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## CHAOTICITY FOR MULTI-CLASS SYSTEMS AND EXCHANGEABILITY WITHIN CLASSES

CARL GRAHAM,\* *École Polytechnique, CNRS*

Email address: carl@cmmapx.polytechnique.fr

### Abstract

We define a natural partial exchangeability assumption for multi-class systems with Polish state spaces, under which we obtain results extending those for exchangeable systems: the conditional law of a finite system given the vector of the empirical measures of its classes corresponds to *independent* uniform permutations within classes, and the convergence in law of this vector is *equivalent* to that of the system. A corollary is that convergence within *each* class to an infinite i.i.d. system implies asymptotic independence between *different* classes. We also extend the Hewitt-Savage 0-1 Law.

*Keywords:* Interacting particle systems; multi-class; multi-type; multi-species; mixtures; partial exchangeability; convergence of empirical measures; de Finetti Theorem; chaoticity; Hewitt-Savage 0-1 Law

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Secondary 60B10; 60G09; 62B05

### 1. Introduction

Among others, Kallenberg [9], Kingman [10], Diaconis and Freedman [5] and Aldous [1] study exchangeable random variables (r.v.) with Polish state space. The related notion of chaoticity (convergence in law to i.i.d. random variables) appears in many contexts, such as statistical estimation and asymptotics for interacting particle systems or communication networks. It is behind many fruitful heuristics, such as the “molecular chaos assumption” (*Stosszahlansatz*) used by Ludwig Boltzmann to derive the Boltzmann equation, see Cercignani *et al.* [4, Sect. 2, 4].

A sequence of exchangeable systems with increasing finite sizes converges in law

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\* Postal address: CMAP, École Polytechnique, CNRS, 91128 Palaiseau France.

if and only if the empirical measures converge to the directing measure of the limit infinite system, given by the de Finetti Theorem. Hence, chaoticity is equivalent to the fact that the empirical measures satisfy a weak law of large numbers, for which A.S. Sznitman developed a compactness-uniqueness method of proof yielding propagation of chaos results for varied models of interest. See Sznitman [13] for a survey and Méléard [11] and Graham [7, 6] for some developments.

Many systems in stratified sampling, statistical mechanics, chemistry, communication networks, biology, etc., involve *dissimilar* objects (which we call particles) classified in a certain number of types, particles of a class being similar and *numerous*. See for instance Cercignani *et al.* [4] (“Mixtures”, Subject index p. 454) and the review papers [2, 7, 8, 12] in a recent book.

The present paper studies natural notions of multi-exchangeability and chaoticity for multi-class systems, which appeared explicitly in Graham [6, pp. 78, 81] without names. We have found no previous such explicit definition, and [4, 2, 8, 12] directly extend the *limit* equations for systems of identical particles to multi-class models.

We prove that the conditional law of a finite multi-exchangeable system given the vector of the empirical measures of the classes corresponds to choosing *independent* uniform permutations within classes, and that the convergence in law of this vector is *equivalent* to that of the system.

As a corollary, for a multi-exchangeable system chaoticity *within* classes implies asymptotic independence *between* classes. This striking result is a major motivation for this paper, since it allows rigorous derivation of limit models from particle dynamics by use of Sznitman’s compactness-uniqueness methods.

We conclude with an extension of a result implying the Hewitt-Savage 0-1 Law.

We state as a “Proposition” any known result, and “Theorem” any result we believe to be new. All state spaces  $\mathcal{S}$  are Polish, and the weak topology is used for the space of probability measures  $\mathcal{P}(\mathcal{S})$  which is then also Polish, and so are products of Polish spaces. For  $k \geq 1$  we denote by  $\Sigma(k)$  the set of permutations of  $\{1, \dots, k\}$ .

