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# A note on the circular law for non-central random matrices

Djalil CHAFAÏ

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## Abstract

Let  $(X_{i,j})_{1 \leq i,j < \infty}$  be an infinite array of i.i.d. complex random variables, with mean  $m = 0$ , variance  $\sigma^2 = 1$ , and say with finite fourth moment. The famous circular law theorem states that the empirical spectral distribution  $\frac{1}{n}(\delta_{\lambda_1(\mathbf{X})} + \dots + \delta_{\lambda_n(\mathbf{X})})$  of  $\mathbf{X} = (n^{-1/2}X_{i,j})_{1 \leq i,j \leq n}$  converges almost surely, as  $n \rightarrow \infty$ , to the uniform law over the unit disc  $\{z \in \mathbb{C}; |z| \leq 1\}$ . For now, most efforts were focused on the improvement of moments hypotheses for the centered case  $m = 0$ . Regarding the non-central case  $m \neq 0$ , Silverstein has already observed that almost surely, the eigenvalue of  $\mathbf{X}$  of largest module goes to  $+\infty$  as  $n \rightarrow \infty$ , while the rest of the spectrum remains bounded. We show in this note that the circular law theorem remains valid when  $m \neq 0$ , by using logarithmic potentials and bounds on extremal singular values.

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**Keywords:** Random matrices; Circle or Circular law; rank-one deformations; singular values; eigenvalues; non-Hermitian matrices; logarithmic potential.

## 1 Introduction

For any square  $n \times n$  matrix  $\mathbf{A}$  with complex entries, let the complex eigenvalues  $\lambda_1(\mathbf{A}), \dots, \lambda_n(\mathbf{A})$  of  $\mathbf{A}$  be labeled so that

$$|\lambda_1(\mathbf{A})| \geq \dots \geq |\lambda_n(\mathbf{A})|.$$

Similarly, we denote by  $s_1(\mathbf{A}) \geq \dots \geq s_n(\mathbf{A})$  the *singular values* of  $\mathbf{A}$ , i.e. the eigenvalues of the positive semi-definite Hermitian matrix  $(\mathbf{A}\mathbf{A}^*)^{1/2}$ . Namely,

$$s_k(\mathbf{A}) = \sqrt{\lambda_k(\mathbf{A}\mathbf{A}^*)}$$

for every  $1 \leq k \leq n$ . In particular, we have the variational formulas

$$s_1(\mathbf{A}) = \max_{\|x\|_2=1} \|\mathbf{A}x\|_2 \quad \text{and} \quad s_n(\mathbf{A}) = \min_{\|x\|_2=1} \|\mathbf{A}x\|_2.$$

In particular,  $s_1(\mathbf{A}) = \|\mathbf{A}\|_2$ . We always have

$$|\lambda_1(\mathbf{A})| \leq s_1(\mathbf{A}) \quad \text{and} \quad s_n(\mathbf{A}) \leq |\lambda_n(\mathbf{A})|. \quad (1)$$

When  $\mathbf{A}$  is a normal matrix, i.e.  $\mathbf{A}\mathbf{A}^* = \mathbf{A}^*\mathbf{A}$ , then  $|\lambda_k(\mathbf{A})| = s_k(\mathbf{A})$  for every  $1 \leq k \leq n$ . The *empirical spectral distribution* (ESD) of  $\mathbf{A}$  is defined by

$$\frac{1}{n}(\delta_{\lambda_1(\mathbf{A})} + \cdots + \delta_{\lambda_n(\mathbf{A})}).$$

The ESD of  $\mathbf{A}$  is a discrete probability distribution on  $\mathbb{C}$  with at most  $n$  atoms. Let  $(\mathbf{X}_n)$  be a sequence of random matrices, defined on the same probability space, where  $\mathbf{X}_n$  is an  $n \times n$  matrix for each  $n$ . For every  $n$ , the ESD of  $\mathbf{X}_n$  is a random probability distribution on  $\mathbb{C}$ . Here the distribution of the entries of  $\mathbf{X}_n$  may depend on  $n$ . Usually, the sequence  $(\mathbf{X}_n)$  is constructed by setting  $\mathbf{X}_n = (X_{i,j})_{1 \leq i,j \leq n}$  where  $(X_{i,j})_{1 \leq i,j < \infty}$  is an infinite array of random variables. A typical topic in Random Matrix Theory is to ask about the almost sure convergence as  $n \rightarrow \infty$  of the ESD of  $\mathbf{X}_n$  to some non-random probability distribution on  $\mathbb{C}$ , called the *limiting spectral distribution* (LSD). In the present article, we will write most of the time  $\mathbf{X}$  instead of  $\mathbf{X}_n$ , omitting the index  $n$  in the notation.

The spectral properties of large dimensional random matrices were explored after the seminal works of the statistician Wishart in the 1930's and of the physicists Wigner and Dyson in the 1950's. Most known results concern Hermitian (or symmetric) random matrices, due to the existence of powerful techniques in that case, see for instance<sup>1</sup> [3], [15], [18], [4], and also [25] and [19]. Most of these techniques (e.g. convergence of moments) are inefficient for non-Hermitian random matrices. One of the most famous result regarding non-Hermitian random matrices is known as the circle (or circular) law theorem, illustrated by figure 1.

**Theorem 1.1 (Circular law).** *Let  $(X_{i,j})_{1 \leq i,j < \infty}$  be an infinite array of i.i.d. complex random variables of common law  $L$ , with mean  $m = 0$ , variance  $\sigma^2 = 1$ , and finite fourth moment. Then almost surely, the ESD of  $\mathbf{X} = (n^{-1/2}X_{i,j})_{1 \leq i,j \leq n}$  tends, as  $n \rightarrow \infty$ , to the uniform distribution on the unit disc  $D(0,1) = \{z \in \mathbb{C}; |z| \leq 1\}$ .*

The extension of theorem 1.1 to the case  $\sigma^2 \in (0, \infty)$  is immediate by rescaling. The LSD is the uniform law on the unit disc of  $\mathbb{C}$ , often referred as the *circle or*

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<sup>1</sup>For the links with Voiculescu's free probability theory, see for instance [15] and [6].

