Global rigidity in CR geometry: the Schoen-Webster theorem.

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GLOBAL RIGIDITY IN CR GEOMETRY: THE SCHOEN–WEBSTER THEOREM

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Abstract. Schoen-Webster theorem asserts a pseudoconvex CR manifold whose automorphism group acts non properly is either the standard sphere or the Heisenberg space. The purpose of this paper is to survey successive works around this result and then provide a short geometric proof in the compact case.

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Among the many aspects of geometric rigidity, the vague principle according to which a given geometry is *rigid* when "few manifolds admit a large automorphism group" has a fairly rich history. In this survey paper, we try to show how strictly pseudoconvex CR geometry fits into this concept of rigidity.

André Lichnerowicz first raised the question in the conformal case. It is well known that the isometry group of a compact Riemannian manifold is compact, due to the compactness of the group $\mathrm{O}(n)$ (see Section 1.2). Since the corresponding group $\mathrm{CO}(n)$ of conformal geometry is not compact, one might expect some compact manifolds to have noncompact conformal groups. There is a simple example: the Euclidean sphere has conformal group $\mathrm{SO}(1, n+1)$. The Lichnerowicz conjecture stating that there are no other examples was settled in the early seventies by Jacqueline Ferrand [LF71] and in a weak form by Morio Obata[3] [Oba72]; it was extended by Ferrand a while later [Fer96].

A few years after Obata and Ferrand's works, it appeared that Lichnerowicz conjecture was not specific to conformal geometry: Sidney Webster extended parts of the proof of Obata in the setting of (strictly pseudoconvex) CR geometry [Web77]. The question raised a lot of interest again in the nineties, several mathematicians trying to work their way out from Webster's result to the full statement. Richard Schoen

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[3] It appeared later that the proof was flawed at some point, but Jacques Lafontaine gave a corrected proof in [Laf88]

gave the first complete proof, using original analytic methods related to the Yamabe problem [Sch95]. In fact, he gave a proof in the conformal case that adapts to CR geometry and obtained the following result.[4]

THEOREM (Schoen–Webster) — *Let M be a strictly pseudoconvex CR manifold, not necessarily compact. If its automorphism group* $\mathrm{Aut}(M)$ *acts non-properly, then M is either the standard CR sphere $\mathcal{S}$ or $\mathcal{S}$ with one point deleted.*

Let us recall that an action of a topological group G is *proper* if for any compact subset K of M, the subset

$$G_K = \{g \in G; g(K) \cap K \neq \varnothing\}$$

of G is compact. In particular if M is compact, $\mathrm{Aut}(M)$ acts properly if and only if it is compact.

The paper is organised as follows. The first section is devoted to preliminaries, including CR geometry, two properties that are important in the sequel and (G, X)-structures. We then survey the successive works on the Schoen–Webster Theorem, trying to give for (almost) each result the flavor of the proof without getting into too much detail. The word "proof" will therefore often be followed by quite imprecise arguments. The last section is devoted to a cleaned geometric proof of the theorem when M is compact, based on some of the ideas exposed.

Before getting started, let us point out that Bun Wong proved a very close theorem for domains of $\mathbb{C}^{n+1}$ [Won77]. Many developments arose from his result and parts of the Schoen–Webster Theorem can be deduced from this work. Indeed, unless $n = 1$, a compact strictly pseudoconvex CR manifold M^{2n+1} can always be embedded as the boundary of a domain of $\mathbb{C}^{n+1}$ and its automorphisms can be extended to automorphisms of the domain. See [Lee96] for details due to Daniel Burns.

However, we will not discuss Wong's theorem and its improvements. First, it cannot be of any help for the least dimensional case. Second, we are interested in more intrinsic methods of proof, independant of any embedding. For further informations on this topic, the reader should refer to [Won03].

For the sake of completeness, note that in [Pan90] Pierre Pansu gave a hint of how one could try to adapt Ferrand's proof to the CR case.

1. Preliminaries

1.1. **Basics of CR geometry.** We only give a glimpse on CR geometry. The interested reader can refer to [D'A93] or [Jac90].

Given a $2n + 1$-dimensional manifold M, a *CR structure* on M is a couple (ξ, J) where:

[4] We chose to name it after both Webster, who initiated the topic, and Schoen, who gave the first complete proof.

(1) ξ is a $2n$-dimensional subbundle of TM,
(2) J is a pseudocomplex operator on ξ:

$$J_x : \xi_x \to \xi_x, \qquad J_x^2 = -\mathrm{Id} \quad \forall x \in M,$$

(3) for all vector fields X, Y tangent to ξ, the vector field $[JX, Y] + [X, JY]$ is tangent to ξ and the following *integrability condition* holds:

$$J([JX, Y] + [X, JY]) = [JX, JY] - [X, Y].$$

Any smooth hypersurface H in a complex manifold X admits a natural CR-structure: denoting by J the complex structure of X, one can define $\xi = TH \cap J(TH)$ so that J acts on ξ; note that the vanishing of the Nijenhuis tensor implies the integrability condition.

A differentiable map between two CR manifolds is a *CR map* if it conjugates the hyperplanes distributions and the pseudocomplex operators. An *automorphism* of a CR manifold M is a diffeomorphism of M that is a CR map. The group of those is denoted by $\mathrm{Aut}(M)$ and its identity component by $\mathrm{Aut}_0(M)$.

1.1.1. *Calibrations, the Levi form and the Webster metric.* Given a CR structure (ξ, J) on M, a (possibly local) 1-form θ such that $\xi = \ker\theta$ is called a (local) *calibration*. One can always find local calibrations. If M is orientable, one can always find a global calibration. However, a calibration need not be preserved by automorphisms.

From now on, all the manifolds under consideration are assumed to be connected and orientable; all the calibrations are assumed to be global.

Given a calibration θ, one defines on ξ the *Levi form*:

$$L_\theta(\cdot) = d\theta(\cdot, J\cdot).$$

As a consequence of the integrability condition, the Levi form is a quadratic form.

A change of calibration induces a linear change in the Levi form:

$$L_{\lambda\theta} = \lambda L_\theta, \tag{1}$$

thus its signature is, up to a change of sign, a CR invariant.

A CR structure is said to be *strictly pseudoconvex* if its Levi form is definite (and then we choose our calibrations so that it is positive definite). It implies that $d\theta$ is nondegenerate on $\ker\theta$, that is θ is a contact form. If the Levi form vanishes at each point, the CR structure is said to be *Levi-flat*. Then $d\theta$ is zero on ξ and the Frobenius Theorem shows that ξ defines a foliation.

One should therefore not think of CR geometry as one geometry : each signature of the Levi form corresponds to a geometry of its own, just like Lorentzian and Riemannian geometry (or foliations and contact structures) are related, but different kind of geometries.

