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► **To cite this version:**

Rémi Carles, Tohru Ozawa. A Poisson formula for the Schrodinger operator. *Journal of Fourier Analysis and Applications*, 2008, 14 (3), pp.475-483. 10.1007/s00041-008-9014-4 . hal-00170775

**HAL Id: hal-00170775**

**<https://hal.science/hal-00170775>**

Submitted on 10 Sep 2007

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## A NONLINEAR POISSON FORMULA FOR THE SCHRÖDINGER OPERATOR

RÉMI CARLES AND TOHRU OZAWA

ABSTRACT. We prove a nonlinear Poisson type formula for the Schrödinger group. Such a formula had been derived in a previous paper by the authors, as a consequence of the study of the asymptotic behavior of nonlinear wave operators for small data. In this note, we propose a direct proof, and extend the range allowed for the power of the nonlinearity to the set of all short range nonlinearities. Moreover,  $H^1$ -critical nonlinearities are allowed.

### 1. INTRODUCTION

For  $n \geq 1$ , define the Schrödinger group as  $U(t) = e^{i\frac{t}{2}\Delta}$ , where  $\Delta$  stands for the Laplacian of  $\mathbb{R}^n$ . We normalize the Fourier transform on  $\mathbb{R}^n$  as follows:

$$(1.1) \quad \mathcal{F}f(\xi) = \widehat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x)e^{-ix \cdot \xi} dx.$$

For  $r \geq 2$ , we define

$$\delta(r) = \frac{n}{2} - \frac{n}{r}.$$

The main result of this note is:

**Theorem 1.1.** *Let  $n \geq 1$ , and fix  $1 + 2/n < p < \infty$  if  $n \leq 2$ ,  $1 + 2/n < p \leq 1 + 4/(n-2)$  if  $n \geq 3$ . Then for every  $\phi \in X_p$ , and almost all  $\xi \in \mathbb{R}^n$ , the following identity holds:*

$$(1.2) \quad \int_0^{\pm\infty} e^{i\frac{t}{2}|\xi|^2} \mathcal{F}(|U(t)\phi|^{p-1}U(t)\phi)(\xi)dt = \int_0^{\pm\infty} |t|^{n(p-1)/2-2} U(t) \left( |U(-t)\widehat{\phi}|^{p-1} U(-t)\widehat{\phi} \right) (\xi)dt,$$

where the space  $X_p$  is defined as follows:

- If  $1 + 2/n < p < 1 + 4/n$ , then  $X_p = \{f \in L^2(\mathbb{R}^n) ; |x|^{\delta(2p)} f \in L^2(\mathbb{R}^n)\}$ .
- If  $p = 1 + 4/n$ , then  $X_p = L^2(\mathbb{R}^n)$ .
- If  $p > 1 + 4/n$ , then  $X_p = H^{\delta(p+1)}(\mathbb{R}^n)$ , the inhomogeneous Sobolev space.

For  $p = 1 + 4/n$ , the above result was proved in [1]. It was also established for  $1 + 2/n < p < 1 + 4/n$  if  $n \leq 2$ , and  $1 + 4/(n+2) < p < 1 + 4/n$  if  $n \geq 3$ , provided that  $\phi \in H^1 \cap \mathcal{F}(H^1)$ . The proof in [1] relies on pseudo-conformal invariances for the nonlinear Schrödinger equation, as well as the explicit computation of the first non-trivial term in the asymptotic expansion of nonlinear wave operators near the origin. In this note, we provide a direct proof of the above identity, which relies on the usual factorization of the Schrödinger group. Moreover, we extend the range of values allowed for  $p$ , and we consider a broader class (when  $p \neq 1 + 4/n$ ) for the function  $\phi$ . We also show that both terms in (1.2) become infinite when  $p = 1 + 2/n$

and  $\phi$  is a Gaussian function (see §3). Note that  $p = 1 + 2/n$  corresponds to the long range case for the scattering theory associated to the nonlinear Schrödinger equation with nonlinearity  $|u|^{p-1}u$ ; see e.g. [2, 3] and references therein.

Let us point out some similarities between (1.2) and the usual Poisson formula. First, if we write  $U(t) = \mathcal{F}^{-1}e^{-i\frac{1}{2}|\xi|^2 t}\mathcal{F}$ , we see that the right hand side of (1.2) has an additional Fourier transform compared to the left hand side. Moreover, we will see in the proof that (1.2) relies on an inversion  $t \mapsto 1/t$ . This is the same as for the Poisson formula associated to the heat equation, or to the Jacobi theta function; see e.g. [5].

