A class of financial products and models where super-replication prices are explicit

Laurence Carassus, Emmanuel Gobet, Emmanuel Temam

A class of financial products and models where super-replication prices are explicit

L. Carassus, E. Gobet, E. Temam

First and third author: Laboratoire de Probabilités et Modèles Aléatoires - Université Paris 7 - Denis Diderot - Case 7012 - 2, place Jussieu - 75251 Paris cedex 05 - France
Second author: Laboratoire Jean Kuntzman - ENSIMAG INP Grenoble - BP 53 - 38041 Grenoble cedex 09 - France

We consider a multidimensional financial model with mild conditions on the underlying asset price process. The trading is only allowed at some fixed discrete times and the strategy is constrained to lie in a closed convex cone. We show how the minimal cost of a super hedging strategy can be easily computed by a backward recursive scheme. As an application, when the underlying follows a stochastic differential equation including stochastic volatility or Poisson jumps, we compute those super-replication prices for a range of European and American style options, including Asian, Lookback or Barrier Options. We also perform some multidimensional computations.
Key words: Closed formula for Super-replication cost; convex cone constraints on portfolio; exotic European and American options.
AMS Classification: 90A09, 60G42, 26B25.

1. Introduction

We consider a financial market consisting of d risky assets with discounted price process denoted by S, and one risk-less bond: the trading is allowed only at fixed discrete times. We assume that the trading strategies are also subject to portfolio constraints. Namely, given a closed convex cone K with vertex in 0, the vector of number of shares invested in the risky assets is constrained to lie in K. Such formalization includes in particular incomplete markets and markets with short-selling constraints. It is well-known that in those contexts, it is not possible to define an unique fair price, i.e the initial cost of a strategy replicating a given contingent claim, as in the context of complete markets. A possible way of defining

a price is to consider the minimal initial wealth needed to hedge without risk the contingent claim. This is called the super-replication cost and has been introduced in the binomial setup for transaction costs by Bensaid-Lesne-Pagès-Scheinkman [1], in a $\mathcal{L}^2$-setup for transaction costs and short-sales constraints by Jouini-Kallal [14, 15] and in the diffusion setup for incomplete markets by El Karoui-Quenez [9]. In the context of convex constraints, this notion has been studied among others by Cvitanić-Karatzas [4], Karatzas-Kou [17], Broadie-Cvitanić-Soner [3] and in a great generality by Föllmer-Kramkov [11]. In those papers a dual formulation is given. Namely, the super-replication cost of an European contingent claim, H, is essentially the supremum over a given set of probability of the expectation of H (or a modification of H). Nevertheless this dual formulation does not enable in general to effectively compute the super-replication price.

Here we combine primal and dual formulations, in order to provide a closed formula for European and American style options under general assumptions on the underlying S (namely, an usual non degeneracy condition), and also to give the hedging strategy. In the case of European vanilla options, finding the super-replication price reduces to compute some concave envelop of the payoff function. For more general options, it involves recursive computations using again kind of concave envelops. The coefficients of the affine function which appears in the concave envelop give the hedging strategy. The application of this algorithm turns to be simple to derive the super-replication prices of all usual options. Patry [20] obtains a similar formula, in the Black-Scholes case, for an European vanilla option.

Our effective computation shows that, when the asset prices can heavily fluctuate, the super-replication prices are trivial in the sense that they correspond to basic strategies such as "Buy and Hold". In particular, for an European call option, the super-replication price is equal to the initial price of the underlying: this result has been already obtained in the context of transaction costs by Cvitanić-Shreve-Soner [7] and Cvitanić-Pham-Touzi [6], and for a continuous time stochastic volatility model by Cvitanić-Pham-Touzi [5]. Our approach emphasizes the fact that the super-replication price depends on the law of the underlying asset price process only through its null sets. These results are presumably not surprising, even if, up to our knowledge, only specific one dimensional cases have been handled in the literature. Finally, this work provides a relatively complete answer to the problem of how to explicitly compute super-replication prices in a general discrete time strategies framework.

The paper is organized as follows. In section 2, we describe the financial model and give the notation of the paper. Then, we recall the notion of No Arbitrage and state a dual formulation for the super-replication

problem: while this result is standard in the European case (see Kabanov-Ràsonyi-Stricker [16] and Schachermayer [23]), it has not been yet stated in the American context (this is Theorem 2.2, which proof is postponed in Appendix). Section 3 is devoted to the closed formulae for the super-replication prices, and their proofs. In Section 4, we effectively compute the super-replication price for European and American style exotic options (including Asian, Lookback or Barrier options), when there is only one risky asset (See table 1 for those explicit computations). We also handle some multidimensional examples. These results hold true if the underlying asset law admits a positive density w.r.t. the Lebesgue measure: it includes for example Black-Scholes model, general stochastic differential equations, stochastic volatility models, or models governed by Brownian motion and Poisson process. We will also see that increasing the number of hedging dates does not modify the super-replication prices.

2. The financial model and super-replication theorem

2.1 Notations and definitions

Let $T > 0$ be a finite time horizon and set $\mathcal{T} = \{0, 1, \ldots, T\}$: the financial market model consists of one risk-less asset with price process normalized to one and d risky assets with price process $S = \{S_t = (S_t^1, ..., S_t^d)^*, t = 0, ..., T\}$ valued in $(0, \infty)^d$. Here the notation * is for the transposition. The stochastic price process $(S_t)_{t\in\mathcal{T}}$ is defined on a complete probability space $(\Omega, \mathcal{F}, P)$ equipped with the filtration $\mathbb{F} = \{\mathcal{F}_t, t \in \mathcal{T}\}$, where the σ-field $\mathcal{F}_t$ is generated by the random variables $S_0, S_1, \cdots, S_t$. We make the usual assumption that $\mathcal{F}_0$ is trivial and $\mathcal{F}_T = \mathcal{F}$.

A trading portfolio is a $\mathbb{R}^d$-valued $\mathbb{F}$-adapted process $\phi = \{\phi_t = (\phi_t^1, \ldots, \phi_t^d)^*, t = 0, ..., T-1\}$, where ϕ_t^i represents the amount of wealth invested in the i-th risky asset at time t. The $\mathbb{R}$-valued F-adapted process $C = \{C_t, t \in \mathcal{T}\}$ represents the cumulative consumption process. We assume that $C_0 = 0$ and that C is non-decreasing. We also use the notation $\Delta S_t = S_t - S_{t-1}$ and $\Delta C_t = C_t - C_{t-1}$, for $t = 1, ..., T$.

