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A limit result for a system of particles in random environment

Pierre Andreoletti *

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Abstract: We consider an infinite system of particles in one dimension, each particle performs independant Sinai's random walk in random environment. Considering an instant t, large enough, we prove a result in probability showing that the particles are trapped in the neighborhood of well defined points of the lattice depending on the random environment the time t and the starting point of the particles.

1 Introduction, definitions and result

Systems of particles have been study a lot in many different directions see for example Ligget [1985], DeMasi and Presutti [1991] and Kipnis and Landim [1998] and the references therein. In this paper we are interested in a system of particles performing independant Sinai's random walk in random environment on the latice $\mathbb{Z}$. At time 0, in each point of the lattice $i \in \mathbb{Z}$ a random number of particles $\eta(i,0)$ is distributed, we assume that the sequence $(\eta(i,0), i \in \mathbb{Z})$ is i.i.d. and that $\eta(0,0)$ is distributed according to a Poisson distribution of parameter λ. Each particle of this system effectuates a Sinai's walk that means a one dimensional random walk in random environment with three conditions on the random environment: two necessaries hypothesis to get a recurrent process (see Solomon [1975]) which is not a simple random walk and an hypothesis of regularity which allows us to have a good control on the fluctuations of the random environment. The asymptotic behavior of such a walk has been understood by Sinai [1982] : it is sub-diffusive, that is, for an instant t it behaves like $(\log t)^2$ and it is localized in the neighborhood of a well defined point of the lattice. We are interested in the general behavior of the system described above in the following sense, let us denote $\eta(x,t)$ the number of particles at time t on x, and let f be a differentiable function with compact support, we study the asymptotic behavior when t becomes large of

$$\frac{1}{(\log t)^2}\left(\sum_{x\in\mathbb{Z}} \eta(x,t) f(x/(\log t)^2)\right). \tag{1.1}$$

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The result we get can been seen as the generalization of Sinai's localization result for a system of particles. It shows that the particles are trapped in the bottom of the deepest valley that the walk can reach within an amount of time t. The coordinates of the bottom of the valleys depend on the starting points of the particles, the random environment and t.

To get the result, first, we make a construction on the random environment closed to the one that Neveu and Pitman [1989] use to study the excursion of the Brownian motion, thanks to this we get a partition in term of valleys of the part of the lattice related to the support of f. Then we prove that there is neither importation of particles from outside of the support of the function f nor exchanged of particles between the (deepest) valleys, finally we we use the accurate result of Andreoletti [2005] about Sinai's localization result extended to a finite number of particles.

In the following section, we define Sinai's walk, then the system of particles and finally, we state the main result. In section 2 we make the construction on the random environment, in section 3 and 4 we prove the main theorem.

1.1 Definition of Sinai's walk

Let $\alpha = (\alpha_i, i \in \mathbb{Z})$ be a sequence of independent and identically distributed (i.i.d.) random variables taking values in $(0,1)$ defined on the probability space $(\Omega_1, \mathcal{F}_1, Q)$, this sequence will be called random environment. A random walk in random environment (denoted R.W.R.E.) $(X_n, n \in \mathbb{N})$ is a sequence of random variable taking value in $\mathbb{Z}$, defined on $(\Omega, \mathcal{F}, \mathbb{P})$ such that

• for every fixed environment α, $(X_n, n \in \mathbb{N})$ is a Markov chain with the following transition probabilities, for all $n \geq 1$ and $i \in \mathbb{Z}$

$$\begin{aligned} &\mathbb{P}^{\alpha,1}\left[X_n = i+1 | X_{n-1} = i\right] = \alpha_i, \\ &\mathbb{P}^{\alpha,1}\left[X_n = i-1 | X_{n-1} = i\right] = 1-\alpha_i. \end{aligned} \tag{1.2}$$

We denote $(\Omega_2, \mathcal{F}_2, \mathbb{P}^{\alpha,1})$ the probability space associated to this Markov chain.

• $\Omega = \Omega_1 \times \Omega_2$, $\forall A_1 \in \mathcal{F}_1$ and $\forall A_2 \in \mathcal{F}_2$, $\mathbb{P}\left[A_1 \times A_2\right] = \int_{A_1} Q(dw_1) \int_{A_2} \mathbb{P}^{\alpha(w_1),1}(dw_2)$.

