{\tau}t)$ is a non trivial trigonometric polynomial of degree p less than N_0, and so is v''. On the other hand if

$d = \deg(g) > 1$, then $g(v)$ is a trigonometric polynomial of degree $dp > p$. It is then trivially impossible to have

$$v'' = -(\frac{\pi}{\tau})^2 g(v).$$

This contradiction concludes the proof of Theorem 5.2, since by Proposition 4.5 it is sufficient to establish that u is not $\frac{2\tau}{N}$, N - cyclic to contradict the $\frac{\tau}{N}$, N - anticyclicity also.

6- Subharmonic solutions of order 3 for equation (1.1).

This section is devoted to a generalization of Theorems 6.2 and 8.1 from [10]. We start with a generalization of Theorem 6.2.

Theorem 6.1. *Assume that g is odd and absolutely monotone on $\mathbb{R}$. Then assuming condition (5.2), for any solution x_0 of (5.1) with smallest anti-period $\tau > 0$ there is a number $\varepsilon_0 > 0$* and a function $\delta : [0, \varepsilon_0] \to \mathbb{R}^+$ such that

$$\lim_{\varepsilon \to 0^+} \delta(\varepsilon) = 0$$

and for each $\varepsilon \in [0, \varepsilon_0]$, $c \in [0, \delta(\varepsilon)]$ the equation

$$x'' + g(x) + cx' = \varepsilon f(t) \tag{6.1}$$

where

$$f(t) = x_0''(t) - x_0''(t + \frac{\tau}{3}) + x_0''(t + 2\frac{\tau}{3})$$

has 3 different τ - antiperiodic solutions with minimal antiperiod τ and a $\frac{\tau}{3}$ - antiperiodic solution.

Proof. This is a direct consequence of [10, Theorem 3.1] applied with

$$f(t) = x_0''(t) - x_0''(t + \frac{\tau}{3}) + x_0''(t + 2\frac{\tau}{3})$$

First we prove that $f \not\equiv 0$. In order to apply Theorem 5.1 we set

$$u(t) = x_0(t), \quad v(t) = -x_0(t + \frac{\tau}{3}), \quad w(t) = x_0(t + 2\frac{\tau}{3})$$

the condition

$$u'' + v'' + w'' \equiv 0$$

would imply, since u, v, w are τ - antiperiodic, that $u' + v' + w' \equiv 0$ and finally $u + v + w \equiv 0$. Then by Theorem 5.1 we would have $x_0 \equiv 0$, a contradiction. Hence $f \not\equiv 0$. Now we have

$$\int_0^\tau f(t)x_0'(t)dt = \sum_{j=0}^{2} \int_{\frac{j\tau}{3}}^{\frac{(j+1)\tau}{3}} f(t)x_0'(t)dt = \sum_{j=0}^{2} \int_0^{\frac{\tau}{3}} f(t + j\frac{\tau}{3})x_0'(t + j\frac{\tau}{3})dt$$

$$= \int_0^{\frac{\tau}{3}} f(t)[x_0'(t) - x_0'(t + \frac{\tau}{3}) + x_0'(t + 2\frac{\tau}{3})]dt = 0$$

since the function

$$h(t) = [x_0'(t) - x_0'(t + \frac{\tau}{3}) + x_0'(t + 2\frac{\tau}{3})]^2$$

is $\frac{\tau}{3}$ - periodic . On the other hand

$$\int_0^\tau f(t)x_0''(t)dt = \sum_{j=0}^{2} \int_{\frac{j\tau}{3}}^{\frac{(j+1)\tau}{3}} f(t)x_0''(t)dt = \int_0^{\frac{\tau}{3}} (f(t))^2 dt > 0$$

