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► **To cite this version:**

Carl Graham. Chaoticity for multi-class systems and exchangeability within classes. 2008. hal-00171668v2

**HAL Id: hal-00171668**

**<https://hal.science/hal-00171668v2>**

Preprint submitted on 27 Feb 2008 (v2), last revised 16 Oct 2008 (v4)

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## CHAOTICITY FOR MULTI-CLASS SYSTEMS AND EXCHANGEABILITY WITHIN CLASSES

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### Abstract

We define a natural partial exchangeability assumption for multi-class systems with Polish state spaces, under which we extend several results for exchangeable systems. We prove that convergence within *each* class to an infinite i.i.d. system implies asymptotic independence between *different* classes, that the vector of the empirical measures of the classes is a sufficient statistic for the law of the system, that the convergence in law of this vector is equivalent to that of the system, and an extension of the Hewitt-Savage 0-1 Law.

*Keywords:* Interacting particles; multi-class, multi-type, or multi-species systems; mixtures; partial exchangeability; chaoticity; convergence of empirical measures; de Finetti Theorem; sufficient statistic; Hewitt-Savage 0-1 Law

2000 Mathematics Subject Classification: Primary 60K35

Secondary 60B10; 60G09; 62B05

### 1. Introduction

Kingman [9], Diaconis and Freedman [5] and Aldous [1] give many examples and results for exchangeable random variables (r.v.). The related notion of chaoticity (convergence in law to independent r.v.) appears in many contexts, such as estimation in statistics, asymptotic models for interacting particle systems, and approximations for invariant laws. It is behind many fruitful heuristics, such as the “molecular chaos assumption” (*Stosszahlansatz*) used by Ludwig Boltzmann in his derivation of the Boltzmann equation, see *e.g.* Cercignani *et al.* [4], notably Sections 2 and 4.

For exchangeable r.v. with Polish state space, chaoticity is equivalent to the fact that the empirical measures satisfy a weak law of large numbers. A.S. Sznitman used

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this fact to obtain propagation of chaos results for varied models of interest, by proving the corresponding laws of large numbers using compactness-uniqueness methods, see Sznitman [12] for a survey and [10, 7, 6] for some developments.

Many systems in stratified sampling, statistical mechanics, chemistry, communication networks, biology, etc., involve dissimilar objects which we call particles, which are classified in a finite number of types, particles of a class being similar and numerous. Some examples can be found in Cercignani *et al.* [4] (see the entry “Mixtures” of the Subject index p. 454) and the review papers [2, 7, 8, 11] in a recent book.

This paper defines natural notions of multi-exchangeability and chaoticity for multi-class systems, so as to extend several classical results in this framework. These notions appear explicitly in Graham [6, pp. 78, 81] (without names), but we have found no definitive source; the references we have read, such as [4, 2, 8, 11], directly extend the *limit* equations for systems of identical particles to multi-class models.

We prove that for a multi-exchangeable system, chaoticity *within* classes implies *global* chaoticity, a result which would allow rigorous derivation of limit models from particle dynamics by use of Sznitman’s compactness-uniqueness methods. We then prove that the vector of the empirical measures of the classes is a sufficient statistic for the law of such a system, and that convergence in law of this vector and of the system are equivalent. We conclude with an extension of the Hewitt-Savage 0-1 Law.

We state as “Proposition” any known result, and “Theorem” any result we believe is new. All state spaces are assumed to be Polish, and the weak topology is used for the space of probability measures, which is then also Polish.

## 2. Classical results for exchangeable systems

A system  $(X_n)_{1 \leq n \leq N}$  of random variables (r.v.) with state space  $\mathcal{S}$  is *exchangeable* if  $\mathcal{L}(X_{\sigma(1)}, \dots, X_{\sigma(N)}) = \mathcal{L}(X_1, \dots, X_N)$  for every permutation  $\sigma$  of  $\{1, \dots, N\}$ . The  $(X_n^N)_{1 \leq n \leq N}$  for  $N \geq 1$  are *P-chaotic* if  $\lim_{N \rightarrow \infty} \mathcal{L}(X_1^N, \dots, X_k^N) = P^{\otimes k}$  for all  $k \geq 1$ , *i.e.*, they converge in law to i.i.d. random variables of law  $P$  on  $\mathcal{S}$ . We recall a classical result, see [12, Prop. 2.2 p. 177] and [10, Prop. 4.2 p. 66].