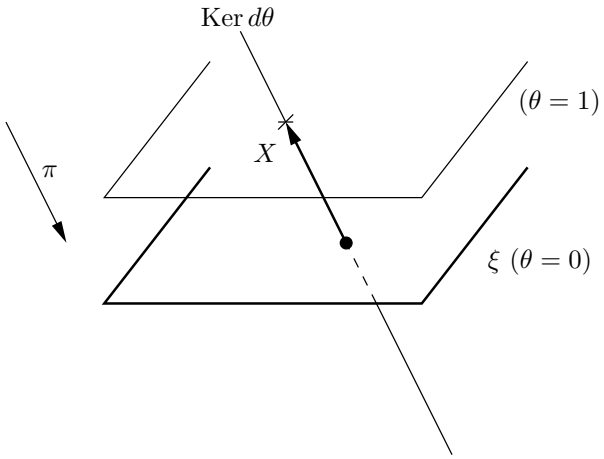

FIGURE 1. The Reeb vector field of a calibration

Given a calibration θ on a strictly pseudoconvex CR manifold, there is a single vector field X, called the *Reeb vector field* of θ, that satisfies:

$$\theta(X) = 1 \quad \text{and} \quad X \lrcorner d\theta = 0. \tag{2}$$

See Figure 1.

Denote by $\pi : TM \to \xi$ the linear projection on ξ along the direction of X. If the Levi form is positive definite, one gets a Riemannian metric on M called the *Webster metric*:

$$W_\theta = L_\theta \circ \pi + \theta^2. \tag{3}$$

A change of calibration $\theta' = \lambda\theta$ changes the metric by a factor λ along ξ and by a factor λ^2 "transversally" that is, on the quotient TM/ξ. Therefore, the Webster metric does not define a canonical conformal structure on a CR manifold.

Note that if M has dimension $2n+1$, the calibration θ defines a volume form $\theta \wedge d\theta^n$ which is compatible with the Webster metric.

1.1.2. *The Webster scalar curvature and the pseudoconformal Laplacian.* There is also a natural metric connection ∇_θ on TM, the so called Tanaka-Webster connection; beware its torsion Tor_θ does not vanish in general. Contracting the curvature Rm_θ of this connection along ξ, we obtain a scalar curvature R_θ. A subelliptic Laplacian Δ_θ arises by taking (minus) the trace over ξ of the Hessian corresponding to ∇_θ; the following integration by parts formula holds:

$$\forall u, v \in C_c^\infty(M), \quad \int_M (\Delta_\theta u) v \theta \wedge d\theta^n = \int_M L_\theta \left(du|_\xi, dv|_\xi\right) \theta \wedge d\theta^n,$$

where ξ and ξ^* are identified thanks to L_θ. To understand the relevance of this operator, consider another calibration θ', which we write $\theta' =$

$u^{\frac{2}{n}}\theta$ for some smooth positive function u. The scalar curvature then transforms according to the following law:

$$R_{\theta'} = b(n)^{-1} u^{-\frac{n+2}{n}} \mathcal{L}_\theta u$$

where $b(n) = \frac{n+1}{4n+2}$ and $\mathcal{L}_\theta = \Delta_\theta + b(n)\,\mathrm{R}_\theta$.

This formula is pretty similar to a conformal one. Indeed, given conformally equivalent metrics g and $h = u^{\frac{4}{n-2}} g$, for some positive function u, the Riemannian scalar curvatures of g and h are related the same kind of formula, where $b(n)$ should be replaced by $\frac{n-2}{4(n-1)}$ and $\mathcal{L}_\theta$ by the conformal Laplacian (and Δ_θ by the Laplace-Beltrami operator). This analogy turns out to be very efficient: it is the key idea behind Schoen's proof (see Section 4).

1.1.3. *The flat models.* The *standard CR sphere* $\mathcal{S}^{2n+1}$ (we will often omit the superscript) is the unit sphere on $\mathbb{C}^{n+1}$:

$$\mathcal{S} = \left\{(z_0, \dots, z_n) \in \mathbb{C}^{n+1}; \sum |z_k|^2 = 1\right\}$$

endowed with the corresponding CR structure. It is a strictly pseudo-convex CR manifold; its automorphism group is $\mathrm{Aut}(S) = \mathrm{PU}(1, n+1)$, a finite quotient of $\mathrm{SU}(1, n+1)$. It is noncompact, connected and acts transitively on $\mathcal{S}$.

The *Heisenberg group* is the CR noncompact manifold $\mathcal{H}$ obtained by removing one point of $\mathcal{S}$. It is therefore diffeomorphic to the Euclidean space $\mathbb{R}^{2n+1}$. Its automorphism group is the stabilizer of the removed point in $\mathrm{Aut}(S)$, it acts non properly and transitively and is connected.

These two CR manifolds are homogeneous and obviously locally isomorphic; they are referred to as the flat models. They play the role of the Euclidean space in Riemannian geometry, or of the sphere and Euclidean space in conformal geometry.

For instance, there are local normal coordinates in any Riemannian manifold, where the metric is very close a Euclidean one. There is an analogous local model for calibrated strictly-pseudoconvex manifolds: [JL89] provides local "normal" coordinates in which the geometry is close to that of the Heisenberg group $\mathcal{H}$. In the Riemannian case, the local model (i.e. the Euclidean space) is global for simply connected complete flat manifolds. The following statement is the CR analogue.

PROPOSITION 1.1 — *A simply connected complete calibrated strictly-pseudoconvex CR manifold with vanishing curvature and torsion is CR equivalent to the Heisenberg group.*

Let us precise what "complete" means. The form θ being contact implies that any two points in M can be connected by a curve that is everywhere tangent to the contact distribution. By minimizing the length of such curves, one defines the *Carnot distance* d_θ. It is a genuine distance, but does not derive from a Riemannian metric. By "balls" of

M, we mean balls with respect to the Carnot distance. A strictly pseudoconvex CR manifold is said to be *complete* if closed balls are compact.

DEFINITION 1.2 — *We say that an open subset U of a strictly pseudoconvex CR manifold M is* flat *if any x in U has a neighborhood which is CR isomorphic to an open subset of $\mathcal{S}$.*

1.2. **Finite order rigidity.** For a general reference on local rigidity, see [Kob95], Theorems 3.2 and 5.1.

Let us start with the well-known rigidity of Riemannian geometry.

PROPOSITION 1.3 — *A Riemannian metric on a manifold M is rigid to order 1, that is: two isometries that have the same value and differential at some point are the same.*

In fact this result follows from a stronger statement. Let OM be the bundle of orthonormal frames on M and $\mathrm{Isom}(M)$ its isometry group. We look at its action on OM. For each element $\mathcal{F}$ of the total space OM ($\mathcal{F}$ is thus the data of a point $x \in M$ and an orthonormal frame of T_xM), one defines the map

$$\begin{array}{ccc} \mathrm{Isom}(M) & \to & OM \\ f & \longmapsto & f(\mathcal{F}). \end{array}$$

Proposition 1.3 asserts that this map is injective. In fact, it is an embedding and its image is a closed submanifold of OM. The group $\mathrm{Isom}(M)$, endowed with the corresponding differential structure, is a Lie group.