*circular law.* If  $Z = re^{i\theta}$  is a complex random variable distributed according to the uniform law on  $D(0, 2\sigma)$ , then the module  $r$  and the argument  $\theta$  of  $Z$  are independent with joint law of density  $(r, \theta) \mapsto (4\pi\sigma^2)^{-1} r I_{[0, 2\pi]}(\theta) I_{[0, 2\sigma]}(r)$ . Both  $\Re(Z)$  and  $\Im(Z)$  follow the Wigner semi-circle law on  $\mathbb{R}$  of density

$$x \mapsto \frac{1}{2\pi\sigma^2} \sqrt{4\sigma^2 - x^2} I_{[-2\sigma, +2\sigma]}(x).$$

More generally, for any angle  $\alpha \in [0, 2\pi)$ , the random variables  $\Re(e^{\sqrt{-1}\alpha} Z)$  and  $\Im(e^{\sqrt{-1}\alpha} Z)$  follow the Wigner semi-circle law mentioned above. Additionally, if a real random variable  $S$  follows the Wigner semi-circle law mentioned above, then its square  $S^2$  follows the Marčenko-Pastur law on  $\mathbb{R}_+$  of density

$$x \mapsto \frac{1}{2\pi x\sigma^2} \sqrt{(4\sigma^2 - x)x} I_{[0, 4\sigma^2]}(x).$$

The circular law theorem (theorem 1.1) has a long history. The first proof goes back to Mehta [17] in the 1960's in the case where  $L$  is a centered complex Gaussian by using the explicit expression of the density of the spectrum provided by Ginibre [10]. The real centered Gaussian case is more “complex” in a way and was investigated by Edelman [9] in the 1990's.

The universal case was attacked by Girko [11] in 1984. However, Girko's proof has been criticised by Bai who pointed out mathematical gaps. In 1997 Bai [2] provided a rigorous proof by using the Fourier-Stieltjes approach suggested by Girko. The reader may find a discussion on Girko's contribution in [2, 3, 4] and [14, p. 3] for instance. More recently, following a suggestion made by Khoruzhenko in 2001, Götze and Tikhomirov [14] used an alternative proof of the circular law theorem based on logarithmic potentials. Later, Pan & Zhou in [19] used this method to provide the version of the circular law theorem given above (theorem 1.1). Very recently, Tao & Vu [28] has shown that the finite fourth moment condition can be relaxed for instance to a finite  $2 + \varepsilon$  moment where  $\varepsilon > 0$ .

Both the Fourier-Stieltjes method and the logarithmic potential method reduce the problem to the control of the singular values of the matrix  $\mathbf{X} - z\mathbf{I}$  where  $z \in \mathbb{C}$ , i.e. to the control of the spectra of the family  $(\mathbf{H}(z))_{z \in \mathbb{C}}$  of positive semi-definite Hermitian matrices where  $\mathbf{H}(z) = ((\mathbf{X} - z\mathbf{I})(\mathbf{X} - z\mathbf{I})^*)^{1/2}$ . Namely, the Stieltjes transform  $S_n$  of the ESD  $\mu_n$  of  $\mathbf{X}$  is given by

$$S_n(z) = \int_{\mathbb{C}} \frac{1}{z' - z} d\mu_{\mathbf{X}}(z') = \frac{1}{n} \sum_{k=1}^n \frac{1}{\lambda_k(\mathbf{X}) - z}.$$

Since  $S_n$  is analytic on  $\mathbb{C}$  except on a finite number of poles  $\lambda_1(\mathbf{X}), \dots, \lambda_n(\mathbf{X})$ , the

function  $\Re(S_n)$  fully determines the spectrum of  $\mathbf{X}$ . We have

$$\begin{aligned}\Re(S_n(z)) &= \frac{1}{n} \sum_{k=1}^n \frac{\Re(\lambda_k(\mathbf{X}) - z)}{|\lambda_k(\mathbf{X}) - z|^2} \\ &= -\frac{1}{2n} \sum_{k=1}^n \partial_{\Re(z)} \log(|\lambda_k(\mathbf{X}) - z|^2) \\ &= \partial_{\Re(z)} U_n(z),\end{aligned}$$

where  $U_n$  is the logarithmic potential of the ESD  $\mu_n$  of  $\mathbf{X}$

$$\begin{aligned}U_n(z) &= - \int_{\mathbb{C}} \log |z - z'| d\mu_n(z') \\ &= -\frac{1}{n} \sum_{k=1}^n \log |\lambda_k(\mathbf{X} - z\mathbf{I})| \\ &= -\frac{1}{n} \log |\det(\mathbf{X} - z\mathbf{I})| \\ &= -\frac{1}{n} \log |\det(\mathbf{H}(z))| \\ &= - \int_0^\infty \log(x) d\nu_{n,z}(x)\end{aligned}$$

where  $\nu_{n,z}$  is the ESD of  $\mathbf{H}(z)$ . The Fourier-Stieltjes approach used by Girko [11], Bai [2], and more recently by Tao & Vu [28] is based on the convergence of the Fourier transform of  $\Re(S_n(z)) = \partial_{\Re(z)} U_n(z)$ , whereas the logarithmic potential approach used by Götze and Tikhomirov [14] and Pan & Zhou [19] is based directly on the convergence of the logarithmic potential  $U_n(z)$ . In both cases, one of the main difficulty is the control of the extremal singular values of  $\mathbf{X} - z\mathbf{I}$ , which correspond to the singularities at 0 and  $\infty$  in the integral above involving  $\nu_{n,z}$ .

While the largest singular value can be relatively easily bounded under moments hypotheses, the smallest singular value of random matrices is hard to bound, and the reader may find some recent developments based on the Littlewood-Offord problem in the works of Rudelson [22], Rudelson & Vershynin [23], Pan & Zhou [19], Tao & Vu [28], and references therein.

The bound on the smallest singular value used in [14] is based on [22], whereas the one used in [19] improves the result of [23]. However, the one used in [28] is of different nature. We need this bound since the preceding ones are not suitable for the non-central case due to the norm of the deformation.

Notice that if  $\mathbf{X}$  is an  $n \times n$  random matrix with i.i.d. entries of absolutely continuous distribution  $L$ , then almost surely,  $\mathbf{X}$  is not normal, has distinct eigenvalues (and is hence diagonalizable), and is invertible. The situation is more

intricate when  $L$  has atoms, for instance when  $L$  is a purely discrete distribution, and this contributes to make difficult the derivation of generic lower bounds on smallest singular values.

Actually, it is conjectured that the circular law theorem is true under the condition  $m = 0$  and  $\sigma^2 = 1$  (or equivalently  $\sigma^2 \in (0, \infty)$ ) without any additional condition. At the time of writing, this conjecture, which goes back to the 1950's, and which is confirmed by numerical experiments, still remains an open problem for mathematicians.