**Proposition 1.** *Let  $(X_n^N)_{1 \leq n \leq N}$  for  $N \geq 1$  be exchangeable systems. These systems are P-chaotic if and only if  $\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \delta_{X_n^N} = P$  in law (and thus in probability).*

We now state the more general result Aldous [1, Prop. 7.20 p. 55]. An infinite system  $(X_n)_{n \geq 1}$  is exchangeable if every finite subsystem  $(X_n)_{1 \leq n \leq N}$  is exchangeable, and the de Finetti Theorem states that such a system is a mixture of i.i.d. sequences: its law is of the form  $\int P^{\otimes \infty} \mathcal{L}_\Lambda(dP)$  where  $\mathcal{L}_\Lambda$  is the law of the *directing measure*  $\Lambda$ , which can be obtained as  $\Lambda = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \delta_{X_n}$  a.s., see [9, 5, 1].

**Proposition 2.** *Let  $(X_n^N)_{1 \leq n \leq N}$  for  $N \geq 1$  be exchangeable systems. These systems converge in law to  $(X_n)_{n \geq 1}$ , which is necessarily exchangeable and has directing measure  $\Lambda$ , if and only if  $\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \delta_{X_n^N} = \Lambda$  in law.*

The proof of Aldous [1] uses [1, Prop. 5.6 p. 39], which in Diaconis and Freedman [5, Thm. 13 p. 749] is subsequently used for proving the de Finetti Theorem.

### 3. Multi-exchangeability and chaoticity

Let  $C \geq 1$  and  $\mathbf{N} = (N_i)_{1 \leq i \leq C} \in \mathbb{N}^C$  and  $(X_{n,i}^{\mathbf{N}})_{1 \leq n \leq N_i, 1 \leq i \leq C}$  be a multi-class system, where  $X_{n,i}^{\mathbf{N}}$  with state space  $\mathcal{S}_i$  is the  $n$ -th particle of class  $i$ . We say that this system is *multi-exchangeable* if its law is invariant under permutation of the particles *within* classes: for all permutations  $\sigma_i$  of  $\{1, \dots, N_i\}$  we have

$$\mathcal{L}((X_{\sigma_i(n),i}^{\mathbf{N}})_{1 \leq n \leq N_i, 1 \leq i \leq C}) = \mathcal{L}((X_{n,i}^{\mathbf{N}})_{1 \leq n \leq N_i, 1 \leq i \leq C}).$$

This expresses that particles of a class are statistically indistinguishable, and implies that  $(X_{n,i}^{\mathbf{N}})_{1 \leq n \leq N_i}$  is exchangeable for  $1 \leq i \leq C$ . It is sufficient for this property to hold that it be true when all  $\sigma_i$  but one are the identity.

We say that the multi-class systems  $(X_{n,i}^{\mathbf{N}})_{1 \leq n \leq N_i, 1 \leq i \leq C}$  are  $(P_1, \dots, P_C)$ -*chaotic*, where  $P_i$  is a law on  $\mathcal{S}_i$ , if  $\lim N_i = \infty$  for all  $i$  and

$$\lim \mathcal{L}((X_{n,i}^{\mathbf{N}})_{1 \leq n \leq k, 1 \leq i \leq C}) = P_1^{\otimes k} \otimes \dots \otimes P_C^{\otimes k} = \bigotimes_{1 \leq i \leq C} P_i^{\otimes k}, \quad k \geq 1,$$

*i.e.*, the multi-class systems converge to an *independent* system, in which particles of class  $i$  have law  $P_i$ . This implies that the  $(X_{n,i}^{\mathbf{N}})_{1 \leq n \leq N_i}$  are  $P_i$ -chaotic.

Even though multi-exchangeability is much weaker than exchangeability (the symmetry orders are  $N_1! \dots N_C! \ll (N_1 + \dots + N_C)! = N!$ ), we prove the following, in which asymptotic independence between particles in *different* classes is remarkable.