The other hypotheses of Theorem 3.1 from [10] are clearly fullfilled. Indeed (5.2) implies that the period is a decreasing function of the amplitude and by performing a time-translation we can always assume that $x_0'(0) = 0$ and $x_0(0) = A > 0$. By [10, Theorem 3.1] we obtain that (6.1) has an antiperiodic solution u with smallest antiperiod equal to τ. It is then clear that $u, -u(. + \frac{\tau}{3}), u(. + \frac{2\tau}{3})$ are 3 solutions of (6.1) since f is $\frac{\tau}{3}$-antiperiodic. And since τ is the minimal antiperiod of u, 2τ is the smallest period and it follows that $u, -u(. + \frac{\tau}{3}), u(. + \frac{2\tau}{3})$ are all different. Finally we know from [12] that (6.1) has a $\frac{\tau}{3}$ - antiperiodic solution, which is necessarily different from $u, -u(. + \frac{\tau}{3})$ and $u(. + \frac{2\tau}{3})$.

The following generalization of Theorem 8.1 from [10] is in fact a consequence of Theorem 6.1.

Theorem 6.2 *Assume that g is odd and absolutely monotone on $\mathbb{R}$. Then assuming condition (5.2), for any solution x_0 of (5.1) with smallest period $T > 0$ there is a number* $\varepsilon_0 > 0$ and a function $\delta : [0, \varepsilon_0] \to \mathbb{R}^+$ such that

$$\lim_{\varepsilon \to 0^+} \delta(\varepsilon) = 0$$

and for each $\varepsilon \in [0, \varepsilon_0]$, $c \in [0, \delta(\varepsilon)]$ the equation (6.1) where

$$f(t) = x_0''(t) + x_0''(t + \frac{T}{3}) + x_0''(t + 2\frac{T}{3})$$

has 3 different periodic solutions with minimal period T. In addition (5.1) has a $\frac{T}{3}$-periodic solution, so that (6.1) has at least 4 different periodic solutions.

Proof . First we recall that x_0 is anti-periodic with smallest anti-period $\tau = \frac{T}{2}$. Then we observe that

$$f(t) = x_0''(t) + x_0''(t + \frac{T}{3}) + x_0''(t + 2\frac{T}{3}) = x_0''(t) + x_0''(t + \frac{2\tau}{3}) + x_0''(t + \frac{4\tau}{3})$$

$$= x_0''(t) + x_0''(t + \frac{2\tau}{3}) - x_0''(t + \frac{\tau}{3})$$

since x_0'' is τ -antiperiodic. The result then follows from Theorem 6.1, since any τ - antiperiodic solution with minimal antiperiod τ is also T - periodic with minimal period $2T$.

7 - Subharmonic solutions of higher order .

This section is devoted to a strong generalization of theorems 6.2 and 8.1 from [10]. However the results are somewhat unprecise. The interesting point is here the existence of subharmonic solutions of arbitrary large order.

Theorem 7.1. *Assume g is an odd polynomial function of degree > 1 on $\mathbb{R}$ which is increasing and strictly convex on $\mathbb{R}^+$. Then for any solution x_0 of (5.1) with smallest anti-period $\tau > 0$ there are infinitely many odd integers N with the following property: there is a number $\varepsilon_N > 0$* and a function $\delta_N : [0, \varepsilon_N] \to \mathbb{R}^+$ such that

$$\lim_{\varepsilon \to 0^+} \delta_N(\varepsilon) = 0$$

and for each $\varepsilon \in [0, \varepsilon_N]$, $c \in [0, \delta_N(\varepsilon)]$ the equation

$$x'' + g(x) + cx' = \varepsilon f_N(t)) \tag{7.1}$$

where

$$f_N(t) = \sum_{j=0}^{N-1} (-1)^j x_0''(t + j\frac{\tau}{N})$$

has N different antiperiodic solutions with minimal antiperiod τ and a $\frac{\tau}{N}$ - antiperiodic solution.