Given an initial wealth $x \in \mathbb{R}$, a trading portfolio ϕ, and a cumulative consumption process C, the wealth process $X^{x,\phi,C}$ is governed by :

$$\begin{aligned} & X_0^{x,\phi,C} = x, \\ (1) \qquad & X_t^{x,\phi,C} = X_{t-1}^{x,\phi,C} + \phi_{t-1}^* \Delta S_t - \Delta C_t, \quad \text{for } t = 1, \ldots, T. \end{aligned}$$

The induction equation (1) leads to

$$X_t^{x,\phi,C} = x + \sum_{u=1}^{t} \phi_{u-1}^* \Delta S_u - C_t, \quad t \in \mathcal{T}.$$

The condition $C = 0$ means that the portfolio ϕ is self-financed. We now impose some constraints on the trading portfolios. Let K be a closed convex cone of $\mathbb{R}^d$ with vertex in 0. For any $x \in [0, \infty)$, we say that a trading strategy (x, ϕ, C) is admissible, and we denote $(x, \phi, C) \in \mathcal{A}$, if for all $t = 0, \ldots, T-1$, $\phi_t \in K$ a.s. Such constraints cover in particular the case of incomplete markets ($K = \{k \in \mathbb{R}^d : k_i = 0, i = 1, ..., n\}$: it is impossible to trade in the n first risky assets) and short-sales constraints ($K = [0, \infty)^d$).

Let H be an European contingent claim, i.e., a $\mathcal{F}_T$-measurable random variable. Following Föllmer and Kramkov [11], we introduce the notion of minimal hedging strategy for H. First, an European H hedging strategy is a strategy $(x, \phi, C) \in \mathcal{A}$ such that $X_T^{x,\phi,C} \geq H$ a.s. We will denote by $\mathcal{A}_H^e$ the set of European H hedging strategies. Then, $(\hat{x}, \hat{\phi}, \hat{C}) \in \mathcal{A}_H^e$ is minimal if for all $(x, \phi, C) \in \mathcal{A}_H^e$ $X_t^{x,\phi,C} \geq X_t^{\hat{x},\hat{\phi},\hat{C}}$ a.s for all $t \in \mathcal{T}$. Note that $\hat{x}$ is then the so-called super-replication cost $p^e(H)$ of H, i.e the minimal initial capital needed for hedging without risk H:

$$p^e(H) = \inf\{x \in \mathbb{R} \ : \ \exists\, (\phi, C) \text{ s.t. } (x, \phi, C) \in \mathcal{A}_H^e\}.$$

It is straightforward that $\hat{x} \geq p^e(H)$. Conversely, set $x \in \mathbb{R}$ such that there exists (Φ, C) with $(x, \Phi, C) \in \mathcal{A}_H^e$, then by minimality of $X^{\hat{x},\hat{\phi},\hat{C}}$, $x \geq \hat{x}$ and taking the infimum over such x, we get the reverse inequality.

We now define the same notion for American contingent claim $(H_t)_{t\in\mathcal{T}}$. An American H hedging strategy is some $(x, \phi, C) \in \mathcal{A}$ such that for all $t \in \mathcal{T}$, $X_t^{x,\phi,C} \geq H_t$ a.s. We will denote by $\mathcal{A}_H^a$ the set of American H hedging strategies. Then $(\hat{x}, \hat{\phi}, \hat{C}) \in \mathcal{A}_H^a$ is minimal if for all $(x, \phi, C) \in \mathcal{A}_H^a$, $X_t^{x,\phi,C} \geq X_t^{\hat{x},\hat{\phi},\hat{C}}$ a.s, for all $t \in \mathcal{T}$. Again $\hat{x}$ is the super-replication cost $p^a(H)$ of H, i.e

$$p^a(H) = \inf\{x \in \mathbb{R} \ : \ \exists\, (\phi, C) \text{ s.t. } (x, \phi, C) \in \mathcal{A}_H^a\}.$$

We now recall the usual notion of No-Arbitrage, which characterization is meaningful for super-replication theorem 2.2.

Definition 2.1. We say that there is no arbitrage opportunity if, for all trading strategies Φ such that $(0, \Phi, 0) \in \mathcal{A}$, we have

$$X_T^{0,\Phi,0} \geq 0 \quad \text{a.s} \implies X_T^{0,\Phi,0} = 0 \quad \text{a.s.}$$

In Pham and Touzi [22], a characterization of this no-arbitrage condition is provide and to state it, we introduce the following two sets:

$$\hat{K} = \left\{x \in \mathbb{R}^d \ : \phi^* x \leq 0, \forall \phi \in K\right\}.$$

$$\mathcal{P} = \Big\{ Q \sim P \ : \ \frac{dQ}{dP} \in \mathcal{L}^\infty, \ \ \Delta S_t \in \mathcal{L}^1(Q) \\ \text{and} \ \ E^Q[\Delta S_t | \mathcal{F}_{t-1}] \ \in \ \hat{K}, \ \ 1 \leq t \leq T \ \ P - \text{a.s.} \Big\}.$$

We also need a non-degeneracy assumption. This assumption is essential to prove Theorem 2.1 below: if it fails to hold, the set of final dominated payoffs may not be closed, see Brannath [2].

Assumption 2.1. *Let $t = 1, \ldots, T$. Then for all $\mathcal{F}_{t-1}$-measurable random variables φ valued in K,*

$$\varphi^* \Delta S_t(\omega) \ = \ 0 \Longrightarrow \varphi(\omega) \ = \ 0 \qquad \textit{for a.e.} \ \ \omega \in \Omega \ .$$

Models studied in section 4 fulfill the above assumption.

Theorem 2.1. *(Pham-Touzi [22]).*
Under Assumption 2.1, the no arbitrage condition is equivalent to $\mathcal{P} \neq \emptyset$.

Without cone constraints, see also Dalang-Morton-Willinger [8]. Let $\mathcal{S}_{t,T}$ be the set of all stopping w.r.t. the filtration $\mathbb{F}$ such that $t \leq \tau \leq T$.