**Theorem 1.** *Let the  $(X_{n,i}^N)_{1 \leq n \leq N_i, 1 \leq i \leq C}$  be multi-exchangeable systems. These systems are  $(P_1, \dots, P_C)$ -chaotic if and only if the  $(X_{n,i}^N)_{1 \leq n \leq N_i}$  are  $P_i$ -chaotic,  $1 \leq i \leq C$ .*

*Proof.* The ‘‘only if’’ result is obvious. Conversely, let  $k \geq 1$  and  $f_i$  be in  $C_b(\mathcal{S}_i^k)$ . With the notation  $(m)_k = m(m-1) \cdots (m-k+1)$ , multi-exchangeability yields

$$\begin{aligned} \mathbf{E} \left[ \prod_{i=1}^C f_i(X_{1,i}^N, \dots, X_{k,i}^N) \right] &= \frac{1}{\prod_{j=1}^C (N_j)_k} \sum_{\substack{1 \leq n_{1,j}, \dots, n_{k,j} \leq N_j \\ \text{distinct}, 1 \leq j \leq C}} \mathbf{E} \left[ \prod_{i=1}^C f_i(X_{n_{1,i}}^N, \dots, X_{n_{k,i}}^N) \right] \\ &= \mathbf{E} \left[ \prod_{i=1}^C \frac{1}{(N_i)_k} \sum_{\substack{1 \leq n_1, \dots, n_k \leq N_i \\ \text{distinct}}} f_i(X_{n_1,i}^N, \dots, X_{n_k,i}^N) \right] \\ &= \mathbf{E} \left[ \prod_{i=1}^C \left\langle f_i, \frac{1}{(N_i)_k} \sum_{\substack{1 \leq n_1, \dots, n_k \leq N_i \\ \text{distinct}}} \delta_{X_{n_1,i}^N, \dots, X_{n_k,i}^N} \right\rangle \right] \end{aligned}$$

which involves the empirical measure for distinct  $k$ -tuples in each class  $i$ . This measure corresponds to sampling without replacement, which is asymptotically equivalent to sampling with replacement, since for  $\Lambda_i^N = \frac{1}{N_i} \sum_{n=1}^{N_i} \delta_{X_{n,i}^N}$  we have

$$(\Lambda_i^N)^{\otimes k} = \frac{1}{N_i^k} \sum_{\substack{1 \leq n_1, \dots, n_k \leq N_i \\ \text{distinct}}} \delta_{X_{n_1,i}^N, \dots, X_{n_k,i}^N} + \frac{1}{N_i^k} \sum_{\substack{1 \leq n_1, \dots, n_k \leq N_i \\ \text{not distinct}}} \delta_{X_{n_1,i}^N, \dots, X_{n_k,i}^N}$$

so that, in total variation norm  $\|\mu\| = \sup\{\langle \phi, \mu \rangle : \|\phi\|_\infty \leq 1\}$ ,

$$\left\| (\Lambda_i^N)^{\otimes k} - \frac{1}{(N_i)_k} \sum_{\substack{1 \leq n_1, \dots, n_k \leq N_i \\ \text{distinct}}} \delta_{X_{n_1,i}^N, \dots, X_{n_k,i}^N} \right\| \leq 2 \frac{N_i^k - (N_i)_k}{N_i^k} \leq \frac{k(k-1)}{N_i}$$

where we bound  $N_i^k - (N_i)_k$  by counting  $k(k-1)/2$  possible positions for two identical indices with  $N_i$  choices and  $N_i^{k-2}$  choices for the other  $k-2$  positions. Using chaoticity of the  $i$ -th class and Proposition 1, we obtain

$$\lim \left\langle f_i, \frac{1}{(N_i)_k} \sum_{\substack{1 \leq n_1, \dots, n_k \leq N_i \\ \text{distinct}}} \delta_{X_{n_1,i}^N, \dots, X_{n_k,i}^N} \right\rangle = \lim \langle f_i, (\Lambda_i^N)^{\otimes k} \rangle = \langle f_i, P_i^{\otimes k} \rangle$$

in law (and in probability), and boundedness implies

$$\lim \mathbf{E} \left[ \prod_{i=1}^C f_i(X_{1,i}^N, \dots, X_{k,i}^N) \right] = \prod_{i=1}^C \langle f_i, P_i^{\otimes k} \rangle$$

and hence  $\lim \mathcal{L}((X_{n,i}^N)_{1 \leq n \leq k, 1 \leq i \leq C}) = P_1^{\otimes k} \otimes \cdots \otimes P_C^{\otimes k}$  (the state spaces are Polish).