Note also that the definition of the space  $X_p$  (which will become natural in the course of the proof of the above result) is reminiscent of the discussion related to scattering theory for the nonlinear Schrödinger equation with nonlinearity  $|u|^{p-1}u$ . The case  $p = 1 + 4/n$  corresponds to the  $L^2$ -critical nonlinearity. For  $p < 1 + 4/n$ , it is usual to work in weighted  $L^2$  spaces, while for  $p > 1 + 4/n$ , Sobolev spaces are more convenient (see e.g. [6]). Also, note that for  $p > 1 + 4/n$ , the upper bound for  $p$  allows  $H^1$ -critical nonlinearities ( $p = 1 + 4/(n-2)$  for  $n \geq 3$ ), thanks to endpoint Strichartz estimates. As mentioned above, when  $p$  reaches the long range case  $p = 1 + 2/n$ , (1.2) becomes irrelevant.

## 2. PROOF OF THEOREM 1.1

We recall the classical factorization of the Schrödinger group:  $U(t) = M_t D_t \mathcal{F} M_t$ , where  $M_t$  is the multiplication by  $e^{i|x|^2/(2t)}$ ,  $\mathcal{F}$  is the Fourier transform (1.1), and  $D_t$  is the dilation operator

$$D_t f(x) = \frac{1}{(it)^{n/2}} f\left(\frac{x}{t}\right).$$

We first prove that both terms in (1.2) are well defined for  $\phi \in X_p$  and almost all  $\xi \in \mathbb{R}^n$ :

**Lemma 2.1.** *Let  $p$  as in Theorem 1.1, and  $\phi \in X_p$ . Let  $F$  denote either of the two terms in (1.2). Then  $F \in L^2(\mathbb{R}^n)$ . More precisely, there exists  $C > 0$  independent of  $\phi \in X_p$  such that:*

$$\|F\|_{L^2} \leq C \begin{cases} \left\| |x|^{\delta(2p)} \phi \right\|_{L^2}^{\theta p} \|\phi\|_{L^2}^{(1-\theta)p} & \text{if } 1 + 2/n < p < 1 + 4/n, \\ \|\phi\|_{L^2} & \text{if } p = 1 + 4/n, \\ \left\| (-\Delta)^{\delta(p+1)/2} \phi \right\|_{L^2}^{(1-\sigma)p} \|\phi\|_{L^2}^{\sigma p} & \text{if } p > 1 + 4/n, \end{cases}$$

where  $\theta$  and  $\sigma$  are given by:

$$\theta = \frac{4}{n(p-1)} - 1 \quad ; \quad \sigma = \frac{n+4-(n-4)p}{np(p-1)}.$$

*Remark 2.2.* We check the following algebraic identities:

- $0 < \theta < 1 \iff 1 + 2/n < p < 1 + 4/n$ , and  $\theta = 0 \iff p = 1 + 4/n$ .
- $\sigma < 1 \iff p > 1 + 4/n$ ;  $\sigma = 1 \iff p = 1 + 4/n$ .
- $\sigma > 0$ , since for  $n \geq 3$ ,  $(n-2)p \leq n+2$ .

*Proof.* By symmetry, we consider only the plus sign in (1.2). We distinguish three cases, according to the value of  $p$ .

**First case:**  $1 + 2/n < p < 1 + 4/n$ . Let  $\psi \in L^2(\mathbb{R}^n)$ ,  $T > 0$ , and  $q$  be defined by  $2/q = \delta(p+1)$ . Note that  $0 < 2/q < 1$ , so the pair  $(q, p+1)$  is Strichartz admissible. By duality, we have:

$$\begin{aligned} \left| \left\langle \int_0^T e^{i\frac{t}{2}|\xi|^2} \mathcal{F}(|U(t)\phi|^{p-1}U(t)\phi) dt, \widehat{\psi} \right\rangle \right| &= \left| \int_0^T \langle |U(t)\phi|^{p-1}U(t)\phi, U(t)\psi \rangle dt \right| \\ &\leq \int_0^T \|U(t)\phi\|_{L^{p+1}}^p \|U(t)\psi\|_{L^{p+1}} dt \\ &\leq T^{1-n(p-1)/4} \|U(\cdot)\phi\|_{L^q(0,T;L^{p+1})}^p \|U(\cdot)\psi\|_{L^q(0,T;L^{p+1})} \\ &\leq CT^{1-n(p-1)/4} \|\phi\|_{L^2}^p \|\psi\|_{L^2}, \end{aligned}$$