### Non-central case

The present article is devoted to the case where  $L$  has mean  $m \neq 0$ . Let us start by the qualitative behavior of the spectrum of  $\mathbf{X}$  for non-central  $L$ . One can write  $\mathbf{X} = \mathbf{X} - \alpha \mathbf{1} + \alpha \mathbf{1}$  where  $\alpha = n^{-1/2}m$  and  $\mathbf{1}$  is the  $n \times n$  matrix with all entries equal to 1. Therefore, if  $m \neq 0$ , the random matrix  $\mathbf{X}$  is a rank one deformation of the centered random matrix  $\mathbf{X} - \alpha \mathbf{1}$ . The rank one positive semi-definite symmetric matrix  $\alpha \mathbf{1}$  has spectrum  $\lambda_1(\alpha \mathbf{1}) = n\alpha$  and  $\lambda_2(\alpha \mathbf{1}) = \dots = \lambda_n(\alpha \mathbf{1}) = 0$ , and in particular

$$\|\alpha \mathbf{1}\|_2 = m\sqrt{n}.$$

Following Andrew [1, p. 167] or Silverstein [27], for every  $n \times n$  complex matrix  $\tilde{\mathbf{A}}$  and complex number  $\alpha \in \mathbb{C}$ , we have by denoting  $r = s_1(\tilde{\mathbf{A}})$  and  $\mathbf{A} = \tilde{\mathbf{A}} + \alpha \mathbf{1}$ ,

$$\{\lambda_1(\mathbf{A}), \dots, \lambda_n(\mathbf{A})\} \subset D(0, r) \cup D(n\alpha, r)$$

where  $D(z, r) = \{z' \in \mathbb{C}; |z - z'| \leq r\}$ . Moreover, if  $|n\alpha| > r$  then

$$\{\lambda_2(\mathbf{A}), \dots, \lambda_n(\mathbf{A})\} \subset D(0, r) \quad \text{and} \quad \lambda_1(\mathbf{A}) \in D(n\alpha, r).$$

The localization result above follows from classical perturbation theory<sup>2</sup>, by seeing  $\mathbf{A} = \tilde{\mathbf{A}} + \alpha \mathbf{1}$  as an additive perturbation of  $\alpha \mathbf{1}$  by  $\tilde{\mathbf{A}}$ . Silverstein came in [27] with more detailed information summarized in theorem 1.2 below.

**Theorem 1.2 (Spectrum localization for non-central entries).** *Let  $(X_{i,j})_{1 \leq i,j < \infty}$  be an infinite array of i.i.d. complex random variables with common law  $L$ , with mean  $m$ , variance  $\sigma^2 = 1$ , and  $\mathbb{E}[(X_{1,1} - m)^4] < \infty$ . Define the random matrices*

$$\mathbf{X} = (n^{-1/2}X_{i,j})_{1 \leq i,j \leq n} \quad \text{and} \quad \tilde{\mathbf{X}} = (n^{-1/2}(X_{i,j} - m))_{1 \leq i,j \leq n}.$$

*Then almost surely, for large enough  $n$ ,*

$$\{\lambda_2(\mathbf{X}), \dots, \lambda_n(\mathbf{X})\} \subset D(0, s_1(\tilde{\mathbf{X}})) \quad \text{and} \quad \lambda_1(\mathbf{X}) \in D(m\sqrt{n}, s_1(\tilde{\mathbf{X}}))$$

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<sup>2</sup>One can for example make use the Bauer-Fike theorem and the Gerschgorin theorem, see for instance [30, p. 87-88], [13, th. 7.2.2 p. 321] and [16, ch. 6 and cor. 7.3.8].

Moreover, if  $m \neq 0$ , then almost surely,  $\lambda_1(\mathbf{X}) \in \mathbb{R}$  for large enough  $n$ , and

$$\lambda_1(\mathbf{X}) \xrightarrow[n \rightarrow \infty]{a.s.} \text{sign}(m) \times \infty \quad \text{and} \quad s_1(\mathbf{X}) \xrightarrow[n \rightarrow \infty]{a.s.} +\infty. \quad (2)$$

and by denoting  $\mathcal{N}(0, 1)$  the standard Gaussian on  $\mathbb{R}$

$$\sqrt{n}(\lambda_1(\mathbf{X}) - m\sqrt{n}) \xrightarrow[n \rightarrow \infty]{D} \mathcal{N}(0, 1). \quad (3)$$

In view of theorem 1.2 above, it is worthwhile to mention that

$$\mathbb{E}[X_{1,1}^4] < \infty \quad \Rightarrow \quad \lim_{n \rightarrow \infty} s_1(\tilde{\mathbf{X}}) = 2 \quad \text{a.s.} \quad (4)$$

whereas

$$\mathbb{E}[X_{1,1}^4] = \infty \quad \Rightarrow \quad \limsup_{n \rightarrow \infty} s_1(\tilde{\mathbf{X}}) = +\infty \quad \text{a.s.} \quad (5)$$

a result which goes back to Bai, Yin, Silverstein, and Krishnaiah [31, 5]. Theorem 1.2 above shows that the largest eigenvalue  $\lambda_1(\mathbf{X})$  and the rest of the spectrum  $\lambda_2(\mathbf{X}), \dots, \lambda_n(\mathbf{X})$  localize on different discs when  $m \neq 0$ . Relaxing the centering condition creates an isolated largest eigenvalue, which goes to infinity, whereas the rest of the spectrum remains roughly near the centered unit disc. A single eigenvalue plays no role in the asymptotic behavior of the ESD of  $\mathbf{X}$ , due to the empirical scaling  $1/n$ . For that reason, one can expect that the circular law theorem remains valid when  $m \neq 0$ , despite the fact that the largest eigenvalue  $\lambda_1(\mathbf{X})$  goes to infinity when  $n \rightarrow \infty$ . This is confirmed by the following theorem, which is our main result, illustrated by figure 2.

**Theorem 1.3 (Circular law for non-central entries).** *Let  $(X_{i,j})_{1 \leq i,j < \infty}$  be an infinite array of i.i.d. complex random variables of law  $L$ , with mean  $m$ , variance  $\sigma^2 = 1$ , and finite centered fourth moment. Then the ESD of  $\mathbf{X} = (n^{-1/2}X_{i,j})_{1 \leq i,j \leq n}$  tends, as  $n \rightarrow \infty$ , to the uniform distribution on the unit disc  $D(0, 1) = \{z \in \mathbb{C}; |z| \leq 1\}$ .*

The proof of theorem 1.3 is given in the next section of this article. More precisely, we show that theorem 1.1 implies theorem 1.3. The last section collects some crucial tools used in the proof of theorem 1.3.

**Example 1.4 (Adjacency matrix of Erdős-Rényi random oriented graphs).**

*An oriented finite simple graph is a set of oriented edges (arrows) between a finite number of points (vertices), with at most one arrow between two vertices. The adjacency matrix of such a graph is defined by  $\mathbf{A}_{i,j} = 1$  if there is an arrow from  $i$  to  $j$  and  $\mathbf{A}_{i,j} = 0$  otherwise. The random version corresponds to fix some parameter  $p \in (0, 1)$ , and to put an arrow between  $i$  and  $j$  with probability  $p$ , independently for*

each couple  $(i, j)$ . Let  $n$  be the number of vertices. By theorem 1.3, almost surely, the ESD of  $\mathbf{X} = (p(1-p)n)^{-1/2} \mathbf{A}$  tends to the uniform distribution on the unit disc  $D(0, 1)$  as  $n \rightarrow \infty$ . Here  $L$  is the Bernoulli distribution  $p\delta_1 + (1-p)\delta_0$  and  $m = p$  and  $\sigma^2 = p(1-p)$ . However, by theorem 1.2, almost surely, the largest eigenvalue  $\lambda_1(\mathbf{X})$  goes to  $+\infty$  at speed  $\sqrt{np/(1-p)}$ . Things remain roughly unchanged if one drops the simple graph hypothesis by allowing at most  $r$  arrows between two nodes (in that case,  $L$  is a discrete probability distribution supported by  $\{0, 1, \dots, r\}$ ).