## 2. Some classical results

### 2.1. Finite and infinite exchangeable systems

For  $N \geq 1$ , a finite system  $(X_n^N)_{1 \leq n \leq N}$  of random variables (r.v.) with state space  $\mathcal{S}$  is *exchangeable* if

$$\mathcal{L}(X_{\sigma(1)}^N, \dots, X_{\sigma(N)}^N) = \mathcal{L}(X_1^N, \dots, X_N^N), \quad \sigma \in \Sigma(N).$$

The conditional law of such a system given its empirical distribution

$$\Lambda^N = \frac{1}{N} \sum_{n=1}^N \delta_{X_n^N}$$

corresponds to a uniform ordering of the  $N$  (possibly repeated) values occurring in  $\Lambda^N$  (its atoms, counted according to their multiplicity), see Aldous [1, Lemma 5.4 p. 38].

An *infinite* system  $(X_n)_{n \geq 1}$  is *exchangeable* if every finite subsystem  $(X_n)_{1 \leq n \leq N}$  is exchangeable. The de Finetti Theorem, see *e.g.* [9, 10, 5, 1], states that such a system is a mixture of i.i.d. sequences: its law is of the form

$$\int P^{\otimes \infty} \mathcal{L}_\Lambda(dP)$$

where  $\mathcal{L}_\Lambda$  is the law of the (random) *directing measure*  $\Lambda$  which can be obtained as

$$\Lambda = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \delta_{X_n} \quad \text{a.s.} \quad (1)$$

These facts lead to the following.

**Proposition 1.** *Let  $(X_n^N)_{1 \leq n \leq N}$  for  $N \geq 1$  be finite exchangeable systems. This sequence converges in law to  $(X_n)_{n \geq 1}$ , which is necessarily infinite exchangeable and thus has a directing measure  $\Lambda$ , if and only if*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \delta_{X_n^N} = \Lambda \quad \text{in law.}$$

For proof, see Kallenberg [9, Theorem 1.3 p. 25] and Aldous [1, Prop. 7.20 p. 55]. The proof in the latter uses [1, Prop. 5.6 p. 39], also Diaconis and Freedman [5, Theorem 13 p. 749] where it was used for proving the de Finetti Theorem.

The sequence  $(X_n^N)_{1 \leq n \leq N}$  for  $N \geq 1$  is *P-chaotic*, where  $P \in \mathcal{P}(\mathcal{S})$ , if

$$\lim_{N \rightarrow \infty} \mathcal{L}(X_1^N, \dots, X_k^N) = P^{\otimes k}, \quad k \geq 1,$$

or equivalently, converges in law to an i.i.d. system of r.v. of law  $P$ . This limit is infinite exchangeable with directing measure  $P$ , and Proposition 1 has the following immediate corollary.

**Proposition 2.** *Let  $(X_n^N)_{1 \leq n \leq N}$  for  $N \geq 1$  be finite exchangeable systems. This sequence is  $P$ -chaotic, where  $P \in \mathcal{P}(\mathcal{S})$ , if and only if*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \delta_{X_n^N} = P \text{ in law}$$

and hence in probability (since the limit is deterministic).

Direct proofs can be found in [13, Prop. 2.2 p. 177] and [11, Prop. 4.2 p. 66].

## 2.2. Multi-exchangeable systems

For a  $C \geq 1$  and multi-index  $\mathbf{N} = (N_i)_{1 \leq i \leq C} \in \mathbb{N}^C$  we consider a multi-class system

$$(X_{n,i}^{\mathbf{N}})_{1 \leq n \leq N_i, 1 \leq i \leq C}, \quad X_{n,i}^{\mathbf{N}} \text{ with state space } \mathcal{S}_i,$$

where  $X_{n,i}^{\mathbf{N}}$  is the  $n$ -th particle, or object, of class  $i$ . We say that this system is *multi-exchangeable* if its law is invariant under permutation of the particles *within* classes:

$$\mathcal{L}((X_{\sigma_i(n),i}^{\mathbf{N}})_{1 \leq n \leq N_i, 1 \leq i \leq C}) = \mathcal{L}((X_{n,i}^{\mathbf{N}})_{1 \leq n \leq N_i, 1 \leq i \leq C}), \quad \sigma_i \in \Sigma(N_i).$$

This natural assumption means that particles of a class are statistically indistinguishable, and implies that  $(X_{n,i}^{\mathbf{N}})_{1 \leq n \leq N_i}$  is exchangeable for  $1 \leq i \leq C$ . It is sufficient to check that it is true when all  $\sigma_i$  but one are the identity.