2.2 Super-replication Theorem

Our starting point to derive closed formulae for super-replication prices is the dual formulation of the super-replication theorem. It states that the super-replication cost of an European (resp. American) contingent claim, H (resp $(H_t)_{t \in \mathcal{T}}$), is essentially the supremum over any probability measure Q in given set $\mathcal{P}$ (resp. and every stopping time τ less than T) of $E^Q(H)$ (resp $E^Q(H_\tau)$): this is given by Theorem 2.2. We give the proof of this non surprising result, since to our knowledge, it has not been done before in the American case.

Indeed, Föllmer and Kramkov [11] obtain, via an Optional Decomposition Theorem, for continuous time asset price process and convex constrained the super-replication Theorem (this is no longer the expectation of H but of a modification of H which takes into account the convex constraints). But to deal with this great generality, they have to assume first that the wealth process is non negative; second, the strategy ϕ has to be chosen so that the set $\{(\sum_{u=1}^t \phi_{u-1}^* \Delta S_u)_{t=1 \ldots T}\}$ is locally bounded from below: in a discrete setup, with say $T = 1$, this boundedness Assumption implies to choose $\phi_0 \geq 0$ or S_1 bounded, which is rather restrictive. Kabanov-Ràsonyi-Stricker [16] and Schachermayer [23] derive a general version of super-replication theorem for European claims, while Schäl [24] have studied the American context for a $\mathcal{L}^2$-setup without constraints on the strategy.

Our proof is different since the result for American claims is obtained thanks to that for European ones.

Theorem 2.2. *Suppose that Assumption 2.1 and the no arbitrage condition hold. Let H be an European contingent claim, assume that*

$$\sup_{Q\in\mathcal{P}} E^Q[H] < \infty.$$

Then, there exists a minimal hedging strategy $(\hat{x},\hat{\phi},\hat{C}) \in \mathcal{A}_H^e$ such that

$$X_t^{\hat{x},\hat{\phi},\hat{C}} = \operatorname{ess\,sup}_{Q\in\mathcal{P}} E^Q[H \mid \mathcal{F}_t].$$

In particular,

$$p^e(H) = \hat{x} = \sup_{Q\in\mathcal{P}} E^Q[H].$$

Let $(H_t)_{t\in\mathcal{T}}$ be an American contingent claim, assume that,

$$\sup_{\tau\in\mathcal{S}_{0,T},Q\in\mathcal{P}} E^Q[H_\tau] < \infty,$$

Then, there exists a minimal hedging strategy $(\hat{x},\hat{\phi},\hat{C}) \in \mathcal{A}_H^a$ such that

$$X_t^{\hat{x},\hat{\phi},\hat{C}} = \operatorname{ess}\sup_{\tau\in\mathcal{S}_{t,T},Q\in\mathcal{P}} E^Q[H_\tau \mid \mathcal{F}_t].$$

In particular,

$$p^a(H) = \hat{x} = \sup_{\tau\in\mathcal{S}_{0,T},Q\in\mathcal{P}} E^Q[H_\tau].$$

Proof. See Appendix.

3. The main results

Our main objective now is to derive closed formulae for the super-replication prices in the mathematical background defined above: while the essential supremum involved in Theorem 2.2 are difficult to be directly evaluated because of the set $\mathcal{P}$, the prices given by formulae from Theorems 3.1 and 3.2 are simple to compute.

Let us introduce two notations:

- we will denote by $\mu_j(S_0,\dots S_{j-1})$, the conditional law of S_j knowing $\mathcal{F}_{j-1}$.
- the law of the vector $(S_0,\dots,S_j)$ will be denoted by P_j.

First we treat the European case. For a measurable function h from $(\mathbb{R}^d)^{T+1}$ into $\mathbb{R}$, we define a sequence of operator, based on kinds of concave envelops, by

$$\Gamma^e_T h(x_0,\dots,x_T) = h(x_0,\dots,x_T) \tag{2}$$

$$\Gamma^e_j h(x_0,\dots,x_j) = \operatorname{ess}\inf_{(\alpha,\beta)\in\mathbb{R}\times K}\{f^{\Gamma^e_{j+1}h}_{\alpha,\beta}\}(x_0,\dots,x_j) \quad 0\le j\le T-1 \tag{3}$$

where, for u from $(\mathbb{R}^d)^{j+2}$ into $\mathbb{R}$, one has

$$f^u_{\alpha,\beta}(x_0,\dots,x_j) = \begin{cases} \alpha+\beta^* x_j \text{ if } \mu_{j+1}(x_0,\dots,x_j)\left\{z:\ \alpha+\beta^* z < u(x_0,\dots,x_j,z)\right\} = 0 \\ +\infty \qquad \text{otherwise.}\end{cases} \tag{4}$$

The essential infimum in (3) is related to the measure P_j. Note that the definition of the operator $\Gamma^e_j h$ is related to a dynamic programming principle: at each time j, $\Gamma^e_j h$ is somehow the minimal value of any strategy which almost everywhere super hedge $\Gamma^e_{j+1}h$. Then, the following theorem holds.

Theorem 3.1. *Assume Assumption 2.1 and the no arbitrage condition. Let $H = h(S_0,\dots,S_T)$ be an European contingent claim, for some measurable function h from $(\mathbb{R}^d)^{T+1}$ into $\mathbb{R}$. Assume that*

$$\sup_{Q\in\mathcal{P}} E^Q[H] < \infty.$$

Then, there exists a minimal hedging strategy $(\hat{x},\hat{\phi},\hat{C}) \in \mathcal{A}^e_H$ and its value at time $t \le T$ is

$$X^{\hat{x},\hat{\phi},\hat{C}}_t = \Gamma^e_t h(S_0,\dots,S_t) \ \ P_t - a.s.$$

In particular,

$$p^e(H) = \Gamma^e_0 h(S_0).$$

We now turn to the American case, by considering $(h_t)_{t\in\mathcal{T}}$ a family of measurable functions such that for $t\in\mathcal{T}$, h_t maps $(\mathbb{R}^d)^{t+1}$ into $\mathbb{R}$. We define a new sequence of operator Γ^a replacing the equations (2) and (3) by

$$\Gamma^a_T h(x_0,\dots,x_T) = h_T(x_0,\dots,x_T) \tag{5}$$

$$\Gamma^a_j h(x_0,\dots,x_j) = \left(\operatorname{ess}\inf_{(\alpha,\beta)\in\mathbb{R}\times K}\{f^{\Gamma^a_{j+1}h}_{\alpha,\beta}\}\vee h_j\right)(x_0,\dots,x_j) \quad 0\le j\le T-1. \tag{6}$$