## Extensions and open problems

Tao & Vu [28] has shown that the centered version of the circular law (theorem 1.1) remains valid if one replaces the finite fourth moment hypothesis  $\mathbb{E}[|X_{1,1}|^4] < \infty$  by  $\mathbb{E}[|X_{1,1}|^{2+\varepsilon}] < \infty$  for some  $\varepsilon > 0$ . In our reduction to the centered case, the finite fourth moment hypothesis is used to bound  $\sigma_1(\tilde{\mathbf{X}})$  via (4) and then  $\lambda_1(\mathbf{X})$  via theorem 1.2. Under the weaker moment condition of Tao & Vu, one can hopefully use instead the probabilistic bound for the condition number [28, cor. 2.10 p. 8]. Recall that the condition number of  $\mathbf{A}$  is  $\kappa(\mathbf{A}) = \|\mathbf{A}\|_2 \|\mathbf{A}^{-1}\|_2 = s_1(\mathbf{A})/s_n(\mathbf{A})$ . However, the spectrum localization given by theorem 1.2 is not meaningful without the finite fourth moment condition. One can also try to obtain a sparse version of the non-central circular law theorem by using the techniques used by Tao & Vu [28] for the centered case.

Another possible extension is to drop the identical distribution of the entries, as already mentioned by Pan & Zhou [19] and Tao & Vu [28] for the centered case. See also the discussion in section 10.8 of Bai-Silverstein's book [4].

Rider [21, 20] has shown recently that when  $L$  is a centered complex Gaussian, the fluctuation of the largest singular value is given by a Gumbel type extreme distribution rather than by a Tracy-Widom distribution as for classical Hermitian random matrices models. The asymptotic behavior (convergence and fluctuations) of the extremal eigenvalues and singular values remains untouched when  $L$  is generic. It seems heuristically that the non-Hermitian nature of the matrices gives more space to the spectrum, which can thus uncorrelate as  $n \rightarrow \infty$  and finally behave roughly like i.i.d. random variables. Quoting Girko [12], “*if all entries of a random matrix which lie on the main diagonal and above are independent and belong to the domain of attraction of an infinitely divisible law then, under some conditions, the normalized real and imaginary parts of eigenvalues behave like some order statistics when  $s, n$  tend to infinity*”. A related natural question for non-central entries is to ask about the asymptotic behavior of the second largest eigenvalue  $\lambda_2(\mathbf{X})$  or singular value  $s_2(\mathbf{X})$  in the case where  $m \neq 0$ , the behavior of  $\lambda_1(\mathbf{X})$  and  $s_1(\mathbf{X})$  being given already by theorem 1.2 (see [27] for more details). Note that it has been shown [8] that the fluctuations can be non-universal in certain centered Hermitian models with low rank deformation. The study of the asymptotic behavior



of  $s_n(\mathbf{X})$  and  $\lambda_n(\mathbf{X})$  is notoriously more difficult than the study of the asymptotic behavior of  $s_1(\mathbf{X})$  and  $\lambda_1(\mathbf{X})$ .

One can ask about the explicit expression of the density of the spectrum when  $L$  is a non-central real or complex Gaussian. This can be done hopefully by using the already known results of Ginibre [10] and Edelman [9] for the centered case.

The non-central version of the Wigner and the Marčenko-Pastur theorems can be easily deduced from the centered versions by using rank lemmas, see for instance [3] and [4, ch. 11] and lemma 3.5. The fact that these rank lemmas do not control the extremal eigenvalues does not pose a problem since the asymptotic behavior of the ESD is not sensitive to a finite number of eigenvalues. In contrast, the Fourier-Stieltjes or the logarithmic potential techniques used in the derivation of the circular law theorem need bounds for the extremal singular values (smallest and largest). One can imagine that a possible solution is to use rank lemmas directly for the complex spectrum in the ESD. Unfortunately, Hermitian rank lemmas are based on the Ky Fan interlacing inequalities for the spectrum of Hermitian matrices, which are consequences of the Courant-Fisher variational formulas for the eigenvalues of Hermitian matrices. For non-Hermitian matrices, no such variational formulas exist for the eigenvalues. It could be very useful to obtain some kind of rank lemmas for complex spectra.

## Notations and conventions

Matrix names are written in bold faced upper case letters. The identity matrix is denoted  $\mathbf{I}$ . The notation  $\mathbf{A}^* = \overline{\mathbf{A}^\top}$  stands for the conjugate transpose of  $\mathbf{A}$ . The notation  $\mathbf{1}$  stands for a matrix with all entries equal to 1. The real and imaginary parts of the complex number  $z$  are written  $\Re(z)$  and  $\Im(z)$ , and in particular  $z\bar{z} = |z|^2 = \Re(z)^2 + \Im(z)^2$ . The letter  $i$  is used for integer indexing and is never used for  $\sqrt{-1}$ . The letter  $m$  is used to denote a mean and is never used for indexing. The symbol  $\mathbb{E}$  denotes the mathematical expectation. If  $Z$  is a complex random variable of mean  $m = \mathbb{E}[Z] \in \mathbb{C}$ , we define its variance by  $\sigma^2 = \mathbb{E}[|Z - m|^2] = \mathbb{E}[|Z|^2] - |m|^2$ , which is the trace of the covariance matrix of  $Z$  viewed as a random vector of  $\mathbb{R}^2$ . The random variables are defined on a probability space  $(\Omega, \mathcal{A}, \mathbb{P})$ , and the notation “*a.s.*” stands for “ $\mathbb{P}$ -almost-surely”.

## 2 Proof of theorem 1.3

This section is devoted to the proof that theorem 1.1 implies theorem 1.3. We will adopt the logarithmic potential approach suggested by Khoruzhenko and used by Götze & Tikhomirov [14] and by Pan & Zhou [19] for the centered case. Lower bounds for the smallest singular value of  $\mathbf{X} - z\mathbf{I}$  are crucial. The ones used by

Götze & Tikhomirov and by Pan & Zhou do not allow to deal with our non-central random matrices. We will circumvent this difficulty by using instead the polynomial bounds obtained recently by Tao & Vu [28].