As a consequence, since the fibers of OM are compact, the isometry group of a compact Riemannian manifold is compact. One even gets:

COROLLARY 1.4 — *Let U be an open set on a manifold M, $K \subset U$ be a compact set with nonempty interior, g be a Riemannian metric defined on U and G be a Lie group acting on M and preserving K and g. Then G is compact.*

Now we turn to the rigidity of strictly pseudoconvex CR geometry.

PROPOSITION 1.5 — *Let M be a strictly pseudoconvex CR manifold. The group $\mathrm{Aut}(M)$ is a Lie group and is rigid to order 2, that is: if two automorphisms f, f' have the same 2-jet (the data of their derivatives up to order 2) at some point, then $f = f'$.*

As before, there is a principal bundle on M in which $\mathrm{Aut}(M)$ embeds, but the fibers are no longer compact and $\mathrm{Aut}(M)$ can thus be noncompact even when M is compact.

As a direct consequence of Proposition 1.5, two CR automorphisms of M that coincide on an open set are the same.

The strict pseudoconvexity condition is of primary importance. For example, the product $S^1 \times \Sigma$ of the circle and any Riemann surface

is a Leviflat CR manifold, and the action of the infinite-dimensional diffeomorphism group of S^1 preserves the CR structure.

1.3. **North-south dynamics.** The following result is a common feature of all "rank 1 parabolic geometries", that is of boundaries of negatively curved symmetric spaces. The standard CR sphere $\mathcal{S}^{2n+1}$ is one of them: it bounds the complex hyperbolic space, seen as the unit ball of $\mathbb{C}^{n+1}$. Note that by an *unbounded sequence* in a topological space, we mean a sequence that is not contained in any compact set.

PROPOSITION 1.6 — *Let $(\phi_k)_k$ be an unbounded sequence in* $\mathrm{Aut}(\mathcal{S})$. *There exists a subsequence, still denoted by $(\phi_k)_k$, and two points (that may be the same) p_+ and p_- on $\mathcal{S}$ such that:*

$$\lim \phi_k(p) = p_+ \qquad \forall p \neq p_- \tag{4}$$

$$\lim \phi_k^{-1}(p) = p_- \qquad \forall p \neq p_+ \tag{5}$$

and the convergences are uniform on compact subsets of $\mathcal{S} - \{p_-\}$, $\mathcal{S} - \{p_+\}$ respectively.

Moreover if the ϕ_k's are powers of a single automorphism ϕ, then $p_\pm$ are fixed point of ϕ. The same result holds for a noncompact flow, which has thus either one or two fixed points.

An unbounded flow or automorphism[5] is said to be *parabolic* if it has one fixed point, *hyperbolic* if it has two of them. A bounded flow or automorphism is said to be *elliptic*.

Proof. The principle is to look at the action of $\mathrm{Aut}(\mathcal{S})$ not only on the sphere $\mathcal{S}$, but also in the complex hyperbolic space it bounds and on the projective space $\mathbb{C}\mathrm{P}^{n+1}$ it is embedded in.

The case when the ϕ_k's are powers of an automorphism ϕ, or the case of a flow, are simple linear algebra results. They are roughly described by Figure 2, which shows the link between negative curvature of the hyperbolic spaces and north-south dynamics: when a geodesic γ is translated, any other geodesic is shrinked toward one of the ends of γ.

The general case can be deduced from the KAK decomposition: every element ϕ of the group $\mathrm{Aut}(\mathcal{S})$ writes down as a product $\phi = k_1 a k_2$ where k_1, k_2 are elements of a maximal compact subgroup $K \subset \mathrm{Aut}(\mathcal{S})$ and a is an element of a maximal noncompact closed abelian subgroup A. The dimension of A is the real rank of $\mathrm{Aut}(S)$, namely 1. More precisely, A corresponds to a hyperbolic flow, that is a noncompact flow with two fixed point on S (one attractive, one repulsive). The general result then follows from the compactness of K. ∎

[5] an automorphism is said to be *unbounded* if the sequence of its powers is unbounded

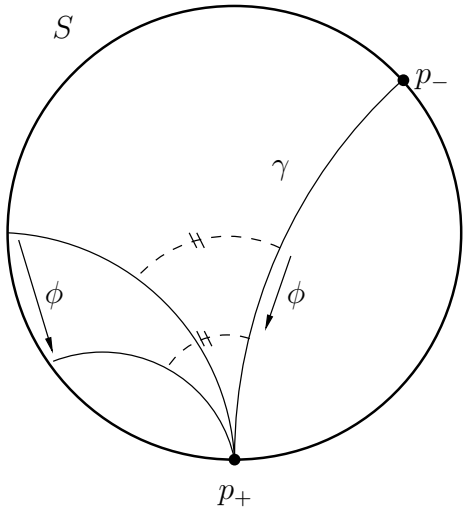

FIGURE 2. North-south dynamics.

1.4. (G, X)**-structures.** The notion of (G, X)-structure is a formalisation of Klein's geometry. In our setting, they arise as a description of flat CR structures in term of the model sphere $\mathcal{S}$ and its automorphism group.

Let X be a manifold and G a Lie group acting transitively on X. Assume that the action is *analytic* in the following sense: an element that acts trivially on an open subset of X acts trivially on the whole of X. A (G, X)*-structure* on a manifold M is an atlas whose charts take their values in X and whose changes of coordinates are restrictions of elements of G. A diffeomorphism of M is an *automorphism* of its (G, X)-structure if it reads in charts as restrictions of elements of G.

Let us consider the case when $G = \mathrm{PU}(1, n+1)$ and $X = \mathcal{S}$. A flat strictly pseudoconvex manifold M carries a (G, X)-structure and its CR automorphisms coincide with its (G, X) automorphisms. Indeed, the flatness of M means that it is locally equivalent to $\mathcal{S}$, thus it is sufficient to prove that any local automorphism of $\mathcal{S}$ can be extended into a global automorphism. This, in turn, follows from the order 2 rigidity and the following fact: any 2-jet of a local automorphism of $\mathcal{S}$ can be realized as the 2-jet of a global automorphism (see e.g. [Spi97]).

