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A POSTERIORI ENERGY-NORM ERROR ESTIMATES FOR ADVECTION-DIFFUSION EQUATIONS APPROXIMATED BY WEIGHTED INTERIOR PENALTY METHODS

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Abstract

We propose and analyze a posteriori energy-norm error estimates for weighted interior penalty discontinuous Galerkin approximations to advection-diffusion-reaction equations with heterogeneous and anisotropic diffusion. The weights, which play a key role in the analysis, depend on the diffusion tensor and are used to formulate the consistency terms in the discontinuous Galerkin method. The error upper bounds, in which all the constants are specified, consist of three terms: a residual estimator which depends only on the elementwise fluctuation of the discrete solution residual, a diffusive flux estimator where the weights used in the method enter explicitly, and a non-conforming estimator which is nonzero because of the use of discontinuous finite element spaces. The three estimators can be bounded locally by the approximation error. A particular attention is given to the dependency on problem parameters of the constants in the local lower error bounds. For moderate advection, it is shown that full robustness with respect to diffusion heterogeneities is achieved owing to the specific design of the weights in the discontinuous Galerkin method, while diffusion anisotropies remain purely local and impact the constants through the square root of the condition number of the diffusion tensor. For dominant advection, it is shown, in the spirit of previous work by Verfürth on continuous finite elements, that the constants are bounded by the square root of the local Péclet number.

Mathematics subject classification: 65N30, 65N15, 76Rxx
Key words: Discontinuous Galerkin, weighted interior penalty, a posteriori error estimate, heterogeneous diffusion, advection-diffusion

1. Introduction

In this work, we are interested in a posteriori energy-norm error estimates for a particular class of discontinuous Galerkin (dG) approximations of the advection-diffusion-reaction equation

$$\begin{cases} -\nabla\cdot(K\nabla u) + \beta\cdot\nabla u + \mu u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where for simplicity homogeneous Dirichlet boundary conditions are considered. Here, Ω is a polygonal domain in $\mathbb{R}^d$ with boundary $\partial\Omega$, $\mu \in L^\infty(\Omega)$, $\beta \in [L^\infty(\Omega)]^d$ with $\nabla\cdot\beta \in L^\infty(\Omega)$,

$\tilde{\mu} := \mu - \frac{1}{2}\nabla\cdot\beta$ is assumed to be uniformly positive, the diffusion tensor K is a symmetric, uniformly positive definite field in $[L^\infty(\Omega)]^{d,d}$ and $f \in L^2(\Omega)$. Owing to the above assumptions, (1.1) is well-posed.

DG methods received extensive interest in the past decade, in particular because of the flexibility they offer in the construction of approximation spaces using non-matching meshes and variable polynomial degrees. For diffusion problems, various DG methods have been analyzed, including the Symmetric Interior Penalty method [5, 6], the Nonsymmetric method with [34] or without [30] penalty, and the Local Discontinuous Galerkin method [16]; see [4] for a unified analysis. For linear hyperbolic problems (e.g., advection–reaction), one of the most common approaches is to use upwind fluxes to formulate the DG method [26, 29]. A unified theory of DG approximations encompassing elliptic and hyperbolic PDE's can be found in [19, 20]. The approximation of the advection-diffusion-reaction problem (1.1) using DG methods has been analyzed in [25] and more recently in [21] with a focus on the high Péclet regime with isotropic and uniform diffusion. The case of high contrasts in the diffusivity poses additional difficulties. Recently, a (Symmetric) Weighted Interior Penalty method has been proposed and analyzed to approximate satisfactorily (1.1) in this situation [23]. The key idea is to use weighted averages (depending on the normal diffusivities at the two mesh elements sharing a given interface) to formulate the consistency terms and to penalize the jumps of the discrete solution by a factor proportional to the harmonic mean of the neighboring normal diffusivities; the idea of using weighted interior penalties in this context can be traced back to [12].

The present paper addresses the a posteriori error analysis of the weighted interior penalty method. Many significant advances in the a posteriori error analysis of dG methods have been accomplished in the past few years. For energy-norm estimates, we refer to the pioneering work of Becker, Hansbo and Larson [8] and that of Karakashian and Pascal [27], while further developments can be found in the work of Ainsworth [2, 3] regarding robustness with respect to diffusivity and that of Houston, Schötzau and Wihler [24] regarding the *hp*-analysis; see also [13, 35]. Furthermore, for L^2-norm estimates, we mention the work of Becker, Hansbo and Stenberg [9], that of Rivière and Wheeler [32], and that of Castillo [15]. Broadly speaking, two approaches can be undertaken to derive a posteriori energy-norm error estimates; in [2, 8, 13], a Helmholtz decomposition of the error is used, following a technique introduced in [17, 14], while the analysis in [24, 27] relies more directly on identifying a conforming part in the discrete solution. The analysis presented herein will be closer to the latter approach. We also mention recent work relying on the reconstruction of a diffusive flux; see [22, 28].

This paper is organized as follows. §2 presents the discrete setting, including the weighted interior penalty bilinear form used to formulate the discrete problem. §3 contains the main results of this work. The starting point is the abstract framework for a posteriori error estimates presented in §3.1 and which is closely inspired from the work of Vohralík for mixed finite element discretizations [42]. Then, §3.2 addresses the case of pure diffusion with heterogeneous and possibly anisotropic diffusivity. We derive an upper bound for the error consisting of three error indicators, i.e. a residual, a diffusive flux and a non-conforming one. This form is similar to that obtained in previous work. The key point however is that the diffusive flux error indicators also provide local lower error bounds that are fully robust with respect to diffusivity heterogeneities and that depend on the local (elementwise) degree of anisotropy; see Propositions 3.1 and 3.2. A key ingredient to obtain this result is the use of weighted averages in writing the consistency term. §3.3 extends the previous analysis to the advection-diffusion-reaction problem. Here, the

focus is set on achieving a certain degree of robustness in the high Péclet regime, namely that achieved by Verfürth [38] for a posteriori energy-norm error estimates with conforming finite elements and SUPG stabilization. Although these estimates are not independent of the Péclet number (see, e.g., [39] for fully robust estimates with suitable norm modification), their present extension to dG methods constitutes the first results of this type. Finally, numerical results are presented in §4.

2. The discrete setting

Let $\{\mathcal{T}_h\}_{h>0}$ be a shape-regular family of affine triangulations covering exactly the polygonal domain Ω. The meshes $\mathcal{T}_h$ may possess hanging nodes, as long as the number of hanging nodes per mesh element is uniformly bounded. A generic element in $\mathcal{T}_h$ is denoted by T, h_T denotes the diameter of T and n_T its outward unit normal. Let an integer $p \geq 1$. We consider the usual dG approximation space

$$V_h = \{v_h \in L^2(\Omega); \forall T \in \mathcal{T}_h, v_h|_T \in \mathbb{P}_p\}, \tag{2.1}$$

where $\mathbb{P}_p$ is the set of polynomials of degree less than or equal to p. The L^2-scalar product and its associated norm on a region $R \subset \Omega$ are indicated by the subscript $0, R$. For $s \geq 1$, a norm (seminorm) with the subscript s, R designates the usual norm (seminorm) in $H^s(R)$. For $s \geq 1$, $H^s(\mathcal{T}_h)$ denotes the usual broken Sobolev space on $\mathcal{T}_h$ and for $v \in H^1(\mathcal{T}_h)$, $\nabla_h v$ denotes the piecewise gradient of v, that is, $\nabla_h v \in [L^2(\Omega)]^d$ and for all $T \in \mathcal{T}_h$, $(\nabla_h v)|_T = \nabla(v|_T)$.

We say that F is an interior face of the mesh if there are $T^-(F)$ and $T^+(F)$ in $\mathcal{T}_h$ such that $F = T^-(F) \cap T^+(F)$. We set $\mathcal{T}(F) = \{T^-(F), T^+(F)\}$ and let n_F be the unit normal vector to F pointing from $T^-(F)$ towards $T^+(F)$. The analysis hereafter does not depend on the arbitrariness of this choice. Similarly, we say that F is a boundary face of the mesh if there is $T^-(F) \in \mathcal{T}_h$ such that $F = T^-(F) \cap \partial\Omega$. We set $\mathcal{T}(F) = \{T^-(F)\}$ and let n_F coincide with the outward normal to $\partial\Omega$. All the interior (resp., boundary) faces of the meshes are collected into the set $\mathcal{F}_h^i$ (resp., $\mathcal{F}_h^{\partial\Omega}$) and we let $\mathcal{F}_h = \mathcal{F}_h^i \cup \mathcal{F}_h^{\partial\Omega}$. Henceforth, we shall often deal with functions that are double-valued on $\mathcal{F}_h^i$ and single-valued on $\mathcal{F}_h^{\partial\Omega}$. This is the case, for instance, of functions in V_h. On interior faces, when the two branches of the function in question, say v, are associated with restrictions to the neighboring elements $T^{\mp}(F)$, these branches are denoted by $v^{\mp}$ and the jump of v across F is defined as

$$[\![v]\!]_F = v^- - v^+. \tag{2.2}$$

We set $[\![v]\!]_F = v|_F$ on boundary faces. On an interior face $F \in \mathcal{F}_h^i$, we also define the standard (arithmetic) average as $\{v\}_F = \frac{1}{2}(v^- + v^+)$. The subscript F in the above jumps and averages is omitted if there is no ambiguity. We define the weighted average of a two-valued function v on an interior face $F \in \mathcal{F}_h^i$ as

$$\{v\}_\omega = \omega^- v^- + \omega^+ v^+, \tag{2.3}$$

where the weights are defined as

$$\omega^- = \frac{\delta_{K+}}{\delta_{K+} + \delta_{K-}}, \qquad \omega^+ = \frac{\delta_{K-}}{\delta_{K+} + \delta_{K-}}, \tag{2.4}$$

with $\delta_{K\mp} = n_F(K|_{T\mp})n_F$. We extend the above definitions to boundary faces by formally letting $\delta_{K+} = +\infty$ so that $\omega^- = 1$ and $\omega^+ = 0$. For the standard average, it is instead more

convenient to set $\{v\}_F = \frac{1}{2}v|_F$ on boundary faces. On interior faces $F \in \mathcal{F}_h^i$, we will also need the conjugate weighted average defined such that

$$\{v\}_{\bar{\omega}} = \omega^+ v^- + \omega^- v^+, \tag{2.5}$$

and make use of the identity $[\![vw]\!] = \{v\}_\omega [\![w]\!] + \{w\}_{\bar{\omega}} [\![v]\!]$.