where  $C$ , independent of  $T$ , is provided by Strichartz inequalities. Note that for the Hölder inequality in time, we have used the formula:

$$1 = \left(1 - \frac{n(p-1)}{4}\right) + \frac{p+1}{q}.$$

We also have directly

$$\begin{aligned} \left\| \int_T^\infty e^{i\frac{t}{2}|\xi|^2} \mathcal{F}(|U(t)\phi|^{p-1}U(t)\phi) dt \right\|_{L^2} &\leq \int_T^\infty \left\| e^{i\frac{t}{2}|\xi|^2} \mathcal{F}(|U(t)\phi|^{p-1}U(t)\phi) \right\|_{L^2} dt \\ &\leq \int_T^\infty \|U(t)\phi\|_{L^{2p}}^p dt \\ &\leq C \int_T^\infty \|U(t)\phi\|_{L^{2p}}^p dt. \end{aligned}$$

Using the factorization for the group  $U$  recalled above, we find:

$$\|U(t)\phi\|_{L^{2p}} = t^{-\delta(2p)} \|\mathcal{F}M_t\phi\|_{L^{2p}} \leq Ct^{-\delta(2p)} \|\mathcal{F}M_t\phi\|_{\dot{H}^{\delta(2p)}},$$

where we have used the critical Sobolev embedding. We infer

$$\begin{aligned} \left\| \int_T^\infty e^{i\frac{t}{2}|\xi|^2} \mathcal{F}(|U(t)\phi|^{p-1}U(t)\phi) dt \right\|_{L^2} &\leq C \int_T^\infty t^{-p\delta(2p)} \left\| |x|^{\delta(2p)} \phi \right\|_{L^2}^p dt \\ &\leq CT^{1-n(p-1)/2} \left\| |x|^{\delta(2p)} \phi \right\|_{L^2}^p. \end{aligned}$$

We have finally, for any  $T > 0$ :

$$\begin{aligned} \left\| \int_0^\infty e^{i\frac{t}{2}|\xi|^2} \mathcal{F}(|U(t)\phi|^{p-1}U(t)\phi) dt \right\|_{L^2} &\leq C \left( T^{1-n(p-1)/4} \|\phi\|_{L^2}^p \right. \\ &\quad \left. + T^{1-n(p-1)/2} \left\| |x|^{\delta(2p)} \phi \right\|_{L^2}^p \right), \end{aligned}$$

where  $C$  is independent of  $T$ . Optimizing in  $T$ , we find:

$$\left\| \int_0^\infty e^{i\frac{t}{2}|\xi|^2} \mathcal{F}(|U(t)\phi|^{p-1}U(t)\phi) dt \right\|_{L^2} \leq C \left\| |x|^{\delta(2p)} \phi \right\|_{L^2}^{\theta p} \|\phi\|_{L^2}^{(1-\theta)p},$$

where  $\theta = \frac{4}{n(p-1)} - 1$ . For the other term involved in (1.2), we proceed in a similar fashion:

$$\begin{aligned}
& \left| \left\langle \int_{1/T}^{\infty} t^{n(p-1)/2-2} U(t) \left( |U(-t)\widehat{\phi}|^{p-1} U(-t)\widehat{\phi} \right) dt, \widehat{\psi} \right\rangle \right| = \\
& = \left| \int_{1/T}^{\infty} t^{n(p-1)/2-2} \left\langle |U(-t)\widehat{\phi}|^{p-1} U(-t)\widehat{\phi}, U(-t)\widehat{\psi} \right\rangle dt \right| \\
& \leq \int_{1/T}^{\infty} t^{n(p-1)/2-2} \left\| |U(-t)\widehat{\phi}|^p \right\|_{L^{p+1}} \left\| U(-t)\widehat{\psi} \right\|_{L^{p+1}} dt \\
& \leq \left( \int_{1/T}^{\infty} t^{(n(p-1)/2-2)/(1-n(p-1)/4)} dt \right)^{1-n(p-1)/4} \left\| U(\cdot)^{-1}\widehat{\phi} \right\|_{L^q L^{p+1}}^p \left\| U(\cdot)^{-1}\widehat{\psi} \right\|_{L^q L^{p+1}} \\
& \leq CT^{1-n(p-1)/4} \|\phi\|_{L^2}^p \|\psi\|_{L^2},
\end{aligned}$$