We say that the multi-class system  $(X_{n,i})_{n \geq 1, 1 \leq i \leq C}$  with *infinite* classes is *multi-exchangeable* if every finite sub-system  $(X_{n,i})_{1 \leq n \leq N, 1 \leq i \leq C}$  is multi-exchangeable. The following remarkable result is attributed to de Finetti by Aldous [1, Cor. 3.9 p. 25].

**Proposition 3.** *Let  $(X_{n,i})_{n \geq 1, 1 \leq i \leq C}$  be an infinite multi-exchangeable system, and  $\Lambda_i$  be the directing measure of  $(X_{n,i})_{n \geq 1}$ . Given  $(\Lambda_i)_{1 \leq i \leq C}$ , the  $X_{n,i}$  are conditionally independent and have regular conditional distributions  $\Lambda_i$ .*

## 3. The extended results

We shall extend to multi-exchangeable systems the main results for exchangeable systems, even though the assumption and resulting structure is much weaker, since for

instance the respective symmetry orders are  $N_1! \cdots N_C! \ll (N_1 + \cdots + N_C)! = N!$ .

A surprising fact in the following extension of [1, Lemma 5.4 p. 38] (stated in words at the beginning of Section 2) is that for a finite multi-exchangeable system, the classes are *conditionally independent* given the empirical measures of each class, and *no further information* can be attained on its law by mixing astutely what happens in different classes. The empirical measure vector is a *sufficient statistic* for the law of the system, the family of all such laws being trivially parameterized by the laws themselves.

**Theorem 1.** *Let  $(X_{n,i}^{\mathbf{N}})_{1 \leq n \leq N_i, 1 \leq i \leq C}$  be a finite multi-exchangeable system. Then its conditional law given the empirical measure vector*

$$(\Lambda_i^{\mathbf{N}})_{1 \leq i \leq C}, \quad \Lambda_i^{\mathbf{N}} = \frac{1}{N_i} \sum_{n=1}^{N_i} \delta_{X_{n,i}^{\mathbf{N}}},$$

*corresponds to  $C$  independent uniform orderings of the  $N_i$  (possibly repeated) particles of class  $i$  which are the atoms of  $\Lambda_i^{\mathbf{N}}$  (counted with their multiplicities).*

*Proof.* Multi-exchangeability implies that for  $g, f_i \geq 0$  we have

$$\begin{aligned} & \mathbf{E} \left[ g \left( (\Lambda_j^{\mathbf{N}})_{1 \leq j \leq C} \prod_{i=1}^C f_i(X_{1,i}^{\mathbf{N}}, \dots, X_{N_i,i}^{\mathbf{N}}) \right) \right] \\ &= \frac{1}{\prod_{k=1}^C N_k!} \sum_{\substack{\sigma_k \in \Sigma(N_k) \\ \text{for } 1 \leq k \leq C}} \mathbf{E} \left[ g \left( \left( \frac{1}{N_j} \sum_{n=1}^{N_j} \delta_{X_{\sigma_j(n),j}^{\mathbf{N}}} \right)_{1 \leq j \leq C} \right) \prod_{i=1}^C f_i(X_{\sigma_i(1),i}^{\mathbf{N}}, \dots, X_{\sigma_i(N_i),i}^{\mathbf{N}}) \right) \right] \\ &= \mathbf{E} \left[ g \left( (\Lambda_j^{\mathbf{N}})_{1 \leq j \leq C} \prod_{i=1}^C \frac{1}{N_i!} \sum_{\sigma \in \Sigma(N_i)} f_i(X_{\sigma(1),i}^{\mathbf{N}}, \dots, X_{\sigma(N_i),i}^{\mathbf{N}}) \right) \right] \\ &= \mathbf{E} \left[ g \left( (\Lambda_j^{\mathbf{N}})_{1 \leq j \leq C} \prod_{i=1}^C \left\langle f_i, \frac{1}{N_i!} \sum_{\sigma \in \Sigma(N_i)} \delta_{X_{\sigma(1),i}^{\mathbf{N}}, \dots, X_{\sigma(N_i),i}^{\mathbf{N}}} \right\rangle \right) \right] \end{aligned} \quad (2)$$