The probability measure $\mathbb{P}^{\alpha,1}\left[.\,|\, X_0 = a\right]$ will be denoted $\mathbb{P}_a^{\alpha,1}\left[.\right]$, the expectation associated to $\mathbb{P}_a^{\alpha,1}$: $\mathbb{E}_a^{\alpha,1}$, and the expectation associated to Q: $\mathbb{E}_Q$.

Now we introduce the hypothesis we will use in all this work. The two following hypothesis are the necessaries hypothesis

$$\mathbb{E}_Q\left[\log \frac{1-\alpha_0}{\alpha_0}\right] = 0, \tag{1.3}$$

$$\mathrm{Var}_Q\left[\log \frac{1-\alpha_0}{\alpha_0}\right] \equiv \sigma^2 > 0. \tag{1.4}$$

Solomon [1975] shows that under 1.3 the process $(X_n, n \in \mathbb{N})$ is $\mathbb{P}$ almost surely recurrent and 1.4 implies that the model is not reduced to the simple random walk. In addition to 1.3 and 1.4 we will consider the following hypothesis of regularity, there exists $0 < \rho_0 < 1/2$ such that

$$\sup\left\{x,\ Q\left[\alpha_0 \geq x\right] = 1\right\} \geq \rho_0 \text{ and } \sup\left\{x,\ Q\left[\alpha_0 \leq 1-x\right] = 1\right\} \geq \rho_0. \tag{1.5}$$

We call *Sinai's random walk* the random walk in random environment previously defined with the three hypothesis 1.3, 1.4 and 1.5. Instead of considering the discrete time process, we will work with the continuous time one $(X_t, t \in \mathbb{R}_+)$ define as follows, $\forall t \in \mathbb{R}_+,\ X_t = X_{[t]}$ where $[t]$ is the integer par of t.

1.2 The system of particles

We recall that $\eta(i) \equiv \eta(i,0)$ is the number of particles in $i \in \mathbb{Z}$ at time 0, we assume that $(\eta(i), i \in \mathbb{Z})$ is i.i.d, and that $\eta(0)$ is distributed according to a Poisson distribution of parameter λ, we denote $\mathbb{P}_1$ the corresponding probability measure. We denote $X_t^{x,i}$ the coordinate of a particle at time t corresponding to the ith particle, which was on site x at time 0 ($X_0^{x,i} = x$). $\eta(x,t)$ can be written in the following useful way:

$$\eta(x,t) = \sum_{y\in\mathbb{Z}} \sum_{i=1}^{\eta(y)} \mathbb{I}_{\{X_t^{x,i}=x\}} \tag{1.6}$$

The distribution of the whole system is denoted P, it is the product of Q and $\mathbb{P}^\alpha$, where $\mathbb{P}^\alpha$ is the product $\mathbb{P}^\alpha \equiv \prod_{x\in\mathbb{Z}} \mathbb{P}_x^\alpha = P_1 \times \prod_{x\in\mathbb{Z}} \prod_{i=1}^{\eta(x)} \mathbb{P}_x^{\alpha,i}$. The measure P generalizes the measure $\mathbb{P}$ defined Section 1.1 for a random number of particles. Here we have three levels of randomness, the environment, the number of particles per sites and the random walks themselves, hence both $\mathbb{P}^\alpha$ and $\mathbb{P}_x^\alpha$ are random measures. Notice that, when the environment is fixed ($\eta(.,t)$) inherits the markov property from the random walks, indeed they all have the same transition probability and they do not interact with each other.

1.3 Main results

First let us state the result in probability, the following Theorem is based on a construction on the random environment (see figure 3). Let C_κ^1 be the set of all differentiable functions with a compact support.

Theorem 1.1. *Assume 1.3, 1.4 and 1.5 hold, for all $f \in C_\kappa^1$, and all t large enough*

$$P\left[\left|\frac{1}{(\log t)^2}\sum_{x\in\mathbb{Z}} \eta(x,t) f(x/(\log n)^2) - \lambda \sum_{i=1}^{n(f)} \frac{|M_{i+1}-M_i|}{(\log t)^2} f(m_i/(\log t)^2)\right| = o(1)\right] = 1 - o(1),$$

where, $(m_i, i \geq 1)$, $(M_i, i \geq 1)$ and $n(f)$ are well defined variables depending only on the random environment α, on the time t and f. $o(1)$ is a positive decreasing function such that $\lim_{t\to+\infty} o(1) = 0$.