for the same  $q$  as above, given by  $2/q = \delta(p+1)$ . We also have directly

$$\begin{aligned}
& \left\| \int_0^{1/T} t^{n(p-1)/2-2} U(t) \left( |U(-t)\widehat{\phi}|^{p-1} U(-t)\widehat{\phi} \right) dt \right\|_{L^2} \leq \\
& \leq \int_0^{1/T} t^{n(p-1)/2-2} \left\| |U(-t)\widehat{\phi}|^{p-1} U(-t)\widehat{\phi} \right\|_{L^2} dt \\
& \leq \int_0^{1/T} t^{n(p-1)/2-2} \left\| U(-t)\widehat{\phi} \right\|_{L^{2p}}^p dt \\
& \leq C \int_0^{1/T} t^{n(p-1)/2-2} \left\| U(-t)\widehat{\phi} \right\|_{\dot{H}^{\delta(2p)}}^p dt \\
& \leq C \int_0^{1/T} t^{n(p-1)/2-2} \left\| \widehat{\phi} \right\|_{\dot{H}^{\delta(2p)}}^p dt = CT^{1-n(p-1)/2} \left\| |x|^{\delta(2p)} \phi \right\|_{L^2}^p.
\end{aligned}$$

We infer:

$$\begin{aligned}
& \left\| \int_0^{\infty} t^{n(p-1)/2-2} U(t) \left( |U(-t)\widehat{\phi}|^{p-1} U(-t)\widehat{\phi} \right) dt \right\|_{L^2} \leq C \left( T^{1-n(p-1)/4} \|\phi\|_{L^2}^p \right. \\
& \quad \left. + T^{1-n(p-1)/2} \left\| |x|^{\delta(2p)} \phi \right\|_{L^2}^p \right).
\end{aligned}$$

We can then conclude as above.

**Second case:**  $p = 1 + 4/n$ . In this case, note that the power of  $t$  in the second term of (1.2) is zero:  $n(p-1)/2 - 2 = 0$ . To prove the result in this case, just notice that the above proof remains valid: for  $\psi \in L^2(\mathbb{R}^n)$  and  $T > 0$ , we now have

$$\begin{aligned}
& \left| \left\langle \int_0^T e^{i\frac{\xi}{2}|\xi|^2} \mathcal{F}(|U(t)\phi|^{p-1} U(t)\phi) dt, \widehat{\psi} \right\rangle \right| \leq CT^{1-n(p-1)/4} \|\phi\|_{L^2}^p \|\psi\|_{L^2} \\
& \leq C \|\phi\|_{L^2}^p \|\psi\|_{L^2},
\end{aligned}$$

where  $C$  is independent of  $T$ . The estimate for the other term in (1.2) is straightforward, by duality.

**Third case:**  $p > 1 + 4/n$ . For  $\psi \in L^2(\mathbb{R}^n)$ , we compute

$$\begin{aligned}
& \left| \left\langle \int_0^{\infty} e^{i\frac{\xi}{2}|\xi|^2} \mathcal{F}(|U(t)\phi|^{p-1} U(t)\phi) dt, \widehat{\psi} \right\rangle \right| \leq \int_0^{\infty} \|U(t)\phi\|_{L^{p+1}}^p \|U(t)\psi\|_{L^{p+1}} dt \\
& \leq \|U(\cdot)\phi\|_{L^\infty L^{p+1}}^{(1-\sigma)p} \|U(\cdot)\phi\|_{L^q L^{p+1}}^{\sigma p} \|U(\cdot)\psi\|_{L^q L^{p+1}},
\end{aligned}$$

for  $2/q = \delta(p+1)$ , where we have used the identity  $1 = \sigma p/q + 1/q$ . We conclude thanks to the Sobolev embedding  $\dot{H}^{\delta(p+1)} \hookrightarrow L^{p+1}$  and Strichartz inequalities. Note that for  $n \geq 3$  and  $p = 1 + 4/(n-2)$ , we use endpoint estimates [4].

