



HAL
open science

Eigenvalue pinching and application to the stability and the almost umbilicity of hypersurfaces

Jean-Francois Grosjean

► **To cite this version:**

Jean-Francois Grosjean. Eigenvalue pinching and application to the stability and the almost umbilicity of hypersurfaces. 2007. hal-00170114v1

HAL Id: hal-00170114

<https://hal.science/hal-00170114v1>

Preprint submitted on 6 Sep 2007 (v1), last revised 14 Feb 2011 (v3)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

EIGENVALUE PINCHING AND APPLICATION TO THE STABILITY AND THE ALMOST UMBILICITY OF HYPERSURFACES

J.-F. GROSJEAN

ABSTRACT. In this paper we give pinching theorems for the first nonzero eigenvalue of the Laplacian on the compact hypersurfaces of ambient spaces with bounded sectional curvature. As application we deduce rigidity results for stable constant mean curvature hypersurfaces M of these spaces N . Indeed, we prove that if M is included in a ball of radius small enough then the Hausdorff-distance between M and a geodesic sphere S of N is small. Moreover M is diffeomorphic and quasi-isometric to S . As other application, we give rigidity results for almost umbilic hypersurfaces.

CONTENTS

1. Introduction	1
2. Preliminaries	4
3. An L^2 -approach	6
4. The proof of the diffeomorphism	17
5. Application to the stability	23
6. Application to the almost umbilic hypersurfaces	25
References	26

1. INTRODUCTION

One way to show that the geodesic spheres are the only stable constant mean curvature hypersurfaces of classical model spaces (i.e. the Euclidean space, the spherical space and the hyperbolic space) is to prove that there is equality in the well-known Reilly's inequality. One of the main points of the present paper is to obtain new stability results for hypersurfaces immersed in more general ambient spaces.

First, let us recall the Reilly's inequality. Let (M^m, g) be a compact, connected and oriented m -dimensional Riemannian manifold without boundary isometrically immersed by ϕ in the simply connected model space $N^{n+1}(c)$ ($c = 0, 1, -1$ respectively for the Euclidean space, the sphere or the hyperbolic space). The Reilly's inequality gives an extrinsic upper bound for the first nonzero eigenvalue $\lambda_1(M)$ of the Laplacian of (M^m, g) in term of the square of the length of the mean curvature H . Indeed we have

Date: 6th September 2007.

2000 Mathematics Subject Classification. 53A07, 53C21.

Key words and phrases. Spectrum, Laplacian, pinching results, hypersurfaces.

$$(1) \quad \lambda_1(M) \leq \frac{m}{V(M)} \int_M (|H|^2 + c) dv$$

where dv and $V(M)$ denote respectively the Riemannian volume element and the volume of (M^m, g) . Moreover in the case of hypersurfaces (i.e. $m = n$), the equality holds if and only if (M^n, g) is immersed as a geodesic sphere of $N^{n+1}(c)$. For $c = 0$ this inequality was proved by Reilly ([11]) and can easily be extended to the spherical case $c = 1$ by considering the canonical embedding of \mathbb{S}^n in \mathbb{R}^{n+1} . For $c = -1$ it has been proved by El Soufi and Ilias in [7].

In the sequel we will consider a weaker inequality due to Heintze ([8]) which generalizes the previous for the case where (M^m, g) is isometrically immersed by ϕ in a $n + 1$ -dimensional Riemannian manifold (N^{n+1}, h) whose sectional curvature K^N is bounded above by δ . Indeed if $\phi(M)$ lies in a convex ball and if the radius of this ball is $\frac{\pi}{4\sqrt{\delta}}$ in the case $\delta > 0$, we have

$$(2) \quad \lambda_1(M) \leq m(\|H\|_\infty + \delta)$$

where $\|H\|_\infty$ denotes the L^∞ -norm of the mean curvature. Now for $m = n$ if we assume that K^N is bounded below by μ and M has a constant mean curvature H and is stable (see section 5) we have

$$n(H^2 + \mu) \leq \lambda_1(M) \leq n(H^2 + \delta)$$

Consequently we see that if N is not of constant sectional curvature we can't conclude as in the model spaces. However, the above inequality is a kind of pinching on the Reilly's inequality, that is a condition of almost equality. Such conditions have been studied for the Reilly's inequality in the Euclidean space in [6]. In the present paper we will generalize the results of [6] to the inequality (2) for hypersurfaces (i.e. $m = n$) of ambient spaces with non constant sectional curvature. That amounts to finding a constant C depending on minimum geometric invariants so that if we have the condition

$$n(\|H\|_\infty^2 + \delta) - C < \lambda_1(M)$$

then M is close to a sphere in a certain sense.

Before giving the main theorems, we precise some notations which will be more convenient. Throughout the paper, we will note $h = (\|H\|_\infty^2 + \delta)^{1/2}$ and $b = \left(\frac{\|B\|_\infty^2}{n} + \delta\right)^{1/2}$ where B denotes the second fundamental form. Moreover if (N^{n+1}, h) is a $n + 1$ -dimensional Riemannian manifold with $K^N \leq \delta$ we will note $\mathcal{H}^*(n, \phi, N)$ the space of all Riemannian compact, connected and oriented n -dimensional Riemannian manifolds without boundary isometrically immersed by ϕ in (N^{n+1}, h) so that the injectivity radius $i(N)$ of N satisfies $i(N) \geq \frac{\pi}{\sqrt{\delta}}$ if $\delta > 0$. We call $\mathcal{H}_C(n, \phi, N)$ the space of all Riemannian manifolds of $\mathcal{H}^*(n, \phi, N)$ satisfying the following convexity hypothesis : $\phi(M)$ lies in a convex ball and the radius of this ball is $\frac{\pi}{4\sqrt{\delta}}$ if $\delta > 0$. Moreover $\mathcal{H}_V(n, \phi, N)$ will be the space of all Riemannian manifolds of $\mathcal{H}^*(n, \phi, N)$ which satisfy the following hypothesis on the volume : $V(M) \leq \frac{c\omega_n}{\delta^{n/2}}$

if $\delta > 0$ and $V(M) \leq (c\omega_n)^{1/n}i(N)$ if $\delta \leq 0$ for some constant c . At last we put $\mathcal{H}(n, \phi, N) = \mathcal{H}_C(n, \phi, N) \cap \mathcal{H}_V(n, \phi, N)$.

Furthermore we need the following function s_δ defined by

$$s_\delta(r) = \begin{cases} \frac{1}{\sqrt{\delta}} \sin \sqrt{\delta}r & \text{if } \delta > 0 \\ r & \text{if } \delta = 0 \\ \frac{1}{\sqrt{|\delta|}} \sinh \sqrt{|\delta|}r & \text{if } \delta < 0 \end{cases},$$

Moreover we will note $B(p, R)$ all geodesic ball in N of center p and radius R .

The first nonzero eigenvalue of the Laplacian is in fact the second eigenvalue since the first $\lambda_1^0(M)$ is vanishing. The previous Reilly's inequalities can be generalized for the second eigenvalue $\lambda_1^L(M)$ of Schrödinger operators $L = \Delta + q$ where $q \in C^\infty(M)$ and we have

$$\lambda_1^L(M) \leq n(\|H\|_\infty^2 + \delta) + \|q\|_\infty$$

The pinching theorems are stated for the second eigenvalue $\lambda_1^L(M)$ of a Schrödinger operators $L = \Delta + q$

Theorem 1.1. *Let (N^{n+1}, h) be a $n + 1$ -dimensional Riemannian manifold whose sectional curvature K^N satisfies $\mu \leq K^N \leq \delta$ and let $M \in \mathcal{H}(n, \phi, N)$. Let $\theta \in (0, 1)$ and $\varepsilon \leq \frac{\theta}{5b}$. Then there exist a point p and positive constants $C_\varepsilon(n, b, h, V(M), \delta)$ and $R(\delta, \mu, \varepsilon b)$ so that if $\phi(M)$ is contained in the ball $B(p, R(\delta, \mu, \varepsilon b))$ and if the pinching condition (P_{C_ε})*

$$n(\|H\|_\infty^2 + \delta) + \|q\|_\infty - C_\varepsilon < \lambda_1^L(M)$$

is satisfied then the Gromov-Hausdorff is so that

$$d_{GH} \left(\phi(M), S \left(p, s_\delta^{-1} \left(\frac{1}{h} \right) \right) \right) < \varepsilon$$

and M is diffeomorphic and θ -quasi-isometric to $S(p, s_\delta^{-1}(\frac{1}{h}))$. Namely there exists a diffeomorphism from M into $S(p, s_\delta^{-1}(\frac{1}{h}))$ so that

$$| |dF_x(u)|^2 - 1 | \leq \theta$$

for any $x \in M$, $u \in T_x M$ and $|u| = 1$.

Moreover, $R(\delta, \mu, \varepsilon b) \rightarrow \infty$ when $\delta - \mu \rightarrow 0$. On the other hand,

- (1) $C_\varepsilon(n, b, h, V(M), \delta) \rightarrow 0$ when $\varepsilon \rightarrow 0$.
- (2) $C_\varepsilon(n, b, h, V(M), \delta) \rightarrow 0$ when $b \rightarrow \infty$.
- (3) if $V(M)^{1/n}b \leq v$ and $\varepsilon = \frac{\kappa}{b}$ for positive constants v and κ , then $C_\varepsilon(n, b, h, V(M), \delta) \rightarrow \infty$ when $b \rightarrow \infty$.

We recall that the Gromov-Hausdorff distance between two compact subsets A and B of a metric space is given by

$$d_{GH}(A, B) = \inf \{ A \subset V_\eta(B) \text{ and } B \subset V_\eta(A) \}$$

where for any subset A , $V_\eta(A)$ is the tubular neighborhood of A defined by $V_\eta(A) = \{x | d(x, A) < \eta\}$.

Remark 1.1. *The point p is not depending on θ , ε , b or h . The point p is nothing but the center of mass of M (see preliminaries).*

Remark 1.2. *We will see in the proofs that we need the dependence on b . However in the case of nonnegative sectional curvature the dependence of $C_\varepsilon(n, b, h, V(M), \delta)$ on h can be omitted.*

As in the euclidean case (see [6]), in the hyperbolic case or spherical case, we can obtain the Hausdorff proximity strictly with a dependence on h . More precisely we have the

Theorem 1.2. *Let $N^{n+1}(\delta)$ with $\delta = -1, 0$ or 1 where $N^{n+1}(-1)$, $N^{n+1}(0)$ and $N^{n+1}(1)$ are respectively the hyperbolic space, the euclidean space and the sphere. Let $M \in \mathcal{H}(n, \phi, N)$. Then for any $\varepsilon > 0$ there exists a positive constant $C_\varepsilon(n, h, V(M))$ so that if the pinching condition (P_{C_ε})*

$$n(\|H\|_\infty^2 + \delta) + \|q\|_\infty - C_\varepsilon < \lambda_1^L(M)$$

is satisfied then the Gromov-Hausdorff satisfies

$$d_{GH} \left(\phi(M), S \left(p, s_\delta^{-1} \left(\frac{1}{h} \right) \right) \right) < \varepsilon$$

The condition (3) of the theorem 1.1 allows to obtain an application for the stable constant mean curvature hypersurfaces. Indeed we have the following stability theorem

Theorem 1.3. *Let (N^{n+1}, h) be a $n + 1$ -dimensional Riemannian manifold whose sectional curvature K^N satisfies $\mu \leq K^N \leq \delta$ and let $M \in \mathcal{H}_C(n, \phi, N)$. Let $v > 0$ so that $V(M)^{1/n} \leq \frac{v}{b}$. For any $\theta \in (0, 1)$, there exists a constant $R_0(\delta, \mu, v, i(N)) > 0$ so that if $\phi(M)$ lies in a ball of radius $R_0(\delta, \mu, v, i(N))$ and ϕ is of constant mean curvature H and is stable then there exists a point p so that*

$$d_{GH} \left(\phi(M), S \left(p, s_\delta^{-1} \left(\frac{1}{h} \right) \right) \right) < \frac{\theta}{5b}$$

and M is diffeomorphic and θ -quasi-isometric to $S(p, s_\delta^{-1}(\frac{1}{h}))$.