The main tool we will need to study (G, X)-structures is the so-called *developping map*. Let M be a manifold endowed with a (G, X)-structure and $\tilde{M}$ be its universal covering. Then there exists a differentiable map

$$\mathcal{D} : \tilde{M} \to X$$

that is a local diffeomorphism and such that for all automorphism f of $\tilde{M}$, there exists some $\phi \in G$ satisfying

$$\mathcal{D} \circ f = \phi \circ \mathcal{D}. \tag{6}$$

Note that in general, this developping map need not be a diffeomorphism onto his image, nor a covering map. It is unique, up to composition with an element of G. More details on (G, X)-structures can for example be found in the classical [Thu97].

2. Webster: a local Theorem

In 1977, Sydney Webster published the first work toward the Schoen–Webster Theorem, [Web77]. Until the end of the paper, M denotes a strictly pseudoconvex CR manifold of dimension $2n+1$.

Theorem 2.1 — *If M is compact and $\mathrm{Aut}_0(M)$ is noncompact, then M is flat.*

There are several reasons why this result has raised a lot of efforts to be improved. First, it is a local statement though Webster gave in the same paper a very specific global result:

Theorem 2.2 — *If M is compact and has finite fundamental group and $\mathrm{Aut}_0(M)$ is noncompact, then M is globally equivalent to the standard sphere.*

Second, he assumes that M is compact and that the *identity component* $\mathrm{Aut}_0(M)$ is noncompact. We refer to these hypotheses as the *compactness assumption* and the *connectedness assumption.*

His paper also contains a result on connected groups of CR automorphisms having a fixed point we shall discuss briefly.

Theorem 2.3 — *If M is compact and $\mathrm{Aut}_0(M)$ admits a noncompact one-parameter Lie subgroup G_1 that has a fixed point p_0, then M is globally equivalent to the standard sphere $\mathcal{S}$.*

Let us turn to the proofs of these three results.

2.1. **Canonical calibration.**

The following result is the key to the local statement.

Lemma 2.4 — *For each calibration θ on M there is a continuous nonnegative function F_θ on M such that:*

1. *F_θ vanishes on a given open set U if and only if U is flat,*
2. *on the open set where F_θ is positive, it is smooth,*
3. *the family $(F_\theta)_\theta$ is homogeneous of degre -1 :*

$$F_{\lambda\theta} = |\lambda|^{-1} F_\theta. \tag{7}$$

Such a family of functions $(F_\theta)_\theta$ is called a *relative invariant* after Cartan's one (see the proof below). Most of the time, the are given by the norm of a curvature tensor.

A point where F_θ vanishes for some (thus for all) calibration θ is said to be *umbilic.*

Let us show the interest of such functions. Pick any calibration θ of M whose Levi form is positive and define

$$\theta^* = F_\theta \theta. \tag{8}$$

Then θ^* is a continuous 1-form that vanishes on the flat part of M and is a smooth calibration everywhere else. It is canonical, for if $\theta' = \lambda\theta$ is another calibration with $\lambda > 0$ (that is, whose Levi form is positive),

$$\begin{aligned} \theta'^* &= F_{\lambda\theta}\lambda\theta \\ &= \lambda^{-1} F_\theta \lambda\theta \\ &= \theta^*. \end{aligned}$$

We call θ^* the *canonical calibration* of M although it is not a genuine calibration unless M contains no umbilic points. If M is flat, θ^* is zero and, therefore, useless.

Proof. Lemma 2.4 follows from the study of invariants of calibrated CR manifolds.

If $n > 1$, one can derive from the Chern-Mother curvature a tensor S on some bundle T over M that only depends upon the CR structure and vanishes on an open set U if and only if U is flat. A calibration θ induces, *via* the Levi form, a metric on T. The corresponding norm $\|S\|_\theta$ of S yields the desired function. See [BS76, page 201] or [Web78, page 35] for details.

If $n = 1$, S is always zero even when M is not flat so that Cartan's relative invariant is needed. It is a function r_θ on M, associated to a calibration θ, that vanishes on an open set if and only if it is flat; the family (r_θ) is homogeneous of order -2, thus $F_\theta = \sqrt{r}$ does the job. For details, one can look at Élie Cartan's work [Car32a, Car32b] or, for a more modern presentation, at the book of Howard Jacobowitz [Jac90]. ∎

2.2. **The local theorem.** Let us give an outline of the proof of Theorem 2.1 given by Webster. We shall see later that a stronger statement can be proved with the same tools.

Proof. Assume M is not flat; we will show that any one-parameter subgroup of $\mathrm{Aut}_0(M)$ has a compact closure, which implies the compactness of $\mathrm{Aut}_0(M)$ by a theorem of Deane Montgomery and Leo Zippin [MZ51].

Let G_1 be a nontrivial one-parameter subgroup of $\mathrm{Aut}_0(M)$ with infinitesimal generator Y on M. Choose some calibration θ; by assumption F_θ is positive on an open set U. Since the vanishing of F_θ is independent of θ, U is invariant under the flow of Y.

Consider the function $\eta = \theta^*(Y)$, on U. Assume it vanishes identically. Then Y lies in the contact distribution. Moreover, since θ^* is a

CR invariant form, $\mathcal{L}_Y\theta^*$ vanishes. Cartan's magic formula yields:

$$0 = \mathcal{L}_Y\theta^* = Y \lrcorner\, d\theta^* + d\eta = Y \lrcorner\, d\theta^*,$$

so Y is identically zero, which contradicts the order two rigidity.

We may therefore assume that $\eta > 0$ somewhere, replacing Y by $-Y$ if necessary. Choosing ε sufficiently small, the set U_ε defined by the inequation $\eta(p) \geqslant \varepsilon$ has non empty interior. It is closed in M, thus is compact, and is invariant under the flow of Y.

The closure $\overline{G_1}$ of G_1 in $\mathrm{Aut}_0(M)$ is a Lie group that preserves the compact U_ε and the Webster metric of θ^* on it, thus is compact (Corollary 1.4). ■

2.3. **The global result.** Webster derives Theorem 2.2 from a weak form of Proposition 1.6 and a (now) standard use of (G, X)-structures.

Proof. By Theorem 2.1, M and its universal covering $\tilde{M}$ are flat. Therefore they can be developped as $(\mathrm{SU}(1, n+1), \mathcal{S})$-structures. Since M has finite fundamental group, $\tilde{M}$ is compact and the developping map $\mathcal{D} : \tilde{M} \to \mathcal{S}$ is a covering map. But $\mathcal{S}$ admits no nontrivial covering and $\tilde{M}$ is globally equivalent to $\mathcal{S}$. By Montgomery-Zippin Theorem [MZ51], there exists some closed noncompact one-parameter subgroup G_1 of $\mathrm{Aut}_0(M)$. This group lifts to a one parameter subgroup $\tilde{G_1}$ acting on $\tilde{M} = \mathcal{S}$.

>From Proposition 1.6, we know that $\tilde{G_1}$ has either one or two fixed points. In both cases G_1 has at least a fixed point.