where the empirical measure  $\frac{1}{N_i!} \sum_{\sigma \in \Sigma(N_i)} \delta_{X_{\sigma(1),i}^{\mathbf{N}}, \dots, X_{\sigma(N_i),i}^{\mathbf{N}}}$  is a function of  $\Lambda_i^{\mathbf{N}}$ , since it corresponds to exhaustive draws without replacement among the  $X_{1,i}^{\mathbf{N}}, \dots, X_{N_i,i}^{\mathbf{N}}$  which are the atoms of  $\Lambda_i^{\mathbf{N}}$  counted according to multiplicity (there are algebraic formulas, see the proof of Theorem 2 below for a step in that direction). The characteristic property of conditional expectation yields

$$\mathbf{E} \left[ \prod_{i=1}^C f_i(X_{1,i}^{\mathbf{N}}, \dots, X_{N_i,i}^{\mathbf{N}}) \mid (\Lambda_i^{\mathbf{N}})_{1 \leq i \leq C} \right] = \prod_{i=1}^C \left\langle f_i, \frac{1}{N_i!} \sum_{\sigma \in \Sigma(N_i)} f_i(X_{\sigma(1),i}^{\mathbf{N}}, \dots, X_{\sigma(N_i),i}^{\mathbf{N}}) \right\rangle$$

which implies, classically for such Polish state spaces, that the conditional law of  $(X_{n,i}^{\mathbf{N}})_{1 \leq n \leq N_i, 1 \leq i \leq C}$  given  $(\Lambda_i^{\mathbf{N}})_{1 \leq i \leq C}$  is the one stated.

The following extension of Proposition 1 is again remarkable in that we do not have to mix what happens in different classes. It follows essentially from Theorem 1 and its proof and Proposition 3 by regularity considerations, and implies the same for families of infinite multi-exchangeable systems.

We say that  $\mathbf{N} = (N_i)_{1 \leq i \leq C}$  goes to infinity when  $\min_{1 \leq i \leq C} N_i$  goes to infinity.

**Theorem 2.** *Consider the family of finite multi-exchangeable systems*

$$(X_{n,i}^{\mathbf{N}})_{1 \leq n \leq N_i, 1 \leq i \leq C}, \quad \mathbf{N} \in \mathbb{N}^C, \quad (3)$$

with empirical measures  $(\Lambda_i^{\mathbf{N}})_{1 \leq i \leq C}$  given in Theorem 1. This family converges in law as  $\mathbf{N}$  goes to infinity to  $(X_{n,i})_{n \geq 1, 1 \leq i \leq C}$ , which is necessarily infinite multi-exchangeable and has directing measure vector  $(\Lambda_i)_{1 \leq i \leq C}$ , if and only if

$$\lim_{\mathbf{N} \rightarrow \infty} (\Lambda_i^{\mathbf{N}})_{1 \leq i \leq C} = (\Lambda_i)_{1 \leq i \leq C} \text{ in law.}$$

*Proof.* Let  $(m)_k = \frac{m!}{(m-k)!} = m(m-1) \cdots (m-k+1)$  for  $m, k \geq 1$ . For bounded continuous  $f_i$  on  $\mathcal{S}_i^k$ , consideration of (2) with  $g = 1$  and the extensions of  $f_i$  on  $\mathcal{S}_i^{N_i}$  depending only on the first  $k$  coordinates yields

$$\mathbf{E} \left[ \prod_{i=1}^C f_i(X_{1,i}^{\mathbf{N}}, \dots, X_{k,i}^{\mathbf{N}}) \right] = \mathbf{E} \left[ \prod_{i=1}^C \left\langle f_i, \frac{1}{(N_i)_k} \sum_{\substack{1 \leq n_1, \dots, n_k \leq N_i \\ \text{distinct}}} \delta_{X_{n_1,i}^{\mathbf{N}}, \dots, X_{n_k,i}^{\mathbf{N}}} \right\rangle \right].$$