Remark 1.3. *If $\delta > 0$, $R_0(\delta, \mu, v, i(N))$ is not depending on the injectivity radius.*

We will end this paper in section 6 by another application of theorems 1.1 and 1.2 for the almost umbilic hypersurfaces of model spaces.

2. PRELIMINARIES

Let (M^n, g) be a compact, connected n -dimensional Riemannian manifold isometrically immersed by ϕ in an $n + 1$ -dimensional Riemannian manifold (N^{n+1}, h) which sectional curvature is bounded by δ . Let $p \in N$ and \exp be the exponential map at this point. If $\delta > 0$, we need to assume that $\phi(M)$ lies in a convex ball around p of radius less than or equal to $\frac{\pi}{2\sqrt{\delta}}$. We consider $(x_i)_{1 \leq i \leq n}$ the normal coordinates of N centered at p and for all $x \in N$, we denote by $r(x) = d(p, x)$, the geodesic distance between p and x on (N^{n+1}, h) .

Moreover we define the function c_δ by $c_\delta = s'_\delta$. Obviously, we have $c_\delta^2 + \delta s_\delta^2 = 1$ and $c'_\delta = -\delta s_\delta$.

The gradient of a function u define on N with respect to h will be denoted by $\nabla^N u$ and the gradient with respect to g of the restriction of u on M will be denoted by $\nabla^M u$.

Briefly, we recall the proof of Heintze ([8]) for the Reilly inequality. This proof can be extend without problem to Schrödinger operators $L = \Delta + q$.

We will use $\frac{s_\delta(r)}{r}x_i$ as test functions in the variational characterization of $\lambda_1(M)$ but these functions must be L^2 -orthogonal to the first eigenfunction u of L which can be chosen positive. For this purpose, we use a standard argument used by Chavel and Heintze ([5] and [8]). Indeed, let Y be a vector field defined by

$$Y_q = \int_M \frac{s_\delta(d(q, x))}{d(q, x)} \exp_q^{-1}(x) u(x) dv(x) \in T_q N, \quad q \in M, \quad ,$$

From the fixed point theorem of Brouwer, there exists a point $p \in N$ such that $Y_p = 0$ and consequently, for a such p , $\int_M \frac{s_\delta(r)}{r} x_i u dv = 0$. But for $\delta > 0$, we must assume $\phi(M)$ is contained in a ball of radius $\frac{\pi}{4\sqrt{\delta}}$. Indeed, in this case $\phi(M)$ lies in a ball of center p (the point p so that $Y_p = 0$) with a radius less or equal to $\frac{\pi}{2\sqrt{\delta}}$.

Now considering the vector field on M , $Z = s_\delta \nabla^N r$ and noting that the coordinates of Z in the normal local frame are $\left(\frac{s_\delta(r)}{r} x_i\right)_{1 \leq i \leq n}$, we have

$$\begin{aligned} \lambda_1^L(M) \int_M s_\delta^2(r) dv &= \lambda_1^L(M) \int_M |Z|^2 dv = \lambda_1^L(M) \int_M \sum_{i=1}^{n+1} \left(\frac{s_\delta(r)}{r} x_i\right)^2 dv \\ &\leq \int_M \sum_{i=1}^{n+1} \left| \nabla^M \left(\frac{s_\delta(r)}{r} x_i\right) \right|^2 dv + \int_M q |Z|^2 dv \end{aligned}$$

Now, Heintze proved that $\sum_{i=1}^{n+1} \left| \nabla^M \left(\frac{s_\delta(r)}{r} x_i\right) \right|^2 \leq n - \delta |Z^T|^2$ and

$$(3) \quad \operatorname{div} (Z^T) \geq n c_\delta - n H \langle Z, \nu \rangle$$

Then

$$\begin{aligned} \lambda_1^L(M) \int_M |Z|^2 dv &\leq \int_M (n - \delta |Z^T|^2) dv + \int_M q |Z|^2 dv \\ &= \int_M (n - \operatorname{div} (Z^T) c_\delta) dv + \int_M q |Z|^2 dv \\ &\leq \int_M (n - n c_\delta^2 + n H \langle Z, \nu \rangle c_\delta) dv + \int_M q |Z|^2 dv \\ &= \int_M (n \delta s_\delta^2 + n H \langle Z, \nu \rangle c_\delta) dv + \int_M q |Z|^2 dv \end{aligned}$$

$$\leq \int_M n\delta s_\delta^2 dv + \|H\|_\infty \int_M ns_\delta c_\delta dv + \int_M q|Z|^2 dv$$

and using again (3) we get

$$\begin{aligned} \lambda_1^L(M) \int_M |Z|^2 dv &\leq n\delta \int_M |Z|^2 dv + \|H\|_\infty \int_M (nH\langle Z, \nu \rangle s_\delta + \operatorname{div} (Z^T) s_\delta) dv + \int_M q|Z|^2 dv \\ &= n\delta \int_M |Z|^2 dv + \|H\|_\infty \int_M (nH\langle Z, \nu \rangle s_\delta - c_\delta s_\delta |\nabla^M r|^2) dv + \int_M q|Z|^2 dv \\ &\leq n\delta \int_M |Z|^2 dv + n\|H\|_\infty^2 \int_M |\langle Z, \nu \rangle| |Z| dv + \int_M q|Z|^2 dv \\ &\leq n\delta \int_M |Z|^2 dv + n\|H\|_\infty^2 \int_M |Z|^2 dv + \int_M q|Z|^2 dv \\ &\leq n(\|H\|_\infty^2 + \delta) \int_M |Z|^2 dv + \|q\|_\infty \int_M |Z|^2 dv \end{aligned}$$

3. AN L^2 -APPROACH

First we recall the standard Sobolev inequality (cf [9], [10], [16] and p 216 in [4]). If f is a smooth function and $f \geq 0$, then

$$(4) \quad \left(\int_M f^{\frac{n}{n-1}} dv \right)^{1-(1/n)} \leq K(n) \int_M (|df| + |H|f) dv$$

where $K(n)$ is a constant depending on n and the volume of the unit ball in \mathbb{R}^n . Taking $f = 1$ on M and $r \in [1, +\infty]$, we deduce that

$$(5) \quad \frac{\|H\|_{2r}^2}{V(M)^{1/r}} \geq \frac{\alpha}{V(M)^{2/n}}$$

An easy computation shows us that

$$(6) \quad h^2 \geq \frac{\alpha}{V(M)^{2/n}} (1 + o(1))$$

Throughout the paper $o(1)$ will denote any function depending on b , h , on the dimension or the sectional curvature of (N^{n+1}, h) but not on the volume of M so that $o(1)$ tends to 0 when b or h goes to the infinity. Moreover α denotes a constant depending on n .

Remark 3.1. *Note that here we can choose $o(1) = 0$ for nonnegative sectional curvature. But we can't deduce such estimate with $o(1) = 0$ or $o(1)$ depending strictly on b for the negative sectional curvature case. This is the reason for which we can omit the dependence on h in the theorem 1.1 as mentioned in the remark 1.2.*

Lemma 3.1. *If the pinching condition (P_C) holds then $\|Z^T\|_2^2 \leq \beta V(M)^{2/n} \|Z\|_2^2 C$.*

Proof. We have

$$\begin{aligned}\|Z^T\|_2^2 &= \int_M |Z|^2 dv - \int_M \langle Z, \nu \rangle^2 dv = \int_M (|Z| + |\langle Z, \nu \rangle|)(|Z| - |\langle Z, \nu \rangle|) dv \\ &\leq 2 \int_M (|Z|^2 - |\langle Z, \nu \rangle| |Z|) dv\end{aligned}$$

and from the proof of Reilly's inequality and the pinching condition, we have

$$n\|H\|_\infty^2 \left(\int_M |Z|^2 dv - \int_M |\langle Z, \nu \rangle| |Z| dv \right) < C\|Z\|_2^2$$

Now the relation (5) allows us to conclude. □

Lemma 3.2. *If $C < \frac{\alpha}{V(M)^{2/n}}(1 + o(1)) < \frac{1}{2} \left(\frac{n^2}{n+2} \right) h^2$, then (P_C) implies that*

$$\|Z\|_2^2 \leq \frac{nV(M)}{nh^2 - \left(\frac{n+2}{n}\right)C} \leq \beta V(M)^{1+2/n}(1 + o(1))$$

Proof. From the previous lemma we have

$$\lambda_1^L(M)\|Z\|_2^2 \leq nV(M) - \delta\|Z^T\|_2^2 + \int_M q|Z|^2 dv$$

If $\delta \geq 0$, we then have $(\lambda_1^L(M) - \|q\|_\infty)\|Z\|_2^2 \leq nV(M)$ and with the pinching condition we get the desired inequality. If $\delta < 0$

$$\lambda_1^L(M)\|Z\|_2^2 \leq nV(M) - \frac{2\delta C}{n\|H\|_\infty^2}\|Z\|_2^2 + \|q\|_\infty\|Z\|_2^2$$

Then

$$\int_M \left(\lambda_1^L(M) - \frac{2|\delta|C}{n\|H\|_\infty^2} - q \right) |Z|^2 dv \leq nV(M)$$

and combining this with the pinching again, we get

$$\left(nh^2 - C - \frac{2|\delta|C}{n\|H\|_\infty^2} \right) \|Z\|_2^2 \leq nV(M).$$

Since $\|H\|_\infty^2 + \delta > 0$, we deduce that $(nh^2 - C - \frac{2C}{n})\|Z\|_2^2 \leq nV(M)$ and taking $C < \frac{1}{2} < \left(\frac{n^2}{n+2}\right)h^2$, we obtain the result of the lemma. □

Lemma 3.3. *If the pinching condition (P_C) holds then*

$$\frac{n(nh^2 - C)}{(nh^2 + \beta V(M)^{2/n}C)^2} V(M) \leq \|Z\|_2^2$$

Proof. From the previous proof, we have

$$0 \leq nV(M) - \delta \int_M |Z^T|^2 dv \leq nh^2 \int_M s_\delta^2 dv$$

Then if $\delta \leq 0$ we have

$$\begin{aligned} \lambda_1^L(M) \int_M |Z|^2 dv &\leq nV(M) - \delta \int_M |Z^T|^2 dv + \int_M q|Z|^2 dv \\ &= \frac{\left(nV(M) - \delta \int_M |Z^T|^2 dv\right)^2}{nV(M) - \delta \int_M |Z^T|^2 dv} + \int_M q|Z|^2 dv \\ &\leq \frac{1}{nV(M)} \left(nh^2 \int_M |Z|^2 dv\right)^2 + \int_M q|Z|^2 dv \end{aligned}$$

From this we deduce that $\int_M (\lambda_1^L(M) - q)|Z|^2 dv \leq \frac{1}{V(M)} nh^4 \|Z\|_2^4$ and from the pinching condition we obtain the desired inequality. For $\delta > 0$, we have

$$\begin{aligned} \lambda_1^L(M) \int_M |Z|^2 dv &\leq \frac{\left(nV(M) - \delta \int_M |Z^T|^2 dv + \delta \int_M |Z^T|^2 dv\right)^2}{nV(M)} + \int_M q|Z|^2 dv \\ &\leq \frac{\left(nh^2 \int_M |Z|^2 dv + \delta \int_M |Z^T|^2 dv\right)^2}{nV(M)} + \int_M q|Z|^2 dv \\ &\leq \frac{(nh^2 \|Z\|_2^2 + \beta V(M)^{2/n} C \|Z\|_2^2)^2}{nV(M)} + \int_M q|Z|^2 dv \\ &= \frac{(nh^2 + \beta V(M)^{2/n} C)^2 \|Z\|_2^4}{nV(M)} + \int_M q|Z|^2 dv \end{aligned}$$

and we conclude as above in the nonpositive case. \square

Remark 3.2. All the results stated now in this section remain true if we replace the dependence on b or $\|B\|_\infty$ respectively by h or $\|H\|_\infty$.