The weak formulation of (1.1) consists of finding $u \in V := H_0^1(\Omega)$ such that

$$B(u, v) = (f, v)_{0,\Omega}, \qquad \forall v \in V, \tag{2.6}$$

with the bilinear form

$$B(v, w) = (K\nabla_h v, \nabla_h w)_{0,\Omega} + (\beta\cdot\nabla_h v, w)_{0,\Omega} + (\mu v, w)_{0,\Omega}. \tag{2.7}$$

Piecewise gradients are used so as to extend the domain of B to functions in $V + V_h$. The energy norm is

$$\|v\|_B^2 = \sum_{T\in\mathcal{T}_h} \|v\|_{B,T}^2, \qquad \|v\|_{B,T}^2 = (K\nabla_h v, \nabla_h v)_{0,T} + (\tilde{\mu} v, v)_{0,T}. \tag{2.8}$$

The discrete problem consists of finding $u_h \in V_h$ such that

$$B_h(u_h, v_h) = (f, v_h)_{0,\Omega}, \qquad \forall v_h \in V_h, \tag{2.9}$$

with the bilinear form

$$\begin{aligned} B_h(v, w) = {} & (K\nabla_h v, \nabla_h w)_{0,\Omega} + ((\mu - \nabla\cdot\beta)v, w)_{0,\Omega} - (v, \beta\cdot\nabla_h w)_{0,\Omega} \\ & + \sum_{F\in\mathcal{F}_h} [(\gamma_F [\![v]\!], [\![w]\!])_{0,F} - (n_F^t \{K\nabla_h v\}_\omega, [\![w]\!])_{0,F} - \theta(n_F^t \{K\nabla_h w\}_\omega, [\![v]\!])_{0,F}] \\ & + \sum_{F\in\mathcal{F}_h} (\beta\cdot n_F \{v\}, [\![w]\!])_{0,F}. \end{aligned} \tag{2.10}$$

The penalty parameter γ_F is defined for all $F \in \mathcal{F}_h$ as $\gamma_F = \gamma_{K,F} + \gamma_{\beta,F}$ with

$$\gamma_{K,F} = \varpi h_F^{-1} \delta_F, \qquad \gamma_{\beta,F} = \tfrac{1}{2}|\beta\cdot n_F|, \tag{2.11}$$

where

$$\delta_F = \frac{\delta_{K+}\delta_{K-}}{\delta_{K+} + \delta_{K-}}, \tag{2.12}$$

and ϖ is a positive parameter (ϖ can also vary from face to face). Note that by the above convention, $\gamma_{K,F} = \varpi h_F^{-1}\delta_{K-}$ on boundary faces. Finally, the parameter θ can take values in $\{-1, 0, +1\}$. The particular value taken by θ plays no role in the subsequent analysis.

To avoid technicalities, the diffusion tensor K is assumed to be piecewise constant on $\mathcal{T}_h$ and its restriction to an element $T \in \mathcal{T}_h$ is denoted by K_T. We will indicate by $\lambda_{m,T}$ and $\lambda_{M,T}$ respectively the minimum and the maximum eigenvalue of K on T. The minimum value of $\tilde{\mu}$ on T is indicated by $\tilde{\mu}_{m,T}$. The degree of diffusion anisotropy on an element T is evaluated by the condition number of K_T, namely $\Delta_T = \frac{\lambda_{M,T}}{\lambda_{m,T}}$.

3. A posteriori error analysis

3.1. Abstract setting

In this section we present the basic abstract framework for our a posteriori error estimates. The following result is directly inspired from the abstract framework introduced by Vohralík [42].

Lemma 3.1. *Let Z and Z_h be two vector spaces. Let A be a bounded bilinear form defined on $Z' \times Z'$ with $Z' := Z + Z_h$. Assume that A can be decomposed into the form $A = A_S + A_{SS}$ where A_S is symmetric and nonnegative on Z' and where A_{SS} is skew-symmetric on Z (but not necessarily on Z'). Then, defining the semi-norm $|\cdot|_* := A_S(\cdot,\cdot)^{1/2}$, the following holds for all $u, s \in Z$ and $u_h \in Z_h$,*

$$|u - u_h|_* \leq |s - u_h|_* + |A(u - u_h, \phi) + A_{SS}(u_h - s, \phi)|, \tag{3.1}$$

where $\phi = \frac{u-s}{|u-s|_*}$.

Proof. Suppose first that $|u - s|_* \leq |u - u_h|_*$. Then,

$$\begin{aligned}
|u - u_h|_*^2 &= A(u - u_h, u - u_h) - A_{SS}(u - u_h, u - u_h) \\
&= A(u - u_h, u - s) + A(u - u_h, s - u_h) - A_{SS}(u - u_h, u - u_h) \\
&= A(u - u_h, u - s) + A_S(u - u_h, s - u_h) + A_{SS}(u - u_h, s - u_h) - A_{SS}(u - u_h, u - u_h) \\
&= A(u - u_h, u - s) + A_S(u - u_h, s - u_h) + A_{SS}(u - u_h, s - u) \\
&= A(u - u_h, u - s) + A_S(u - u_h, s - u_h) + A_{SS}(u_h - s, u - s),
\end{aligned}$$

where we have used $A_{SS}(u - s, u - s) = 0$ since $(u - s) \in Z$. Introducing ϕ yields

$$|u - u_h|_*^2 \leq |u - s|_* A(u - u_h, \phi) + |u - u_h|_* |s - u_h|_* + |u - s|_* A_{SS}(u_h - s, \phi). \tag{3.2}$$

Having hypothesized that $|u - s|_* \leq |u - u_h|_*$, we infer

$$|u - u_h|_* \leq |s - u_h|_* + |A(u - u_h, \phi) + A_{SS}(u_h - s, \phi)|. \tag{3.3}$$

Consider now the case $|u - u_h|_* \leq |u - s|_*$. Since $A_{SS}(u - s, u - s) = 0$,

$$\begin{aligned}
|u - s|_*^2 &= A(u - s, u - s) = A(u - u_h, u - s) + A_S(u_h - s, u - s) + A_{SS}(u_h - s, u - s) \\
&\leq |u - s|_* A(u - u_h, \phi) + |u_h - s|_* |u - s|_* + |u - s|_* A_{SS}(u_h - s, \phi).
\end{aligned}$$

Thus

$$|u - u_h|_* \leq |u - s|_* \leq A(u - u_h, \phi) + |s - u_h|_* + A_{SS}(u_h - s, \phi). \tag{3.4}$$

Combining the results we obtain (3.1).

3.2. Pure diffusion

Let $\beta = 0$ and $\mu = 0$ in (1.1), i.e., we consider a diffusion problem with anisotropic and heterogeneous diffusivity:

$$\begin{cases} -\nabla\cdot(K\nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.5}$$

The bilinear form B defined by (2.7) becomes

$$B(v,w) = (K\nabla_h v, \nabla_h w)_{0,\Omega}, \tag{3.6}$$

while the definition of the (semi-)norm $\|\cdot\|_B$ involves only the diffusive contribution, i.e., $\|v\|_{B,T}^2 = (K\nabla_h v, \nabla_h v)_{0,T}$. The discrete problem is still (2.9) with bilinear form B_h defined by

$$\begin{aligned} B_h(v,w) = (K\nabla_h v, \nabla_h w)_{0,\Omega} + \sum_{F\in\mathcal{F}_h} [(\gamma_{K,F}[\![v]\!], [\![w]\!])_{0,F} \\ - (n_F^t\{K\nabla_h v\}_\omega, [\![w]\!])_{0,F} - (n_F^t\{K\nabla_h w\}_\omega, [\![v]\!])_{0,F}]. \end{aligned} \tag{3.7}$$

Lemma 3.1 can be applied by letting $Z := V$, $Z_h := V_h$, $A = A_S := B$ and $A_{SS} := 0$. The semi-norm $|\cdot|_*$ coincides with $\|\cdot\|_B$. This yields

$$\|u - u_h\|_B \le \inf_{s\in V} \|u_h - s\|_B + \sup_{\phi\in V, \|\phi\|_B=1} |B(u-u_h,\phi)|. \tag{3.8}$$

We now proceed to estimate the second term in the right-hand side of (3.8). Let $\Pi_h : L^2(\Omega) \to V_h$ denote the L^2-orthogonal projection onto V_h. It is well-known that for $v \in L^2(\Omega)$, $\Pi_h v$ coincides on each mesh element with the mean value of v on the corresponding element. The projector Π_h satisfies the following approximation properties: For all $T \in \mathcal{T}_h$ and for all $\phi \in H^1(T)$,

$$\|\phi - \Pi_h\phi\|_{0,T} \le C_p^{\frac{1}{2}} h_T \|\nabla\phi\|_T, \tag{3.9}$$

$$\|\phi - \Pi_h\phi\|_{0,\partial T} \le C_T^{\frac{1}{2}} h_T^{\frac{1}{2}} \|\nabla\phi\|_{0,T}. \tag{3.10}$$