**Remark.** Theorem 1 allows for the proof of  $(P_1, \dots, P_C)$ -chaoticity by use of Proposition 1 and Sznitman's compactness-uniqueness methods for proving that the empirical measures  $\Lambda_i^{\mathbf{N}} = \frac{1}{N_i} \sum_{n=1}^{N_i} \delta_{X_{n,i}}$  converge to  $P_i$  for  $1 \leq i \leq C$ .

#### 4. Some further extensions

In the following, independence between different classes is again remarkable.

**Theorem 2.** *Let  $(X_{n,i})_{1 \leq n \leq N_i, 1 \leq i \leq C}$  be a multi-exchangeable system. The vector  $(\Lambda_i)_{1 \leq i \leq C}$  of the empirical measures  $\Lambda_i = \frac{1}{N_i} \sum_{n=1}^{N_i} \delta_{X_{n,i}}$  is a sufficient statistic for the law of the system. The conditional law of the system given  $(\Lambda_i)_{1 \leq i \leq C}$  corresponds to  $C$  independent uniform permutations, or exhaustive drawings without replacement, among the  $N_i$  (possibly repeated) drawings of particles of class  $i$ .*

*Proof.* A simple extension of the proof of Theorem 1 yields for  $g, f_i \geq 0$  that

$$\begin{aligned} & \mathbf{E} \left[ g((\Lambda_i)_{1 \leq i \leq C}) \prod_{i=1}^C f_i(X_{1,i}, \dots, X_{N_i,i}) \right] \\ &= \mathbf{E} \left[ g((\Lambda_i)_{1 \leq i \leq C}) \prod_{i=1}^C \left\langle f_i, \frac{1}{N_i!} \sum_{\substack{1 \leq n_1, \dots, n_{N_i} \leq N_i \\ \text{distinct}}} \delta_{X_{n_1,i}, \dots, X_{n_{N_i},i}} \right\rangle \right] \end{aligned}$$

where the empirical measure for distinct  $N_i$ -tuples corresponds to exhaustive draws without replacement among  $X_{1,i}, \dots, X_{N_i,i}$  and thus can be written in terms of  $\Lambda_i$ . Hence

$$\mathbf{E} \left[ \prod_{i=1}^C f_i(X_{1,i}, \dots, X_{N_i,i}) \mid (\Lambda_i)_{1 \leq i \leq C} \right] = \prod_{i=1}^C \left\langle f_i, \frac{1}{(N_i)_{N_i}} \sum_{\substack{1 \leq n_1, \dots, n_{N_i} \leq N_i \\ \text{distinct}}} \delta_{X_{n_1,i}, \dots, X_{n_{N_i},i}} \right\rangle$$

and we may easily conclude.

We now state the related remarkable result Aldous [1, Cor. 39 p. 25]. We say the multi-class system  $(X_{n,i})_{n \geq 1, 1 \leq i \leq C}$  with *infinite* classes is multi-exchangeable if every sub-system  $(X_{n,i})_{1 \leq n \leq N, 1 \leq i \leq C}$  is multi-exchangeable.

**Proposition 3.** *Let  $(X_{n,i})_{n \geq 1, 1 \leq i \leq C}$  be a multi-exchangeable system, and  $(X_{n,i})_{n \geq 1}$  have directing measure  $\Lambda_i$ . Given the directing measure vector  $(\Lambda_i)_{1 \leq i \leq C}$ , the  $X_{n,i}$  are conditionally independent and have regular conditional distributions  $\Lambda_i$ .*

This leads to the following extension of Proposition 2.