Proof. First we observe that as a consequence of Theorem 5.2, there are arbitrarily large odd integers N for which u is not $\frac{\tau}{N}$, N - anticyclic. For such a value of N we have

$$\sum_{j=0}^{N-1}(-1)^j x_0(t+j\frac{\tau}{N}) \not\equiv 0$$

and because this function is τ-antiperiodic this implies $f \not\equiv 0$. Now we set

$$F_N(t) = \sum_{j=0}^{N-1}(-1)^j x_0'(t+j\frac{\tau}{N})$$

A simple calculation shows that

$$\int_0^\tau f_N(t)x_0'(t)dt = \sum_{j=0}^{N-1}\int_{\frac{j\tau}{N}}^{\frac{(j+1)\tau}{N}} f_N(t)x_0'(t)dt = \int_0^{\frac{\tau}{N}} F_N(t)f_N(t)dt = 0$$

while

$$\int_0^\tau f_N(t)x_0''(t)dt = \sum_{j=0}^{N-1}\int_{\frac{j\tau}{N}}^{\frac{(j+1)\tau}{N}} f_N(t)x_0''(t)dt = \int_0^{\frac{\tau}{N}} (f_N(t))^2dt > 0$$

The other hypotheses of Theorem 3.1 from [10] are clearly fullfilled. Indeed the increasing and strictly convex character of g implies that the period is a decreasing function of the amplitude and by performing a time-translation we can always assume that $x_0'(0) = 0; \quad x_0(0) = A > 0$. By [10 , Theorem 3.1] we obtain that (7.1) has an antiperiodic solution u with smallest antiperiod equal to τ. It is then clear that $(-1)^j u(. + \frac{j\tau}{N})$ is a solution of of (7.1)for each $j \in \mathbb{N}$ since f_N is $\frac{\tau}{N}$-antiperiodic. And since τ is the minimal antiperiod of u, 2τ is the smallest period and it follows that the functions$(-1)^j u(. + \frac{j\tau}{N})$ are all different for $j \in 0, ...N-1$. Finally since f_N is $\frac{\tau}{N}$-antiperiodic we know from [Haraux] that (7.1) has a $\frac{\tau}{N}$ - antiperiodic solution, which has to be different from $(-1)^j u(. + \frac{j\tau}{N})$ for all $j \in \mathbb{N}$.

The following generalization of Theorem 8.1 from [10] is in fact a consequence of Theorem 7.1.

Theorem 7.2. *Assume g is an odd polynomial function of degree > 1 on $\mathbb{R}$ which is increasing and convex on $\mathbb{R}^+$. Then for any solution x_0 of (5.1) with smallest period*

$T>0$ there are infinitely many odd integers N with the following property: there is a number $\varepsilon_N>0$ and a function $\gamma_N:[0,\varepsilon_N]\to\mathbb{R}^+$ such that

$$\lim_{\varepsilon\to 0^+}\gamma_N(\varepsilon)=0$$

and for each $\varepsilon\in[0,\varepsilon_N]$, $c\in[0,\gamma_N(\varepsilon)]$ the equation

$$x''+g(x)+cx'=\varepsilon f_N(t)$$

where

$$f_N(t)=\sum_{j=0}^{N-1}x_0''(t+j\frac{T}{N})$$

has N different periodic solutions with minimal period T .

Proof. Same argument as for passing from Theorem 6.1 to Theorem 6.2. The only point to check is the identity

$$f_N(t)=\sum_{j=0}^{N-1}(-1)^j x_0''(t+j\frac{\tau}{N})=\sum_{k=0}^{\frac{N-1}{2}}x_0''(t+2k\frac{\tau}{N})-\sum_{k=0}^{\frac{N-3}{2}}x_0''(t+(2k+1)\frac{\tau}{N})$$

$$=\sum_{k=0}^{\frac{N-1}{2}}x_0''(t+2k\frac{\tau}{N})+\sum_{k=0}^{\frac{N-3}{2}}x_0''(t+(2k+1+N)\frac{\tau}{N})=\sum_{k=0}^{N-1}x_0''(t+2k\frac{\tau}{N})$$

$$=\sum_{k=0}^{N-1}x_0''(t+k\frac{T}{N})$$

8 - Subharmonic solutions for sinusoidal forcing terms.

This section is devoted to a (partial) generalization of Theorem 7.1 from [10].