The constant C_p in the Poincaré-type inequality (3.9) can be bounded for each convex T by d/π, see [7, 31], while it follows from [40] that the constant C_T in the trace inequality (3.10) is given by $C_T = 3d\rho_T$ with $\rho_T = h_T|\partial T|/|T|$ where $|\partial T|$ denotes the $(d-1)$-measure of ∂T and $|T|$ the d-measure of T; note that ρ_T is uniformly bounded owing to the shape-regularity of the mesh family. For all $T \in \mathcal{T}_h$, define on T the volumetric residual

$$R(u_h) = f + \nabla_h\cdot(K\nabla_h u_h), \tag{3.11}$$

and on ∂T the boundary residual such that for $F \subset \partial T$,

$$J_K(u_h)|_F = \omega_{T,F} n_T^t [\![K\nabla_h u_h]\!] + \gamma_{K,F}[\![u_h]\!], \tag{3.12}$$

where

$$\omega_{T,F} = \frac{n_F^t K_T n_F}{n_F^t K_T n_F + n_F^t K_{T'} n_F}, \tag{3.13}$$

with $F = T \cap T'$. Note that the convention regarding δ_{K+} yields $\omega_{T,F} = 0$ on boundary faces.

Lemma 3.2. *The following holds:*

$$\sup_{\phi\in V, \|\phi\|_B=1} |B(u-u_h,\phi)| \le \left(\sum_{T\in\mathcal{T}_h} (\eta_T + \zeta_T)^2\right)^{\frac{1}{2}}, \tag{3.14}$$

where the residual error indicator η_T is

$$\eta_T = C_p^{\frac{1}{2}} h_T \lambda_{m,T}^{-\frac{1}{2}} \|(I - \Pi_h)R(u_h)\|_T, \tag{3.15}$$

and the diffusive flux error indicator is

$$\zeta_T = C_T^{\frac{1}{2}} h_T^{\frac{1}{2}} \lambda_{m,T}^{-\frac{1}{2}} \|J_K(u_h)\|_{0,\partial T}. \tag{3.16}$$

Proof. Let $\phi \in V$ such that $\|\phi\|_B = 1$. Using $B(u,\phi) = (f,\phi)_{0,\Omega}$ and integrating by parts we obtain

$$B(u - u_h, \phi) = \sum_{T\in\mathcal{T}_h} (f + \nabla_h\cdot(K\nabla_h u_h), \phi)_{0,T} - \sum_{F\in\mathcal{F}_h^i} (n_F^t [\![K\nabla_h u_h]\!], \phi)_{0,F}$$

since $\phi \in V = H_0^1(\Omega)$. Testing the discrete equations with $\Pi_h\phi$ yields

$$\sum_{F\in\mathcal{F}_h} (\gamma_{K,F}[\![u_h]\!] - n_F^t\{K\nabla_h u_h\}_\omega, [\![\Pi_h\phi]\!])_{0,F} = (f, \Pi_h\phi)_{0,\Omega}.$$

Observe that

$$\sum_{T\in\mathcal{T}_h} (\nabla_h\cdot(K\nabla_h u_h), \Pi_h\phi)_{0,T} = \sum_{F\in\mathcal{F}_h} (n_F^t\{K\nabla_h u_h\}_\omega, [\![\Pi_h\phi]\!])_{0,F} + \sum_{F\in\mathcal{F}_h^i} (n_F^t [\![K\nabla_h u_h]\!], \{\Pi_h\phi\}_{\bar\omega})_{0,F}.$$

Combining the above equations and using $[\![\phi]\!] = 0$ leads to

$$\begin{aligned} B(u - u_h, \phi) &= \sum_{T\in\mathcal{T}_h} (f + \nabla_h\cdot(K\nabla_h u_h), \phi - \Pi_h\phi)_{0,T} - \sum_{F\in\mathcal{F}_h} (\gamma_{K,F}[\![u_h]\!], [\![\phi - \Pi_h\phi]\!])_{0,F} \\ &\quad - \sum_{F\in\mathcal{F}_h^i} (n_F^t [\![K\nabla_h u_h]\!], \{\phi - \Pi_h\phi\}_{\bar\omega})_{0,F} \\ &= \sum_{T\in\mathcal{T}_h} (R(u_h), \phi - \Pi_h\phi)_{0,T} - \sum_{T\in\mathcal{T}_h} \sum_{F\subset\partial T} n_T\cdot n_F (J_K(u_h), \phi - \Pi_h\phi|_T)_{0,F}. \end{aligned}$$

The conclusion is straightforward using (3.9)–(3.10) and the fact that $\Pi_h(R(u_h))$ and $(\phi - \Pi_h\phi)$ are L^2-orthogonal on each $T \in \mathcal{T}_h$.

Remark 3.1. *Taking off the mean value of $R(u_h)$ in the residual error estimator is possible because the discrete space contains piecewise constant functions. This is a feature of dG approximations, but not, for instance, of continuous finite element approximations.*

Theorem 3.1. *Pick any $s_h \in V$ and define the non-conforming error indicator ι_T as*

$$\iota_T = \|u_h - s_h\|_{B,T}. \tag{3.17}$$

Then, the following holds

$$\|u - u_h\|_B \le \left(\sum_{T\in\mathcal{T}_h} (\eta_T + \zeta_T)^2\right)^{\frac{1}{2}} + \left(\sum_{T\in\mathcal{T}_h} \iota_T^2\right)^{\frac{1}{2}}. \tag{3.18}$$

Proof. Direct consequence of Lemma 3.2 and of (3.8).

We now investigate the local efficiency of the above error indicators η_T, ζ_T and ι_T. Here, $x \lesssim y$ indicates the inequality $x \leq cy$ with positive c independent of the mesh and of the diffusion tensor. To simplify, the data f is assumed to be a polynomial; otherwise, the usual data oscillation term has to be added to the estimates. The following two propositions establish that the error indicators η_T and ζ_T are fully robust with respect to heterogeneities in the diffusion tensor, while the dependency on anisotropies remains local, i.e., only the square root of the condition numbers $\Delta_{\tilde{T}}$ on T and neighboring elements appears in the local lower bounds, but not the ratios of two diffusion tensor eigenvalues from different elements.

Proposition 3.1. *For all $T \in \mathcal{T}_h$,*

$$\eta_T \lesssim \Delta_T^{\frac{1}{2}} \|u - u_h\|_{B,T}. \tag{3.19}$$

Proof. Since $\|(I - \Pi_h)R(u_h)\|_{0,T} \leq \|R(u_h)\|_{0,T}$, we simply bound $\|R(u_h)\|_{0,T}$. To this purpose, we use the technique of element bubble functions introduced by Verfürth [36, 37]; the arguments, which are fairly standard, are only briefly sketched. Let $T \in \mathcal{T}_h$, let b_T be a suitable local bubble function in T vanishing on ∂T and set $\nu_T = b_T R(u_h)$. Then,

$$\|R(u_h)\|_{0,T}^2 \lesssim (R(u_h), \nu_T)_{0,T} = (K\nabla_h(u - u_h), \nabla\nu_T)_{0,T} \lesssim \lambda_{M,T}^{\frac{1}{2}} h_T^{-1} \|u - u_h\|_{B,T} \|R(u_h)\|_{0,T},$$

from which follows (3.19).

Proposition 3.2. *For all $T \in \mathcal{T}_h$,*

$$\zeta_T \lesssim \Delta_T^{\frac{1}{2}} \sum_{\tilde{T} \in \mathcal{N}_T} \Delta_{\tilde{T}}^{\frac{1}{2}} \|u - u_h\|_{B,\tilde{T}}, \tag{3.20}$$

where $\mathcal{N}_T$ is the set of elements sharing a face with the element T.

Proof. Let $T \in \mathcal{T}_h$. Observe that

$$|\zeta_T| \lesssim \lambda_{m,T}^{-\frac{1}{2}} \sum_{F \subset \partial T} \delta_F h_F^{-\frac{1}{2}} \|[\![u_h]\!]\|_F + \lambda_{m,T}^{-\frac{1}{2}} h_T^{\frac{1}{2}} \sum_{F \subset \partial T} \omega_{T,F} \|[\![K\nabla_h u_h]\!]\|_F \equiv X + Y,$$

and let us bound X and Y.
(i) Bound on X. Let $F \subset \partial T$. We use the result obtained by Achdou, Bernardi and Coquel [1]:

$$h_F^{-\frac{1}{2}} \|[\![u_h]\!]\|_{0,F} \lesssim \sum_{T' \in \mathcal{T}(F)} \|\nabla_h(u - u_h)\|_{0,T'}, \tag{3.21}$$

to infer

$$\begin{aligned} X &\lesssim \lambda_{m,T}^{-\frac{1}{2}} \sum_{F \subset \partial T} \sum_{T' \in \mathcal{T}(F)} \delta_F \|\nabla_h(u - u_h)\|_{0,T'} \\ &= \Delta_T^{\frac{1}{2}} \sum_{F \subset \partial T} \sum_{T' \in \mathcal{T}(F)} (\lambda_{M,T}^{-\frac{1}{2}} \lambda_{M,T'}^{-\frac{1}{2}} \delta_F) \Delta_{T'}^{\frac{1}{2}} \|u - u_h\|_{B,T'} \lesssim \Delta_T^{\frac{1}{2}} \sum_{\tilde{T} \in \mathcal{N}_T} \Delta_{\tilde{T}}^{\frac{1}{2}} \|u - u_h\|_{B,\tilde{T}}, \end{aligned}$$

since $\delta_F \leq \min(n_F K_T n_F, n_F K_{T'} n_F)$, $n_F K_T n_F \leq \lambda_{M,T}$ and $n_F K_{T'} n_F \leq \lambda_{M,T'}$.
(ii) Bound on Y. Let $F \subset \partial T$. Using the technique of edge bubble functions introduced by Verfürth [36, 37], it is shown that

$$h_F^{\frac{1}{2}} \|n_F^t [\![K \nabla_h u_h]\!]\|_{0,F} \lesssim \sum_{T' \in \mathcal{T}(F)} \lambda_{M,T'}^{\frac{1}{2}} \|u - u_h\|_{B,T'}.$$