This Theorem says that at time t the particles are localized in the neighborhood of well defined points of the lattice. These points have the coordinates of the bottoms of the valleys where the particles started their walk. In other words it says that the number of particles at time t, at the points m_i is equal to $\lambda|M_{i+1} - M_i|$ which is the total number of particles at the instant $t = 0$ present in the valley $(\{M_i, m_i, M_{i+1}\})$. $n(f)$ is the number of valleys deep enough within the support of f, notice that $n(f)$ can be equal to zero if the support of f is too small. The following Proposition give the limit distributions of the random variables present in the Theorem.

Proposition 1.2. *Assume 1.3, 1.4 and 1.5 hold, for all $i \in \mathbb{N}^*$, in distribution when t goes to infinity*

$$\frac{\sigma^2 M_i}{(\log t)^2} \to L_i, \tag{1.7}$$

$$\frac{\sigma^2 m_i}{(\log t)^2} \to L_i, \tag{1.8}$$

The $(L_{i+1} - L_i, i \in \mathbb{Z})$ are independent and equidistributed random variables with Laplace transform given by $E(\exp(-\lambda(L_2 - L_1))) = 1/(\cosh(\sqrt{(}2\lambda)))^2$, for all $\lambda > 0$. Moreover when the support of f is large enough the distribution of $n(f)$ is given by a normal law with mean $\sigma^2 supp(f)/2$ and variance $3 * \sigma^4 supp(f)/4$.

2 Construction for the random environment

In this section we begin with some basic notions on the random environment of Sinai's random walk, then we construct the key random variables $(M_i, m_i, i \leq n(f))$ appearing in our result. The reader can follow the different steps of the construction with the Figures 2-5.

2.1 Definition and basic notions of valleys

For completeness we begin with some basic notions originally introduced by Sinai [1982].
The random potential and the valleys
Let

$$\epsilon_i \equiv \log \frac{1-\alpha_i}{\alpha_i}, \ i \in \mathbb{Z}, \tag{2.1}$$

define :

Definition 2.1. The random potential $(S_k, \ k \in \mathbb{R})$ associated to the random environment α is defined in the following way:

$$S_k = \begin{cases} \sum_{1 \leq i \leq k} \epsilon_i, & \text{if } \ k = 1, 2, \cdots, \\ -\sum_{k+1 \leq i \leq 0} \epsilon_i, & \text{if } \ k = -1, -2, \cdots, \end{cases}$$
$$S_0 = 0.$$

for the other $k \in \mathbb{R} \setminus \mathbb{Z}$, S_k is defined by linear interpolation.

Definition 2.2. We will say that the triplet $\{M', m, M''\}$ is a *valley* if

$$S_{M'} = \max_{M' \leq t \leq m} S_t, \tag{2.2}$$

$$S_{M''} = \max_{m \leq t \leq M''} S_t, \tag{2.3}$$

$$S_m = \min_{M' \leq t \leq M''} S_t \ . \tag{2.4}$$

If m is not unique we choose the one with the smallest absolute value.

Definition 2.3. We will call *depth of the valley* $\{M', m, M''\}$ and we will denote it $d([M', M''])$ the quantity

$$\min(S_{M'} - S_m, S_{M''} - S_m). \tag{2.5}$$

Now we define the operation of *refinement*

Definition 2.4. Let $\{M', m, M''\}$ be a valley and let M_1 and m_1 be such that $m \leq M_1 < m_1 \leq M''$ and

$$S_{M_1} - S_{m_1} = \max_{m \leq t' \leq t'' \leq M''} (S_{t'} - S_{t''}). \tag{2.6}$$

We say that the couple (m_1, M_1) is obtained by a *right refinement* of $\{M', m, M''\}$. If the couple (m_1, M_1) is not unique, we will take the one such that m_1 and M_1 have the smallest absolute value. In a similar way we define the *left refinement* operation.