Lemma 3.4. Let $X = nHc_\delta \nu - \|H\|_\infty^2 Z$. If $C < \frac{\alpha}{V(M)^{2/n}}(1 + o(1)) < \frac{1}{2} \left(\frac{n^2}{n+2}\right) h^2$, then the pinching condition (P_C) implies

$$\|X\|_2^2 \leq \beta b^2 V(M)^{1+2/n} (1 + o(1)) C$$

where β is a constant depending only on the dimension.

Proof. Using again (3) we have

$$\begin{aligned}
\|X\|_2^2 &= n^2 \int_M H^2 c_\delta^2 dv - 2n^2 \|H\|_\infty^2 \int_M H \langle Z, \nu \rangle c_\delta dv + n^2 \|H\|_\infty^4 \|Z\|_2^2 \\
&\leq n^2 \int_M H^2 c_\delta^2 dv + 2n \|H\|_\infty^2 \int_M (\operatorname{div} (Z^T) c_\delta - n c_\delta^2) dv + n^2 \|H\|_\infty^4 \|Z\|_2^2 \\
&= n^2 \int_M H^2 c_\delta^2 dv + 2n \|H\|_\infty^2 \int_M \delta |Z^T|^2 dv - 2n^2 \|H\|_\infty^2 \int_M c_\delta^2 dv + n^2 \|H\|_\infty^4 \|Z\|_2^2 \\
&\leq -n^2 \|H\|_\infty^2 \int_M c_\delta^2 dv + n^2 \|H\|_\infty^4 \|Z\|_2^2 + 2n |\delta| \|H\|_\infty^2 \|Z^T\|_2^2 \\
&= -n^2 \|H\|_\infty^2 V(M) + n^2 \|H\|_\infty^2 \delta \|Z\|_2^2 + n^2 \|H\|_\infty^4 \|Z\|_2^2 + 2n |\delta| \|H\|_\infty^2 \|Z^T\|_2^2 \\
&= n^2 \|H\|_\infty^2 (-V(M) + h^2 \|Z\|_2^2) + 2n |\delta| \|H\|_\infty^2 \|Z^T\|_2^2
\end{aligned}$$

Now by applying the lemma 3.2 and lemma 3.1 we obtain

$$\|X\|_2^2 \leq n^2 \|H\|_\infty^2 \left(\frac{\binom{n+2}{n}}{nh^2 - \binom{n+2}{n} C} \right) V(M)C + \frac{4n|\delta|}{nh^2 - \binom{n+2}{n} C} V(M)C$$

and we end the proof by using the fact that $C < \frac{1}{2} \left(\frac{n^2}{n+2} \right) h^2$ and the relation (6). \square

Lemma 3.5. *Let $Y = |Z|^{1/2} \left(\delta Z + H c_\delta \nu - h \frac{Z}{|Z|} \right)$. If $C < \frac{\alpha}{V(M)^{2/n}} (1 + o(1)) < \frac{1}{2} \left(\frac{n^2}{n+2} \right) h^2$, then the condition (P_C) implies*

$$\begin{aligned}
\|Y\|_2^2 &\leq [\beta V(M)^{1/n} (1 + o(1)) + \beta' b^\gamma V(M)^{\frac{\gamma+3}{n}} (1 + o(1)) \\
&\quad + \beta'' b^\gamma V(M)^{\frac{\gamma+5}{n}} (1 + o(1))] V(M)C
\end{aligned}$$

where $\gamma \in (e^{n/2} - 1, e^n - 1)$ and β, β' and β'' are constants depending only on n and δ .

Proof. First we have

$$\begin{aligned}
\|Y\|_2^2 &\leq \int_M (|Z| |\delta Z + H c_\delta \nu|^2 - 2h \langle \delta Z + H c_\delta \nu, Z \rangle + h^2 |Z|) dv \\
(7) \quad &\leq \int_M (|Z| |\delta Z + H c_\delta \nu|^2 - 2h \langle \delta Z + H c_\delta \nu, Z \rangle) dv + h^2 \|Z\|_2 V(M)^{1/2}
\end{aligned}$$

Let us compute the first term

$$\begin{aligned}
\int_M |Z| |\delta Z + H c_\delta \nu|^2 dv &= \int_M |Z| (\delta^2 |Z|^2 + 2\delta c_\delta H \langle Z, \nu \rangle + H^2 c_\delta^2) dv \\
&= \int_M |Z| (H^2 - \delta H^2 s_\delta^2 + 2\delta c_\delta H \langle Z, \nu \rangle + \delta - \delta c_\delta^2) dv \\
&= \int_M |Z| (H^2 + \delta - \delta |HZ - c_\delta \nu|^2) dv \\
&\leq h^2 \|Z\|_2 V(M)^{1/2} - \delta \int_M |Z| |HZ - c_\delta \nu|^2 dv \\
&\leq h^2 \|Z\|_2 V(M)^{1/2} + |\delta| \|Z\|_\infty \int_M |HZ - c_\delta \nu|^2 dv
\end{aligned}$$

On the other hand,

$$\int_M |HZ - c_\delta \nu|^2 dv \leq \|H\|_\infty^2 \int_M s_\delta^2 dv - 2 \int_M H \langle Z, \nu \rangle c_\delta dv + \int_M c_\delta^2 dv$$

Now the pinching implies that

$$\|H\|_\infty^2 \int_M s_\delta^2 dv - \int_M H \langle Z, \nu \rangle c_\delta dv \leq \frac{C}{n} \int_M s_\delta^2 dv = \frac{C}{n} \|Z\|_2^2$$

and

$$\int_M c_\delta^2 dv - \int_M H \langle Z, \nu \rangle c_\delta dv \leq \frac{1}{n} \int_M \operatorname{div} (Z^T) c_\delta dv = \frac{|\delta|}{n} \|Z^T\|_2^2$$

Then we have proved

$$(8) \quad \int_M |Z| |\delta Z + H c_\delta \nu|^2 dv \leq h^2 \|Z\|_2 V(M)^{1/2} + \frac{|\delta| \|Z\|_\infty}{n} (C \|Z\|_2^2 + |\delta| \|Z^T\|_2^2)$$

Now let us compute the two last terms of (7)

$$\begin{aligned}
&-2h \int_M \langle \delta Z + H c_\delta \nu, Z \rangle dv + h^2 \|Z\|_2 V(M)^{1/2} \\
&\leq -2\delta h \int_M s_\delta^2 dv + \frac{2h}{n} \int_M \operatorname{div} (Z^T) c_\delta dv - 2h \int_M c_\delta^2 dv + h^2 \|Z\|_2 V(M)^{1/2} \\
&= -2hV(M) + \frac{2h\delta}{n} \int_M |Z^T|^2 dv + h^2 \|Z\|_2 V(M)^{1/2} \\
&\leq -2hV(M) + h^2 V(M)^{1/2}
\end{aligned}$$

Therefore reporting this and (8) in (7), we get

$$\|Y\|_2^2 \leq 2h^2 \|Z\|_2 - 2hV(M) + \frac{|\delta| \|Z\|_\infty}{n} (C \|Z\|_2^2 + |\delta| \|Z^T\|_2^2)$$

and using the estimates given in the lemmas 3.2 and 3.1 with the fact that $C < \frac{1}{2} \left(\frac{n^2}{n+2} \right) h^2$, we get

$$\begin{aligned}
\|Y\|_2^2 &\leq \frac{2n^{1/2}h^2V(M)}{(nh^2 - (\frac{n+2}{n})C)^{1/2}} - 2hV(M) + (\beta + \beta'V(M)^{2/n})\|Z\|_\infty\|Z\|_2^2C \\
&= 2hV(M) \left(\frac{n^{1/2}h}{(nh^2 - (\frac{n+2}{n})C)^{1/2}} - 1 \right) + (\beta + \beta'V(M)^{2/n})V(M)^{1+2/n}(1 + o(1))\|Z\|_\infty C \\
&\leq 2hV(M) \left(\frac{(\frac{n+2}{n})C}{2(nh^2 - (\frac{n+2}{n})C)} \right) + (\beta V(M)^{1+2/n} + \beta'V(M)^{1+4/n})(1 + o(1))\|Z\|_\infty C \\
&\leq \frac{\beta}{h}V(M)C + (\beta'V(M)^{1+2/n} + \beta''V(M)^{1+4/n})(1 + o(1))\|Z\|_\infty C
\end{aligned}$$

Finally we have proved

$$\|Y\|_2^2 \leq (\beta V(M)^{1+1/n}(1 + o(1)) + (\beta'V(M)^{1+2/n} + \beta''V(M)^{1+4/n})(1 + o(1))\|Z\|_\infty) C$$

Now the researched inequality is a straightforward consequence of the following lemma

Lemma 3.6. *If $C < \frac{\alpha}{V(M)^{2/n}}(1 + o(1)) < \frac{1}{2} \left(\frac{n^2}{n+2} \right) h^2$, then the pinching condition (P_C) implies*

$$\|Z\|_\infty \leq \beta b^\gamma V(M)^{\frac{\gamma+1}{n}}(1 + o(1))$$

where β is a constant depending only on n and δ and $\gamma \in (e^{n/2} - 1, e^n - 1)$.

□

The proof of the above lemma is providing from a result stated in the following proposition using a Nirenberg-Moser type of proof (see [6]).

Proposition 3.1. *Let (N^{n+1}, h) be a Riemannian manifold and $M \in \mathcal{H}_V(n, \phi, N)$. Let ξ be a nonnegative continuous function so that ξ^k is smooth for $k \geq 2$. Let $0 \leq r < s \leq 2$ so that*

$$\frac{1}{2} \int_M \Delta \xi^2 \xi^{2k-2} dv \leq (A_1 + kA_2) \int_M \xi^{2k-r} dv + (B_1 + kB_2) \int_M \xi^{2k-s} dv$$

where A_1, A_2, B_1, B_2 are nonnegative constants. Then for any $\eta > 0$, if $\|\xi\|_\infty > \eta$ then

$$\|\xi\|_\infty \leq L(n, A_1, A_2, B_1, B_2, \|B\|_\infty, V(M), \eta) \|\xi\|_2$$

where

$$\begin{aligned}
&L(n, A_1, A_2, B_1, B_2, \|B\|_\infty, V(M), \eta) \\
&= K(n) \left(\frac{4A_1^{1/2} + 4A_2^{1/2}}{\eta^{r/2}} + \frac{4B_1^{1/2} + 4B_2^{1/2}}{\eta^{s/2}} + \|B\|_\infty \right)^\gamma V(M)^{\frac{\gamma}{n} - \frac{1}{2}}
\end{aligned}$$

and $\gamma \in (e^{n/2} - 1, e^n - 1)$.