In this section,  $\mathbf{X}$  is the  $n \times n$  random matrix of theorem 1.3, and we assume from now that  $m \neq 0$ . The ESD of  $\mathbf{X}$  is denoted

$$\mu_n = \frac{1}{n}(\delta_{\lambda_1(\mathbf{X})} + \cdots + \delta_{\lambda_n(\mathbf{X})}).$$

We also define, for every complex number  $z \in \mathbb{C}$ ,

$$\nu_{n,z} = \frac{1}{n}(\delta_{s_1(\mathbf{X}-z\mathbf{I})} + \cdots + \delta_{s_n(\mathbf{X}-z\mathbf{I})}),$$

which is the ESD of the positive semi-definite Hermitian matrix

$$\mathbf{H}(z) = \sqrt{(\mathbf{X} - z\mathbf{I})(\mathbf{X} - z\mathbf{I})^*}.$$

The logarithmic potential  $U_n$  of the ESD  $\mu_n$  of  $\mathbf{X}$  is given by (see section 3.1)

$$\begin{aligned} U_n(z) &= - \int_{\mathbb{C}} \log |z - z'| d\mu_n(z') \\ &= - \frac{1}{n} \sum_{k=1}^n \log |\lambda_k(\mathbf{X} - z\mathbf{I})| \\ &= - \frac{1}{n} \log |\det(\mathbf{X} - z\mathbf{I})| \\ &= - \frac{1}{n} \log |\det(\mathbf{H}(z))| \\ &= - \int_0^\infty \log(x) d\nu_{n,z}(x). \end{aligned}$$

Define the  $n \times n$  random matrix  $\tilde{\mathbf{X}}$  as in theorem 1.2 by

$$\tilde{\mathbf{X}} = \mathbf{X} - n^{-1/2}m\mathbf{1} = n^{-1/2}(X_{i,j} - m)_{1 \leq i,j \leq n}$$

where  $\mathbf{1}$  is the  $n \times n$  matrix with all entries equal to 1. Define also similarly

$$\tilde{\mu}_n = \frac{1}{n}(\delta_{\lambda_1(\tilde{\mathbf{X}})} + \cdots + \delta_{\lambda_n(\tilde{\mathbf{X}})})$$

and

$$\tilde{\nu}_{n,z} = \frac{1}{n}(\delta_{s_1(\tilde{\mathbf{X}}-z\mathbf{I})} + \cdots + \delta_{s_n(\tilde{\mathbf{X}}-z\mathbf{I})})$$

and

$$\begin{aligned}\tilde{U}_n(z) &= - \int_{\mathbb{C}} \log |z - z'| d\tilde{\mu}_n(z') \\ &= -\frac{1}{n} \log \left| \det(\tilde{\mathbf{X}} - z\mathbf{I}) \right| \\ &= - \int_0^\infty \log(x) d\tilde{\nu}_{n,z}(x).\end{aligned}$$

Let  $\mu$  be the circular law, i.e the uniform law on  $D(0, 1)$ , and let  $U$  be its logarithmic potential. We first take a look at the behavior of  $\tilde{U}_n$ . By (1) and (4),

$$|\lambda_1(\tilde{\mathbf{X}})| \leq s_1(\tilde{\mathbf{X}}) \xrightarrow[n \rightarrow \infty]{a.s.} 2 < \infty.$$

Thus, almost surely, the sequence of probability distributions  $(\tilde{\mu}_n)$  are supported in a common compact subset of  $\mathbb{C}$ . Since by the centered circular law (theorem 1.1),  $\mu_n \rightarrow \mu$  weakly as  $n \rightarrow \infty$ , the Lower Envelope Theorem 3.3 gives

$$\liminf_{n \rightarrow \infty} \tilde{U}_n(z) = U(z)$$

for quasi-every  $z \in \mathbb{C}$ . Set now

$$\Delta_n(z) := U_n(z) - \tilde{U}_n(z) = \int_0^\infty \log(x) d\nu_{n,z}(x) - \int_0^\infty \log(x) d\tilde{\nu}_{n,z}(x).$$

Let us denote by  $F_{n,z}$  and  $\tilde{F}_{n,z}$  the cumulative distribution functions of  $\nu_{n,z}$  and  $\tilde{\nu}_{n,z}$  respectively defined by

$$F_{n,z}(x) = \nu_{n,z}((-\infty, x]) \quad \text{and} \quad \tilde{F}_{n,z}(x) = \tilde{\nu}_{n,z}((-\infty, x])$$

for every  $x \in \mathbb{R}$ . Let  $(\alpha_n)$  and  $(\beta_n)$  be as in lemma 2.1 below. By an integration by parts (lemma 3.1), almost surely, for every  $z \in \mathbb{C}$  and large enough  $n$ ,

$$\Delta_n = \int_{\alpha_n}^{\beta_n} \log(x) (d\nu_{n,z}(x) - d\tilde{\nu}_{n,z}(x)) = \int_{\alpha_n}^{\beta_n} \frac{1}{x} (F_{n,z}(x) - \tilde{F}_{n,z}(x)) dx$$

and hence

$$|\Delta_n| \leq (\log(\beta_n) - \log(\alpha_n)) \left\| F_{n,z} - \tilde{F}_{n,z} \right\|_\infty.$$

Now, the Bai-Thomson rank Lemma 3.5 gives

$$\left\| F_{n,z} - \tilde{F}_{n,z} \right\|_\infty \leq \frac{1}{n} \text{rank}(\mathbf{X} - \tilde{\mathbf{X}}) = \frac{1}{n} \text{rank}(n^{-1/2} m \mathbf{1}) = \frac{1}{n}$$

and therefore, almost surely, for every  $z \in \mathbb{C}$ , and for large enough  $n$ ,

$$|\Delta_n| \leq \frac{1}{n}(\log(\beta_n) - \log(\alpha_n)) \xrightarrow[n \rightarrow \infty]{} 0.$$

Consequently, almost surely, for quasi-every  $z \in \mathbb{C}$ ,

$$\liminf_{n \rightarrow \infty} U_n(z) = \liminf_{n \rightarrow \infty} \tilde{U}_n(z) = U(z) \quad (6)$$

The probability distributions  $(\mu_n)$  are not supported in a common compact set, due to the behavior of the atom  $a_n = \lambda_1(\mathbf{X})$  in  $\mu_n$ . One can however make use of (6) with theorem 3.2. Since  $m \neq 0$ , theorem 1.2 gives  $|a_n| \rightarrow \infty$  a.s. and  $\max(|\lambda_2(\mathbf{X})|, \dots, |\lambda_n(\mathbf{X})|) \leq s_1(\tilde{\mathbf{X}}) \rightarrow 2 < \infty$  a.s. Additionally,  $p_n \log |a_n| \rightarrow 0$  a.s. Finally, theorem 3.2 used with  $p_n = 1/n$  implies that almost surely,  $\mu_n \rightarrow \mu$  weakly as  $n \rightarrow \infty$ . This achieves the proof of theorem 1.3.