Theorem 8.1. *Assume that g is odd and absolutely monotone on $\mathbb{R}$. Then assuming condition (5.2), for any solution x_0 of (5.1) with smallest anti-period $\tau>0$ and $x_0'(0)=0$ there is $k\in\mathbb{N}$, a number $\varepsilon_0>0$* and a function $\delta:[0,\varepsilon_0]\to\mathbb{R}^+$ such that

$$\lim_{\varepsilon\to 0^+}\delta(\varepsilon)=0$$

and for each $\varepsilon\in[0,\varepsilon_0]$, $c\in[0,\delta(\varepsilon)]$ the equation

$$x''+g(x)+cx'=\varepsilon\cos(3(2k+1)\frac{\pi}{\tau}t) \tag{8.1}$$

has 3 different τ - antiperiodic solutions with minimal antiperiod τ and a $\frac{\tau}{3}$ - antiperiodic solution.

Proof. This is a direct consequence of Theorem 3.1 from [10] applied with

$$f(t) = \cos(3(2k+1)\frac{\pi}{\tau}t)$$

where $k \in \mathbb{N}$ is chosen such that

$$\int_0^\tau f(t)x_0''(t)dt \neq 0$$

Such an integer k exists as a consequence of Theorem 5.1 and Proposition 4.3. On the other hand we have

$$\int_0^\tau f(t)x_0'(t)dt = 0$$

because x_0' is a pure sine series. The rest of the proof is identical to that of Theorem 6.1.

Theorem 8.2. *Assume g is an odd polynomial function of degree > 1 on $\mathbb{R}$ which is increasing and strictly convex on $\mathbb{R}^+$. Then for any solution x_0 of (5.1) with smallest anti-period $\tau > 0$ and $x_0'(0) = 0$ there are infinitely many odd integers N with the following property: there is $k_N \in \mathbb{N}$, a number $\varepsilon_N > 0$* and a function $\delta_N : [0, \varepsilon_N] \to \mathbb{R}^+$ such that

$$\lim_{\varepsilon \to 0^+} \delta_N(\varepsilon) = 0$$

and for each $\varepsilon \in [0, \varepsilon_N]$, $c \in [0, \delta_N(\varepsilon)]$ the equation

$$x'' + g(x) + cx' = \varepsilon \cos(N(2k+1)\frac{\pi}{\tau}t) \tag{8.2}$$

has N different antiperiodic solutions with minimal antiperiod τ and a $\frac{\tau}{N}$ - antiperiodic solution.

Proof. This is a direct consequence of Theorem 3.1 from [10] applied with

$$f(t) = \cos(N(2k+1)\frac{\pi}{\tau}t)$$

where $k \in \mathbb{N}$ is chosen such that

$$\int_0^\tau f(t)x_0''(t)dt \neq 0$$

Such an integer k exists as a consequence of Theorem 5.1 and Proposition 4.3. On the other hand we have

$$\int_0^\tau f(t)x_0'(t)dt = 0$$

because x_0' is a pure sine series. The rest of the proof is identical to that of Theorem 7.1.

9 - Conclusions and final remarks.

The main results of this paper enable us to find an arbitrary large number N of anti-periodic solutions of (1.1), actually close from N different time-translates of a non constant τ-antiperiodic solution x_0 of (1.2), with f a small $\frac{\tau}{N}$ -antiperiodic forcing term and c a small positive constant. The function f can be a sinusoid, a function built from x_0 which consequently has an infinite number of non-trivial Fourier components, or a different finite or infinite Fourier series built from the Fourier expansion of x_0 . The problem that we solved is quite different from finding many solutions for (1.1) with a fixed forcing term f since here the forcing term is essentially computed from x_0 and the exact computation of x_0 itself is in general impossible. Our results are interesting in the anti-periodic case, in the periodic framework the results of [11] are much stronger. However, our method also gives some (weak) results at no expense in the periodic setting, under general conditions on g.

The following questions are still apparently open after this work

1) Can we get an anti-periodic theorem similar to [1] for odd undamped anti-periodic systems? (This seems highly probable.)