Remark 3.3. *In particular we see that*

- (1) *If $\|\xi\|_2 \leq \frac{\eta}{L(n, A_1, A_2, B_1, B_2, \|B\|_\infty, V(M), \eta)}$, then $\|\xi\|_\infty \leq \eta$.*
(2) *If $\|\xi\|_2 \leq A$, then for any $\eta > 0$,*

$$\|\xi\|_\infty \leq \max(\eta, L(n, A_1, A_2, B_1, B_2, \|B\|_\infty, V(M), \eta)A)$$

In [6] this proposition has been proved for hypersurfaces of the Euclidean space. The proof is similar for hypersurfaces of some ambient space with bounded sectional curvature. This proof uses a Sobolev inequality due to Hoffman and Spruck (see [9] and [10]) which is available under the conditions on the injectivity radius of N and the volume of M contained in the definition of $\mathcal{H}_V(n, \phi, N)$.

PROOF OF THE LEMMA 3.6: First we compute the Laplacian of $|Z|^2$. An easy computation shows that $\Delta|Z|^2 = (-2c_\delta^2 + 2\delta s_\delta^2)|\nabla^M r|^2 + 2s_\delta c_\delta \Delta r$.

Since $r \leq \frac{\pi}{4\sqrt{\delta}}$ for $\delta > 0$, the first term is nonpositif. Now let us consider $(e_i)_{1 \leq i \leq n+1}$ an orthonormal frame in a neighborhood of the point $p \in M$ where we compute the Laplacian and so that $e_{n+1} = \nu$. Then

$$\begin{aligned} \Delta|Z|^2 &\leq 2s_\delta c_\delta \left(-\sum_{i=1}^n \nabla^N dr(e_i, e_i) + nH \langle \nu, \nabla^N r \rangle \right) \\ &= 2s_\delta c_\delta \left(-\sum_{i=1}^n \nabla^N dr(e_i - \langle \nabla^N r, e_i \rangle \nabla^N r, e_i - \langle \nabla^N r, e_i \rangle \nabla^N r) + nH \langle \nu, \nabla^N r \rangle \right) \\ &\leq 2s_\delta c_\delta \left(-\frac{c_\delta}{s_\delta} \sum_{i=1}^n |e_i - \langle \nabla^N r, e_i \rangle \nabla^N r|^2 + n\|H\|_\infty \right) \\ &\leq 2n\|H\|_\infty s_\delta c_\delta \leq \frac{2n\|H\|_\infty}{\sqrt{|\delta|}} c_\delta^2 = \frac{2n\|H\|_\infty}{\sqrt{|\delta|}} (1 - \delta s_\delta^2) \\ &\leq \frac{2n\|H\|_\infty}{\sqrt{|\delta|}} + 2n\|H\|_\infty \sqrt{|\delta|} |Z|^2 \leq \frac{2\sqrt{n}\|B\|_\infty}{\sqrt{|\delta|}} + 2\sqrt{n}\|B\|_\infty \sqrt{|\delta|} |Z|^2 \end{aligned}$$

And from the remark 3.3 about the proposition 3.1 and lemma 3.2 we deduce that

$$\begin{aligned} \|Z\|_\infty &\leq \max \left(1, \beta L(n, 2\sqrt{n}\|B\|_\infty \sqrt{|\delta|}, 0, \frac{2\sqrt{n}\|B\|_\infty}{\sqrt{|\delta|}}, 0, \|B\|_\infty, 1) V(M)^{1/2+1/n} (1 + o(1)) \right) \\ &= \beta b^\gamma V(M)^{\gamma/n} (1 + o(1)) \end{aligned}$$

□

Let's introduce now the function $\varphi = |Z| \left(|Z| - \frac{1}{h} \right)^2 = |Z| \left| Z - \frac{1}{h} \frac{Z}{|Z|} \right|^2$. In the following lemma, we give an L^2 -estimate of φ

Lemma 3.7. *If $C < \frac{\alpha}{V(M)^{2/n}} (1 + o(1)) < \frac{1}{2} \left(\frac{n^2}{n+2} \right) h^2$, then (P_C) implies that*

$$\|\varphi\|_2 \leq \|\varphi\|_\infty^{3/4} [\beta(bV(M)^{1/n})^{1/2}(1+o(1)) + \beta'(bV(M)^{1/n})^{\gamma/4}V(M)^{1/2n}(1+o(1)) \\ + \beta''(bV(M)^{1/n})^{\gamma/4}V(M)^{1/n}(1+o(1)) + \beta'''(1+o(1))] V(M)^{1/2+5/4n}C^{1/4}$$

where β , β' and β'' are constants depending only on n and δ .

Proof. First we have $\left(\int_M \varphi^2 dv\right)^{1/2} \leq \|\varphi\|_\infty^{3/4} \left(\int_M \varphi^{1/2} dv\right)^{1/2}$. Moreover

$$\varphi^{1/2} = |Z|^{1/2} \left| \frac{1}{h^2}(h^2Z - \delta Z - Hc_\delta\nu) + \frac{1}{h^2} \left(\delta Z + Hc_\delta\nu - h\frac{Z}{|Z|} \right) \right| \\ \leq \frac{|Z|^{1/2}}{nh^2}|X| + \frac{1}{h^2}|Y|$$

Then

$$\left(\int_M \varphi^{1/2} dv\right)^{1/2} \leq \frac{1}{h} \left(\int_M |Z|^{1/2}|X| dv\right)^{1/2} + \frac{1}{h} \left(\int_M |Y| dv\right)^{1/2} \\ \leq \frac{1}{hn^{1/2}} \left(\int_M |Z| dv\right)^{1/4} \|X\|^{1/2} + \frac{1}{h} \|Y\|_2^{1/2} V(M)^{1/4} \\ \leq \frac{V(M)^{1/8}}{hn^{1/2}} \|Z\|_2^{1/4} \|X\|_2^{1/2} + \frac{1}{h} \|Y\|_2^{1/2} V(M)^{1/4}$$

Since we have choose $C < \frac{\alpha}{V(M)^{2/n}}(1+o(1)) < \frac{1}{2} \left(\frac{n^2}{n+2}\right) h^2$, we deduce from the lemma 3.2 and (6) that

$$\left(\int_M \varphi^{1/2} dv\right)^{1/2} \leq \left(\beta V(M)^{1/4n}(1+o(1))\|X\|_2^{1/2} \\ + \beta'(1+o(1))\|Y\|_2^{1/2}\right) V(M)^{1/4+1/n}$$

and the lemmas 3.4 and 3.5 give us

$$\left(\int_M \varphi^{1/2} dv\right)^{1/2} \leq \left[\beta b^{1/2}V(M)^{3/4n}(1+o(1)) + \beta' b^{\gamma/4}V(M)^{\frac{\gamma+3}{4n}}(1+o(1)) \\ + \beta'' b^{\gamma/4}V(M)^{\frac{\gamma+5}{4n}}(1+o(1)) + \beta'''V(M)^{1/4n}(1+o(1))\right] V(M)^{1/2+1/n}C^{1/4}$$

□

Lemma 3.8. *There exists a constant $A(n, b, h, \delta, V(M), \eta)$ so that for any $\eta > 0$, the pinching condition (P_C) with $C = \min\left(\frac{\eta}{A}, \frac{\alpha}{V(M)^{2/n}}(1+o(1))\right)$ implies*

$$\|\varphi\|_\infty \leq \eta$$

Moreover,

$$A(n, b, h, \delta, V(M), \eta)$$

$$= \left(\frac{A_1(n, b, h, \delta, V(M), \eta)}{\eta} + \frac{A_2(n, b, h, \delta, V(M), \eta)}{\eta^{1/2}} + \frac{\|B\|_\infty}{\sqrt{n}} \right)^{4\gamma} \\ \times A_3(n, b, h, \delta, V(M), \eta) V(M)^{4\gamma/n}$$

where the constants $A_i(n, b, h, \delta, V(M))$ are of the form

$$A_i = \sum_{j=1}^{k_i} \beta_{ij} (bV(M)^{1/n})^{\theta_{ij}} V(M)^{\nu_{ij}}, \quad \beta_{ij} \text{ are constants depending only on } n \text{ and } \delta \\ \text{and } \theta_i \text{ and } \nu_i \text{ are positive reals depending on } n \text{ so that } \nu_{1j} \geq 4/n, \nu_{2j} \geq 1/n \text{ and } \nu_{3j} \geq 5/n.$$

Proof. We have for any $k \geq 2$

$$(9) \quad \frac{1}{2} \int_M \Delta \varphi^2 \varphi^{2k-2} dv = \frac{1}{2} \int_M \langle \nabla^M \varphi^2, \nabla^M \varphi^{2k-2} \rangle dv \leq 2k \int_M |\nabla^M \varphi|^2 \varphi^{2k-2} dv$$

Let us compute $|\nabla^M \varphi|^2$

$$\begin{aligned} |\nabla^M \varphi|^2 &= \left| \nabla^M \left(|Z| \left(|Z| - \frac{1}{h} \right)^2 \right) \right|^2 \\ &= \left| \nabla^M |Z| \left(|Z| - \frac{1}{h} \right)^2 + 2|Z| \left(|Z| - \frac{1}{h} \right) \nabla^M |Z| \right|^2 \\ &= \left[\left(|Z| - \frac{1}{h} \right)^4 + 4|Z| \left(|Z| - \frac{1}{h} \right)^3 + 4|Z|^2 \left(|Z| - \frac{1}{h} \right)^2 \right] |\nabla^M |Z||^2 \\ &\leq \left[\left(\|Z\|_\infty + \frac{1}{h} \right)^4 + 4 \left(\left(\|Z\|_\infty + \frac{1}{h} \right)^2 + \|Z\|_\infty \right) \varphi \right] |\nabla^M |Z||^2 \end{aligned}$$

A straightforward computation shows that $|\nabla^M |Z||^2 \leq c_\delta^2 \leq 1 + |\delta| \|Z\|_\infty^2$. Now using the lemma 3.6 we get

$$\begin{aligned} |\nabla^M \varphi|^2 &\leq \left[V(M)^{4/n} (\beta(bV(M)^{1/n})^\gamma (1 + o(1)) + \beta'(1 + o(1)))^4 \right. \\ &\quad + 4\varphi \left(V(M)^{2/n} [\beta(bV(M)^{1/n})^\gamma (1 + o(1)) + \beta'(1 + o(1))]^2 \right. \\ &\quad \left. \left. + \beta V(M)^{1/n} (bV(M)^{1/n})^\gamma (1 + o(1)) \right) \right] (1 + |\delta| \beta'' V(M)^{2/n} (bV(M)^{1/n})^{2\gamma} (1 + o(1))) \\ &\leq A_1(n, b, h, \delta, V(M)) + A_2(n, b, h, \delta, V(M)) \varphi \end{aligned}$$

where the constants $A_i(n, b, h, \delta, V(M))$ are of the form

$$A_i = \sum_{j=1}^{k_i} \beta_{ij} (bV(M)^{1/n})^{\theta_{ij}} V(M)^{\nu_{ij}} \text{ with } \nu_{1j} \geq 4/n, \nu_{2j} \geq 1/n.$$

Then reporting this in (9) we get

$$\frac{1}{2} \int_M \Delta \varphi^2 \varphi^{2k-2} dv \leq 2k A_1 \int_M \varphi^{2k-2} dv + 2k A_2 \int_M \varphi^{2k-1} dv$$

Now, applying the lemma 3.1 we see that if $\|\varphi\|_\infty > \eta$ then

$$\|\varphi\|_\infty \leq \left(\frac{A_1}{\eta} + \frac{A_2}{\eta^{1/2}} + \frac{\|B\|_\infty}{\sqrt{n}} \right)^\gamma V(M)^{\frac{\gamma}{n}-\frac{1}{2}} \|\varphi\|_2$$

Combining this with the inequality of the lemma 3.7, we deduce that

$$\|\varphi\|_\infty \leq A(n, b, h, \delta, V(M), \eta)C$$

with

$$\begin{aligned} & A(n, b, h, \delta, V(M), \eta) \\ &= \left(\frac{A_1(n, b, h, \delta, V(M))}{\eta} + \frac{A_2(n, b, h, \delta, V(M))}{\eta^{1/2}} + \frac{\|B\|_\infty}{\sqrt{n}} \right)^{4\gamma} \\ & \times A_3(n, b, h, \delta, V(M))V(M)^{4\gamma/n} \end{aligned}$$

where the constant $A_3(n, b, h, \delta, V(M))$ is of the form

$$A_3 = \sum_{j=1}^{k_3} \beta_{3j} (bV(M)^{1/n})^{\theta_{3j}} V(M)^{\nu_{3j}} (1 + o(1)).$$

It is easy to see that and $\nu_{3j} \geq 5/n$.