For the other term, write

$$\begin{aligned} & \int_0^\infty t^{n(p-1)/2-2} \left\| U(-t)\widehat{\phi} \right\|_{L^{p+1}}^p \left\| U(-t)\widehat{\psi} \right\|_{L^{p+1}} dt \leq \\ & \leq \left( \sup_{t>0} t^{n(p-1)/2-2} \left\| U(-t)\widehat{\phi} \right\|_{L^{p+1}}^{(1-\sigma)p} \right) \left\| U(\cdot)^{-1}\widehat{\phi} \right\|_{L^q L^{p+1}}^{\sigma p} \left\| U(\cdot)^{-1}\widehat{\psi} \right\|_{L^q L^{p+1}} \\ & \leq C \left( \sup_{t>0} t^{n(p-1)/2-2} \left\| U(-t)\widehat{\phi} \right\|_{L^{p+1}}^{(1-\sigma)p} \right) \|\phi\|_{L^2}^{\sigma p} \|\psi\|_{L^2}. \end{aligned}$$

We then remark that

$$\begin{aligned} \left\| U(-t)\widehat{\phi} \right\|_{L^{p+1}} &= \left\| M_t U(-t)\widehat{\phi} \right\|_{L^{p+1}} = \left\| D_{-t} \mathcal{F} M_{-t} \widehat{\phi} \right\|_{L^{p+1}} = \frac{1}{|t|^{\delta(p+1)}} \left\| \mathcal{F} M_{-t} \widehat{\phi} \right\|_{L^{p+1}} \\ &\leq \frac{C}{|t|^{\delta(p+1)}} \left\| \mathcal{F} M_{-t} \widehat{\phi} \right\|_{\dot{H}^{\delta(p+1)}} = \frac{C}{|t|^{\delta(p+1)}} \left\| |x|^{\delta(p+1)} M_{-t} \widehat{\phi} \right\|_{L^2} \\ &\leq \frac{C}{|t|^{\delta(p+1)}} \left\| |x|^{\delta(p+1)} \widehat{\phi} \right\|_{L^2} = \frac{C}{|t|^{\delta(p+1)}} \|\phi\|_{\dot{H}^{\delta(p+1)}}. \end{aligned}$$

In view of the identity  $(1-\sigma)p\delta(p+1) = n(p-1)/2 - 2$ , this yields

$$\left\| \int_0^{\pm\infty} |t|^{n(p-1)/2-2} U(t) \left( \left| U(-t)\widehat{\phi} \right|^{p-1} U(-t)\widehat{\phi} \right) dt \right\|_{L^2} \leq C \|\phi\|_{\dot{H}^{\delta(p+1)}}^{(1-\sigma)p} \|\phi\|_{L^2}^{\sigma p},$$

which completes the proof of the lemma.  $\square$

We can now prove Theorem 1.1.

*Proof of Theorem 1.1.* Recall the decomposition  $U(t) = M_t D_t \mathcal{F} M_t$ . Direct computations yield:

$$(2.1) \quad \mathcal{F} D_t = D_{1/t} \mathcal{F},$$

$$(2.2) \quad D_t^{-1} = i^n D_{1/t},$$

$$(2.3) \quad \mathcal{F}^{-1} D_t^{-1} = i^n D_t \mathcal{F}^{-1}.$$

We infer

$$U(-t) = U(t)^{-1} = M_{-t} \mathcal{F}^{-1} D_t^{-1} M_{-t} = i^n M_{-t} D_t \mathcal{F}^{-1} M_{-t}.$$

Since  $U(t) = \mathcal{F}^{-1} M_{-1/t} \mathcal{F}$ , we deduce

$$U(-t)\mathcal{F} = i^n M_{-t} D_t U \left( \frac{1}{t} \right),$$

which in turn implies

$$\begin{aligned} \left| U(-t)\widehat{\phi} \right|^{p-1} U(-t)\widehat{\phi} &= i^n M_{-t} \left| D_t U \left( \frac{1}{t} \right) \phi \right|^{p-1} D_t U \left( \frac{1}{t} \right) \phi \\ &= i^n t^{-n(p-1)/2} M_{-t} D_t \left( \left| U \left( \frac{1}{t} \right) \phi \right|^{p-1} U \left( \frac{1}{t} \right) \phi \right). \end{aligned}$$

Using (2.1) and (2.2) again, we have then:

$$(2.4) \quad U(t) \left( \left| U(-t)\widehat{\phi} \right|^{p-1} U(-t)\widehat{\phi} \right) = t^{-n(p-1)/2} M_t \mathcal{F} \left( \left| U\left(\frac{1}{t}\right)\phi \right|^{p-1} U\left(\frac{1}{t}\right)\phi \right).$$

Theorem 1.1 follows by integrating the above identity on a half line, and using the change of variable  $t \mapsto 1/t$ .  $\square$

*Remark 2.3.* The identity (2.4) can also be considered as a Poisson formula, by writing  $U(t)$  on the left hand side, and  $\mathcal{F}$  on the right hand side, as integrals.