2) Is it possible to get many anti-periodic solutions with the same smaller anti-period as the forcing term? (It is probably possible to obtain that at least in some cases by a perturbation argument.)

3) Is the number of anti-periodic solutions always finite? The result might be different if we consider subharmonic solutions or only anti-periodic solutions with the same minimal anti-period as the forcing term.

All these questions seem to deserve future investigation, the most difficult being probably the last one.

References

1. A. Bahri, H. Berestycki, Forced vibrations of superquadratic Hamiltonian systems, Acta Math. (1984)
2. A. Beaulieu, Etude de l'unicité des solutions anti-périodiques pour des équations d'évolution non linéaires, Portug. Math. 49, 4 (1992).
3. S. Bernstein, Sur les fonctions absolument monotones, Acta Mathematica 51 (1928), 1-66.
4. L. Brull & J. Mawhin, Finiteness of the set of solutions of some boundary value problems for ordinary differential equations, Arch Mathematicum (BRNO), 24, 4 (1988), 163-172
5. M.L. Cartwright & J.E. Littlewood, On non-linear differential equations of the second order, Ann. Math. 48 (1947), 472-494.
6. E.A. Coddington & N. Levinson, *Theory of ordinary differential equations*, New York, 1955.
7. T. Ding, Boundedness of solutions of Duffing's equation, J. Differential Equations 61 (1986), no. 2, 178–207.
8. K.O. Friedrichs, *Advanced ordinary differential equations*, New York, 1954.
9. K.O. Friedrichs & J.J. Stoker, Forced vibrations of systems with nonlinear restoring force, Quarterly oh Applied Math. vol. 1, 97-115 (1943).
10. S. Gasmi, Solutions anti-périodiques multiples de $x'' + cx' + g(x) = \varepsilon f(t)$, to appear.
11. J.K. Hale & P.Z. Taboas, Interaction of damping and forcing in a second order evolution equation, Nonlinear analysis, T.M.A. 2, 1 (1978), 77-84.
12. A. Haraux, Antiperiodic solutions of some nonlinear evolution equations. Manuscripta Math. 63, 479-505 (1989).
13. M.E. Levenson, Harmonic and subharmonic response for the Duffing equation $x'' + \alpha x + \beta x^3 = Fcos(\omega t)$ $(\alpha > 0)$, J. Appl. Phy. vol. 20, 1045-1051 (1949).
14. N. Levinson, Transformation Theory of Nonlinear Differential Equations of the second order, Annals of Mathematics, vol 45 (1944) p723-737.
15. W. S. Loud, On periodic solutions of Duffing's equation with damping, Journal of Mathematics and Physics 34 (1955), 173-178
16. W. S. Loud, Boundedness and convergence of solutions of x"+cx' +g(x) = e(t), Duke Math. J. 24, no. 1 (1957), 63-72

17. W. S. Loud, Periodic solutions of x"+cx' +g(x) = f(t), Mem. Amer. Math. Soc., 31, (1959), 1-57.
18. J. Mawhin, Bound sets and Floquet boundary value problems for nonlinear differential equations. In *Topological Methods in Differential Equations and Dynamical Systems (Krakow-Przegorzały, 1996)*, Univ. Iagel. Acta Math. No. 36 (1998), 41-53.
19. G. Morris, A case of boundedness in Littlewood's problem on oscillatory differential equations, Bull. Austral.Soc. 14 (1976), 71-93.
20. H. Okochi, On the existence of periodic solutions to nonlinear abstract parabolic equations, J. Math. Soc. Japan 40 (3), 541-553 (1988).
21. V. A. Pliss, *Nonlocal problems in the theory of nonlinear oscillations*, Academic Press, New-York, 1966
22. Ph. Souplet, Uniqueness and nonuniqueness results for the antiperiodic solutions of some second-order nonlinear evolution equations, Nonlinear Analysis T.M.A. 26, 26 (1996), 1511 - 1525
23. J.J. Stoker, *Nonlinear vibrations in mechanical and electrical systems*, New York, 1950.