Now taking $C = \min\left(\frac{\eta}{A}, \frac{\alpha}{V(M)^{2/n}}(1 + o(1))\right)$, we see that $\|\varphi\|_\infty \leq \eta$. □

Lemma 3.9. *For any $\varepsilon < \frac{1}{4b}$, there exists a function $\psi(\varepsilon, b)$ and a constant depending on n so that the pinching condition (P_{Γ_ε}) with*

$$\Gamma_\varepsilon(n, b, h, \delta, V(M)) = \min\left(\frac{\alpha(1 + o(1))}{V(M)^{4/n}}, \frac{\alpha'(1 + o(1))}{V(M)^{2/n}}, \frac{\psi(\varepsilon, b)}{A(n, b, h, \delta, V(M), \psi(\varepsilon, b))}\right)$$

implies

$$\left| |Z| - \frac{1}{h} \right| \leq \varepsilon \quad \text{and} \quad \left| r - s_\delta^{-1} \left(\frac{1}{h} \right) \right| \leq \varepsilon$$

Moreover for a fixed ε or if $\varepsilon = \frac{\kappa}{b}$ for a positive real κ then $\psi(\varepsilon, b) = \frac{\beta}{b^3}(1 + o(1))$.

Proof. Consider the function $f(t) = t(t - \frac{1}{h})^2$ and $g(t) = (\frac{1}{b} - t)t^2$. The function f is increasing on $[0, \frac{1}{3h}]$ and $[\frac{1}{h}, +\infty)$ and decreasing on $[\frac{1}{3h}, \frac{1}{h}]$. For any $\varepsilon > 0$, put

$$\psi_1(\varepsilon, b) = \min \left[g \left(\min \left(\varepsilon, \frac{1}{3b} \right) \right), g \left(-\min \left(\varepsilon, \frac{1}{3b} \right) \right) \right]$$

and

$$\Gamma_{1,\varepsilon}(n, b, h, \delta, V(M)) = \min\left(\frac{\alpha(1 + o(1))}{V(M)^{4/n}}, \frac{\alpha'(1 + o(1))}{V(M)^{2/n}}, \frac{\psi_1(\varepsilon, b)}{A(n, b, h, \delta, V(M), \psi_1(\varepsilon, b))}\right)$$

First note that $\psi_1(\varepsilon, b) \leq \min \left[f \left(\frac{1}{h} - \min \left(\varepsilon, \frac{1}{3b} \right) \right), f \left(\frac{1}{h} + \min \left(\varepsilon, \frac{1}{3b} \right) \right) \right]$, and since $\frac{1}{h} - \min \left(\varepsilon, \frac{1}{3b} \right) > \frac{1}{3h}$ we deduce that $\psi_1(\varepsilon, b) < f \left(\frac{1}{3h} \right)$. On the other hand we choose the constants α and α' so that

$$\frac{\alpha(1+o(1))}{V(M)^{4/n}} \leq \frac{nh^2}{2\beta V(M)^{2/n}}$$

where β is the constant of the lemma 3.1 and

$$\frac{\alpha'(1+o(1))}{V(M)^{2/n}} \leq \min\left(\frac{n}{2}h^2, \frac{1}{2}\left(\frac{n^2}{n+2}\right)h^2\right)$$

From the definition of $\Gamma_{1,\varepsilon}$ and the lemma 3.8 it follows that the pinching condition $(P_{\Gamma_{1,\varepsilon}})$ implies that $\|\varphi\|_\infty \leq \psi_1(\varepsilon, b)$, that is $f(|Z|) \leq \psi_1(\varepsilon, b) < f\left(\frac{1}{3h}\right)$. Now, because of the choice of α and α' , we deduce from the lemma 3.3 that

$$\|Z\|_2^2 \geq \frac{2}{9h^2}V(M)$$

Then there exists a point x_0 of M so that $|Z|_{x_0} > \frac{1}{3h}$. Then by the connexity of M , it follows that $\left||Z| - \frac{1}{h}\right| \leq \varepsilon$.

Now a straightforward computation shows that $\left|r - s_\delta^{-1}\left(\frac{1}{h}\right)\right| \leq (1+o(1))\varepsilon$ (for $\delta \leq 0$ we can choose $o(1) = 0$ whilst $o(1) > 0$ for $\delta > 0$). Indeed

$$\left|r - s_\delta^{-1}\left(\frac{1}{h}\right)\right| \leq \frac{\varepsilon}{\sqrt{1-\delta\left(\varepsilon + \frac{1}{h}\right)^2}} \leq \frac{\varepsilon}{\sqrt{1-\delta\frac{25}{16h^2}}}$$

The last term is well defined since the extrinsic radius R of M satisfies $s_\delta(R) \geq \frac{1}{h}$ and $R < \frac{\pi}{4\sqrt{\delta}}$ if $\delta > 0$ (see for instance [2]). Then $s_\delta^{-1}\left(\frac{1}{h}\right) \in [0, \frac{\pi}{4\sqrt{\delta}})$ for $\delta > 0$. Putting $\psi(\varepsilon, b) = \psi_1\left(\frac{\varepsilon}{1+o(1)}, b\right)$, we see that the pinching (P_{Γ_ε}) implies the desired result. □

We are now in a position to prove the theorem 1.2.

Proof of Theorem 1.2: As mentioned in remark 3.2 all the previous results remain available by replacing b or $\|B\|_\infty$ respectively by h or $\|H\|_\infty$. The case $\delta = 0$ is a particular case of [6]. From the lemma above, we know that for any $\varepsilon > 0$, there exists $C_\varepsilon(n, \|H\|_\infty, V(M))$ so that $\phi(M) \subset \overline{B}_p\left(s_\delta^{-1}\left(\frac{1}{h}\right) + \varepsilon\right) \setminus B_p\left(s_\delta^{-1}\left(\frac{1}{h}\right) - \varepsilon\right)$. Now putting $R = s_\delta^{-1}\left(\frac{1}{h}\right) + \varepsilon$ and $\eta = 2\varepsilon$, then $\phi(M) \subset \overline{B}_p(R) \setminus B_p(R - \varepsilon)$. Let $x_0 \in S_p\left(s_\delta^{-1}\left(\frac{1}{h}\right)\right)$ so that $\phi(M) \subset (\overline{B}_p(R) \setminus B_p(R - \varepsilon)) \setminus B_{x_0}(\rho)$ where ρ satisfies

$$t_\delta\left(\frac{R+\rho}{2}\right) - t_\delta\left(\frac{R}{2}\right) = c\eta$$

if $\delta \leq 0$ and

$$t_\delta\left(\frac{R}{2}\right) - t_\delta\left(\frac{R-\rho}{2}\right) = c\eta$$

if $\delta > 0$, where $t_\delta = \frac{s_\delta}{c_\delta}$ and c is a constant depending on n . Taking $\varepsilon \leq 1$, from the lemma 4.3 of [12] we deduce that there exist constants D_1 and D_2 depending on n , $\|H\|_\infty$ and δ so that if $\eta \leq D_1$ then there exists $y_0 \in M$ so that

$$|H(y_0)| \geq \frac{c'D_2}{\eta} = \frac{c'D_2}{2\varepsilon}$$

where c' is a constant depending on n . Now if we assume $\varepsilon \leq \frac{c'D_2}{2\|H\|_\infty}$ we obtain a contradiction and we conclude that for all $x_0 \in S_p(s_\delta^{-1}(\frac{1}{h}))$, $\phi(M) \cap B_{x_0}(\rho) \neq \emptyset$ and $S_p(s_\delta^{-1}(\frac{1}{h})) \subset V_{2\rho}(M)$. If $\delta < 0$,

$$\begin{aligned} \frac{\rho}{2} &= t_\delta^{-1} \left(t_\delta \left(\frac{R+\rho}{2} \right) \right) - t_\delta^{-1} \left(t_\delta \left(\frac{R}{2} \right) \right) \leq \frac{c\eta}{1 + \delta t_\delta \left(\frac{R}{2} \right)} \\ &\leq \frac{2c\varepsilon}{1 + \delta(c\eta + t_\delta(R/2))} \leq a(n, \|H\|_\infty)\varepsilon \end{aligned}$$

Similarly we obtain the same estimate for the case $\delta > 0$ with $a(n, \|H\|_\infty) \geq 1$ (for $\delta > 0$ we can choose $a(n, \|H\|_\infty) = 1$). Finally considering the pinching $(P_{C'_\varepsilon})$ with $C'_\varepsilon(n, \|H\|_\infty, V(M)) = C_{\frac{\varepsilon}{2a(n, \|H\|_\infty)}}(n, \|H\|_\infty, V(M))$ we obtain the desired conclusion. \square

4. THE PROOF OF THE DIFFEOMORPHISM

Let us consider

$$\begin{aligned} F : M &\longrightarrow S(p, s_\delta^{-1}(\frac{1}{h})) \\ x &\longmapsto \exp_p \left(s_\delta^{-1}(\frac{1}{h}) \frac{X}{|X|} \right) \end{aligned}$$

where $X = \exp_p^{-1}(x)$. For more convenience we will put $\varrho = s_\delta^{-1}(\frac{1}{h}) \frac{X}{|X|}$.