Then we get

Theorem 3.2. *Assume Assumption 2.1 and the no arbitrage condition. Let $H = (H_t)_{t\in\mathcal{T}}$ be an American contingent claim, such that*

$$\sup_{\tau\in\mathcal{S}_{0,T},Q\in\mathcal{P}} E^Q[H_\tau] < \infty.$$

For $t \in \mathcal{T}$, we denote by h_t a measurable function from $(\mathbb{R}^d)^{t+1}$ into $[0,\infty)$ such that $H_t = h_t(S_0,\dots,S_t)$ a.s.. Then, there exists a minimal hedging strategy $(\hat{x},\hat{\phi},\hat{C}) \in \mathcal{A}_H^a$ and its value at time $t \le T$ is

$$X_t^{\hat{x},\hat{\phi},\hat{C}} = \Gamma_t^a h(S_0,\dots,S_t) \ \ a.s. \tag{7}$$

In particular,

$$p^a(H) = \Gamma_0^a h(S_0).$$

Theorem 2.2 proves the existence of an optimal strategy and thus, from Theorems 3.1 and 3.2, we can easily deduce that the essential infima, involved in the definition of operators Γ^e and Γ^a, are attained. It turns that the optimal portfolio is the optimal β from (3) and (6), which is easy to compute in the practical examples (see section 4).

Proof of Theorems 3.1 and 3.2. We only give the proof for American contingent claims, since the European case is very similar. In the following we will denote

$$I_u(x_0,\dots,x_j) = \{(\alpha,\beta)\in\mathbb{R}\times K \mid \ \mu_{j+1}(x_0,\dots,x_j)\{z \mid \alpha+\beta^* z < u(x_0,\dots,x_j,z)\} = 0\}. \tag{8}$$

First, it is easy to check that the measurability of u implies that of the functions $f^u_{\alpha,\beta}$. Thus, recursively, by definition of the essential infimum and remembering that each h_t is measurable, we can prove that each $\Gamma_t^a h$ is also measurable.

First step: $X_t^{\hat{x},\hat{\Phi},\hat{C}} \le \Gamma_t^a h(S_0,\cdots S_t) \quad P_t - a.s.$

Conditionally on $\mathcal{F}_{T-1}$, let $(\alpha,\beta) \in I_{h_T}(S_0,\dots,S_{T-1})$; then, by (8)

$$h_T(S_0,\dots,S_{T-1},z) \le \alpha + \beta^* z\ , \qquad \mu_T(S_0,\dots,S_{T-1}) - a.e.$$

Let $Q \in \mathcal{P}$; since Q is in particular equivalent to P on $\mathcal{F}_{T-1}$, one gets

$$\begin{aligned} E^Q[h_T(S_0,\dots,S_T) \mid \mathcal{F}_{T-1}] &\le E^Q[\alpha+\beta^* S_T \mid \mathcal{F}_{T-1}] \\ &\le \alpha + \beta^* S_{T-1} \qquad P_{T-1} - a.s. \end{aligned}$$

using that $E^Q[\Delta S_t|\mathcal{F}_{t-1}] \ \in \ \hat{K}$. By (4), it follows that

$$E^Q[h_T(S_0,\dots,S_T) \mid \mathcal{F}_{T-1}] \le f^{h_T}_{\alpha,\beta}(S_0,\dots,S_{T-1}) \qquad P_{T-1}\text{–}a.s. \qquad \forall\ \alpha,\beta\in\mathbb{R}\times K.$$

and thus,

$$E^Q\left[h_T(S_0,\dots,S_T)\mid\mathcal{F}_{T-1}\right]\le \operatorname{ess}\inf_{(\alpha,\beta)\in\mathbb{R}\times K}\{f^{h_T}_{\alpha,\beta}\}(S_0,\dots,S_{T-1})\qquad P_{T-1}-a.s.$$

$$\le \Gamma^a_{T-1}h(S_0,\dots,S_{T-1})\qquad P_{T-1}-a.s.\,. \tag{9}$$

Let $\tau\in\mathcal{S}_{t,T}$. Writing $H_\tau=H_\tau\mathbf{1}_{\tau\le T-1}+H_T\mathbf{1}_{\tau>T-1}$, it follows from (6) and (9), that P_{T-1} *a.s.* one has

$$\begin{aligned}E^Q\left[H_\tau\mid\mathcal{F}_{T-1}\right]&\le \mathbf{1}_{\tau\le T-1}\Gamma^a_\tau h(S_0,\dots,S_\tau)+\mathbf{1}_{\tau>T-1}\Gamma^a_{T-1}h(S_0,\dots,S_{T-1})\\&\le \Gamma^a_{(T-1)\wedge\tau}h(S_0,\dots,S_{(T-1)\wedge\tau}).\end{aligned}$$

Recursively, repeating the same kinds of arguments with $\Gamma^a_{T-1}h,\cdots,\Gamma^a_{t+1}h$, we get

$$E^Q[H_\tau\mid\mathcal{F}_t]\le\Gamma^a_{t\wedge\tau}h(S_0,\dots,S_{t\wedge\tau})=\Gamma^a_th(S_0,\dots,S_t)\qquad P_t\ a.s.\,.$$

Now, take the essential supremum on $Q\in\mathcal{P}$ and $\tau\in\mathcal{S}_{t,T}$, and recall that by Theorem 2.2, there exists $(\hat{x},\hat{\phi},\hat{C})\in\mathcal{A}^a_H$ such that $X^{\hat{x},\hat{\phi},\hat{C}}_t=\operatorname{ess}\sup_{\tau\in\mathcal{S}_{t,T},Q\in\mathcal{P}}E^Q\left[H_\tau\mid\mathcal{F}_t\right]$: the first inequality is completed.

Second step: $X^{x,\Phi,C}_t\ge\Gamma^a_th(S_0,\cdots S_t)\quad P_t-a.s.$, for any $(x,\Phi,C)\in\mathcal{A}^a_H$.