The empirical measures for distinct  $k$ -tuples in each class, appearing in the last term, correspond to sampling without replacement, which is asymptotically equivalent to sampling with replacement: more precisely,

$$(\Lambda_i^{\mathbf{N}})^{\otimes k} = \frac{1}{N_i^k} \sum_{\substack{1 \leq n_1, \dots, n_k \leq N_i \\ \text{distinct}}} \delta_{X_{n_1,i}^{\mathbf{N}}, \dots, X_{n_k,i}^{\mathbf{N}}} + \frac{1}{N_i^k} \sum_{\substack{1 \leq n_1, \dots, n_k \leq N_i \\ \text{not distinct}}} \delta_{X_{n_1,i}^{\mathbf{N}}, \dots, X_{n_k,i}^{\mathbf{N}}}$$

so that, in total variation norm  $\|\mu\| = \sup\{\langle \phi, \mu \rangle : \|\phi\|_{\infty} \leq 1\}$ ,

$$\left\| (\Lambda_i^{\mathbf{N}})^{\otimes k} - \frac{1}{(N_i)_k} \sum_{\substack{1 \leq n_1, \dots, n_k \leq N_i \\ \text{distinct}}} \delta_{X_{n_1,i}^{\mathbf{N}}, \dots, X_{n_k,i}^{\mathbf{N}}} \right\| \leq 2 \frac{N_i^k - (N_i)_k}{N_i^k} \leq \frac{k(k-1)}{N_i}$$

where we bound  $N_i^k - (N_i)_k$  by counting  $k(k-1)/2$  possible positions for two identical indices with  $N_i$  choices and  $N_i^{k-2}$  choices for the other  $k-2$  positions.

Hence, with bounds depending only on the  $\|f_i\|_\infty$  for  $1 \leq i \leq C$ , we have

$$\mathbf{E} \left[ \prod_{i=1}^C f_i(X_{1,i}^{\mathbf{N}}, \dots, X_{k,i}^{\mathbf{N}}) \right] = \mathbf{E} \left[ \prod_{i=1}^C \langle f_i, (\Lambda_i^{\mathbf{N}})^{\otimes k} \rangle \right] + O \left( \frac{k(k-1)}{\min_{1 \leq i \leq C} N_i} \right). \quad (4)$$

If  $\lim_{\mathbf{N} \rightarrow \infty} (\Lambda_i^{\mathbf{N}})_{1 \leq i \leq C} = (\Lambda_i)_{1 \leq i \leq C}$  then continuity and Proposition 3 imply

$$\lim_{\mathbf{N} \rightarrow \infty} \mathbf{E} \left[ \prod_{i=1}^C f_i(X_{1,i}^{\mathbf{N}}, \dots, X_{k,i}^{\mathbf{N}}) \right] = \mathbf{E} \left[ \prod_{i=1}^C \langle f_i, \Lambda_i^{\otimes k} \rangle \right] = \mathbf{E} \left[ \prod_{i=1}^C f_i(X_{1,i}, \dots, X_{k,i}) \right]$$

so that, since the state spaces are Polish,

$$\lim_{\mathbf{N} \rightarrow \infty} \mathcal{L}((X_{1,i}^{\mathbf{N}}, \dots, X_{k,i}^{\mathbf{N}})_{1 \leq i \leq C}) = \mathcal{L}((X_{1,i}, \dots, X_{k,i})_{1 \leq i \leq C}), \quad k \geq 1,$$

and  $(X_{n,i}^{\mathbf{N}})_{1 \leq n \leq N_i, 1 \leq i \leq C}$  converges in law to  $(X_{n,i})_{n \geq 1, 1 \leq i \leq C}$ . Conversely, this convergence, (4), continuity, and Proposition 3 imply that