Hence,

$$\begin{aligned} Y &\lesssim \lambda_{m,T}^{-\frac{1}{2}} \sum_{F \subset \partial T} \omega_{T,F} \sum_{T' \in \mathcal{T}(F)} \lambda_{M,T'}^{\frac{1}{2}} \|u - u_h\|_{B,T'} \\ &\lesssim \Delta_T^{\frac{1}{2}} \sum_{F \subset \partial T} \sum_{T' \in \mathcal{T}(F)} (\lambda_{M,T}^{-\frac{1}{2}} \lambda_{m,T'}^{\frac{1}{2}} \omega_{T,F}) \Delta_{T'}^{\frac{1}{2}} \|u - u_h\|_{B,T'} \lesssim \Delta_T^{\frac{1}{2}} \sum_{F \subset \partial T} \sum_{T' \in \mathcal{T}(F)} \Delta_{T'}^{\frac{1}{2}} \|u - u_h\|_{B,T'}, \end{aligned}$$

since

$$\lambda_{M,T}^{-\frac{1}{2}} \lambda_{m,T'}^{\frac{1}{2}} \omega_{T,F} \leq \frac{(n_F K_T n_F)^{\frac{1}{2}} (n_F K_{T'} n_F)^{\frac{1}{2}}}{(n_F K_T n_F) + (n_F K_{T'} n_F)} \leq \frac{1}{2}.$$

The proof is complete.

To analyze the local efficiency of the non-conforming error indicator ι_T, a particular choice must be made for $s_h \in V$. Presently, one of the state-of-the-art approaches consists in considering the so-called Oswald interpolate of the discrete solution u_h. For $v_h \in V_h$, its Oswald interpolate $\mathcal{I}_{Os}(v_h) \in V_h \cap V$ is defined by prescribing its values at the usual Lagrange interpolation nodes on each mesh element by taking the average of the values of v_h at the node,

$$\mathcal{I}_{Os}(v_h)(s) = \frac{1}{|\mathcal{T}_s|} \sum_{T \in \mathcal{T}_s} v_h|_T(s), \tag{3.22}$$

where $\mathcal{T}_s$ is the set of mesh elements that contain the node s and where $|\mathcal{T}_s|$ denotes the cardinal of that set. On boundary nodes, $\mathcal{I}_{Os}(v_h)(s)$ is set to zero. The Oswald interpolation operator $\mathcal{I}_{Os}$ yields the following local approximation properties [1, 27]: For all $v_h \in V_h$ and for all $T \in \mathcal{T}_h$,

$$\|v_h - \mathcal{I}_{Os}(v_h)\|_{0,T}^2 \leq C \sum_{F \in \mathcal{F}_h, F \cap \partial T \neq \emptyset} h_F \|[\![v_h]\!]\|_{0,F}^2, \tag{3.23}$$

$$\|\nabla_h (v_h - \mathcal{I}_{Os}(v_h))\|_{0,T}^2 \leq C \sum_{F \in \mathcal{F}_h, F \cap \partial T \neq \emptyset} h_F^{-1} \|[\![v_h]\!]\|_{0,F}^2, \tag{3.24}$$

where the constant C depends on the space dimension, the polynomial degree p used to construct the space V_h, and the shape-regularity parameter associated with the mesh $\mathcal{T}_h$; the dependency of the constant C on p has been recently explored in [11]. Setting $s_h := \mathcal{I}_{Os}(u_h)$ to evaluate ι_T, it is inferred using (3.21) and (3.24) that

$$\iota_T \lesssim \lambda_{M,T}^{\frac{1}{2}} \sum_{T' \in \mathcal{R}_T} \lambda_{m,T'}^{-\frac{1}{2}} \|u - u_h\|_{B,T'}, \tag{3.25}$$

where $\mathcal{R}_T = \{T' \in \mathcal{T}_h; T \cap T' \neq \emptyset\}$. Clearly, the above estimate is not robust with respect to heterogeneities and/or anisotropies in the diffusion tensor. In the isotropic case, the result

can be improved by using weighted averages in (3.22) to define the nodal values of the Oswald interpolate. The weights depend on the diffusivity and a robust bound can be inferred on ι_T when evaluated with this modified Oswald interpolate provided a monotonicity property of the diffusivity around vertices is assumed to hold; see [2, 10, 18]. To the authors' knowledge, no fully satisfactory result on a modified Oswald interpolation operator is yet available in the case of anisotropic diffusivity. We will not explore this issue further here. Finally, we point out that the local efficiency of the error indicator ι_T has to be weighted against the computational costs required for its evaluation. Indeed, since any reconstructed function $s_h \in V$ can be chosen to evaluate it and since

$$\inf_{s\in V} \|u_h - s\|_{B,T} \leq \|u_h - u\|_{B,T}, \tag{3.26}$$

the local efficiency properties of ι_T can be improved at the expense of solving more detailed local problems. Developments along this line go beyond the present scope.

Remark 3.2. *Using a triangle inequality, the flux error indicator ζ_T can be split into two contributions, one associated with the jump of the diffusive flux and the other associated with the jump of the discrete solution itself, and the latter can be regrouped with the non-conforming error indicator ι_T. Both contributions are locally efficient and fully robust with respect to heterogeneities in the diffusivity, as shown in the proof of Proposition* 3.2 *where the quantities X and Y are bounded separately. By proceeding this way, the error upper bound is somewhat less sharp because a triangle inequality has been used, but the final form of the a posteriori error estimate takes a more familiar form.*

3.3. Advection-diffusion-reaction

In this section we turn to the general case of an advection-diffusion-reaction problem. Our purpose is to extend the a posteriori error indicators derived in Lemma 3.2 and in Theorem 3.1 to this situation, with a particular emphasis on the robustness of the estimates in the high-Péclet regime in the sense of Verfürth [38]. The starting point is again the abstract estimate derived in Lemma 3.1 which is now applied with $Z := V$, $Z_h := V_h$,

$$A_S(v,w) = (K\nabla_h v, \nabla_h w)_{0,\Omega} + (\tilde{\mu} v, w)_{0,\Omega}, \tag{3.27}$$

$$A_{SS}(v,w) = (\beta\cdot\nabla_h v, w)_{0,\Omega} + \tfrac{1}{2}((\nabla\cdot\beta)v, w)_{0,\Omega}, \tag{3.28}$$

and $A = A_S + A_{SS} = B$ as defined by (2.7). Observe that A_S is symmetric and nonnegative on $Z + Z_h$, that $|\cdot|_*$ coincides with $\|\cdot\|_B$, and that A_{SS} is skew-symmetric on Z (but not on $Z + Z_h$). As a first step, we rewrite the quantity $B(u - u_h, \phi) + A_{SS}(u_h - s, \phi)$ in a more convenient form.

Lemma 3.3. *Let $s \in V$. For all $T \in \mathcal{T}_h$, define on T the volumetric residual*

$$R(u_h) = f + \nabla_h\cdot(K\nabla_h u_h) - \beta\cdot\nabla_h u_h - \mu u_h, \tag{3.29}$$

let $J_K(u_h)$ be defined on ∂T by (3.12)*, and let $J_\beta(u_h - s)$ be defined such that for $F \subset \partial T$,*

$$J_\beta(u_h - s)|_F = \langle \gamma_\beta [\![u_h - s]\!] + \beta\cdot n_F\{u_h - s\}\rangle_F, \tag{3.30}$$

where $\langle\cdot\rangle_F$ denotes the mean value over F. Then, for all $\phi \in V$,

$$B(u - u_h, \phi) + A_{SS}(u_h - s, \phi) = X_1 + X_2 + X_3, \tag{3.31}$$

with

$$X_1 = \sum_{T\in\mathcal{T}_h} ((I-\Pi_h)R(u_h), \phi - \Pi_h\phi)_{0,T}, \tag{3.32}$$

$$X_2 = -\sum_{T\in\mathcal{T}_h}\sum_{F\subset\partial T} n_T{\cdot}n_F(J_K(u_h), \phi - \Pi_h\phi|_T)_{0,F}, \tag{3.33}$$

$$\begin{aligned} X_3 = &\sum_{T\in\mathcal{T}_h} [((I-\Pi_h)(\beta{\cdot}\nabla_h(u_h-s)), \phi-\Pi_h\phi)_{0,T} + \tfrac{1}{2}(\nabla{\cdot}\beta(u_h-s), \phi - 2\Pi_h\phi)_{0,T}] \\ &+ \sum_{F\in\mathcal{F}_h} (J_\beta(u_h-s), [\![\Pi_h\phi]\!])_{0,F}. \end{aligned} \tag{3.34}$$

Proof. Let $\phi\in V$. Using $B(u,\phi) = (f,\phi)_{0,\Omega}$ and integrating by parts, we infer

$$B(u-u_h,\phi) = \sum_{T\in\mathcal{T}_h}(R(u_h),\phi)_{0,T} - \sum_{F\in\mathcal{F}_h^i}(n_F^t[\![K\nabla_h u_h]\!],\phi)_{0,F}.$$