**Lemma 2.1.** *Almost surely, for every  $z \in \mathbb{C}$ , there exist real numbers  $\alpha, \beta$  and positive real numbers  $c_\alpha > 0$  and  $c_\beta > 0$  such that for large enough  $n$ ,*

$$\alpha_n := c_\alpha n^\alpha \leq \min(s_n(\mathbf{X} - z\mathbf{I}), s_n(\tilde{\mathbf{X}} - z\mathbf{I}))$$

and

$$\beta_n := c_\beta n^\beta \geq \max(s_1(\mathbf{X} - z\mathbf{I}), s_1(\tilde{\mathbf{X}} - z\mathbf{I})).$$

*Proof.* First, by (4), almost surely,  $s_1(\tilde{\mathbf{X}}) \rightarrow 2$  as  $n \rightarrow \infty$ . Consequently, almost surely, for every  $z \in \mathbb{C}$ ,

$$s_1(\tilde{\mathbf{X}} - z\mathbf{I}) \leq s_1(\tilde{\mathbf{X}}) + s_1(z\mathbf{I}) = s_1(\tilde{\mathbf{X}}) + |z|^2 = \mathcal{O}_z(1)$$

and also almost surely, for every  $z \in \mathbb{C}$ ,

$$s_1(\mathbf{X} - z\mathbf{I}) \leq s_1(\tilde{\mathbf{X}} + z\mathbf{I}) + s_1(n^{-1/2}m\mathbf{1}) = s_1(\tilde{\mathbf{X}} + z\mathbf{I}) + n^{1/2}m = \mathcal{O}_z(1) + \mathcal{O}(n^{1/2}).$$

On the other hand, by the Tao-Vu theorem 3.6 and the Borel-Cantelli lemma, there exist  $B > 0$  and  $C_2 > 0$  such that for every  $z \in \mathbb{C}$ , almost surely,

$$\min(s_n(\mathbf{X} - z\mathbf{I}), s_n(\tilde{\mathbf{X}} - z\mathbf{I})) \geq C_2 n^B$$

for large enough  $n$ . Since  $s_n(\cdot)$  is continuous, it is locally uniformly continuous, and one can then commute the universal quantifiers on  $z$  and  $\omega$ .  $\square$

**Remark 2.2.** *Since  $m \neq 0$ , we have for every  $z \in \mathbb{C}$ , by using (1) and theorem 1.2,*

$$s_1(\mathbf{X} - z\mathbf{I}) \geq |\lambda_1(\mathbf{X} - z\mathbf{I})| \xrightarrow[n \rightarrow \infty]{a.s.} +\infty$$

*Thompson's interlacing inequalities (see discussion after lemma 3.5) and (4) give*

$$s_2(\mathbf{X} - z\mathbf{I}) \leq s_1(\tilde{\mathbf{X}} - z\mathbf{I}) \leq s_1(\tilde{\mathbf{X}}) + s_1(z\mathbf{I}) = s_1(\tilde{\mathbf{X}}) + |z|^2 \xrightarrow[n \rightarrow \infty]{a.s.} 2 + |z|^2 < \infty.$$

### 3 Tools

This section gathers some tools used in our proof of the non-central version of the circular law theorem (theorem 1.3). The following lemma is a special case of the integration by parts formula for the Lebesgue-Stieltjes integral (with atoms).

**Lemma 3.1 (Integration by parts).** *Let  $a_1, \dots, a_n$  and  $b_1, \dots, b_n$  be real numbers in some interval  $[\alpha, \beta]$  of  $\mathbb{R}$ , and let  $F_\mu$  and  $F_\nu$  be the cumulative distribution functions of  $\mu = \frac{1}{n}(\delta_{a_1} + \dots + \delta_{a_n})$  and  $\nu = \frac{1}{n}(\delta_{b_1} + \dots + \delta_{b_n})$ . Then for any smooth  $f : [\alpha, \beta] \rightarrow \mathbb{R}$ ,*

$$\int_\alpha^\beta f(x) d\mu(x) - \int_\alpha^\beta f(x) d\nu(x) = \int_\alpha^\beta f'(x)(F_\mu(x) - F_\nu(x)) dx.$$

*In particular, when  $f$  is non decreasing,*

$$\left| \int_\alpha^\beta f(x) d\mu(x) - \int_\alpha^\beta f(x) d\nu(x) \right| \leq (f(\beta) - f(\alpha)) \|F_\mu - F_\nu\|_\infty.$$

*Proof.* One can assume by continuity that  $a_1, \dots, a_n, b_1, \dots, b_n$  are all pairwise different. Let us reorder the sequence  $a_1, \dots, a_n, b_1, \dots, b_n$  into  $c_1 \leq \dots \leq c_{2n}$ . For every  $1 \leq k \leq 2n$ , set  $\varepsilon_k = +1$  if  $c_k \in \{a_1, \dots, a_n\}$  and  $\varepsilon_k = -1$  if  $c_k \in \{b_1, \dots, b_n\}$ . We have

$$\int_\alpha^\beta f(x) d\mu(x) - \int_\alpha^\beta f(x) d\nu(x) = \frac{1}{n} \sum_{k=1}^n (f(a_i) - f(b_i)) = \frac{1}{n} \sum_{k=1}^{2n} \varepsilon_k f(c_k).$$

By an Abel transform, we get by denoting  $S_k = \varepsilon_1 + \dots + \varepsilon_k$ ,

$$\sum_{k=1}^{2n} \varepsilon_k f(c_k) = - \sum_{k=1}^{2n-1} S_k (f(c_{k+1}) - f(c_k)) + S_{2n} f(c_{2n}).$$

Since  $F_\mu - F_\nu$  is constant and equal to  $S_k$  on  $[c_k, c_{k+1}[$ ,

$$S_k (f(c_{k+1}) - f(c_k)) = \int_{c_k}^{c_{k+1}} f'(x)(F_\mu(x) - F_\nu(x)) dx.$$

It remains to notice that  $S_{2n} = F_\mu(c_{2n}) - F_\nu(c_{2n}) = 0$ . □

#### 3.1 Logarithmic potential tools

We give here some logarithmic potentials tools, already used by Götze & Tikhomirov [14] and Pan & Zhou [19] in order to prove centered versions of the circular law theorem, following a suggestion made by Khoruzhenko. The reader may take a look at [15, sec. 5.3] for the relation with the free entropy of Voiculescu.