$$\lim_{\mathbf{N} \rightarrow \infty} \mathbf{E} \left[ \prod_{i=1}^C \langle f_i, (\Lambda_i^{\mathbf{N}})^{\otimes k} \rangle \right] = \mathbf{E} \left[ \prod_{i=1}^C f_i(X_{1,i}, \dots, X_{k,i}) \right] = \mathbf{E} \left[ \prod_{i=1}^C \langle f_i, \Lambda_i^{\otimes k} \rangle \right]$$

and classical characterizations of convergence in law for r.v. with the Polish state space  $\otimes_{1 \leq i \leq C} \mathcal{P}(\mathcal{S}_i)$  imply that  $\lim_{\mathbf{N} \rightarrow \infty} (\Lambda_i^{\mathbf{N}})_{1 \leq i \leq C} = (\Lambda_i)_{1 \leq i \leq C}$ .

We have the following corollary, in which the striking fact is asymptotic independence between particles in *different* classes. We say that the family of finite multi-class systems (3) is  $(P_1, \dots, P_C)$ -chaotic, where  $P_i \in \mathcal{P}(\mathcal{S}_i)$ , if

$$\lim_{\mathbf{N} \rightarrow \infty} \mathcal{L}((X_{n,i}^{\mathbf{N}})_{1 \leq n \leq k, 1 \leq i \leq C}) = P_1^{\otimes k} \otimes \dots \otimes P_C^{\otimes k}, \quad k \geq 1.$$

This means that the multi-class systems converge to an *independent* system, in which particles of class  $i$  have law  $P_i$ .

**Theorem 3.** *The family of finite multi-exchangeable systems (3) is  $(P_1, \dots, P_C)$ -chaotic, where  $P_i \in \mathcal{P}(\mathcal{S}_i)$ , if and only if the  $(X_{n,i}^{\mathbf{N}})_{1 \leq n \leq N_i}$  are  $P_i$ -chaotic for  $1 \leq i \leq C$ .*

*Proof.* The “only if” result is obvious. Conversely, if the  $(X_{n,i}^{\mathbf{N}})_{1 \leq n \leq N_i}$  are  $P_i$ -chaotic, then Proposition 2 implies that  $\lim_{\mathbf{N} \rightarrow \infty} (\Lambda_i^{\mathbf{N}})_{1 \leq i \leq C} = (P_i)_{1 \leq i \leq C}$  in probability, and we conclude using Theorem 2 and Proposition 3.



**Remark 1.** Theorem 3 allows for the proof of  $(P_1, \dots, P_C)$ -chaoticity by use of Proposition 2 and Sznitman's compactness-uniqueness methods for proving that the empirical measures  $\Lambda_i^{\mathbf{N}}$  converge in law to  $P_i$  for  $1 \leq i \leq C$ . This was a main goal of this study.

We finish with the following extension of Aldous [1, Cor. 3.10 p. 26] and of the Hewitt-Savage 0-1 Law. For  $k \geq 1$ , we say that a set

$$B \subset \mathcal{S}_1^{\otimes \infty} \otimes \dots \otimes \mathcal{S}_C^{\otimes \infty}$$

is  $k$ -multi-exchangeable if

$$(x_{n,i})_{n \geq 1, 1 \leq i \leq C} \in B \Leftrightarrow (x_{\sigma_i(n),i})_{n \geq 1, 1 \leq i \leq C} \in B, \quad \sigma_i \in \Sigma(k),$$

and define the multi-exchangeable  $\sigma$ -algebra

$$\mathcal{E} = \bigcap_{k \geq 1} \mathcal{E}_k, \quad \mathcal{E}_k = \{ \{(X_{n,i})_{n \geq 1, 1 \leq i \leq C} \in B\} : B \text{ is } k\text{-multi-exchangeable} \},$$

and multi-tail  $\sigma$ -algebra

$$\mathcal{T} = \bigcap_{k \geq 1} \mathcal{T}_k, \quad \mathcal{T}_k = \sigma((X_{n,i})_{n \geq k, 1 \leq i \leq C}).$$

Clearly,  $\mathcal{T}_k \subset \mathcal{E}_k$  and hence  $\mathcal{T} \subset \mathcal{E}$ .