Let $(x,\Phi,C)\in\mathcal{A}^a_H$. Put $\bar{\alpha}=x+\sum_{i=1}^{T-1}\Phi^*_{i-1}\Delta S_i-\Phi^*_{T-1}S_{T-1}$ and $\bar{\beta}=\Phi_{T-1}$: remark that conditionally on $\mathcal{F}_{T-1}$, $(\bar{\alpha},\bar{\beta})$ belongs to $I_{h_T}(S_0,\dots,S_{T-1})$. Thus, one has $P_{T-1}-a.s.$

$$\begin{aligned}X^{x,\Phi,C}_{T-1}&\ge x+\sum_{i=1}^{T-1}\Phi^*_{i-1}\Delta S_i=\bar{\alpha}+\bar{\beta}^*S_{T-1}=f^{h_T}_{\bar{\alpha},\bar{\beta}}(S_0,\dots,S_{T-1})\\&\ge\operatorname{ess}\inf_{(\alpha,\beta)\in\mathbb{R}\times K}\{f^{h_T}_{\alpha,\beta}\}(S_0,\dots,S_{T-1}),\end{aligned}$$

and by definition of a H hedging portfolio of an American contingent claim, we conclude

$$X^{x,\Phi,C}_{T-1}\ge\Gamma^a_{T-1}h(S_0,\dots,S_{T-1})\qquad P_{T-1}-a.s.\,.$$

Repeating this process, one gets the result for the second step. In particular, this holds true for the minimal strategy and Theorem 3.2 is proved.

4. Application: some super-replication prices

4.1 Specification of the models

The explicit prices given in the sequel are available if for each $j\in\{1,...,T\}$, the measure $\mu_j(S_0,...,S_{j-1})$ is equivalent to the Lebesgue measure

on $(0,\infty)^d$. In that case, it is easy to check that there is no arbitrage opportunity. Note also that all the measures involved in the essential infima can be taken as the Lebesgue measure. Actually, the existence of a positive density for the corresponding law is very often satisfied: we list some examples, illustrating by the way that the results cover a wide class of financial models. Note that tree models do not satisfy this condition of existence of a density w.r.t. the Lebesgue measure (however, up to very tedious computations, it is possible to get super-replication prices for binomial and trinomial models). In the following, W is a q-dimensional Brownian motion.

- The well-known stochastic differential equation of Black-Scholes in its multidimensional version:

$$\frac{dS_t^i}{S_t^i} = \mu_i dt + \sum_{j=1}^{q} \sigma_{i,j} dW_t^j.$$

  If $\sigma\sigma^*$ is invertible, it is clear that this process satisfies the required condition.

- A non Markovian generalized version of the model above:

$$\frac{dS_t^i}{S_t^i} = \mu_i(t,(S_s)_{0\le s\le t})dt + \sum_{j=1}^{q} \sigma_{i,j}(t,(S_s)_{0\le s\le t})dW_t^j,$$

  with the non degeneracy condition $[\sigma\sigma^*](.,.) \ge \sigma_0^2$ Id for some $\sigma_0 \neq 0$. For the existence of the positive density, see Kusuoka-Stroock [18].

- A stochastic volatility model:

$$\frac{dS_t^i}{S_t^i} = \mu_i dt + \sum_{j=1}^{q} \sigma_{i,j,t} dW_t^j$$

  where $(\sigma_t\sigma_t^*)_{t\ge 0}$ is a matrix-valued continuous time process, which we assume to be positive definite and independent of the Brownian motion W. It is easy to check the existence of the positive density.

- Merton's model with jumps [19]: this is a generalization of Black-Scholes model including Poisson type jumps. It may be defined by

$$S_t^i = S_0^i \left(\prod_{j=1}^{N_t^i} (f(Y_j^i) + 1) \right) e^{\sum_{j=1}^q \sigma_{i,j} W_t^j + (\mu_i - \sum_{j=1}^q \sigma_{i,j}^2/2)t},$$

where $(f(Y_j^i))_{1\leq i\leq d, j\geq 1}$ are i.i.d. random variables, strictly greater than -1, $(N_t^i; 1 \leq i \leq d)$ are Poisson processes with arrival rate λ_i. All processes and random variables defining this multidimensional model are independent. For this homogeneous Markov process, it is easy to prove the existence of a positive density w.r.t. Lebesgue measure on $(0,\infty)^d$ assuming that $\sigma\sigma^*$ is invertible.

4.2 Computation of the prices in dimension one

Here, we restrict to one risky asset ($d = 1$) starting with the unconstrained case ($K = \mathbb{R}$). We sketch the proofs of some results of table 1: it somehow reduces to compute iterative concave envelops (w.r.t. the Lebesgue measure on $(0,\infty)$), which is easy for the usual options.

4.2.1 Vanilla Options.

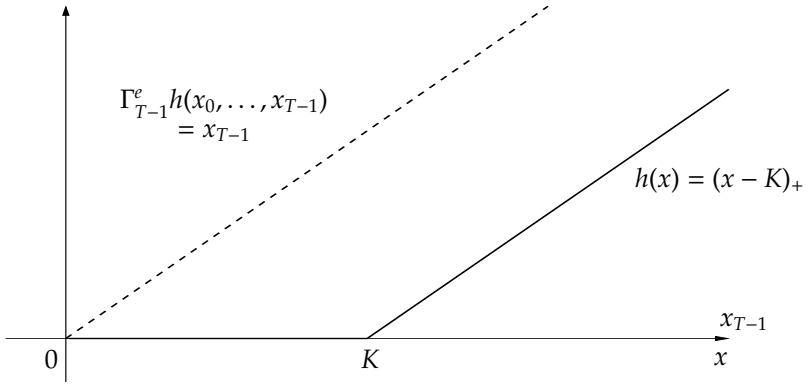

Figure 1. Computation of concave envelops for the Call option.

We first consider the case of an European Call option whose payoff is $h(x_0,\ldots,x_T) = (x_T - K)_+$. Applying formulae (3), one first gets $\Gamma^e_{T-1}h(x_0,\ldots,x_{T-1}) = x_{T-1}$ (see figure 1); by a straightforward iteration, it follows that $\Gamma^e_j h(x_0,\ldots,x_j) = x_j$, and thus $p^e(H) = \Gamma^e_0 h(S_0) = S_0$. Analogously, for the European Put $h(x_0,\ldots,x_T) = (K - x_T)_+$, one gets $\Gamma^e_j h(x_0,\ldots,x_j) = K$, and thus $p^e(H) = K$. These results have already been obtained by Patry (2001).

For the American style options, analogous computations provide the same prices as above.