Testing the discrete equations with $\Pi_h\phi$ yields

$$\sum_{F\in\mathcal{F}_h}(\gamma[\![u_h]\!] - n_F^t\{K\nabla_h u_h\}_\omega + \beta{\cdot}n_F\{u_h\}, [\![\Pi_h\phi]\!])_{0,F} + ((\mu-\nabla{\cdot}\beta)u_h, \Pi_h\phi)_{0,\Omega} = (f,\Pi_h\phi)_{0,\Omega}.$$

Combining the two above equations and proceeding as in the proof of Lemma 3.2 for the diffusive term leads to

$$B(u-u_h,\phi) = X_1 + X_2 + \sum_{F\in\mathcal{F}_h}(\gamma_\beta[\![u_h]\!], [\![\Pi_h\phi]\!])_{0,F} - \sum_{F\in\mathcal{F}_h}(\beta{\cdot}n_F[\![u_h]\!], \{\Pi_h\phi\})_{0,F}.$$

Using the relation

$$\begin{aligned} &-\sum_{T\in\mathcal{T}_h}((\nabla{\cdot}\beta)(u_h-s), \Pi_h\phi)_{0,T} - \sum_{T\in\mathcal{T}_h}(\beta{\cdot}\nabla_h(u_h-s), \Pi_h\phi)_{0,T} \\ &+ \sum_{F\in\mathcal{F}_h}(\beta{\cdot}n_F[\![u_h]\!], \{\Pi_h\phi\})_{0,F} + \sum_{F\in\mathcal{F}_h}(\beta{\cdot}n_F\{u_h-s\}, [\![\Pi_h\phi]\!])_{0,F} = 0, \end{aligned}$$

and adding $A_{SS}(u_h-s,\phi)$ as evaluated from (3.28), (3.31) is inferred. Note that the upwind related term $J_\beta(u_h-s)$ can be evaluated as a mean value over each face because it is tested against a piecewise constant function and that the mean value of $\beta{\cdot}\nabla_h(u_h-s)$ can be taken off on each element because it is tested against $\phi-\Pi_h\phi$.

Remark 3.3. *The idea of evaluating the upwind related term as a mean value over each face has been proposed by Vohralík* [41]. *Since for any function $\psi\in L^2(F)$, $\|\langle\psi\rangle_F\|_{0,F} \le \|\psi\|_{0,F}$, this modification can only sharpen the a posteriori error estimate.*

The next step is to control $\phi-\Pi_h\phi$ for $\phi\in V$ in terms of the energy norm $\|\phi\|_B$. To obtain bounds that behave satisfactorily when the Péclet number is large, a sharper version of inequalities (3.9)–(3.10) needs to be used. Observing that on all $T\in\mathcal{T}_h$, $\|\phi-\Pi_h\phi\|_{0,T} \le \|\phi\|_{0,T}$ and letting

$$m_T = \min\left(C_p^{\frac{1}{2}} h_T \lambda_{m,T}^{-\frac{1}{2}}, \tilde{\mu}_{m,T}^{-\frac{1}{2}}\right), \tag{3.35}$$

the bound (3.9) can be sharpened as follows:

$$\|\phi - \Pi_h\phi\|_{0,T} \leq m_T\|\phi\|_{B,T}. \tag{3.36}$$

Furthermore, owing to the trace inequality

$$\forall v \in H^1(T), \quad \|v\|_{0,\partial T} \leq C_{*\mathcal{T}}^{\frac{1}{2}}[h_T^{-\frac{1}{2}}\|v\|_{0,T} + \|v\|_{0,T}^{\frac{1}{2}}\|\nabla v\|_{0,T}^{\frac{1}{2}}], \tag{3.37}$$

where the constant $C_{*\mathcal{T}}$ depends on the space dimension, the polynomial degree p, and the shape-regularity of the mesh $\mathcal{T}_h$, (3.10) can be sharpened as follows:

$$\|\phi - \Pi_h\phi\|_{0,\partial T} \leq C_{*\mathcal{T}}^{\frac{1}{2}}[h_T^{-\frac{1}{2}}m_T + \lambda_{m,T}^{-\frac{1}{4}}m_T^{\frac{1}{2}}]\|\phi\|_{B,T} \leq \tilde{C}_{\mathcal{T}}^{\frac{1}{2}}\lambda_{m,T}^{-\frac{1}{4}}m_T^{\frac{1}{2}}\|\phi\|_{B,T}, \tag{3.38}$$

where we have set

$$\tilde{C}_{\mathcal{T}}^{\frac{1}{2}} = C_{*\mathcal{T}}^{\frac{1}{2}}(1 + C_p^{\frac{1}{4}}). \tag{3.39}$$

Estimate (3.38) will be used to bound the term X_2 introduced in Lemma 3.3. However, this estimate turns out not be sharp enough when bounding the last term in X_3. In this case, we will use the trace inequality

$$\forall \phi_h \in V_h, \quad \|\phi_h\|_{0,\partial T} \leq \rho_T^{\frac{1}{2}}h_T^{-\frac{1}{2}}\|\phi_h\|_{0,T}, \tag{3.40}$$

and we define for all $F \in \mathcal{F}_h$,

$$\tilde{m}_F^2 = \min\left(\max_{T'\in\mathcal{T}(F)}(C_{\mathcal{T}}h_{T'}\lambda_{m,T'}^{-1}), \max_{T'\in\mathcal{T}(F)}(\rho_{T'}h_{T'}^{-1}\tilde{\mu}_{m,T'}^{-1})\right). \tag{3.41}$$

Finally, let $\kappa_{\mu,\beta,T} = \frac{1}{2}\|\nabla\cdot\beta\|_{L^\infty(T)}\tilde{\mu}_{m,T}^{-\frac{1}{2}}$.

Lemma 3.4. *Let $s \in V$. The following holds*

$$\sup_{\substack{\phi\in V\\ \|\phi\|_B=1}} |B(u-u_h,\phi) + A(u_h - s,\phi)| \leq \left(\sum_{T\in\mathcal{T}_h}(\eta_T + \zeta_T + \iota'_T)^2\right)^{\frac{1}{2}}, \tag{3.42}$$

where the residual error indicator η_T is

$$\eta_T = m_T\|(I - \Pi_h)R(u_h)\|_T, \tag{3.43}$$

the diffusive flux error indicator ζ_T is

$$\zeta_T = \tilde{C}_{\mathcal{T}}^{\frac{1}{2}}\lambda_{m,T}^{-\frac{1}{4}}m_T^{\frac{1}{2}}\|J_K(u_h)\|_{0,\partial T}, \tag{3.44}$$

and the non-conforming error indicator ι'_T is

$$\iota'_T = m_T\|(I - \Pi_h)(\beta\cdot\nabla_h(u_h - s))\|_{0,T} + \kappa_{\mu,\beta,T}\|u_h - s\|_{0,T} + \sum_{F\subset\partial T} 2\tilde{m}_F\|J_\beta(u_h - s)\|_{0,F}. \tag{3.45}$$

Proof. Let $\phi \in V$ such that $\|\phi\|_B = 1$. We bound the three terms X_1, X_2 and X_3 introduced in Lemma 3.3. Owing to (3.36) and (3.38), it is clear that

$$|X_1 + X_2| \leq \sum_{T \in \mathcal{T}_h} (\eta_T + \zeta_T)\|\phi\|_{B,T}.$$

Decompose X_3 into $X_3 = X_{3,1} + X_{3,2}$ where $X_{3,1}$ denotes the sum over elements and where $X_{3,2}$ denotes the sum over faces. Observing that $\|\phi - 2\Pi_h\phi\|_{0,T} = \|\phi\|_{0,T}$ and using again (3.36), we obtain

$$|X_{3,1}| \leq \sum_{T \in \mathcal{T}_h} (m_T\|(I - \Pi_h)(\beta \cdot \nabla_h(u_h - s))\|_{0,T} + \kappa_{\mu,\beta,T}\|u_h - s\|_{0,T})\|\phi\|_{B,T}.$$

To bound $X_{3,2}$, let $F \in \mathcal{F}_h$. On the one hand, owing to (3.10),

$$\begin{aligned}
|(J_\beta(u_h - s), [\![\Pi_h\phi]\!])_{0,F}| &= |(J_\beta(u_h - s), [\![\Pi_h\phi - \phi]\!])_{0,F}| \\
&\leq \sum_{T' \in \mathcal{T}(F)} |(J_\beta(u_h - s), \Pi_h\phi|_{T'} - \phi)_{0,F}| \\
&\leq \|J_\beta(u_h - s)\|_{0,F} \max_{T' \in \mathcal{T}(F)} (C_{\mathcal{T}}^{\frac{1}{2}} h_{T'}^{\frac{1}{2}} \lambda_{m,T'}^{-\frac{1}{2}}) \sum_{T' \in \mathcal{T}(F)} \|\phi\|_{B,T'}.
\end{aligned}$$

On the other hand, owing to (3.40),

$$\begin{aligned}
|(J_\beta(u_h - s), [\![\Pi_h\phi]\!])_{0,F}| &\leq \sum_{T' \in \mathcal{T}(F)} |(J_\beta(u_h - s), \Pi_h\phi|_T)_{0,F}| \\
&\leq \|J_\beta(u_h - s)\|_{0,F} \max_{T' \in \mathcal{T}(F)} (\tilde{C}_{\mathcal{T}}^{\frac{1}{2}} h_{T'}^{-\frac{1}{2}} \tilde{\mu}_{m,T'}^{-\frac{1}{2}}) \sum_{T' \in \mathcal{T}(F)} \|\phi\|_{B,T'}.
\end{aligned}$$

Hence,

$$|(J_\beta(u_h - s), [\![\Pi_h\phi]\!])_{0,F}| \leq \tilde{m}_F\|J_\beta(u_h - s)\|_{0,F} \sum_{T' \in \mathcal{T}(F)} \|\phi\|_{B,T'},$$

and therefore,

$$|X_{3,2}| \leq \sum_{T \in \mathcal{T}_h} \left(\sum_{F \subset \partial T} 2\tilde{m}_F\|J_\beta(u_h - s)\|_{0,F} \right) \|\phi\|_{B,T}.$$

The conclusion is straightforward.