**Theorem 4.** *Let  $(X_{n,i})_{n \geq 1, 1 \leq i \leq C}$  be an infinite multi-exchangeable system, and  $\Lambda_i$  be the directing measure of  $(X_{n,i})_{n \geq 1}$ . Then*

$$\sigma((\Lambda_i)_{1 \leq i \leq C}) = \mathcal{T} = \mathcal{E} \text{ a.s.}$$

*If moreover the  $X_{n,i}$  are independent, then  $A \in \mathcal{E} \Rightarrow P(A) \in \{0, 1\}$ .*

*Proof.* Consideration of (1) yields  $\sigma((\Lambda_i)_{1 \leq i \leq C}) \subset \mathcal{T}$ , a.s., and we have seen that  $\mathcal{T} \subset \mathcal{E}$ . Hence, the first statement is true if  $\mathcal{E} \subset \sigma((\Lambda_i)_{1 \leq i \leq C})$ , a.s.

For all  $k \geq 1$  and  $\sigma_i \in \Sigma(k)$  for  $1 \leq i \leq C$  and  $A = \{(X_{n,j})_{n \geq 1, 1 \leq j \leq C} \in B\} \in \mathcal{E}$ , where  $B$  is multi-exchangeable, this last property implies

$$(\mathbf{1}_A, X_{\sigma_i(n),i})_{n \geq 1, 1 \leq i \leq C} = (\mathbf{1}_{\{(X_{\sigma_j(n),j})_{n \geq 1, 1 \leq j \leq C} \in B\}}, X_{\sigma_i(n),i})_{n \geq 1, 1 \leq i \leq C}$$

so that the multi-exchangeability of  $(X_{n,i})_{n \geq 1, 1 \leq i \leq C}$  implies

$$\mathcal{L}((\mathbf{1}_A, X_{\sigma_i(n),i})_{n \geq 1, 1 \leq i \leq C}) = \mathcal{L}((\mathbf{1}_A, X_{n,i})_{n \geq 1, 1 \leq i \leq C}).$$

Hence  $(\mathbf{1}_A, X_{n,i})_{n \geq 1, 1 \leq i \leq C}$  is infinite multi-exchangeable, and Proposition 3 implies that the  $(\mathbf{1}_A, X_{n,i})$  are conditionally independent given  $(\hat{\Lambda}_i)_{1 \leq i \leq C}$  and have conditional laws  $\hat{\Lambda}_i$  given by

$$\hat{\Lambda}_i = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \delta_{\mathbf{1}_A, X_{n,i}} = \delta_{\mathbf{1}_A} \otimes \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \delta_{X_{n,i}} = \delta_{\mathbf{1}_A} \otimes \Lambda_i \text{ a.s.},$$

so that for  $k \geq 1$  and Borel sets  $B_{n,i} \subset \mathcal{S}_i$  we have

$$\mathbf{P}(X_{n,i} \in B_{n,i} : 1 \leq n \leq k, 1 \leq i \leq C \mid A, (\Lambda_i)_{1 \leq i \leq C}) = \prod_{1 \leq n \leq k, 1 \leq i \leq C} \Lambda_i(B_{n,i})$$

which is a function of  $(\Lambda_i)_{1 \leq i \leq C}$ , and thus  $\mathcal{E}$  and  $(X_{n,i})_{n \geq 1, 1 \leq i \leq C}$  are conditionally independent given  $(\Lambda_i)_{1 \leq i \leq C}$ . Since  $\mathcal{E} \subset \sigma((X_{n,i})_{n \geq 1, 1 \leq i \leq C})$  this can only happen if  $\mathcal{E} \subset \sigma((\Lambda_i)_{1 \leq i \leq C})$ , a.s., and we conclude to the first statement.

The Kolmogorov 0-1 Law yields that  $\mathcal{T}$  is a.s. trivial if the  $X_{n,i}$  are independent.

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