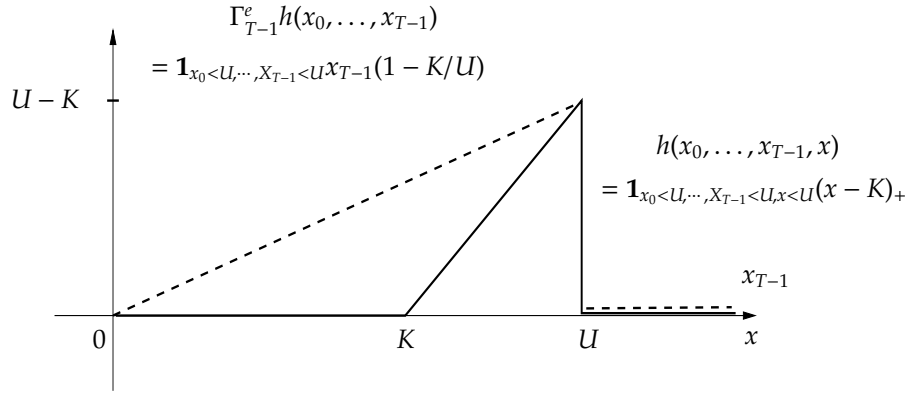

Figure 2. Computation of concave envelops for the Up and Out Call option.

4.2.2 Barrier Options.

Let us consider, for example, the case of an European Up and Out Call whose payoff is $h(x_0,\ldots,x_T) = \prod_{i=0}^{T} \mathbf{1}_{x_i<U}(x_T-K)_+$, assuming $S_0 < U$ and $K < U$. For given $x_0,\ldots,x_{T-1}$ less than U, the concave envelop of the function $(x_T-K)_+\mathbf{1}_{x_T<U}$ is given by the function $x \mapsto (x \wedge U)(1-K/U)$; hence, one has $\Gamma^e_{T-1}h(x_0,\ldots,x_{T-1}) = \prod_{i=0}^{T-1} \mathbf{1}_{x_i<U} x_{T-1}(1-K/U)$ (see figure 2). For the associated American claim for which $h_j(x_0,\ldots,x_j) = \prod_{i=0}^{j} \mathbf{1}_{x_i<U}(x_j-K)_+$, one also gets $\Gamma^a_{T-1}h(x_0,\ldots,x_{T-1}) = \prod_{i=0}^{T-1} \mathbf{1}_{x_i<U} x_{T-1}(1-K/U)$. Iteratively, one obtains $\Gamma^e_j h(x_0,\ldots,x_j) = \Gamma^a_j h(x_0,\ldots,x_j) = \prod_{i=0}^{j} \mathbf{1}_{x_i<U} x_j(1-K/U)$. Finally, this proves $p^e(H) = p^a(H) = S_0(1-K/U)$.

4.2.3 Extension.

Assume that the contingent claim $H = h(S_0,\cdots,S_T)$ can be trade at some extra dates. Then, the definition of the super-replication price should imply more rebalancing dates. But, it is easy to prove, in our context of conditional laws equivalent to the Lebesgue measure, that the super-replication prices are unchanged. For example, consider a monthly monitored barrier option with expiration date equal to one year: if we are allowed to hedge each month, or each day, or even each hour, the super-replication price will be the same.

Name	Payoff	European Price	American Price
Call	$(S_T - K)_+$	S_0	S_0
Put	$(K - S_T)_+$	K	K
Asian Call (Fixed strike)	$\left(\sum_{i=1}^{T} a_i S_i - K\right)_+$ $0 \le a_i$	$S_0\left(\sum_{i=1}^{T} a_i\right)$	$\left(\sum_{i=2}^{T-1} \frac{1}{i} + \frac{2}{T}\right) S_0$ $(a_i = 1/T)$
Asian Call (Floating Strike)	$\left(\sum_{i=1}^{T} a_i S_i - S_T\right)_+$ $0 \le a_i \le 1$	$S_0\left(\sum_{i=1}^{T-1} a_i\right)$	$\left(\sum_{i=2}^{T} \frac{1}{i}\right) S_0$ $(a_i = 1/T)$
Asian Put (Fixed strike)	$\left(K - \sum_{i=1}^{T} a_i S_i\right)_+$ $0 \le a_i$	K	K
Asian Put (Floating Strike)	$\left(S_T - \sum_{i=1}^{T} a_i S_i\right)_+$ $0 \le a_i \le 1$	$S_0(1 - a_T)$	$S_0(1 - a_T)$
Partial Lookback Call	$(S_T - \lambda \min(S_1, \ldots, S_T))_+$ $\lambda \in [0,1]$	S_0	S_0
Call on maximum	$(\max(S_1, \ldots, S_T) - K)_+$	$T\ S_0$	$T\ S_0$
Barrier Up and Out Call	$\prod \mathbf{1}_{S_i<U}(S_T - K)_+$ $(K < U, S_0 < U)$	$S_0(1 - K/U)$	$S_0(1 - K/U)$
Barrier Up and Out Put	$\prod \mathbf{1}_{S_i<U}(K - S_T)_+$ $(S_0 < U)$	K	K
Barrier Up and In Call	$\mathbf{1}_{\exists i / S_i > U}(S_T - K)_+$	S_0	S_0
Barrier Up and In Put	$\mathbf{1}_{\exists i / S_i > U}(K - S_T)_+$ $(S_0 < U)$	$S_0 K/U$	$S_0 K/U$
Barrier Down and Out Call	$\prod \mathbf{1}_{S_i>L}(S_T - K)_+$ $(S_0 > L)$	S_0	S_0
Barrier Down and Out Put	$\prod \mathbf{1}_{S_i>L}(K - S_T)_+$ $(S_0 > L, K > L)$	$K - L$	$K - L$
Barrier Down and In Call	$\mathbf{1}_{\exists i / S_i < L}(S_T - K)_+$ $(S_0 > L)$	$L\mathbf{1}_{L<K}$ $+S_0\mathbf{1}_{L>K}$	$L\mathbf{1}_{L<K}$ $+S_0\mathbf{1}_{L>K}$
Barrier Down and In Put	$\mathbf{1}_{\exists i / S_i < L}(K - S_T)_+$ $(S_0 > L)$	K	K

Table 1 Explicit super-replication prices of some options

Remark that in dimension 1, the only relevant cones are $\mathbb{R}^+$ and $\mathbb{R}^-$. In the first case, the prices given in table 1 are unchanged. In the second one, for bounded payoffs, results still hold: otherwise, prices become infinite.