### 3. THE LONG RANGE CASE

When  $\phi$  is a Gaussian function, the value in (1.2) can be computed explicitly. For  $\operatorname{Re} z > 0$ , define:

$$g_z(x) = e^{-z\frac{|x|^2}{2}}, \quad x \in \mathbb{R}^n.$$

We have:

$$\int_{\mathbb{R}^n} g_z(x) dx = \left( \frac{2\pi}{z} \right)^{n/2}.$$

We compute:

$$\mathcal{F}g_z(\xi) = z^{-n/2} e^{-\frac{|\xi|^2}{2z}},$$

and

$$U(t)g_z(x) = (1+itz)^{-n/2} e^{-\frac{z}{1+itz}\frac{|x|^2}{2}}.$$

Note that if  $z = a + ib$ ,

$$\operatorname{Re} \left( \frac{z}{1+itz} \right) = \frac{a}{(1-bt)^2 + a^2t^2} > 0.$$

For  $p > 1$  and  $z = a + ib$ , we find:

$$|U(t)g_z|^{p-1} U(t)g_z = \frac{e^{-\frac{(p-1)a}{(1-bt)^2 + (at)^2}\frac{|x|^2}{2}}}{((1-bt)^2 + (at)^2)^{n(p-1)/4}} (1+itz)^{-n/2} e^{-\frac{z}{1+itz}\frac{|x|^2}{2}}.$$

Set

$$\zeta = \frac{(p-1)a}{(1-bt)^2 + (at)^2} + \frac{z}{1+itz}.$$

We have:

$$\mathcal{F} \left( |U(t)g_z|^{p-1} U(t)g_z \right) = \frac{1}{((1-bt)^2 + (at)^2)^{n(p-1)/4}} (1+itz)^{-n/2} \zeta^{-n/2} e^{-\frac{|x|^2}{2\zeta}}.$$

Consider the case  $z \in \mathbb{R}$ :  $b = 0$ . We find:

$$\zeta = \frac{a}{1+(at)^2} (p - iat).$$

We infer:

$$e^{i\frac{1}{2}|x|^2} \mathcal{F} \left( |U(t)g_z|^{p-1} U(t)g_z \right) = \frac{1}{(1+(at)^2)^{n(p-1)/4}} (\zeta(1+ita))^{-n/2} e^{(it-\frac{1}{\zeta})\frac{|x|^2}{2}}.$$

We compute

$$it - \frac{1}{\zeta} = \frac{iatp-1}{a(p-iat)} \xrightarrow{t \rightarrow \infty} -\frac{p}{a}.$$

Also,

$$\zeta(1 + ita) = \frac{a}{1 + (at)^2} (p - iat)(1 + ita) \xrightarrow{t \rightarrow \infty} a.$$

We have finally:

$$e^{i\frac{t}{2}|x|^2} \mathcal{F} \left( |U(t)g_z|^{2\sigma} U(t)g_z \right) \underset{t \rightarrow \infty}{\sim} \frac{1}{(1 + (at)^2)^{n(p-1)/4}} \frac{1}{a^{n/2}} e^{-p\frac{|x|^2}{2a}}.$$

Integrating with respect to  $t$ , the integral is convergent if and only if  $p > 1 + 2/n$ . Since we also have

$$\begin{aligned} U(t) \left( |U(-t)\widehat{g}_a|^{p-1} U(-t)\widehat{g}_a \right) &= \frac{a^2 + itp}{a + it} \frac{1}{(a^2 + t^2)^{n(p-1)/4}} e^{-\frac{p+it}{a^2+itp}\frac{|x|^2}{2}} \\ &\xrightarrow{t \rightarrow 0} a^{1-n(p-1)/2} e^{-\frac{p}{a^2}\frac{|x|^2}{2}}, \end{aligned}$$

we check that both terms in (1.2) become infinite for  $p = 1 + 2/n$ , due to a logarithmic divergence.

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