Theorem 3.2. *Pick any $s_h \in V$ and define the non-conforming error indicator ι''_T as*

$$\iota''_T = \|u_h - s_h\|_{B,T}, \tag{3.46}$$

and let ι'_T be evaluated from (3.45) *using s_h. Then,*

$$\|u - u_h\|_B \leq \left(\sum_{T \in \mathcal{T}_h} (\eta_T + \zeta_T + \iota'_T)^2 \right)^{\frac{1}{2}} + \left(\sum_{T \in \mathcal{T}_h} (\iota''_T)^2 \right)^{\frac{1}{2}}. \tag{3.47}$$

Proof. Apply Lemmata 3.1 and 3.4.

Remark 3.4. *The non-conforming error indicators ι'_T and ι''_T can be regrouped into a single non-conforming error indicator ι_T by setting*

$$\iota_T^2 = 4(\iota'_T)^2 + 2(\iota''_T)^2. \tag{3.48}$$

Then, (3.47) *becomes*

$$\|u - u_h\|_B \leq \left(2 \sum_{T \in \mathcal{T}_h} (\eta_T + \zeta_T)^2\right)^{\frac{1}{2}} + \left(\sum_{T \in \mathcal{T}_h} \iota_T^2\right)^{\frac{1}{2}}, \tag{3.49}$$

which is less sharp but has a more familiar form.

We now investigate the local efficiency of the above error indicators η_T, ζ_T and ι_T. Here, $x \lesssim y$ indicates the inequality $x \leq cy$ with positive c independent of the mesh and of the parameters K, β, and μ. Again, the data f is assumed to be a polynomial; otherwise, the usual data oscillation term has to be added to the estimates. As in the pure diffusion case, we will not take advantage of the presence of the operator $(I - \Pi_h)$ in η_T and in the first term of ι'_T to derive the bounds below.

Proposition 3.3. *For all $T \in \mathcal{T}_h$,*

$$\eta_T \lesssim m_T[\lambda_{M,T}^{\frac{1}{2}} h_T^{-1} + \min(\alpha_{1,T}, \alpha_{2,T})]\|u - u_h\|_{B,T}, \tag{3.50}$$

where

$$\alpha_{1,T} = \frac{\|\mu\|_{L^\infty(T)}}{\tilde{\mu}_{m,T}^{\frac{1}{2}}} + \frac{\|\beta\|_{L^\infty(T)}}{\lambda_{m,T}^{\frac{1}{2}}}, \qquad \alpha_{2,T} = \frac{\|\mu - \nabla\cdot\beta\|_{L^\infty(T)} + \|\beta\|_{L^\infty(T)} h_T^{-1}}{\tilde{\mu}_{m,T}^{\frac{1}{2}}}.$$

Proof. Let $T \in \mathcal{T}_h$, let b_T be a suitable local bubble function in T vanishing on ∂T and set $\nu_T = b_T R(u_h)$. Then,

$$\begin{aligned}
\|R(u_h)\|_{0,T}^2 &\lesssim (R(u_h), \nu_T)_{0,T} = (K\nabla_h(u - u_h), \nabla_h \nu_T)_{0,T} + (\mu(u - u_h), \nu_T)_{0,T} \\
&\quad + (\beta\cdot\nabla_h(u - u_h), \nu_T)_{0,T} \\
&\lesssim \lambda_{M,T}^{\frac{1}{2}} h_T^{-1} \|u - u_h\|_{B,T} \|R(u_h)\|_{0,T} + \min(\alpha_{1,T}, \alpha_{2,T}) \|u - u_h\|_{B,T} \|R(u_h)\|_{0,T},
\end{aligned}$$

where the min is obtained by integrating by parts or not the advective derivative. The conclusion is straightforward.

Proposition 3.4. *For all $T \in \mathcal{T}_h$,*

$$\zeta_T \lesssim \Delta_T^{\frac{1}{2}} \lambda_{m,T}^{\frac{1}{4}} m_T^{\frac{1}{2}} \sum_{\tilde{T} \in \mathcal{N}_T} (\Delta_{\tilde{T}}^{\frac{1}{2}} + m_{\tilde{T}} \alpha_{1,\tilde{T}}) m_{\tilde{T}}^{-\frac{1}{2}} \lambda_{m,\tilde{T}}^{-\frac{1}{4}} \|u - u_h\|_{B,\tilde{T}}. \tag{3.51}$$

Proof. Let $T \in \mathcal{T}_h$. Observe that

$$|\zeta_T| \lesssim \lambda_{m,T}^{-\frac{1}{4}} m_T^{\frac{1}{2}} \sum_{F \subset \partial T} \delta_F h_F^{-1} \|[\![u_h]\!]\|_F + \lambda_{m,T}^{-\frac{1}{4}} m_T^{\frac{1}{2}} \sum_{F \subset \partial T} \omega_{T,F} \|n_F^t [\![K\nabla_h u_h]\!]\|_F \equiv X + Y,$$

and let us bound X and Y by the right-hand side of (3.51).
(i) Bound on X. Owing to (3.21) and the definition of δ_F,

$$\begin{aligned} X &\lesssim \lambda_{m,T}^{-\frac{1}{4}} m_T^{\frac{1}{2}} \sum_{F\subset\partial T} \sum_{T'\in\mathcal{T}(F)} \delta_F h_F^{-\frac{1}{2}} \lambda_{m,T'}^{-\frac{1}{2}} \|u-u_h\|_{B,T'} \\ &\lesssim \Delta_T^{\frac{1}{2}} m_T^{\frac{1}{2}} \lambda_{m,T}^{\frac{1}{4}} \sum_{F\subset\partial T} \sum_{T'\in\mathcal{T}(F)} (\lambda_{M,T}^{-\frac{1}{2}} \lambda_{M,T'}^{-\frac{1}{2}} \delta_F) \Delta_{T'}^{\frac{1}{2}} h_F^{-\frac{1}{2}} \|u-u_h\|_{B,T'} \\ &\lesssim \Delta_T^{\frac{1}{2}} m_T^{\frac{1}{2}} \lambda_{m,T}^{\frac{1}{4}} \sum_{\tilde{T}\in\mathcal{N}_T} \Delta_{\tilde{T}}^{\frac{1}{2}} h_{\tilde{T}}^{-\frac{1}{2}} \|u-u_h\|_{B,\tilde{T}}, \end{aligned}$$

since $\lambda_{M,T}^{-\frac{1}{2}} \lambda_{M,T'}^{-\frac{1}{2}} \delta_F \le 1$. Owing to the obvious bound $h_{\tilde{T}}^{-\frac{1}{2}} \le m_{\tilde{T}}^{-\frac{1}{2}} \lambda_{m,\tilde{T}}^{-\frac{1}{4}}$, it is inferred that X is bounded by the right-hand side of (3.51).
(ii) Bound on Y. Let $F \subset \partial T$. Following the ideas of Verfürth [38], let b_F be a suitable bubble function with support in F and let ℓ_F be the lifting of $(n_F^t [\![K\nabla_h u_h]\!])b_F$ in $\mathcal{T}(F)$ with cut-off parameter

$$\theta_{T'} = m_{T'} C_p^{-\frac{1}{2}} h_{T'}^{-1} \lambda_{m,T'}^{\frac{1}{2}} \le 1,$$

on each $T' \in \mathcal{T}(F)$. Then,

$$\begin{aligned} \|n_F^t [\![K\nabla_h u_h]\!]\|_{0,F}^2 &\lesssim (n_F^t [\![K\nabla_h u_h]\!], \ell_F)_{0,F}, \\ \|\ell_F\|_{0,T'} &\lesssim h_{T'}^{\frac{1}{2}} \theta_{T'}^{\frac{1}{2}} \|n_F^t [\![K\nabla_h u_h]\!]\|_{0,F} \lesssim m_{T'}^{\frac{1}{2}} \lambda_{m,T'}^{\frac{1}{4}} \|n_F^t [\![K\nabla_h u_h]\!]\|_{0,F}, \\ \|\nabla\ell_F\|_{0,T'} &\lesssim h_{T'}^{-\frac{1}{2}} \theta_{T'}^{-\frac{1}{2}} \|n_F^t [\![K\nabla_h u_h]\!]\|_{0,F} \lesssim m_{T'}^{-\frac{1}{2}} \lambda_{m,T'}^{-\frac{1}{4}} \|n_F^t [\![K\nabla_h u_h]\!]\|_{0,F}. \end{aligned}$$

Observe that

$$B(u-u_h, \ell_F) = (R(u_h), \ell_F)_{0,\mathcal{T}(\mathcal{F})} + (n_F^t [\![K\nabla_h u_h]\!], \ell_F)_{0,F},$$

and that

$$|B(u-u_h, \ell_F)| \lesssim \sum_{T'\in\mathcal{T}(F)} (\lambda_{M,T'}^{\frac{1}{2}} m_{T'}^{-\frac{1}{2}} \lambda_{m,T'}^{-\frac{1}{4}} + m_{T'}^{\frac{1}{2}} \lambda_{m,T'}^{\frac{1}{4}} \alpha_{1,T'}) \|u-u_h\|_{B,T'} \|n_F^t [\![K\nabla_h u_h]\!]\|_{0,F}.$$