4.3 Some prices in a multidimensional setting

We may consider an option written on d assets with payoff equal to

$$H = \left(\sum_{i=1}^{d} a_i S_T^i + a_0 \right)_+ ,$$

for some real numbers $(a_i)_{0 \leq i \leq d}$: it includes exchange options with payoff equal to $(S_T^1 - KS_T^2)_+$, or Call options on index with payoff equal to $(\sum_{i=1}^d p_i S_T^i - K)_+$ (with $p_i \geq 0$ and $\sum_{i=1}^d p_i = 1$). It is not hard to check that without cone constraints, the super-replication price is given by

$$p^e(H) = (a_0)_+ + \sum_{i=1}^{d} (a_i)_+ S_0^i.$$

The optimal strategy is again static and equals $\phi_t^* = ((a_1)_+, \cdots, (a_d)_+)$.

If there is a cone constraint defined by K, we can easily see that the price is unchanged if K contains any vector e_i (the i-th element of the canonical base of $\mathbb{R}^d$) for which $a_i > 0$. Otherwise, one has $p^e(H) = +\infty$.

5. Conclusion

Taking mainly advantage of the primal formulation, we give a recursive formula to compute the super-replication price and the optimal strategy for European and American contingent claims. For this, we have assumed that mild conditions on the underlying assets hold (Assumption 2.1), and that the trading is discrete and constrained to lie in a closed convex cone. When the conditional law of the asset process is equivalent to the Lebesgue measure, we perform explicit computations for the usual options. What clearly happens is that the super-replication prices are somehow very high. It is already known that in the context of imperfect continuous financial markets, the super-replication price of an European call is equal to S_0. Our results show that this feature of high prices remains true for a large class of financial products and models, when only discrete time strategies are allowed. It reinforces the necessity to turn to other concepts to price options such as minimization of shortfall risk or prices based on utility functions (see Föllmer-Leukert [12], and Hodges-Neuberger [13] among others).

Appendix: Proof of theorem 2.2

The proof of the European case is standard (see Pham [21], Kabanov-Ràsonyi-Stricker [16] and Schachermayer [23]).

For the American case, let $(H_t)_{t\in\mathcal{T}}$ be an American payoff such that

$$\sup_{Q\in\mathcal{P},\tau\in\mathcal{S}_{0,T}} E^Q[H_\tau] < \infty. \tag{10}$$

By analogy with the usual dynamic programming equation, we introduce the process Y_t defined by

$$\begin{aligned} Y_T &= H_T \\ Y_t &= H_t \vee \operatorname*{ess\,sup}_{Q\in\mathcal{P}} E^Q\left[Y_{t+1} \mid \mathcal{F}_t\right] \quad \text{for } t = 0,\dots,T-1. \end{aligned}$$

Set $A_t = \{H_t \geq \operatorname*{ess\,sup}_{Q\in\mathcal{P}} E^Q\left[Y_{t+1} \mid \mathcal{F}_t\right]\}$ and

$$\begin{aligned} \tau_T &= T \\ \tau_t &= t\mathbf{1}_{A_t} + \tau_{t+1}\mathbf{1}_{A_t^c}. \end{aligned}$$

Note that each τ_t belongs to $\mathcal{S}_{t,T}$: τ_0 will play the role of an optimal stopping time. Actually, the proof of the American part of Theorem 2.2 follows from the following lemma.

Lemma 5.1. *With the above notation and Assumption 10, there exists a minimal strategy $(Y_0, \hat{\phi}, \hat{C}) \in \mathcal{A}_H^a$ such that*

$$X_t^{Y_0,\hat{\phi},\hat{C}} = Y_t = \operatorname*{ess\,sup}_{Q\in\mathcal{P}} E^Q[H_{\tau_t} \mid \mathcal{F}_t] = \operatorname*{ess\,sup}_{Q\in\mathcal{P},\tau\in\mathcal{S}_{t,T}} E^Q[H_\tau \mid \mathcal{F}_t].$$

We now turn to its proof.

Step 1: $X_t^{x,\phi,C} \geq \operatorname*{ess\,sup}_{Q\in\mathcal{P},\tau\in\mathcal{S}_{t,T}} E^Q[H_\tau \mid \mathcal{F}_t]$ for any $(x,\phi,C) \in \mathcal{A}_H^a$. The result is clear using the admissibility of the American strategy and the supermartingale property of $X^{x,\phi,C}$ under any $Q \in \mathcal{P}$.

Step 2: $\operatorname*{ess\,sup}_{Q\in\mathcal{P}} E^Q[H_{\tau_t} \mid \mathcal{F}_t] \geq Y_t$. We proceed by induction. Clearly, the property holds when $t = T$. Assume now that it holds for $t+1$, and let denote by $(p^e(H_{\tau_{t+1}}), \tilde{\phi}^{t+1}, \tilde{C}^{t+1})$ the minimal strategy for the European claim

$H_{\tau_{t+1}}$, using Theorem 2.2. Thus, one deduces from the definition of Y_t that

$$\begin{aligned}
Y_t &= \mathbf{1}_{A_t} H_t + \mathbf{1}_{A_t^c} \operatorname*{ess\,sup}_{Q\in\mathcal{P}} E^Q[Y_{t+1} \mid \mathcal{F}_t] \\
&\le \mathbf{1}_{A_t} H_t + \mathbf{1}_{A_t^c} \operatorname*{ess\,sup}_{Q\in\mathcal{P}} E^Q[\operatorname*{ess\,sup}_{Q\in\mathcal{P}} E^Q[H_{\tau_{t+1}} \mid \mathcal{F}_{t+1}] \mid \mathcal{F}_t] \text{ by induction} \\
&\le \mathbf{1}_{A_t} H_t + \mathbf{1}_{A_t^c} \operatorname*{ess\,sup}_{Q\in\mathcal{P}} E^Q[X_{t+1}^{p^e(H_{\tau_{t+1}}),\tilde{\phi}^{t+1},\tilde{C}^{t+1}} \mid \mathcal{F}_t] \\
&\le \mathbf{1}_{A_t} H_t + \mathbf{1}_{A_t^c} X_t^{p^e(H_{\tau_{t+1}}),\tilde{\phi}^{t+1},\tilde{C}^{t+1}} \text{ using the super-martingale property} \\
&\le \mathbf{1}_{A_t} H_t + \mathbf{1}_{A_t^c} \operatorname*{ess\,sup}_{Q\in\mathcal{P}} E^Q[H_{\tau_{t+1}} \mid \mathcal{F}_t] = \operatorname*{ess\,sup}_{Q\in\mathcal{P}} E^Q[H_{\tau_t} \mid \mathcal{F}_t].
\end{aligned}$$