Lemma 4.1. *Let $u \in T_x M$ and $v = u - \langle u, \nabla^M r \rangle \nabla^N r$. We have*

$$\frac{1}{h^2 s_\mu(r)^2} |v|^2 \leq |dF_x(u)|^2 \leq \frac{s_\mu(s_\delta^{-1}(\frac{1}{h}))^2}{s_\delta(r)^2} |v|^2$$

Proof. An easy computation shows that

$$d \left(\frac{X}{|X|} \right) \Big|_x(u) = \frac{1}{r} d \exp_p^{-1} \Big|_x(u) - \frac{dr(u)}{r^2} \exp_p^{-1}(x)$$

Then we deduce that

$$\begin{aligned} dF_x(u) &= d \exp_p \Big|_\varrho \left(s_\delta^{-1} \left(\frac{1}{h} \right) d \left(\frac{X}{|X|} \right) \Big|_x(u) \right) \\ &= \frac{s_\delta^{-1}(\frac{1}{h})}{r} d \exp_p \Big|_\varrho \left(d \exp_p^{-1} \Big|_x(u) \right) \\ &\quad - \frac{s_\delta^{-1}(\frac{1}{h})}{r^2} dr(u) d \exp_p \Big|_\varrho \left(\exp_p^{-1}(x) \right) \\ &= \frac{s_\delta^{-1}(\frac{1}{h})}{r} d \exp_p \Big|_\varrho \left(d \exp_p^{-1} \Big|_x(u) \right) - \frac{s_\delta^{-1}(\frac{1}{h})}{r} dr(u) \nabla^N r \Big|_{F(x)} \end{aligned}$$

Now let us compute the norm of $dF_x(u)$. We have

$$|dF_x(u)|^2 = \frac{s_\delta^{-1}\left(\frac{1}{h}\right)^2}{r^2} \left[|d \exp_p|_\varrho (d \exp_p^{-1}|_x(u))|^2 - 2\langle d \exp_p|_\varrho (d \exp_p^{-1}|_x(u)), \nabla^N r \rangle_{F(x)} dr(u) + dr(u)^2 \right]$$

Now since \exp_p is a radial isometry (see for instance [13]), we have

$$\langle d \exp_p|_\varrho (d \exp_p^{-1}|_x(u)), \nabla^N r \rangle_{F(x)} = \langle d \exp_p^{-1}|_x(u), \frac{X}{|X|} \rangle = \langle u, \nabla^N r \rangle_x$$

and it follows that

$$(10) \quad |dF_x(u)|^2 = \frac{s_\delta^{-1}\left(\frac{1}{h}\right)^2}{r^2} \left[|d \exp_p|_\varrho (d \exp_p^{-1}|_x(u))|^2 - \langle \nabla^M r, u \rangle^2 \right]$$

Now

$$|d \exp_p|_\varrho (d \exp_p^{-1}|_x(u))|^2 = |d \exp_p|_\varrho (d \exp_p^{-1}|_x(v))|^2 + \langle u, \nabla^M r \rangle^2 |d \exp_p|_\varrho (d \exp_p^{-1}|_x(\nabla^N r))|^2$$

where $v = u - \langle u, \nabla^M r \rangle \nabla^N r$. Developing this expression we get

$$\begin{aligned} & |d \exp_p|_\varrho (d \exp_p^{-1}|_x(u))|^2 = \\ & |d \exp_p|_\varrho (d \exp_p^{-1}|_x(v))|^2 + \langle u, \nabla^M r \rangle^2 |d \exp_p|_\varrho (d \exp_p^{-1}|_x(\nabla^N r))|^2 \\ & + 2\langle u, \nabla^M r \rangle \langle d \exp_p|_\varrho (d \exp_p^{-1}|_x(v)), d \exp_p|_\varrho (d \exp_p^{-1}|_x(\nabla^N r)) \rangle \\ & = |d \exp_p|_\varrho (d \exp_p^{-1}|_x(v))|^2 + \langle u, \nabla^M r \rangle^2 \end{aligned}$$

where in the last equality we have used again the radial isometry property of the exponential map. And reporting this in (10) we obtain

$$|dF_x(u)|^2 = \frac{s_\delta^{-1}\left(\frac{1}{h}\right)^2}{r^2} |d \exp_p|_\varrho (d \exp_p^{-1}|_x(v))|^2$$

Since $\mu \leq K^N \leq \delta$ the standard Jacobi field estimates (see for instance corollary 2.8, p 153 of [13]) say that for any vector w orthogonal to $\nabla^N r$ at y we have

$$|w|^2 \frac{r^2}{s_\mu(r)^2} \leq |d \exp_p^{-1}|_y(w)|^2 \leq |w|^2 \frac{r^2}{s_\delta(r)^2}$$

This gives

$$\frac{s_\delta(s_\delta^{-1}\left(\frac{1}{h}\right))^2}{r^2} |d \exp_p^{-1}|_x(v)|^2 \leq |dF_x(u)|^2 \leq \frac{s_\mu(s_\delta^{-1}\left(\frac{1}{h}\right))^2}{r^2} |d \exp_p^{-1}|_x(v)|^2$$

and applying again the standard Jacobi field estimates we obtain the desired inequalities of the lemma. \square

Lemma 4.2. *Let $u \in T_x M$ so that $|u| = 1$. Then for any $\eta > 0$, there exists a constant $\rho(\delta, \mu, \eta) > 0$ so that if M is contained in the ball $B(p, \rho(\delta, \mu, \eta))$, then*

$$\frac{(1 - \eta)^2}{h^2 s_\delta^2(r)} (1 - |\nabla^M r|^2) \leq |dF_x(u)|^2 \leq \frac{(1 + \eta)^2}{h^2 s_\delta^2(r)}$$

Moreover $\rho(\delta, \mu, \eta) \rightarrow \infty$ when $\delta - \mu \rightarrow 0$ and $\rho(\delta, \mu, \eta) \rightarrow 0$ when $\eta \rightarrow 0$.

Proof. Let $r \geq 0$. For $t \in (-\infty, \frac{\pi^2}{16r^2})$, consider the function $\sigma(t) = s_t(r)$. An easy verification yields that σ is C^1 on $(-\infty, \frac{\pi^2}{16r^2})$ and

$$\sigma'(t) = \begin{cases} \frac{r^3 c_t(r)}{2} \left(\frac{\sqrt{tr} - \tan(\sqrt{tr})}{(\sqrt{tr})^3} \right) & \text{if } t \in (0, \frac{\pi^2}{16r^2}) \\ -\frac{r^3}{6} & \text{if } t = 0 \\ \frac{r^3 c_t(r)}{2} \left(\frac{-\sqrt{-tr} + \tanh(\sqrt{-tr})}{(\sqrt{-tr})^3} \right) & \text{if } t \in (-\infty, 0) \end{cases}$$

It follows that σ is decreasing on $(-\infty, \frac{\pi^2}{16r^2})$ and that there exists a constant D so that $|\sigma'(t)| \leq Dr^3 c_t(r)$, for any $t \in (-\infty, \frac{\pi^2}{16r^2})$. It follows that

$$(11) \quad 0 \leq s_\mu(r) - s_\delta(r) \leq Dr^3 c_\mu(r)(\delta - \mu)$$

Now we have

$$\begin{aligned} \frac{1}{hs_\mu(r)} &\geq \frac{1}{h(s_\delta(r) + Dr^3 c_\mu(r)(\delta - \mu))} \\ &\geq \frac{1}{hs_\delta(r)(1 + D \left(\frac{r}{s_\delta(r)}\right) r^2 c_\mu(r)(\delta - \mu))} \end{aligned}$$

The function $t \mapsto \frac{t}{s_\delta(t)}$ being bounded on $[0, \infty)$ and on $[0, \frac{\pi}{4\sqrt{\delta}})$ for $\delta > 0$ there exists a constant D' so that

$$(12) \quad \frac{1}{hs_\mu(r)} \geq \frac{1}{hs_\delta(r)(1 + D'r^2 c_\mu(r)(\delta - \mu))}$$

On the other hand, as we have seen it in the proof of the lemma 3.9, $s_\delta^{-1}(\frac{1}{h}) \in [0, \frac{\pi}{4\sqrt{\delta}})$ for $\delta > 0$ and we can apply the inequality (11) which gives

$$\begin{aligned} s_\mu \left(s_\delta^{-1} \left(\frac{1}{h} \right) \right) &\leq \frac{1}{h} + D \left(s_\delta^{-1} \left(\frac{1}{h} \right) \right)^3 c_\mu \left(s_\delta^{-1} \left(\frac{1}{h} \right) \right) (\delta - \mu) \\ &\leq \frac{1}{h} \left[1 + D \left(\frac{s_\delta^{-1}(\frac{1}{h})}{1/h} \right) s_\delta^{-1} \left(\frac{1}{h} \right)^2 c_\mu \left(s_\delta^{-1} \left(\frac{1}{h} \right) \right) (\delta - \mu) \right] \end{aligned}$$

And using the same arguments concerning the function $t \mapsto \frac{t}{s_\delta(t)}$, we have

$$(13) \quad s_\mu \left(s_\delta^{-1} \left(\frac{1}{h} \right) \right) \leq \frac{1}{h} (1 + D'R^2 \tilde{c}_\mu(R)(\delta - \mu))$$

where

$$\tilde{c}_\mu(r) = \begin{cases} 1 & \text{if } \mu \geq 0 \\ c_\mu(r) & \text{if } \mu < 0 \end{cases}$$

From the two inequalities (12) and (13) we deduce that there exists a constant $\rho(\delta, \mu, \eta)$ so that if $R \leq \rho(\delta, \mu, \eta)$ then $\frac{1}{hs_\mu(r)} \geq \frac{1-\eta}{hs_\delta(r)}$ and $s_\mu(s_\delta^{-1}(\frac{1}{h})) \leq \frac{1}{h}(1+\eta)$. Finally from the lemma 4.1 we deduce that

$$\frac{(1-\eta)^2}{h^2 s_\delta^2(r)} |v|^2 \leq |dF_x(u)|^2 \leq \frac{(1+\eta)^2}{h^2 s_\delta^2(r)} |v|^2$$

Since we have assumed that $|u| = 1$ and $v = u - \langle u, \nabla^M r \rangle \nabla^N r$ we get the desired result. \square

We can now give the proof of the theorem 1.1

Proof of Theorem 1.1: Let $\varepsilon < \frac{1}{4b} \leq \frac{1}{4h}$. From the lemma 3.9 there exists a constant $\Gamma_\varepsilon(n, b, h, \delta, V(M))$ so that if (P_{Γ_ε}) holds then $||Z| - \frac{1}{h}| < \varepsilon$. From this and the lemma 4.2 we deduce that if M is contained in the ball $B(p, \rho(\delta, \mu, \eta))$ then

$$\left[\frac{(1-\eta)^2}{(1+\varepsilon h)^2} - 1 \right] - \frac{(1-\eta)^2}{(1+\varepsilon h)^2} \|\nabla^M r\|_\infty^2 \leq |dF_x(u)|^2 - 1 \leq \frac{(1+\eta)^2}{(1-\varepsilon h)^2} - 1$$

To complete the proof of the theorem 1.1 we need the following lemma

Lemma 4.3. *There exists $\rho'(\delta, \mu)$ so that for any $\varepsilon < \frac{1}{4b}$ and $\eta > 0$, the pinching condition $(P_{\tilde{\Gamma}_{\varepsilon, \eta}})$ with*

$$\tilde{\Gamma}_{\varepsilon, \eta}(n, b, h, \delta, V(M)) = \min(\Gamma_\varepsilon(n, b, h, \delta, V(M)), \Lambda_\eta(n, b, h, \delta, V(M)))$$

and

$$\Lambda_\eta(n, b, h, \delta, V(M)) = \frac{\eta^2}{\left(\frac{\beta b(1+o(1))}{\eta} + \beta' b(1+o(1)) \right)^{2\gamma} V(M)^{\frac{2(\gamma+2)}{n}} b^2(1+o(1))}$$

implies that for any immersion ϕ so that $\phi(M) \subset B(p, \rho'(\delta, \mu))$ we have

$$\|\nabla^M r\|_\infty \leq \eta$$

Proof. As usually by computing the Laplacian of $|\nabla^M r|^2$ and using the Bochner formula we get

$$\frac{1}{2} \int_M \Delta |\nabla^M r|^2 |\nabla^M r|^{2k-2} dv \leq \int_M (\langle \Delta dr, dr \rangle - \text{Ric}(\nabla^M r, \nabla^M r)) |\nabla^M r|^{2k-2} dv$$

Now integrating by part and using the Gauss formula we obtain

$$\begin{aligned}
\frac{1}{2} \int_M \Delta |\nabla^M r|^2 |\nabla^M r|^{2k-2} dv &\leq \int_M ((\Delta r)^2 - \bar{R}^\phi(\nabla^M r, \nabla^M r) - nHB(\nabla^M r, \nabla^M r) \\
&\quad + |B\nabla^M r|^2) |\nabla^M r|^{2k-2} dv - \int_M \Delta r \langle \nabla^M r, \nabla^M |\nabla^M r|^{2k-2} \rangle dv \\
(14) \quad &\leq \int_M ((\Delta r)^2 |\nabla^M r|^{2k-2} dv + ((n-1)|\mu| + (\sqrt{n}+1)\|B\|_\infty^2)) \int_M |\nabla^M r|^{2k} dv \\
&\quad - 2(k-1) \int_M \Delta r \nabla^M dr(\nabla^M r, \nabla^M r) |\nabla^M r|^{2k-4} dv
\end{aligned}$$