Let p be a fixed point of G_1. The lifts of p are fixed points of $\tilde{G_1}$ of the same type (attractive, repulsive or both). But $\tilde{G_1}$ has at most one fixed point of a given type, thus $\tilde{M} \to M$ must be a one-sheeted covering. ■

2.4. **One-parameter subgroups with a fixed point.** Theorem 2.3 is based on a principle of extension of local conjugacy, making use of the dynamics on the model space. We detail a similar argument at the end of the paper, using it in the proof of the compact case of the Schoen–Webster Theorem.

Proof. Let Y be an infinitesimal generator of G_1. According to Theorem 2.1, M is flat so there is an isomorphism between a neighborhood U of p_0 and an open set U' of $\mathcal{S}$. Denote by Y' the vector field on U' corresponding to the restriction of Y to U. Then Y' extends uniquely to a CR vector field on S, which has a fixed point p'_0.

If Y' is elliptic, then it follows from the finite order rigidity that G_1 is compact, in contradiction with the assumptions. If Y' is parabolic, then one can use it to extend the conjugacy between U and U' to the basins of attraction and repulsion of p'_0, therefore M is globally equivalent to $\mathcal{S}$. If Y' is hyperbolic, the same argument shows that there is an open

set $V \subset M$ that is conjugate to either $\mathcal{S}$ or $\mathcal{S}$ with a point (namely the second fixed point of Y') deleted. Since Y is a complete vector field with isolated zeros, M itself must be globally equivalent to either $\mathcal{S}$ or $\mathcal{S}$ with a point deleted. ∎

3. Kamishima and Lee: two ways from local flatness to global rigidity

Yoshinobu Kamishima seems to be the first to prove the local to global statement (under both the compactness and connectedness assumptions) in a workshop in honor of Obata held at Keio University in 1991. He announced the result in the proceedings [Kam93] and the complete proof appeared a while after [Kam96].

Theorem 3.1 — *If M is flat, compact and $\mathrm{Aut}_0(M)$ is noncompact then M is globally equivalent to the standard sphere.*

We will not detail his proof at all, but let us quote an interesting corollary he gave in relation with the so-called *Seifert conjecture*. This celebrated conjecture states that any non singular vector field on the 3-dimensional sphere has at least one closed orbit. It was disproved for $\mathscr{C}^1$ vector fields by Paul Schweitzer [Sch74], then in $\mathscr{C}^\infty$ regularity by Krystyna Kuperberg [Kup94]. The question was then raised for vector fields preserving some geometric structure. Kamishima's following result gives an answer for vector fields preserving a CR structure.

Corollary 3.2 — *If M is a rational homology sphere endowed with a strictly pseudoconvex CR structure, then any nonsingular CR vector field on M has a closed orbit.*

In [Lee96], John Lee proved Theorem 3.1 independently of Kamishima. His method relies on Webster's Theorem 2.3: he proves

Theorem 3.3 — *If M is compact and $\mathrm{Aut}_0(M)$ admit a closed noncompact one-parameter subgroup G_1, then G_1 has a fixed point.*

Once again, the Montgomery–Zippin Theorem is used to deduce Theorem 3.1 from Theorems 3.3, 2.1 and 2.3.

Proof. Let Y be an infinitesimal generator of G_1 and assume by contradiction that Y has no zero on M.

Note first that Y must be somewhere tangent to the contact distribution ξ: otherwise Y would be the Reeb vector field of a unique calibration, thus would preserve the associated Webster metric.

The first part of the proof consists in understanding the set of points where $Y \in \xi$; it is a classical computation that involves only the contact structure on M: pick any calibration θ of M and define $\eta = \theta(Y)$. Then one can show, using that Y has no zero, that 0 is a regular value of

η. Therefore $H = \{\eta = 0\} \subset M$ is a nonempty, compact, embedded hypersurface along which Y is tangent to both H and ξ.

The next step consists in proving that one can find a new calibration such that $\mathcal{L}_Y\theta = 0$ and $\mathcal{L}_Y d\theta = 0$ at every point of H. It is easy to get the first condition by rescaling θ; then a rather tedious computation allows Lee to refine the rescaling in order to get the second condition.

These two conditions imply that the Webster metric on TM is preserved by the flow of Y along H. It follows for any sequence (f_i) in G_1, $(f_i|_H)$ converges in $\mathscr{C}^\infty$ topology. Using the complex operator J, it is then possible to prove that the 2-jets of the sequence (f_i) converge at all points of H. By the order 2 rigidity, (f_i) is convergent in G_1, a contradiction. ∎

4. Schoen: Yamabe problem methods

The aim of this section is to survey the proof of Schoen–Webster theorem by R. Schoen in [Sch95]. For convenience, we only deal with the compact case, even though [Sch95] also considers the non-compact case with the same kind of techniques, based on global analysis. R. Schoen first proves that the conformal group is compact for any closed Riemannian manifold which is not conformally equivalent to the standard sphere. Then he explains how to adapt the proof in a CR setting, which is what we want to develop below. Another proof of the conformal group compactness is given in [Heb97] : it is a bit shorter but relies on the positive mass theorem, which makes it less elementary than what follows.

4.1. **Yamabe theorem.**

The celebrated Yamabe problem is basic in conformal geometry: is there a metric with constant scalar curvature in each conformal class of a given closed manifold ? This question was the beginning of a long story: see the excellent [Heb97] or [LP87] for an exhaustive account. The answer to the problem is yes and the proof relies on a careful study of the conformal Laplacian.

As explained in [JL87], there is a deep analogy between conformal and CR geometry. In particular, Yamabe theory has a counterpart in the CR realm, which enables R. Schoen to extend his conformal geometry arguments to the CR case.

In order to develop a Yamabe theory in the CR setting, one needs a Sobolev-like analysis. In the conformal case, the natural conformal operator is elliptic, so that its analysis is rather standard. In the CR case, the corresponding natural operator $\mathcal{L}_\theta$ is only subelliptic. G. Folland and E. Stein [FS74] (see also paragraph 5 of [JL87]) have nonetheless developped a powerful theory which yields the necessary tools.

As in conformal geometry, given a calibration θ, we define the CR Yamabe invariant $Q(M,\theta)$ as the infimum of the functional

$$\int_M \phi\, \mathcal{L}_\theta\, \phi\, \theta \wedge d\theta^n$$

over the elements ϕ of the unit sphere in the Lebesgue space $L^{\frac{2n+2}{n}}(M)$. The choice of this exponent is related to the transformation law for the volume form: if $\theta' = u^{\frac{2}{n}}\theta$ for some positive function u, then

$$\theta' \wedge (d\theta')^n = u^{\frac{2n+2}{n}}\theta \wedge d\theta^n.$$

It turns out that $Q(M,\theta)$ is a CR invariant.