1. The *logarithmic potential*  $U_\mu$  of a probability measure  $\mu$  on  $\mathbb{C}$  is the function defined for every  $z \in \mathbb{C}$  by

$$U_\mu(z) = - \int_{\mathbb{C}} \log |z - z'| d\mu(z').$$

2. The *logarithmic energy*  $\mathcal{E}(\mu)$  of a probability distribution  $\mu$  on  $\mathbb{C}$  is

$$\mathcal{E}(\mu) = \int_{\mathbb{C}} U_\mu(z) d\mu(z) = - \int_{\mathbb{C}} \int_{\mathbb{C}} \log |z - z'| d\mu(z) d\mu(z').$$

3. The *energy*  $\mathcal{E}(K)$  of a compact subset  $K$  of  $\mathbb{C}$  is

$$\mathcal{E}(K) = \inf\{\mathcal{E}(\mu); \mu \text{ is a probability distribution supported in } K\}$$

4. The *logarithmic capacity* of a compact subset  $K$  of  $\mathbb{C}$  is defined by

$$\text{Cap}(K) = e^{-\mathcal{E}(K)}.$$

5. The *capacity* of a Borel subset  $B$  of  $\mathbb{C}$  is defined by

$$\text{Cap}(B) = \sup\{\text{Cap}(K); K \text{ is a compact subset of } \mathbb{C} \text{ included in } B\}.$$

If  $\text{Cap}(B) = 0$  then  $B$  has zero Lebesgue measure. A property is said to hold *quasi-everywhere* on  $B$  if the set where it does not hold has zero capacity.

**Theorem 3.2.** *Let  $(\nu_n)$  be a sequence of probability distributions on  $\mathbb{C}$  with support in a common compact subset of  $\mathbb{C}$ ,  $(a_n)$  be a sequence of complex numbers such that  $|a_n| \rightarrow +\infty$ , and  $(p_n)$  be a sequence of real numbers in  $[0, 1]$  such that  $p_n \rightarrow 0$ . Define  $\mu_n = (1 - p_n)\nu_n + p_n\delta_{a_n}$ . If  $p_n \log |a_n| \rightarrow 0$  and  $\liminf_{n \rightarrow \infty} U_{\mu_n} = U_\mu$  quasi-everywhere on  $\mathbb{C}$  for some compactly supported probability distribution  $\mu$  on  $\mathbb{C}$ , then  $\mu_n \rightarrow \mu$  and  $\nu_n \rightarrow \mu$  weakly as  $n \rightarrow \infty$ .*

*Proof.* We have for every fixed  $z \in \mathbb{C}$  and large enough  $n$ ,

$$U_{\mu_n}(z) = -p_n \log |z - a_n| + (1 - p_n)U_{\nu_n}(z).$$

By hypothesis, we have for every fixed  $z \in \mathbb{C}$  and large enough  $n$ ,

$$p_n \log |z - a_n| = p_n \log |a_n| + p_n \log \left| \frac{z}{a_n} - 1 \right| \rightarrow 0.$$

Consequently,  $\liminf_{n \rightarrow \infty} U_{\nu_n} = \liminf_{n \rightarrow \infty} U_{\mu_n} = U_\mu$  quasi-everywhere on  $\mathbb{C}$ . The sequence  $(\nu_n)$  is tight (i.e. weakly relatively compact) since it is supported in a

common compact subset of  $\mathbb{C}$ . Let  $\mu_{n_k} \rightarrow \eta$  be a weakly converging subsequence, to some probability distribution  $\eta$  (necessarily supported in the same compact subset). By the Lower Envelope Theorem 3.3,  $\liminf_{n \rightarrow \infty} U_{\nu_n} = U_\eta$  quasi-everywhere in  $\mathbb{C}$ . In particular,  $U_\mu = U_\eta$  quasi-everywhere in  $\mathbb{C}$ , and thus almost everywhere for the Lebesgue measure. Now, by the Unicity Theorem 3.4, we get  $\eta = \mu$ . Therefore,  $\mu$  is the unique weak adherence value of  $(\nu_n)$ . By a standard theorem of Prohorov [7], the set of probability distributions on a compact space equipped with the weak topology is metrizable and compact. It follows then that  $\nu_n \rightarrow \mu$  weakly. Since  $p_n \rightarrow 0$ , we get also that  $\mu_n = (1 - p_n)\nu_n + p_n\delta_{a_n} \rightarrow \mu$  weakly.  $\square$

Theorems 3.3 and 3.4 can be found in [24, th. 6.9 p. 73 and cor. 2.2 p. 98].

**Theorem 3.3 (Lower Envelope Theorem).** *Let  $(\mu_n)$  be a sequence of probability distributions on  $\mathbb{C}$  with support in a common compact subset of  $\mathbb{C}$ . If  $\mu_n \rightarrow \mu$  weakly as  $n \rightarrow \infty$ , then  $\liminf_{n \rightarrow \infty} U_{\mu_n} = U_\mu$  quasi-everywhere on  $\mathbb{C}$ .*

**Theorem 3.4 (Unicity Theorem).** *If  $\mu$  and  $\nu$  are two compactly supported probability distributions on  $\mathbb{C}$  and if their logarithmic potential  $U_\mu$  and  $U_\nu$  coincide almost everywhere with respect to the Lebesgue measure on  $\mathbb{R}^2$ , then  $\mu = \nu$ .*

## 3.2 Singular values tools

For an  $n \times n$  Hermitian matrix  $\mathbf{H}$ , we denote by  $F_{\mathbf{H}} : \mathbb{R} \rightarrow [0, 1]$  the cumulative distribution function of the ESD  $\mu_{\mathbf{H}}$  of  $\mathbf{H}$  defined by

$$F_{\mathbf{H}}(x) = \mu_{\mathbf{H}}((-\infty, x]) = \frac{1}{n} \text{card}\{k \in \{1, \dots, n\} \text{ s.t. } \lambda_k(\mathbf{H}) - x \leq 0\}$$

for any  $x \in \mathbb{R}$ . The following lemma, due to Bai, shows that the rank of the deformation controls the effect on singular values.

**Lemma 3.5 (Rank inequality).** *For any  $n \times m$  complex matrices  $\mathbf{A}$  and  $\mathbf{B}$ ,*

$$\|F_{\sqrt{\mathbf{A}\mathbf{A}^*}} - F_{\sqrt{\mathbf{B}\mathbf{B}^*}}\|_\infty \leq \frac{1}{n} \text{rank}(\mathbf{A} - \mathbf{B}).$$

See [3, Lemma 2.6 page 621] or [4, Theorem 11.43 page 362] for a proof based on interlacing inequalities derived from the Courant-Fischer min-max variational formulas for singular values. Bai states this lemma with  $F_{\mathbf{A}\mathbf{A}^*}$  and  $F_{\mathbf{B}\mathbf{B}^*}$  instead of  $F_{\sqrt{\mathbf{A}\mathbf{A}^*}}$  and  $F_{\sqrt{\mathbf{B}\mathbf{B}^*}}$  but the proof are identical since  $\sqrt{\cdot}$  is non decreasing. The rank lemma 3.5 appears actually in a different form in the article [29, Theorem 3 page 76] published in 1976 by Thompson, who showed that for any  $n \times n$  complex matrices  $\mathbf{A}$  and  $\mathbf{B}$  with  $\text{rank}(\mathbf{A} - \mathbf{B}) \leq k$ ,

$$s_{i+k}(\mathbf{A}) \leq s_i(\mathbf{B}) \leq s_{i-k}(\mathbf{A}) \tag{7}$$

for any  $1 \leq i \leq n$ , where  $s_r = +\infty$  if  $r \leq 0$  and  $s_r = 0$  if  $r \geq n + 1$ . In particular,

$$[s_{n-k}(\mathbf{B}), s_{k+1}(\mathbf{B})] \subset [s_n(\mathbf{A}), s_1(\mathbf{A})].$$

Conversely, Thompson showed additionally that any couple of sequences of non negative real numbers which satisfy to the interlacing inequalities (7) are the singular values of two respective matrices  $\mathbf{A}$  and  $\mathbf{B}$  with  $\text{rank}(\mathbf{A} - \mathbf{B}) \leq k$ .