Furthermore, since

$$\begin{aligned} |(R(u_h), \ell_F)_{0,\mathcal{T}(\mathcal{F})}| &\le \sum_{T'\in\mathcal{T}(F)} \|R(u_h)\|_{0,T'} \|\ell_F\|_{0,T'} \\ &\lesssim \sum_{T'\in\mathcal{T}(F)} [\lambda_{M,T'}^{\frac{1}{2}} h_{T'}^{-1} + \min(\alpha_{1,T'}, \alpha_{2,T'})] \|u-u_h\|_{B,T'} \|\ell_F\|_{0,T'} \\ &\lesssim \sum_{T'\in\mathcal{T}(F)} [\lambda_{M,T'}^{\frac{1}{2}} h_{T'}^{-1} + \alpha_{1,T'}] m_{T'}^{\frac{1}{2}} \lambda_{m,T'}^{\frac{1}{4}} \|u-u_h\|_{B,T'} \|n_F^t [\![K\nabla_h u_h]\!]\|_{0,F}, \end{aligned}$$

and since $h_{T'}^{-1} m_{T'}^{\frac{1}{2}} \lambda_{m,T'}^{\frac{1}{4}} \le m_{T'}^{-\frac{1}{2}} \lambda_{m,T'}^{-\frac{1}{4}}$, it is inferred that $|(R(u_h), \ell_F)_{0,\mathcal{T}(\mathcal{F})}|$ can be bounded as $|B(u-u_h, \ell_F)|$, whence

$$\|n_F^t [\![K\nabla_h u_h]\!]\|_{0,F} \lesssim \sum_{T'\in\mathcal{T}(F)} (\lambda_{M,T'}^{\frac{1}{2}} m_{T'}^{-\frac{1}{2}} \lambda_{m,T'}^{-\frac{1}{4}} + m_{T'}^{\frac{1}{2}} \lambda_{m,T'}^{\frac{1}{4}} \alpha_{1,T'}) \|u-u_h\|_{B,T'}.$$

As a result,

$$
\begin{aligned}
Y &\lesssim \lambda_{m,T}^{-\frac{1}{4}} m_T^{\frac{1}{2}} \sum_{F\subset\partial T} \sum_{T'\in\mathcal{T}(F)} \omega_{T,F}(\lambda_{M,T'}^{\frac{1}{2}} m_{T'}^{-\frac{1}{2}} \lambda_{m,T'}^{-\frac{1}{4}} + m_{T'}^{\frac{1}{2}} \lambda_{m,T'}^{\frac{1}{4}} \alpha_{1,T'}) \|u-u_h\|_{B,T'} \\
&\lesssim \Delta_T^{\frac{1}{2}} \lambda_{m,T}^{\frac{1}{4}} m_T^{\frac{1}{2}} \sum_{F\subset\partial T} \sum_{T'\in\mathcal{T}(F)} (\lambda_{M,T}^{-\frac{1}{2}} \lambda_{m,T'}^{\frac{1}{2}} \omega_{T,F})(\Delta_{T'}^{\frac{1}{2}} + m_{T'}\alpha_{1,T'}) m_{T'}^{-\frac{1}{2}} \lambda_{m,T'}^{-\frac{1}{4}} \|u-u_h\|_{B,T'} \\
&\lesssim \Delta_T^{\frac{1}{2}} \lambda_{m,T}^{\frac{1}{4}} m_T^{\frac{1}{2}} \sum_{\tilde{T}\in\mathcal{N}_T} (\Delta_{\tilde{T}}^{\frac{1}{2}} + m_{\tilde{T}}\alpha_{1,\tilde{T}}) m_{\tilde{T}}^{-\frac{1}{2}} \lambda_{m,\tilde{T}}^{-\frac{1}{4}} \|u-u_h\|_{B,\tilde{T}}.
\end{aligned}
$$

The conclusion is straightforward.

Finally, we investigate the local efficiency of the non-conforming error estimator ι_T. To this purpose, we pick $s_h = \mathcal{I}_{Os}(u_h)$. As discussed at the end of §3.2, a modified Oswald interpolation operator can be considered in the case of isotropic and heterogeneous diffusivity with a monotonicity property around vertices to sharpen the result.

Proposition 3.5. *Set $s_h = \mathcal{I}_{Os}(u_h)$. Let $T \in \mathcal{T}_h$. Then,*

$$
\begin{aligned}
\iota_T \lesssim &\left(\lambda_{M,T}^{\frac{1}{2}} + h_T \|\tilde{\mu}\|_{L^\infty(T)}^{\frac{1}{2}} + m_T \|\beta\|_{L^\infty(T)} + h_T \kappa_{\mu,\beta,T} + \sum_{F\subset\partial T} \tilde{m}_F \|\beta\|_{L^\infty(F)} h_F^{\frac{1}{2}} \right) \\
&\times \sum_{T'\in\mathcal{R}_T} \lambda_{m,T'}^{-\frac{1}{2}} \|u-u_h\|_{B,T'}.
\end{aligned} \tag{3.52}
$$

Proof. Let $T \in \mathcal{T}_h$. Observe first that using (3.23)–(3.24),

$$
\|u_h - s_h\|_{B,T} \lesssim (\lambda_{M,T}^{\frac{1}{2}} + h_T \|\tilde{\mu}\|_{L^\infty(T)}^{\frac{1}{2}}) \sum_{T'\in\mathcal{R}_T} \lambda_{m,T'}^{-\frac{1}{2}} \|u-u_h\|_{B,T'},
$$

where $\mathcal{R}_T = \{T' \in \mathcal{T}_h; T\cap T' \neq \emptyset\}$. Furthermore, still using (3.23)–(3.24), the first two terms in ι_T' (see (3.45)) are bounded by

$$
(m_T \|\beta\|_{L^\infty(T)} + h_T \kappa_{\mu,\beta,T}) \sum_{T'\in\mathcal{R}_T} \lambda_{m,T'}^{-\frac{1}{2}} \|u-u_h\|_{B,T'},
$$

and it remains to bound the last term, namely $\sum_{F\subset\partial T} 2\tilde{m}_F \|J_\beta(u_h - s_h)\|_{0,F}$. For $F \subset \partial T$, it can be shown that for all $v_h \in V_h$,

$$
\|v_h - \mathcal{I}_{Os}(v_h)\|_{0,F} \lesssim \sum_{F'\in\mathcal{F}_h, F'\cap F\neq\emptyset} \|[\![v_h]\!]\|_{0,F'}.
$$

Applying this estimate with $v_h := u_h$, the conclusion is straightforward.

To illustrate by a simple example, assume that β and μ are of order unity, that β is solenoidal (or that its divergence is uniformly bounded by $\tilde{\mu}$ locally), and that the diffusion is homogeneous and isotropic, i.e., $K = \epsilon I_d$ with real parameter $0 < \epsilon \leq 1$ and where I_d denotes the identity matrix in $\mathbb{R}^d$. Then, $m_T = \min(h_T\epsilon^{-\frac{1}{2}}, 1)$, $\alpha_{1,T} = 1 + \epsilon^{-\frac{1}{2}}$, $\alpha_{2,T} = 1 + h_T^{-1}$, and it is readily verified that all the constants appearing in the upper bounds for η_T, ζ_T, and ι_T are of the form $(1 + \epsilon^{-\frac{1}{2}} \min(h_T\epsilon^{-\frac{1}{2}}, 1))$, which corresponds to the result derived in [38] for continuous finite elements with vanishing, isotropic, and homogeneous diffusion.

4. Numerical results

In this section, the present a posteriori error estimators are assessed on two test cases. The first one is a pure diffusion problem with heterogeneous isotropic diffusion; its aim is to verify numerically the sharpness of the diffusion flux error indicator ζ_T when evaluated with the proper weights. The second test case is an advection–diffusion-reaction problem with homogeneous diffusion; its aim is to verify the behavior of the a posteriori error estimates in the low- and high-Péclet regimes. We have always taken $\varpi = 4$ and $\theta = 1$ in (2.11) and (2.10), respectively. The corresponding dG method is the so-called Symmetric Weighted Interior Penalty method analyzed recently in [23]. Moreover, we have set $p = 1$, i.e., used piecewise linears. In all cases, the non-conforming error indicators have been evaluated using the standard Oswald interpolate of the discrete solution; see (3.22).