Step 3: there exists a strategy $(Y_0, \hat{\phi}, \hat{C}) \in \mathcal{A}_H^a$ such that $X_t^{Y_0,\hat{\phi},\hat{C}} = Y_t$. Let (x_t, ϕ^t, C^t) be the minimal strategy associated to the European contingent claim Y_t: we can take $\phi_u^t = \Delta C_{u+1}^t = 0$ for $u \ge t$, so that $X_t^{x_t,\phi^t,C^t} = X_T^{x_t,\phi^t,C^t} \ge Y_t$. Set

$$\begin{aligned}
\hat{\phi}_t &= \phi_t^{t+1} \quad \text{for } t = 0, \dots, T-1 \\
\Delta\hat{C}_t &= Y_{t-1} - Y_t + \phi_{t-1}^t{}^* \Delta S_t \quad \text{for } t = 1, \dots, T.
\end{aligned}$$

We first prove that $\Delta\hat{C}_t$ is non negative. Theorem 2.2 for the European claim Y_t yields

$$\Delta\hat{C}_t \ge -X_{t-1}^{x_t,\phi^t,C^t} + Y_{t-1} + \Delta C_t^t = -\operatorname*{ess\,sup}_{Q\in\mathcal{P}} E^Q[Y_t \mid \mathcal{F}_{t-1}] + Y_{t-1} + \Delta C_t^t$$

which is non negative by definition of Y_{t-1}. This proves that $(Y_0, \hat{\phi}, \hat{C}) \in \mathcal{A}$. We now show by induction that $X_t^{Y_0,\hat{\phi},\hat{C}} = Y_t$: this will also complete the proof of $(Y_0, \hat{\phi}, \hat{C}) \in \mathcal{A}_H^a$. For $t = 0$, this is obvious. If the property holds true at time t, we deduce that

$$X_{t+1}^{Y_0,\hat{\phi},\hat{C}} = X_t^{Y_0,\hat{\phi},\hat{C}} + \hat{\phi}_t^* \Delta S_{t+1} - \Delta\hat{C}_{t+1} = Y_t + \hat{\phi}_t^{t+1}{}^* \Delta S_{t+1} - \Delta\hat{C}_{t+1} = Y_{t+1},$$

by definition of the consumption $\Delta\hat{C}_{t+1}$.

The combination of the three steps leads to the equality of Lemma 5.1; taking into account Step 1, we prove the minimality of the strategy $(Y_0, \hat{\phi}, \hat{C})$.

Acknowledgments. This work has been partly done while authors were affiliated to Ecole Nationale des Ponts et Chaussées and Ecole Polytechnique.

References

1. B , B., L , J., P , H. S , J. (1992). Derivative asset pricing with transaction costs. *Math. Finance* **2,** 63–86.
2. B , W. (1997). No-arbitrage amd martingale measures in option pricing. *PhD thesis*. Universität Wien, Vienna, Austria.
3. B , M., C , J. S , M. (1998). Optimal replication of contingent claims under portfolio constraints. *Rev. of Financial Studies* **11,** 59–79.
4. C ´, J. K , I. (1993). Hedging contingent claims with constrained portfolios. *Ann. Appl. Probab.* **3,** 652–681.
5. C ´, J., P , H. T , N. (1999). A closed formula for the problem of super-replication under transaction costs. *Finance Stoch.* **3,** 35–54.
6. C ´, J., P , H. T , N. (1999). Super-replication in stochastic volatility models under portfolio constraints. *J. Appl. Probab.* **36,** 523–545.
7. C ´, J., S , S. S , H. (1995). There is no nontrivial hedging portfolio for option pricing with transaction costs. *Ann. Appl. Probab.* **5,** 327–355.
8. D , R.C., M , A. W , W. (1990). Equivalent martingale measures and no-arbitrage in stochastic securities market models. *Stochastics Stochastics Rep.*, **29(2),** 185–201.
9. E . K , N. Q , M. (1995). Dynamic programming and pricing of contingent claims in an incomplete market. *SIAM J. Control Optim.* **33,** 29–66.
10. F¨ , H. K , Y. (1998). Optional decomposition and Lagrange multipliers. *Finance Stoch.* **2,** 69–81.
11. F¨ , H. K , D. (1997). Optional decompositions under constraints. *Probab. Theory Related Fields* **109,** 1–25.
12. F¨ , H. L , P. (1999). Quantile hedging. *Finance Stoch.* **3,** 251–273.
13. H , S. N , A. (1989). Optimal replication of contingent claims under transaction costs. *Review of Futures Markets* **8,** 222–239.
14. J , E. K , H. (1995). Arbitrage in securities markets with short-sales constraints. *Math. Finance* **5,** 197–232.
15. J , E. K , H. (1995). Martingales and arbitage in securities markets with transaction costs. *J. Econ. Theory* **66,** 178–197.
16. K , Y., R` , M. S , C. (2003). On the closedness of sums of convex cones in $\mathcal{L}^0$ and the robust no-arbitrage property. *Finance and Stochastics.* **7(3),** 403–411.
17. K , I. K , S. (1996). On the pricing of contingent claims under constraints. *Ann. Appl. Probab.* **6,** 321–369.
18. K , S. S , D. (1984). Applications of the Malliavin calculus I. *in: K.Itô, ed., Stochastic Analysis, Proc. Taniguchi Internatl. Symp. Katata and Kyoto 1982, Kinokuniya, Tokyo* 271–306.
19. M , R. (1976). Option pricing when underlying stock returns are discontinuous. *Journal of Financial Economics* **3,** 125–144.
20. P , C. (2001). Couverture apporchée optimale des options européennes. *PhD thesis*. Université Paris Dauphine, Paris, France.
21. P , H. (2000). Dynamic $\mathcal{L}^p$-hedging in discrete time under cone constraints. *SIAM J. Control Optimization* **38,** 665–682.
22. P , H. T , N. (1999). The fundamental theorem of asset pricing with

cone constraints. *J. Math. Econom.* **31,** 265–279.
23. S , W. (2004). The Fundamental Theorem of Asset Pricing under Proportional Transaction Costs in Finite Discrete Time. *Mathematical Finance* **14(1),** 19–48.
24. S ¨ , M. (1999). Martingale measures and hedging for discrete-time financial markets. *Math. Oper. Res.* **24,** 509–528.