It is worthwhile to mention that the Bai-Thompson inequalities give neither an upper bound for the largest singular values  $s_1(\mathbf{B}), \dots, s_k(\mathbf{B})$  nor a lower bound for the smallest singular values  $s_{n-k+1}(\mathbf{B}), \dots, s_n(\mathbf{B})$ , even in the case  $k = 1$ .

The following theorem is nothing else but Theorem 2.1 by Tao & Vu in [28, p. 6]. It gives a probabilistic polynomial bound for the smallest singular value of a random matrix deformed by a deterministic matrix.

**Theorem 3.6 (Polynomial bounds for smallest singular values).** *Let  $L$  be a probability distribution on  $\mathbb{C}$  with finite and non-zero variance. For every constants  $A > 0$  and  $C_1 > 0$ , there exists constants  $B > 0$  and  $C_2 > 0$  such that for every  $n \times n$  random matrix  $\mathbf{X}$  with i.i.d. entries of law  $L$  and every  $n \times n$  deterministic matrix  $\mathbf{C}$  with  $s_1(\mathbf{C}) \leq n^{C_1}$ , we have*

$$\mathbb{P}(s_n(\mathbf{X} + \mathbf{C}) \leq n^{-B}) \leq C_2 n^{-A}. \quad (8)$$

**Remark 3.7 (The matrix inverse of a low rank deformation).** *Concerning the behavior of the smallest singular value under a rank 1 deformation, it is natural to ask about the expression of the inverse of the deformed matrix. The Sherman-Morrison formula [26] states that if  $\mathbf{A}$  is an  $n \times n$  invertible matrix and if  $u$  and  $v$  are two column  $n$ -vectors such that  $1 + v^\top \mathbf{A}^{-1}u \neq 0$ , then the rank 1 deformation  $\mathbf{A} + uv^\top$  of  $\mathbf{A}$  is invertible and*

$$(\mathbf{A} + uv^\top)^{-1} = \mathbf{A}^{-1} - \frac{\mathbf{A}^{-1}uv^\top \mathbf{A}^{-1}}{1 + v^\top \mathbf{A}^{-1}u}.$$

*In particular, it shows that the inverse of a rank one deformation is a rank one deformation of the inverse. In the same spirit, we have under the same assumptions*

$$\det(\mathbf{A} + uv^\top) = (1 + v^\top \mathbf{A}^{-1}u) \det(\mathbf{A}).$$

*Unfortunately, this gives, to the author knowledge, nothing really useful for the control of logarithmic potentials in the non-central case. The Sherman-Morrison formula is a particular case of the Woodbury formula which states that if  $\mathbf{A}, \mathbf{C}, \mathbf{U}, \mathbf{V}$  are matrices of suitable dimensions then*

$$(\mathbf{A} + \mathbf{UCV}^\top)^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{U}(\mathbf{C}^{-1} + \mathbf{V}^\top \mathbf{A}^{-1}\mathbf{U})^{-1}\mathbf{V}^\top \mathbf{A}^{-1}.$$

*In the same spirit, and under the same assumptions,*

$$\det(\mathbf{A} + \mathbf{UCV}^\top) = \det(\mathbf{C}^{-1} + \mathbf{V}^\top \mathbf{A}^{-1}\mathbf{U}) \det(\mathbf{C}) \det(\mathbf{A}).$$



The initial motivation of the author in studying the non-central version of the circular law theorem comes from the study of the LSD of uniform random Markov matrices, which is still a work in progress.

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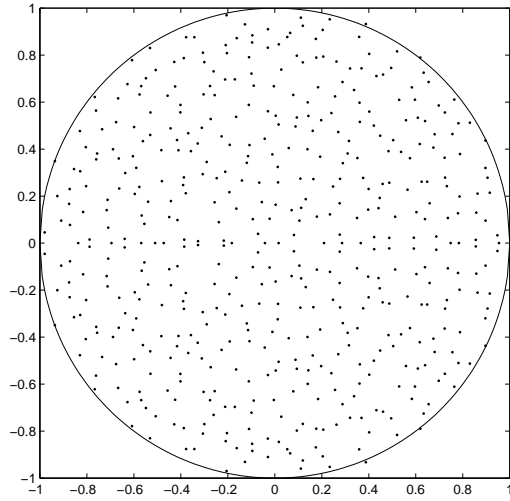


Figure 1: The spectrum of a single realization of  $\mathbf{X}$ . Here  $n = 500$  and  $L$  is a standard Gaussian law  $\mathcal{N}(0, 1)$  of mean  $m = 0$  and variance  $\sigma^2 = 1$ . The plot illustrate well the circular law theorem in the central case  $m = 0$  (theorem 1.1).

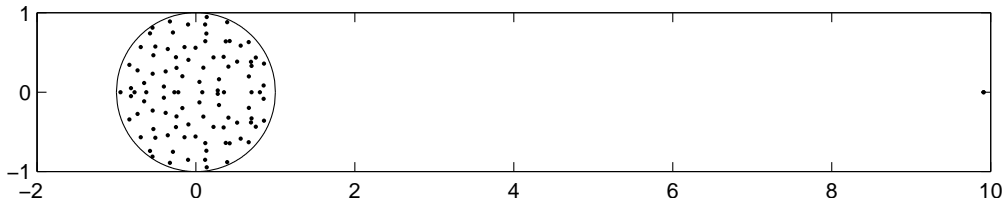


Figure 2: The typical behavior of the non-central case. The graphic shows the spectrum of a single realisation of  $\mathbf{X}$ , in the case where  $n = 100$  and  $L$  is the Gaussian distribution  $\mathcal{N}(1, 1)$  of mean  $m = 1$  and variance  $\sigma^2 = 1$ . The plot is in accordance with theorem 1.2, which gives  $\lambda_1(\mathbf{X}) \approx m\sqrt{n} = 10$ , whereas the rest of the spectrum remains localized near the unit disc. The largest eigenvalue  $\lambda_1(\mathbf{X})$  has weight  $1/n \rightarrow 0$  in the ESD of  $\mathbf{X}$ , which explains why the non-central circular theorem is possible (theorem 1.3).

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