**Theorem 3.** *Let the  $(X_{n,i}^{\mathbf{N}})_{1 \leq n \leq N_i, 1 \leq i \leq C}$  be multi-exchangeable systems with empirical measure vectors  $(\Lambda_i^{\mathbf{N}})_{1 \leq i \leq C}$  and such that  $\lim N_i = \infty$ . These systems converge in law to  $(X_{n,i})_{n \geq 1, 1 \leq i \leq C}$ , which is necessarily multi-exchangeable and has directing measure vector  $(\Lambda_i)_{1 \leq i \leq C}$ , if and only if  $\lim(\Lambda_i^{\mathbf{N}})_{1 \leq i \leq C} = (\Lambda_i)_{1 \leq i \leq C}$  in law.*

*Proof.* The proof for Theorem 1 extends in a straightforward way.

For a multi-exchangeable system  $(X_{n,i})_{n \geq 1, 1 \leq i \leq C}$  with infinite classes, we define the tail  $\sigma$ -algebra  $\mathcal{T} = \bigcap_{k \geq 1} \sigma((X_{n,i})_{n \geq k, 1 \leq i \leq C})$  and the multi-exchangeable  $\sigma$ -algebra

$$\mathcal{E} = \left\{ \left\{ (X_{n,i})_{n \geq 1, 1 \leq i \leq C} \in B \right\} : (x_{n,i})_{n \geq 1, 1 \leq i \leq C} \in B \Rightarrow (x_{\sigma_i(n),i})_{n \geq 1, 1 \leq i \leq C} \in B \right\},$$

where the  $\sigma_i$  are allowed to vary over all finite permutations, and obtain the following extension of Aldous [1, Cor. 3.10 p. 26] and of the Hewitt-Savage 0-1 Law.

**Theorem 4.** *Let  $(X_{n,i})_{n \geq 1, 1 \leq i \leq C}$  be a multi-exchangeable system with infinite classes, and  $(X_{n,i})_{n \geq 1}$  have directing measure  $\Lambda_i$ . Then  $\sigma((\Lambda_i)_{1 \leq i \leq C}) \subset \mathcal{T} \subset \mathcal{E}$  are a.s. equal. If moreover the  $X_{n,i}$  are independent, then  $A \in \mathcal{E} \Rightarrow P(A) \in \{0, 1\}$ .*

*Proof.* Clearly  $\sigma((\Lambda_i)_{1 \leq i \leq C}) \subset \mathcal{T} \subset \mathcal{E}$ . If  $A \in \mathcal{E}$  then  $(\mathbf{1}_A, X_{n,i})_{n \geq 1, 1 \leq i \leq C}$  is multi-exchangeable, and Proposition 3 implies that the  $(\mathbf{1}_A, X_{n,i})$  are conditionally independent given  $(\hat{\Lambda}_i)_{1 \leq i \leq C}$  and have conditional laws  $\hat{\Lambda}_i$  where, a.s.,

$$\hat{\Lambda}_i = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \delta_{\mathbf{1}_A, X_{n,i}} = \delta_{\mathbf{1}_A} \otimes \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \delta_{X_{n,i}} = \delta_{\mathbf{1}_A} \otimes \Lambda_i.$$

Hence, for  $N \geq 1$  and Borel sets  $B_{n,i} \subset \mathcal{S}_i$  we have

$$\mathbf{P}(X_{n,i} \in B_{n,i} : 1 \leq n \leq N, 1 \leq i \leq C \mid A, (\Lambda_i)_{1 \leq i \leq C}) = \prod_{1 \leq n \leq N, 1 \leq i \leq C} \Lambda_i(B_{n,i})$$

which is a function of  $(\Lambda_i)_{1 \leq i \leq C}$ . Hence  $(X_{n,i})_{n \geq 1, 1 \leq i \leq C}$  and  $\mathcal{E} \subset \sigma((X_{n,i})_{n \geq 1, 1 \leq i \leq C})$  are independent conditional on  $(\Lambda_i)_{1 \leq i \leq C}$ , which implies  $\mathcal{E} \subset \sigma((\Lambda_i)_{1 \leq i \leq C})$ , a.s. The Kolmogorov 0-1 Law yields that  $\mathcal{T}$  is a.s. trivial if the  $X_{n,i}$  are independent.

### Acknowledgment

We thank the referee for stimulating remarks leading to a much improved paper.

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