4.1. Heterogeneous diffusion

We consider the following test problem proposed in [33]. The domain $\Omega = (-1,1) \times (-1,1)$ is split into four subregions: $\Omega_1 = (0,1) \times (0,1)$, $\Omega_2 = (-1,0) \times (0,1)$, $\Omega_3 = (-1,0) \times (-1,0)$, and $\Omega_4 = (0,1) \times (-1,0)$. The source term f is zero. The diffusion tensor is isotropic, i.e., of the form $K = \epsilon_i I$ with constant value within each subregion. Letting $\epsilon_1 = \epsilon_3 = 100$ and $\epsilon_2 = \epsilon_4 = 1$, the exact solution written in polar coordinates is

$$u|_{\Omega_i} = r^\alpha \left(a_i \sin(\alpha\theta) + b_i \cos(\alpha\theta)\right), \tag{4.1}$$

with $\alpha = 0.12690207$ and

$$\begin{aligned} a_1 &= 0.100000000 & b_1 &= 1.000000000, \\ a_2 &= -9.603960396 & b_2 &= 2.960396040, \\ a_3 &= -0.480354867 & b_3 &= -0.882756593, \\ a_4 &= 7.701564882 & b_4 &= -6.456461752. \end{aligned}$$

Non-homogeneous Dirichlet boundary conditions as given by (4.1) are enforced on $\partial\Omega$. The exact solution possesses a singularity at the origin, and its regularity depends on the constant α, namely $u \in H^\alpha(\Omega)$. The expected convergence order of the error in the L^2-norm is 2α, while the expected convergence order in the energy norm is α. Table 4.1 presents the results on a series of quasi-uniform unstructured triangulations (that are compatible with the above partition of the domain Ω). The last line of this table displays the convergence orders evaluated on the last two meshes. The convergence orders for the error both in the L^2-norm and in the energy norm are in good agreement with the theoretical predictions. The same conclusion is reached for the a posteriori error estimators based on ζ_T and ι_T (observe that in the present case, $\eta_T = 0$ because $f = 0$ and $p = 1$). Note that $\|u_h - s_h\|_B$ is actually lower than the actual error norm $\|u - u_h\|_B$, which indicates that although the lower bound (3.26) can be invoked to guarantee the efficiency of the non-conforming error estimators, there may be functions in $V \cap V_h$ (here the Oswald interpolate of the discrete solution) that are actually closer to the discrete solution than is the exact solution. Furthermore, the column labelled "est" in Table 4.1 reports the total a posteriori error estimator derived in Theorem 3.1, and the column labelled "eff" reports the efficiency of the estimator, namely the ratio of the a posteriori error estimator to the actual approximation error. The efficiency is about 4 on all meshes. Notice that all

Table 4.1: Heterogeneous diffusion with parameter $\alpha = 0.13$

h	$\|u-u_h\|_{0,\Omega}$	$\|u-u_h\|_B$	$(\sum_{T\in\mathcal{T}_h}\zeta_T^2)^{\frac{1}{2}}$	$(\sum_{T\in\mathcal{T}_h}\iota_T^2)^{\frac{1}{2}}$	est.	eff.
9.43e-2	4.99e-2	12.37	37.58	11.25	48.83	3.9
4.71e-2	4.24e-2	11.43	35.42	10.60	46.01	4.0
2.36e-2	3.63e-2	10.52	33.24	9.94	43.18	4.1
order	0.22	0.12	0.09	0.09	0.09	–

Table 4.2: Heterogeneous diffusion with parameter $\alpha = 0.54$

h	$\|u-u_h\|_{0,\Omega}$	$\|u-u_h\|_B$	$(\sum_{T\in\mathcal{T}_h}\zeta_T^2)^{\frac{1}{2}}$	$(\sum_{T\in\mathcal{T}_h}\iota_T^2)^{\frac{1}{2}}$	est.	eff.
9.43e-2	2.35e-3	1.06e-0	5.78	3.48e-1	6.13	5.8
4.71e-2	8.29e-4	8.29e-1	4.12	2.40e-1	4.36	5.3
2.36e-2	2.95e-4	6.17e-1	2.93	1.66e-1	3.10	5.0
order	1.5	0.43	0.49	0.53	0.53	–

the constants in the estimators are explicitly evaluated. It is interesting to compare the results of Table 4.1 to those obtained using the more conventional dG method based on arithmetic averages (i.e., weights equal to $\frac{1}{2}$ on all faces) and a penalty term $\gamma_{K,F}$ equal to the arithmetic mean of the normal diffusivities on each face. In this case, the efficiency is equal to 28, i.e., 7 times larger.

We have also examined a similar test case with a less singular solution corresponding to milder contrasts in the diffusion, namely $\epsilon_1 = \epsilon_3 = 5$ and $\epsilon_2 = \epsilon_4 = 1$. In this case, the exact solution is still given by (4.1) with $\alpha = 0.53544095$ and

$$\begin{aligned} a_1 &= 0.44721360 & b_1 &= 1.00000000,\\ a_2 &= -0.74535599 & b_2 &= 2.33333333,\\ a_3 &= -0.94411759 & b_3 &= 0.55555556,\\ a_4 &= -2.40170264 & b_4 &= -0.48148148. \end{aligned}$$

Table 4.2 presents the results. The conclusions are similar to those reached with the previous test case. The efficiency is between 5 and 6 on all meshes, and thus takes comparable values to those taken in the previous test case, confirming the robustness of the estimates with respect to diffusion heterogeneities. If the more conventional dG method with arithmetic averages is used instead, the efficiencies are about 7, hinting at a dependency on diffusion heterogeneities.

4.2. Advection-diffusion-reaction

Consider the domain $\Omega = (0,1)\times(0,1)$, the advection field $\beta = (1,0)^t$, the reaction coefficient $\mu = 1$, and an isotropic homogeneous diffusion tensor $K = \epsilon I$. We run tests with $\epsilon = 1$ and $\epsilon = 10^{-4}$ to examine the difference between dominant diffusion and dominant advection regimes. Since the diffusion is homogeneous and isotropic, the SWIP method coincides with the more conventional Interior Penalty dG method. The source term f is designed so that the exact solution is

$$u(x,y) = 0.5\left(1 - \tanh\left(\frac{0.5-x}{\gamma}\right)\right). \tag{4.2}$$

Table 4.3: Advection-diffusion with $\epsilon = 1$

h	$\|u-u_h\|_B$	$(\sum_{T\in\mathcal{T}_h}\eta_T^2)^{\frac{1}{2}}$	$(\sum_{T\in\mathcal{T}_h}\zeta_T^2)^{\frac{1}{2}}$	$(\sum_{T\in\mathcal{T}_h}\iota_T^{'2})^{\frac{1}{2}}$	$(\sum_{T\in\mathcal{T}_h}\iota_T^{''2})^{\frac{1}{2}}$	est.	eff.
9.43e-2	7.47e-1	3.45e-0	2.37e-0	5.64e-2	9.83e-2	5.92	7.9
4.71e-2	4.04e-1	1.74e-0	1.05e-1	1.79e-2	7.07e-2	2.87	7.1
2.36e-2	2.05e-1	8.69e-1	3.83e-1	3.64e-3	3.37e-2	1.29	6.3
order	0.98	1.00	1.45	2.30	1.09	1.18	–

Table 4.4: Advection-diffusion with $\epsilon =$ 1e-4

h	$\|u-u_h\|_B$	$(\sum_{T\in\mathcal{T}_h}\eta_T^2)^{\frac{1}{2}}$	$(\sum_{T\in\mathcal{T}_h}\zeta_T^2)^{\frac{1}{2}}$	$(\sum_{T\in\mathcal{T}_h}\iota_T^{'2})^{\frac{1}{2}}$	$(\sum_{T\in\mathcal{T}_h}\iota_T^{''2})^{\frac{1}{2}}$	est.	eff.
9.43e-2	2.57e-2	6.62e-1	8.46e-3	1.93e-0	9.40e-3	2.55e-0	99
4.71e-2	9.34e-3	4.09e-1	7.64e-3	1.29e-0	3.63e-3	1.70e-0	181
2.36e-2	2.96e-3	2.08e-1	5.22e-3	6.26e-1	1.40e-3	8.36e-1	283
order	1.66	0.98	0.55	1.04	0.82	1.02	–

Here, the parameter $\gamma = 0.05$ controls the thickness of the internal layer at $x = 0.5$. On the left and right boundaries of Ω ($x = 0$ and $x = 1$), non-homogeneous Dirichlet boundary conditions as given by (4.2) are enforced, while on the lower and upper boundaries ($y = 0$ and $y = 1$), homogeneous Neumann conditions are enforced.

In Table 4.3 we present the results for the dominant diffusion regime. The estimator and the error converge at the same order, and the global efficiency is comparable with that obtained for a pure diffusion problem. The dominant contributions to the total a posteriori error estimate are the residue and the diffusive flux error indicators. When the advection becomes dominant, the error $\|u-u_h\|_B$ converges at 1.5 (because it is dominated by the L^2-contribution), while the total a posteriori error estimate (see column labelled "est") maintains the order of convergence equal to one, as can be seen in Table 4.4. This is because the cut-off coefficients m_T and the like are equal to one with dominant advection. As a result, the global efficiency increases (roughly as $h^{-\frac{1}{2}}$) as the mesh is refined. The trend will only be reversed once the mesh is sufficiently fine to resolve the diffusion. We notice that the dominant error indicators here are the non-conforming error indicator $\iota_T^{'}$ and the residue η_T, as expected.

5. Conclusions

In this work, we have proposed and analyzed a posteriori energy-norm error estimates for weighted interior penalty dG approximations to advection-diffusion-reaction equations with heterogeneous and anisotropic diffusion. All the constants in the error upper bounds have been specified, so that the present estimates can be used for actual control over the error in practical simulations. Local lower error bounds in which all the dependencies on model parameters are explicitly stated, have been derived as well. In the case of pure diffusion, full robustness is achieved with respect to diffusion heterogeneities owing to the use of suitable diffusion-dependent weights to formulate the consistency terms in the dG method. This feature has been verified numerically and stands in contrast to the results obtained with more conventional interior penalty dG approximations. Furthermore, diffusion anisotropies enter the lower error bounds only through the square root of the condition number of the diffusion tensor on a given mesh cell and its neighbors. The current state-of-the-art available results have been used to

evaluate the non-conforming error estimators through the use of so-called Oswald interpolates; further work in this direction is needed to investigate the robustness with respect to diffusion heterogeneities and anisotropies. In the presence of advection, we have shown, in the spirit of the work of Verfürth for continuous finite element approximations with SUPG stabilization, that the lower error bounds involve constants that are bounded by the square root of the local Péclet numbers.

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