D. Jerison, J. M. Lee [JL87], N. Gamara and R. Yacoub [Gam01], [GY01] adapted the proof of the conformal Yamabe theorem to prove the

THEOREM 4.1 — *A closed strictly pseudoconvex CR manifold admits a calibration with constant scalar curvature* 1 *(resp.* 0 *and* -1*) if its CR Yamabe invariant is positive (resp. zero and negative).*

We will only need the nonpositive (and easiest) case, which was settled by [JL87].

4.2. **The proof.** Theorem 4.1 leads to the

PROPOSITION 4.2 — *When the CR Yamabe invariant is nonpositive, the CR automorphism group is compact.*

Proof. We prove that CR automorphisms are isometries for the Webster metric of a calibration; since the isometry group of a closed Riemannian manifold is compact, the result will follow. Endow M with a calibration θ.

If $Q(M) = 0$, we can assume θ has vanishing scalar curvature (Yamabe). A CR automorphism F of M then obeys $F^*\theta = u^{\frac{2}{n}}\theta$ with $\mathcal{L}_\theta\, u = \Delta_\theta u = 0$ ($F^*\theta$ has scalar curvature $F^*R_\theta = 0$). An integration by parts yields

$$0 = \int u\Delta_\theta u = \int L_\theta\,(du|_\xi, du|_\xi)\,\theta \wedge d\theta^n.$$

So we can write $du = f\theta$, which implies $0 = df \wedge \theta + f d\theta$. Since $d\theta$ is definite on the kernel ξ of θ, f vanishes, so u is constant. Since

$$\mathrm{vol}_\theta(M) = \mathrm{vol}_\theta(F(M)) = \mathrm{vol}_{F^*\theta}(M) = u^{\frac{2n+2}{n}}\,\mathrm{vol}_\theta(M),$$

u is constant to 1: F preserves θ hence W_θ.

If $Q(M) < 0$, we can make a similar argument: we are left to show that a solution u of

$$\Delta_\theta u = b(n)\left(u - u^{\frac{n+2}{n}}\right)$$

is constant to 1. It follows from a weak maximum principle. At a maximum point, $\Delta_\theta u$ is nonnegative so that the equation ensures $u \leq 1$.

At a minimum point, one finds $u \geq 1$ for the same reason. Therefore u is constant to 1. ∎

The following lemma is the key to complete the proof. We denote by D_r the ball of radius r in $\mathbb{R}^{2n+1}$. To avoid technical details, we do not give the precise statement (cf. [Sch95]).

LEMMA 4.3 — *Let $F : (D_1, \theta) \to (N, \sigma)$ be a CR diffeomorphism. We assume θ is close to the Heisenberg calibration and σ has vanishing scalar curvature. If $\lambda := \sqrt{(F^*\sigma/\theta)(0)}$ denotes the dilation factor at 0, then:*

- *the dilation factor is almost constant to λ, i.e. $F^*\sigma/\theta \approx \lambda$ on $D_{1/2}$;*
- *images of balls have moderate eccentricity, i.e. $F(D_{1/2}) \approx B(F(0), \lambda/2)$;*
- *the total curvature and the torsion are small when the dilation factor is large, i.e. $|\mathrm{Rm}_\sigma| \lesssim \lambda^{-2}$ and $|\mathrm{Tor}_\sigma| \lesssim \lambda^{-2}$ on $B(F(0), \lambda/2)$.*

Proof. By scaling σ, we can assume $\lambda = 1$. Write $F^*\sigma = u^{\frac{2}{n}}\theta$ and observe that $\mathrm{R}_\sigma = 0$ implies $\mathcal{L}_\theta\, u = 0$. Since θ is close to the Heisenberg calibration, $\mathcal{L}_\theta$ is close to the Heisenberg subelliptic Laplacian, so that u satisfies a Harnack inequality ([JL87], 5.12) : $\sup u \leq C \inf u$, with a controlled constant. The first and second assertions follow. Subelliptic regularity ([JL87], 5.7) also yields a C^2 bound on u, hence the third assertion. ∎

Now we can finish the proof of the

THEOREM 4.4 (Schoen–Webster) — *The CR automorphism group of a closed strictly pseudo-convex CR manifold which is not CR equivalent to a standard sphere is compact.*

Here, we only deal with C^0 topology. Thanks to a bootstrap argument, [Sch95] proves that all C^k topologies, $k \geq 0$, are the same. They also coincide with the Lie group topology.

Proof. Assume M^{2n+1} is a closed strictly pseudo-convex CR manifold with non-compact conformal group and choose a calibration θ. Ascoli theorem yields CR automorphisms F_i and points x_i such that the dilation factors

$$\lambda_i := \sqrt{(F_i^*\theta/\theta)(x_i)} = \max \sqrt{(F_i^*\theta/\theta)}$$

go to infinity.

The rough idea of the proof consists in multiplying the calibration θ by suitable Green functions so as to build a sequence of conformal scalar flat blow ups; then lemma 4.3 will enable us to find a sequence of larger and larger balls endowed with a calibration of smaller and

smaller curvature and torsion: taking a limit, we will realize M minus a point as a Heisenberg group; a last effort will seal the fate of the missing point.

To begin with, we can choose a small $\epsilon > 0$ such that the geometry of all the balls of radius ϵ in M is close to that of the Heisenberg group. Then we choose points y_i outside $F_i(B(x_i, \epsilon))$ and use a standard trick in Yamabe theory. Since the CR Yamabe invariant is positive (4.2), the operator $\mathcal{L}_\theta$ is positive. Therefore, there are Green functions G_i, i.e. preimages of Dirac distributions δ_{y_i} (cf. [Gam01] for instance): outside y_i, they are smooth, satisfy $\mathcal{L}_\theta G_i = 0$ and we can normalize them so that their minimum value is 1. Put $z_i := F_i(x_i)$ and consider the calibration

$$\theta_i := \left(\frac{G_i}{G_i(z_i)}\right)^{\frac{2}{n}} \theta,$$

defined outside y_i. It has vanishing scalar curvature.

We can assume y_i converges to y, z_i converges to z and G_i converges to G on compact sets of $M - \{y\}$. Besides, one can show that $G_i(z_i)$ remains bounded, that is $y \neq z$: it stems from a convenient use of lemma 4.3 and from a Harnack inequality for the dilation factor between $F_i^* \theta_i$ and θ. So we can assume $G_i(z_i)$ converges.

Therefore θ_i tends to a calibration $\theta_\infty = cG^{\frac{2}{n}}\theta$ on the compact sets of $M - \{y\}$. Now lemma 4.3 ensures that, roughly, θ_i has curvature and torsion of magnitude λ_i^{-2} on $F_i(B_\theta(x_i, \epsilon/2)) \approx B_{\theta_i}(z_i, \lambda_i \epsilon/2)$, so that letting i go to infinity, we conclude our manifold, outside y, is CR equivalent to a calibrated strictly pseudo-convex CR manifold with vanishing curvature and torsion; and it happens to be complete and simply connected (it is a nondecreasing union of topological balls), so that it is $\mathcal{H}^{2n+1}$.