Now

$$\begin{aligned}
\nabla^M dr(\nabla^M r, \nabla^M r) &= \nabla^N dr(\nabla^M r, \nabla^M r) - B(\nabla^M r, \nabla^M r) \langle \nabla^N r, \nu \rangle \\
&= \nabla^N dr(\nabla^M r - |\nabla^M r|^2 \nabla^N r, \nabla^M r - |\nabla^M r|^2 \nabla^N r) \\
&\quad - B(\nabla^M r, \nabla^M r) \langle \nabla^N r, \nu \rangle
\end{aligned}$$

From the comparisons theorems (see for instance [13] p 153) we deduce that

$$(15) \quad |\nabla^M dr(\nabla^M r, \nabla^M r)| \leq \left(\frac{c_\mu}{s_\mu} + \|B\|_\infty \right) |\nabla^M r|^2$$

Similarly, $(e_i)_{1 \leq i \leq n}$ being an orthonormal frame in a neighborhood of the point where we are computing we have

$$\begin{aligned}
|\Delta r| &\leq \sum_{i=1}^n |\nabla^N r(e_i, e_i)| + nH \langle \nabla^N r, \nu \rangle \\
&= \sum_{i=1}^n |\nabla^N (e_i - \langle e_i, \nabla^N r \rangle \nabla^N r, e_i - \langle e_i, \nabla^N r \rangle \nabla^N r)| + \sqrt{n} \|B\|_\infty \\
&\leq n \frac{c_\mu}{s_\mu} + \sqrt{n} \|B\|_\infty
\end{aligned}$$

and reporting this and (15) in (14) we get

$$\begin{aligned}
\frac{1}{2} \int_M \Delta |\nabla^M r|^2 |\nabla^M r|^{2k-2} dv &\leq 2k \left(n \frac{c_\mu}{s_\mu} + \sqrt{n} \|B\|_\infty \right)^2 \int_M |\nabla^M r|^{2k-2} dv \\
&\quad + ((n-1)|\mu| + (\sqrt{n}+1)\|B\|_\infty^2) \int_M |\nabla^M r|^{2k} dv
\end{aligned}$$

Now we choose $\rho'(\delta, \mu)$ so that in the ball of radius $\rho'(\delta, \mu)$, $c_\mu(r) \leq 1$. Now for $\varepsilon < \frac{1}{2b} \leq \frac{1}{2h}$ the pinching condition $((P_{\Gamma_\varepsilon}))$ implies that $\frac{c_\mu(r)}{s_\mu(r)} \leq \frac{1}{s_\delta(r)} \leq 2h \leq 2b$.

Finally we have

$$\begin{aligned} \frac{1}{2} \int_M \Delta |\nabla^M r|^2 |\nabla^M r|^{2k-2} dv &\leq \beta b^2 (1 + o(1)) k \int_M |\nabla^M r|^{2k-2} dv \\ &+ \beta' b^2 (1 + o(1)) \int_M |\nabla^M r|^{2k} dv \end{aligned}$$

where β and β' are constants depending only on n or μ .

Now let $\varepsilon < 1/2h$. Then note that $|\nabla^M r|^2 \leq \frac{1}{s_\delta^2(r)} |Z^T|^2 \leq \frac{h^2}{(1-\varepsilon h)^2} |Z^T|^2 \leq 4b^2 |Z^T|^2$ and $\|\nabla^M r\|_2 \leq 2b \|Z^T\|_2 \leq \beta b V(M)^{1/n} \|Z\|_2 \Gamma_\varepsilon^{1/2} \leq \beta b V(M)^{2/n+1/2} (1 + o(1)) \Gamma_\varepsilon^{1/2}$. Again from the proposition 3.1, if $\|\nabla^M r\|_\infty > \eta$ then

$$\|\nabla^M r\|_\infty \leq \left(\frac{\beta b (1 + o(1))}{\eta} + \beta' b (1 + o(1)) \right)^\gamma V(M)^{\gamma/n-1/2} \|\nabla^M r\|_2$$

From this and the estimate of $\|\nabla^M r\|_2$ above, we have the desired result. \square

Now we are allowed to complete the proof of theorem 1.1. Let us recall that we have chosen $\eta = \varepsilon b < \frac{1}{4}$ and put $\eta' = (\varepsilon b)^{1/2}$. Then if $\phi(M) \subset B(p, R(\delta, \mu, \eta))$ (with $R(\delta, \mu, \eta) = \min(\rho(\delta, \mu, \eta), \rho'(\delta, \mu))$) and if $(P_{\tilde{\Gamma}_{\varepsilon, \eta'}})$ is satisfied then

$$\left(\frac{(1-\varepsilon b)^2}{(1+\varepsilon b)^2} - 1 \right) - (1-\varepsilon b)^2 \varepsilon b \leq |dF_x(u)|^2 - 1 \leq \frac{(1+\varepsilon b)^2}{(1-\varepsilon b)^2} - 1$$

noting that $\left(\frac{(1-\varepsilon b)^2}{(1+\varepsilon b)^2} - 1 \right) - (1-\varepsilon b)^2 \varepsilon b = \frac{-4\varepsilon b}{(1+\varepsilon b)^2} - (1-\varepsilon b)^2 \varepsilon b \geq -5\varepsilon b$ and $\frac{(1+\varepsilon b)^2}{(1-\varepsilon b)^2} - 1 = \frac{4\varepsilon b}{(1-\varepsilon b)^2} \leq \frac{16}{9} \varepsilon b$, we deduce that $||dF_x(u)|^2 - 1| \leq 5\varepsilon b$. From the choice of θ and ε , we deduce that F is a diffeomorphism and from the definition of the Gromov-Hausdorff distance it follows that we have also

$$d_{GH} \left(\phi(M), S \left(p, s_\delta^{-1} \left(\frac{1}{h} \right) \right) \right) < \varepsilon$$

On the other hand the we have

$$\begin{aligned} C_\varepsilon(n, b, h, \delta, V(M)) &= \tilde{\Gamma}_{\varepsilon, \eta'}(n, b, h, \delta, V(M)) \\ &= \min(\Gamma_\varepsilon(n, b, h, \delta, V(M)), \Lambda_{(\varepsilon b)^{1/2}}(n, b, h, \delta, V(M))) \end{aligned}$$

where

$$\Gamma_\varepsilon(n, b, h, \delta, V(M)) = \min \left(\frac{\alpha(1+o(1))}{V(M)^{4/n}}, \frac{\alpha'(1+o(1))}{V(M)^{2/n}}, \frac{\psi(\varepsilon, b)}{A(n, b, h, \delta, V(M), \psi(\varepsilon, b))} \right)$$

and

$$\Lambda_{(\varepsilon b)^{1/2}}(n, b, h, \delta, V(M)) = \frac{\varepsilon}{\left(\frac{\beta b^{1/2}(1+o(1))}{\varepsilon^{1/2}} + \beta' b (1 + o(1)) \right)^{2\gamma} V(M)^{\frac{2(\gamma+2)}{n}} b (1 + o(1))}$$

From the definition of A and ψ we see that $C_\varepsilon(n, b, h, V(M), \delta) \rightarrow 0$ when $\varepsilon \rightarrow 0$ and $C_\varepsilon(n, b, h, V(M), \delta) \rightarrow 0$ when $b \rightarrow \infty$.

Moreover if $V(M)^{1/n}b \leq v$ and $\varepsilon = \frac{\kappa}{b}$ for positive constants v and κ , we have

$$\begin{aligned} & \frac{\psi(\varepsilon, b)}{A(n, b, h, \delta, V(M), \psi(\varepsilon, b))} \\ &= \frac{\beta}{(\beta_1 A_1 b^3 (1 + o(1)) + \beta_2 A_2 b^{3/2} (1 + o(1)) + \beta_3 b (1 + o(1)))^{4\gamma} b^3 A_3 V(M)^{4\gamma/n}} \\ &\geq \frac{\beta}{(\beta_1 A_1 b^2 v (1 + o(1)) + \beta_2 A_2 b^{1/2} v (1 + o(1)) + \beta_3 (1 + o(1)))^{4\gamma} b^3 A_3} \end{aligned}$$

and from the definition of A_1 , A_2 and A_3 , we see that $A_1 b^2 \rightarrow 0$, $A_2 b^{1/2} \rightarrow 0$ and $A_3 b^3 \rightarrow 0$ when $b \rightarrow \infty$ and then $\Gamma_\varepsilon(n, b, h, \delta, V(M))$ too. Moreover

$$\Lambda_{(\varepsilon b)^{1/2}}(n, b, h, \delta, V(M)) \geq \frac{\kappa}{\left(\frac{\beta(1+o(1))}{\kappa^{1/2}} + \beta'(1+o(1))\right)^{2\gamma} v^{2\gamma+2} V(M)^{2/n} (1+o(1))}$$

This completes the proof. \square

5. APPLICATION TO THE STABILITY

Briefly, we recall the problem of the stability of hypersurfaces with constant mean curvature (see for instance [3]).

Let (M^n, g) be an oriented compact n -dimensional hypersurface isometrically immersed by ϕ in a $n+1$ -dimensional oriented manifold (N^{n+1}, h) . We assume that M is oriented by the global unit normal field ν so that ν is compatible with the orientations of M and N . Let $F : (-\varepsilon, \varepsilon) \times M \rightarrow N$ be a variation of ϕ so that $F(0, \cdot) = \phi$. We recall that the balance volume is the function $V : (-\varepsilon, \varepsilon) \rightarrow \mathbb{R}$ defined by

$$\int_{[0,t] \times M} F^* dv_h$$

where dv_h is the element volume associated to the metric h . It is well known that

$$V'(0) = \int_M f dv$$

where $f(x) = \langle \frac{\partial F}{\partial t}(0, x), \nu \rangle$. Moreover the area function $A(t) = \int_M dv_{F_t^* h}$ satisfies

$$A'(0) = -n \int_M H f dv$$

The balance volume V is said to be preserving volume if $V(t) = V(0)$ in a neighborhood of 0 ; in this case we have $\int_M f dv = 0$. Conversely, for all smooth function f so that $\int_M f dv = 0$, there exists a preserving volume variation so that $f = \langle \frac{\partial F}{\partial t}(0, x), \nu \rangle$. The following assertions are equivalent

- (1) The immersion ϕ is a critical point of the area (i.e. $A'(0) = 0$) for all variation with preserving volume.
- (2) $\int_M H f dv = 0$ for any smooth function so that $\int_M f dv = 0$.
- (3) There exists a constant H_0 so that $A'(0) + nH_0 V'(0) = 0$ for any variation.
- (4) ϕ is of constant mean curvature H_0 .

An immersion with constant mean curvature H_0 will be said stable if $A''(0) \geq 0$ for all preserving volume variation. Now we consider the function $J(t)$ defined by

$$J(t) = A(t) + nH_0 V(t)$$

Then $J''(0)$ is depending only on f and we have

$$J''(0) = \int_M |df|^2 dv - \int_M (Ric^N(\nu, \nu) + |B|^2) f^2 dv$$

where Ric^N is the Ricci curvature of N with respect to the metric h . It is known that ϕ is a stable constant mean curvature immersion if and only if $J''(0) \geq 0$ for any smooth function so that $\int_M f dv = 0$.

Now let us give a proof of the theorem 1.3

Proof of Theorem 1.3: Let f be the first eigenfunction associated to $\lambda_1(M)$. Since $\int_M f dv = 0$ then $J''(0) \geq 0$ and

$$\lambda_1(M) \int_M f^2 dv - \int_M (Ric^N(\nu, \nu) + nH^2 + |\tau|^2) f^2 dv \geq 0$$

where τ is the umbilicity tensor (i.e. $\tau = nHg - B$). Since $\mu \leq K^N \leq \delta$, we deduce that

$$n(H^2 + \mu) \leq \lambda_1(M) \leq n(H^2 + \delta)$$

In other words, we have the pinching condition

$$n(H^2 + \delta) - n(\delta - \mu) \leq \lambda_1(M) \leq n(H^2 + \delta)$$

Now fix $\varepsilon = \frac{\theta}{5b}$ and let $R > 0$ so that $\phi(M)$ lies on a ball of radius R . Let ρ be the extrinsic radius of M (i.e. the radius of the smallest ball containing $\phi(M)$). Then $s_\delta(R) \geq s_\delta(\rho)$. On the other hand, we know that $s_\delta(\rho) \geq \frac{1}{h} \geq \frac{1}{b}$ (see [2]). If we assume that $V(M)^{1/n} \leq \frac{v}{b}$, we see that $M \in \mathcal{H}(n, \phi, N)$ for R small enough. On the other hand from the theorem 1.1 it follows that $C_{\theta/5b}(n, b, h, V(M), \delta) \rightarrow \infty$ when $R \rightarrow 0$ and there exists $R'(\delta, \mu, v, i(N))$ so that if $\phi(M)$ lies in a ball of radius $R'(\delta, \mu, v, i(N))$ then $C_{\theta/5b}(n, b, h, V(M), \delta) \geq n(\delta - \mu)$. Now we conclude by putting $R_0(\delta, \mu, v, i(N)) = \min(R(\delta, \mu, \theta/5), R'(\delta, \mu, v, i(N)))$ and the point p is nothing but the center of mass of M (the quantity $R(\delta, \mu, \theta/5)$ is defined in the theorem 1.1).

□

6. APPLICATION TO THE ALMOST UMBILIC HYPERSURFACES

In this section we give an other application of eigenvalue pinching theorems for almost umbilic hypersurfaces. We consider only the case where the ambient space is of constant sectional curvature.

Theorem 6.1. *Let (N^{n+1}, h) be a $n + 1$ -dimensional Riemannian manifold with constant sectional curvature δ and let $M \in \mathcal{H}(n, \phi, N)$. Let p be the center of mass of M . Let $q > \frac{n}{2}$, $\theta \in (0, 1)$ and $\varepsilon \leq \frac{\theta}{5b}$. Then there exist positive constants $\eta_{1,\varepsilon}(n, b, h, V(M), \delta)$ and $\eta_{2,\varepsilon}(n, b, h, V(M), \delta)$ so that if*

- (1) $\|\tau\|_{2q} \leq \eta_{1,\varepsilon}$.
- (2) $\|H^2 - \|H\|_\infty^2\|_q \leq \eta_{2,\varepsilon}$.

Then

$$d_{GH} \left(\phi(M), S \left(p, s_\delta^{-1} \left(\frac{1}{h} \right) \right) \right) < \varepsilon$$

and M is diffeomorphic and θ -quasi-isometric to $S(p, s_\delta^{-1}(\frac{1}{h}))$.

Remark 6.1. *In the hyperbolic case we can obtain the Hausdorff proximity with the dependence on the mean curvature.*

In the Euclidean case providing from the pinching theorem proved in [6] we can improve the condition 2)

Theorem 6.2. *Let (M^n, g) be a compact, connected and oriented n -dimensional Riemannian manifold without boundary isometrically immersed by ϕ in \mathbb{R}^{n+1} . Let p be the center of mass of M . Then for any $\varepsilon > 0$, there exist two constants $\eta_{1,\varepsilon}(n, \|H\|_\infty, V(M))$ and $\eta_{2,\varepsilon}(n, \|H\|_\infty, V(M))$ so that if*

- (1) $\|\tau\|_{2q} \leq \eta_{1,\varepsilon}$.
- (2) $\left\| H^2 - \frac{\|H\|_{2r}^2}{V(M)^{1/r}} \right\|_q \leq \eta_{2,\varepsilon}$ for $r \geq 2$.

Then

$$d_{GH} \left(\phi(M), S \left(p, \frac{V(M)^{1/2r}}{\|H\|_{2r}} \right) \right) < \varepsilon$$

Moreover for any $\theta \in (0, 1)$, there exist two constants $\eta_{1,\varepsilon}(n, \|B\|_\infty, V(M))$ and $\eta_{2,\varepsilon}(n, \|B\|_\infty, V(M))$ so that if the two conditions 1) and 2) are satisfied then M is diffeomorphic and θ -quasi-isometric to $S \left(p, \frac{V(M)^{1/2r}}{\|H\|_{2r}} \right)$.

These theorems are obtained by combining the theorem 1.1 and the eigenvalue pinching theorems of [6] with an eigenvalue pinching result in almost positive Ricci curvature due to E. Aubry ([1]).

In the following theorem we denote $\underline{\text{Ric}}(x)$ the lowest eigenvalue of the Ricci tensor $\text{Ric}(x)$ at $x \in M$. Moreover for any function f , we put $f_- = \min(-f, 0)$.

Theorem 6.3. (E.AUBRY) *Let (M^n, g) be a complete n -dimensional Riemannian manifold and $q > n/2$. If M has finite volume and*

$$\rho_q = \frac{1}{kV(M)^{1/q}} \left(\int_M (\underline{\text{Ric}} - (n-1)k)_-^q dv \right)^{1/q} \leq C(q, n)^{-1/q}$$

then M is compact and

$$\lambda_1(M) \geq nk(1 - C(n, q)\rho_q)$$

Proof of Theorems: Using Gauss formula and the fact that N is of constant sectional curvature δ , we have

$$\begin{aligned} \frac{\|\text{Ric} - (n-1)(H^2 + \delta)g\|_q}{V(M)^{1/q}} &= \frac{\|\overline{R}^\phi + nHB - B^2 - (n-1)H^2g - (n-1)\delta g\|_q}{V(M)^{1/q}} \\ &= \frac{\|(n-2)H\tau - \tau^2\|_q}{V(M)^{1/q}} \\ &\leq \frac{(n-2)\|H\|_\infty\|\tau\|_{2q}}{V(M)^{1/2q}} + \frac{\|\tau\|_{2q}^2}{V(M)^{1/q}} \end{aligned}$$

Now, putting $k = \frac{\|H\|_{2r}^2}{V(M)^{1/r}} + \delta$ we get

$$\begin{aligned} \frac{\|\text{Ric} - (n-1)kg\|_q}{V(M)^{1/q}} &\leq \frac{\|\text{Ric} - (n-1)(H^2 + \delta)g\|_q}{V(M)^{1/q}} + \frac{(n-1)\sqrt{n}}{V(M)^{1/q}} \left\| H^2 - \frac{\|H\|_{2r}^2}{V(M)^{1/r}} \right\|_q \\ &\leq \frac{(n-2)\|H\|_\infty\|\tau\|_{2q}}{V(M)^{1/2q}} + \frac{\|\tau\|_{2q}^2}{V(M)^{1/q}} + \frac{(n-1)\sqrt{n}}{V(M)^{1/q}} \left\| H^2 - \frac{\|H\|_{2r}^2}{V(M)^{1/r}} \right\|_q \end{aligned}$$

There exists two constants $\eta_{1,\varepsilon}(n, b, h, V(M), \delta)$ and $\eta_{2,\varepsilon}(n, b, h, V(M), \delta)$ so that if $\|\tau\|_{2q} \leq \eta_{1,\varepsilon}$ and $\left\| H^2 - \frac{\|H\|_{2r}^2}{V(M)^{1/r}} \right\|_q \leq \eta_{2,\varepsilon}(n, b, h, V(M), \delta)$ so that

$$\frac{\|\text{Ric} - (n-1)kg\|_q}{V(M)^{1/q}} \leq A_\varepsilon(n, b, h, V(M))$$

where $A_\varepsilon(n, b, h, V(M)) = \min \left(\left(\frac{\alpha}{V(M)^{2/n}} + \delta \right) C(q, n)^{-1/q}, \frac{C_\varepsilon(n, b, h, V(M))}{nC(n, q)} \right)$. The estimate (5) and the theorem 6.3 allows us to conclude that

$$\lambda_1(M) \geq n \left(\frac{\|H\|_{2r}^2}{V(M)^{1/r}} + \delta \right) - C_\varepsilon$$

Now the conclusion is immediate from the pinching theorems of this paper and [6]. \square

REFERENCES

- [1] AUBRY E. Diameter pinching in almost positive Ricci curvature, *Preprint*.
- [2] BAIKOSSIS C., KOUFOGIORGOS T. The diameter of an immersed Riemannian manifold with bounded mean curvature, *J. Austral. Soc.*, (Series A), **31**, (1981), 189-192.
- [3] BARBOSA J.L., DO CARMO M. Stability of Hypersurfaces with Constant Mean Curvature, *Math. Z.*, **185**, No. 3, (1984), 339-353.
- [4] BURAGO D., ZALGALLER V. A. Geometric Inequalities, *Springer-Verlag*.
- [5] CHAVEL I.: Riemannian geometry-A modern introduction, *Cambridge university press*.
- [6] COLBOIS B., GROSJEAN J.F. A pinching theorem for the first eigenvalue of the Laplacian on hypersurfaces of the Euclidean space, *Comment. Math. Helv.*, **82**, (2007), 175-195.

- [7] EL SOUFI A., ILIAS S., Une inégalité du type "Reilly" pour les sous-variétés de l'espace hyperbolique, *Comm. Math. Helv.*, **67**, (1992), 167-181.
- [8] HEINTZE E.: Extrinsic upper bound for λ_1 , *Math. Ann.*, **280**, (1988), 389-402.
- [9] HOFFMAN D., SPRUCK J., Sobolev and isoperimetric inequalities for Riemannian submanifolds, *Comm. Pure and Appl. Math.*, **27**, (1974), 715-727.
- [10] HOFFMAN D., SPRUCK J., Sobolev and isoperimetric inequalities for Riemannian submanifolds, Erratum *Comm. Pure and Appl. Math.*, **28**, (1975), 765-766.
- [11] REILLY R., On the first eigenvalue of the Laplacian for compact submanifolds of Euclidean space, *Comment. Math. Helv.*, **52**, (1977), 525-533.
- [12] ROTH J., Extrinsic radius pinching for hypersurfaces of space forms, to appear in *Differ. Geom. Appl.*
- [13] SAKAI T.: Riemannian geometry, *A.M.S. translations of Math. Monographs*, **149**, (1996).
- [14] SHIOHAMA K., XU H., Rigidity and sphere theorems for submanifolds, *Kyushu J. Math.*, **48**, (1994), no 2, 291-306.
- [15] SHIOHAMA K., XU H., Rigidity and sphere theorems for submanifolds II, *Kyushu J. Math.*, **54**, (2000), no 1, 103-109.
- [16] XU H.W., Some results on geometry of Riemannian submanifolds, *Ph. D. dissertation, Fudan University*, (1990).

(J.-F. Grosjean) INSTITUT ÉLIE CARTAN (MATHÉMATIQUES), UNIVERSITÉ HENRI POINCARÉ NANCY I, B.P. 239,
F-54506 VANDŒUVRE-LES-NANCY CEDEX, FRANCE
E-mail address: grosjean@iecn.u-nancy.fr