Thus there is a CR diffeomorphism F between M minus y and the standard sphere minus some point, ∞. In the neighbourhood of ∞ in $\mathcal{S}^{2n+1}$, consider a CR equivalent Heisenberg calibration σ. Writing $F^*\sigma = u^{\frac{2}{n}}\theta$, we obtain $\mathcal{L}_\theta u = 0$ outside y, since σ has vanishing scalar curvature. Extending F at y amounts to show that u has a removable singularity at y. But the integral of $u^{\frac{2n+2}{n}}$ over some ball is exactly the volume of the image of this ball through F, which is bounded by the volume of the standard sphere; it follows (proposition 5.17 in [JL87]) that u is a weak solution of the equation $\mathcal{L}_\theta u = 0$ over a neighborhood of y so that it extends as a smooth function in the neighborhood of y (5.10, 5.15 in [JL87]). Thus (M, θ) is CR equivalent to the standard sphere. ∎

5. Frances: a unified dynamical proof

Charles Frances recently gave a unified proof of the Ferrand-Obata and Schoen–Webster Theorems [Fra06]; in fact he also proves analogous results for quaternionic-contact and octonionic-contact geometries.

To obtain these results, he uses the setting of Cartan geometries (see [Sha97] for a detailed account on this topic). Given a model homogeneous space $X = G/P$, a *Cartan geometry modelled on* X on a manifold M consists of:

- a P-principal bundle $B \to M$ and
- a 1-form ω on the total space B with values in the Lie algebra $\mathfrak{g}$.

The form ω is called the *Cartan connection* of the structure and is supposed to satisfy some compatibility conditions we do not detail.

The points of B play the role of "adapted" frames (like orthonormal frames for Riemannian geometry). The Cartan connection is used to identify infinitesimally B with G: in particular, it is asked that at each point $p \in B$, ω_p is an isomorphism between T_pB and $\mathfrak{g}$.

The geometries Frances is concerned with are modelled on the homogeneous spaces $\partial\mathbb{K}\mathrm{H}^d = G/P$ where $\mathbb{K}\mathrm{H}^d$ is the hyperbolic space based on $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$, G is the isometry group of $\mathbb{K}\mathrm{H}$ and P is the stabilizer of a boundary point. Note that when $\mathbb{K} = \mathbb{C}$, $X = \mathcal{S}$.

For each of these Cartan geometries, the "equivalence problem" has been solved, that is: there exists a construction that gives for any conformal, strictly pseudoconvex CR, etc. structure on M a corresponding Cartan structure B, ω such that isomorphisms of the original structure induce isomorphisms of the Cartan structure and reciprocally. The Cartan structure is not unique, one can impose further assumptions. In particular the Cartan connection can be chosen "regular" (a technical condition involving the curvature of ω) for the geometries considered here.

We can now state the result of Frances.

Theorem 5.1 — *Let (M, B, ω) be a Cartan geometry modelled on $X = \partial\mathbb{K}\mathrm{H}^d$, with regular connection. If $\mathrm{Aut}(M, \omega)$ acts nonproperly on M, then M is isomorphic to either X or X with a point deleted.*

Proof. The first and main step is to prove that any sequence (f_k) of automorphisms of M that acts nonproperly admit a subsequence that shrinks an open set $U \subset M$ onto a point p. The principle is to use some sort of developping map from the space of curves on M passing through p to the space of curves on X passing through a given base point o. Then, choosing an appropriate family of curves in the model and its north-south dynamics, one gets the desired property on M.

Then one proves that an open set that collapses to a point must be flat. Note that in the CR case, one could use the Webster metric of

the canonical calibration (see Section 2). As a consequence, one can choose U to be of the form

$$U = \Gamma\backslash(X - \{o\})$$

where Γ is a discrete subgroup of the stabilizer P of $o \in X$.

The final step is a result on geometrical rigidity of embeddings: if a flat manifold $\Gamma\backslash(X-\{o\})$ embeds in M, then either $M = \Gamma\backslash(X-\{o\})$ or $\Gamma = \{\mathrm{Id}\}$. In the latter case, $M = X$ or $M = X - \{o\}$.

The conclusion follows since the automorphism group of $\Gamma\backslash(X-\{o\})$ acts properly when Γ is not trivial. ∎

6. Gathering a geometric proof in the compact case

In this last section, we give a geometric proof of the Schoen–Webster Theorem under the compactness assumption. It is not elementary, as it makes use of Lemma 2.4. However: it is a geometric proof, thus gives an alternative to Schoen's techniques; it do not rely on the Montgomery-Zippin Theorem, holds without the connectedness assumption and is quite short, which makes it an improvement of those of Webster, Kamishima and Lee together.

It does not pretend to originality, since it relies on arguments of Webster [Web77] and Frances and Tarquini [FT02], rephrased.

6.1. **The local statement.**

Theorem 6.1 — *If M is compact and* $\mathrm{Aut}(M)$ *is noncompact, then M is flat.*

Proof. Suppose that M is not flat; then the canonical calibration θ^* defined thanks to Lemma 2.4 does not vanish identically. Denote by W the Webster metric associated with θ^*: it is continuous on M, smooth and positive definite on the open set U of nonumbilic points and zero on its complementary F. For all x and y in M let

$$d(x,y) = \inf_{\gamma} \int_{\gamma} \sqrt{W(\dot\gamma)}$$

define the natural semimetric associated to W (not to be confused with the Carnot metric of Section 1.1.3 : here the infimum is taken on all curves connecting x to y). We have $d(x,y) = 0$ if and only if x and y are in F, in particular d is a genuine metric on U.

If $U = M$, then $\mathrm{Aut}(M)$ preserves a Riemannian metric, thus is compact. Otherwise, F is nonempty, the distance $d(x,F)$ is finite for every $x \in M$ and we can define the set $U_\varepsilon = \{x \in U; d(x,F) \geqslant \varepsilon\}$ for any positive ε. This set is compact and has nonempty interior for ε small enough.

Now $\mathrm{Aut}(M)$ preserves U_ε and its Webster metric, thus is compact. ∎

6.2. **The local-to-global statement.**

THEOREM 6.2 — *If M is flat and $\mathrm{Aut}(M)$ acts nonproperly, then M is globally equivalent to the standard CR sphere $\mathcal{S}$ or to $\mathcal{S}$ with a point deleted.*

This result follows, by a principle of "extension of local conjugacy", from the dynamics of unbounded sequences of $\mathrm{Aut}(\mathcal{S})$. Note that we do not use the compactness assumption for this part.

The end of the section is dedicated to the proof of Theorem 6.2. Note that it holds as it is for any "rank-one parabolic" (G, X)-structure (namely $X = \partial\mathbb{K}\mathrm{H}^n$ with $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$).

6.2.1. *Set up: developping the dynamics.* We assume that M is flat, thus carries a $(\mathrm{SU}(1, n+1), \mathcal{S})$-structure, and that $\mathrm{Aut}(M)$ acts nonproperly: there is a convergent sequence $x_i \in M$ and a sequence $f_i \in \mathrm{Aut}(M)$ going to infinity (that is, having no convergent subsequence), such that $y_i = f_i(x_i)$ converges in M. We set $x_\infty = \lim x_i$ and $y_\infty = \lim y_i$.

Let $\tilde{M}$ be the universal cover of M. There are lifts $(\tilde{x}_i)_{i\in\mathbb{N}\cup\{\infty\}}$, $(\tilde{y}_i)_{i\in\mathbb{N}\cup\{\infty\}}$ and $\tilde{f}_i$ such that $\lim \tilde{x}_i = \tilde{x}_\infty$, $\lim \tilde{y}_i = \tilde{y}_\infty$ and $\tilde{y}_i = \tilde{f}_i(\tilde{x}_i)$. Moreover, the sequence $(\tilde{f}_i)$ has no convergent subsequence in $\mathrm{Aut}(\tilde{M})$.

Let $\mathcal{D} : \tilde{M} \to \mathcal{S}$ be the developping map of M and ϕ_i be a sequence of $\mathrm{Aut}(\mathcal{S})$ such that $\mathcal{D}\tilde{f}_i = \phi_i\mathcal{D}$. If (ϕ_i) had a convergent subsequence, by the order 2 rigidity and since ϕ_i and $\tilde{f}_i$ are locally conjugated, so would $(\tilde{f}_i)$. Thus (ϕ_i) is unbounded and admit a North-South dynamics, whose poles are denoted by p_+ and p_-.

Since $\mathcal{D}(\tilde{y}_i) = \phi_i\mathcal{D}(\tilde{x}_i)$, we have either $\mathcal{D}(\tilde{y}_\infty) = p_+$ or $\mathcal{D}(\tilde{x}_\infty) = p_-$. Up to inverting the f_i's and exchanging the x_i's and the y_i's, we assume that $\mathcal{D}(\tilde{y}_\infty) = p_+$.

6.2.2. *Stretching injectivity domains.* A subset of $\tilde{M}$ is said to be an *injectivity domain* if the developping map is one-to-one on its closure.

We denote by U_0 an open connected injectivity domain containing $\tilde{y}_\infty$ and we let $V_0 = \mathcal{D}(U_0)$. We choose an open connected injectivity domain Ω containing $\tilde{x}_\infty$ and having connected boundary $\mathrm{Bd}\,\Omega$ whose image $\mathcal{D}(\mathrm{Bd}\,\Omega)$ does not contain p_-. Up to extracting a subsequence, we can assume that for all i, $\tilde{x}_i \in \Omega$ and $\tilde{y}_i \in U_0$.

According to Proposition 1.6, there is an increasing sequence of open sets $V_i \subset \mathcal{S}$ $(i > 0)$ such that, extracting a subsequence if necessary:

(1) for all i, $\mathcal{D}(\mathrm{Bd}\,\Omega) \subset V_i$,
(2) $\bigcup V_i = S - \{p_-\}$,
(3) for all i, $\phi_i(V_i) \subset V_0$.

Let $\delta : U_0 \to V_0$ be the restriction of $\mathcal{D}$ and define the following open connected injectivity domains: $U_i = \tilde{f}_i^{-1} \circ \delta^{-1} \circ \phi_i(V_i)$. Since we

assumed $\tilde{x}_i \in \Omega$ and $\tilde{y}_i \in U_0$, we get

$$U_i \cap \Omega \neq \varnothing \quad \forall i \tag{9}$$

and by construction we have

$$\mathcal{D}(\operatorname{Bd}\Omega) \subset \mathcal{D}(U_i) = V_i \subset \mathcal{D}(U_{i+1}) = V_{i+1} \quad \forall i. \tag{10}$$

6.2.3. *Monotony and consequences.* We prove that (U_i) (or a subsequence) is an increasing sequence.

If we can extract a subsequence such that $U_i \subset \Omega$ for all i, since Ω is an injectivity domain and $(\mathcal{D}U_i)$ is increasing, (U_i) must be increasing.

Otherwise we use the following Lemma.

LEMMA 6.3 — *Let A, B be two injectivity domains such that B is open, A is connected and $A \cap B \neq \varnothing$. If $\mathcal{D}(A) \subset \mathcal{D}(B)$, then $A \subset B$.*

Proof. Since A is connected, we only have to prove that $A \cap B$ is open and closed in A.

First, B is open so that $A \cap B$ is open in A. Second, let y be a point in $A \cap \overline{B}$. Since $\mathcal{D}(A) \subset \mathcal{D}(B)$, there is a $z \in B$ such that $\mathcal{D}(z) = \mathcal{D}(y)$. But since B is an injectivity domain and y belongs to the closure of B, $z = y$ and $y \in B$. Therefore, $A \cap B$ is closed in A. ∎

When $U_i \not\subset \Omega$, $\operatorname{Bd}\Omega \cap U_i \neq \varnothing$ and we can apply Lemma 6.3: $\operatorname{Bd}\Omega \subset U_i$. But then $U_i \cap U_{i+1} \neq \varnothing$ thus by the same argument: $U_i \subset U_{i+1}$.

Now let $U_\infty = \bigcup U_n$; $\mathcal{D}$ is a diffeomorphism from U_∞ to $\mathcal{S} - \{p_-\}$.

If $U_\infty \neq \tilde{M}$, let x be a point of the boundary of U_∞. If $\mathcal{D}(x)$ were in $\mathcal{S} - \{p_-\}$, for any neighborhood W of x, $\mathcal{D}(W \cap U_\infty)$ would meet any neighborhood of any inverse image of $\mathcal{D}(x)$, contradicting the injectivity of $\mathcal{D}$ on U_∞. Therefore the boundary of U_∞ consists of x alone and $\mathcal{D}$ is a global diffeomorphism from $\tilde{M} = U_\infty \cup \{x\}$ to $\mathcal{S}$.

We thus proved that $\tilde{M}$ is equivalent to either $\mathcal{S}$ or $\mathcal{S} - \{p_-\}$. Moreover, any inverse image of y_∞ in $\tilde{M}$ is an attracting point, thus is $\tilde{y}_\infty$: $\tilde{M}$ is one-sheeted and M is itself equivalent to either $\mathcal{S}$ or $\mathcal{S} - \{p_-\}$.

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