We denote $\log_2 \equiv \log \log$, in all this work we will suppose that t is large enough such that $\log_2 t$ is positive.

Definition 2.5. Let $\gamma > 0$, define $\Gamma_t \equiv \log t + \gamma \log_2 t$, we say that a valley $\{M', m, M''\}$ is of depth larger than Γ_t if and only if $d([M', M'']) \geq \Gamma_t$.

2.2 Construction of a random cover for $(supp(f)) * (\log t)^2$

We start with the constructions of a sequence of valleys with identical properties, the different steps of the construction are shown in Figures 1 to 2. Notice that the variables in greek letters we use below are not the fundamental one, they just help us to construct $m_\cdot^+, m_\cdot^-, M_\cdot^+$, $M_\cdot^-$ and M_0.
Let $u, v \in \mathbb{R}$, $u < v$, define

$$S_{u,v}^- = \min\{S_s,\ u \le s \le v\}, \tag{2.7}$$

$$S_{u,v}^+ = \max\{S_s,\ u \le s \le v\}, \tag{2.8}$$

$$\tau_0^+ = \inf\{s > 0,\ S_s - S_{0,s}^- \ge \Gamma_t\}, \tag{2.9}$$

$$m_0^+ = \sup\{s < \tau_0^+,\ S_s = S_{0,\tau_0^+}^-\}, \tag{2.10}$$

$$\sigma_0^+ = \inf\{s > m_0^+,\ S_{m_0^+,s}^+ - S_s \ge \Gamma_t\}, \tag{2.11}$$

$$\tau_0^- = \sup\{s < 0,\ S_s - S_{s,0}^- \ge \Gamma_t\}, \tag{2.12}$$

$$m_0^- = \sup\{s > \tau_0^-,\ S_s = S_{0,\tau_0^+}^-\}, \tag{2.13}$$

$$\sigma_0^- = \sup\{s < m_0^-,\ S_{m_0^-,s}^+ - S_s \ge \Gamma_t\}, \tag{2.14}$$

$$M_0 = \inf\{s > m_0^-,\ S_s = S_{m_0^-,m_0^+}^+\}. \tag{2.15}$$

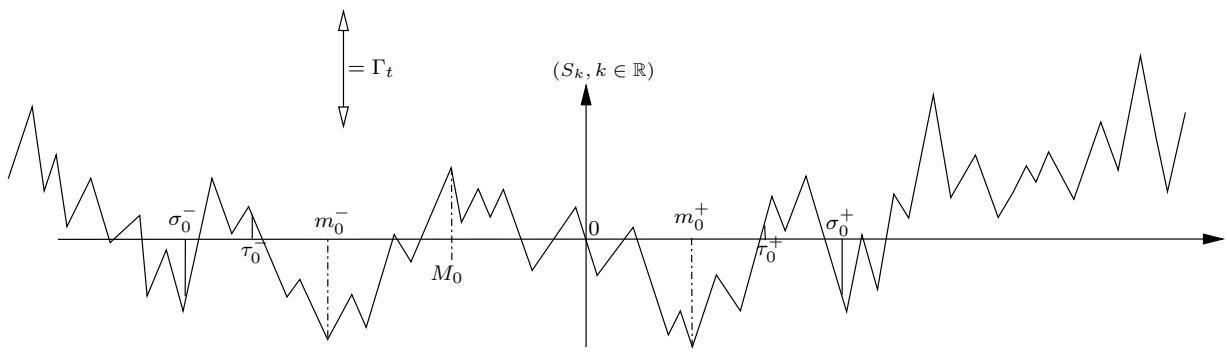

Figure 1: Construction of the valleys 1/2

Now define recursively the following variables, let $i \ge 1$

$$\tau_i^+ = \inf\{s > \sigma_{i-1}^+,\ S_s - S_{\sigma_{i-1}^+,s}^+ \ge \Gamma_t\}, \tag{2.16}$$

$$m_i^+ = \sup\{s > \sigma_{i-1}^+,\ S_s = S_{\sigma_{i-1}^+,\tau_i^+}^+\}, \tag{2.17}$$

$$M_i^+ = \inf\{s > m_{i-1}^+,\ S_s = S_{m_{i-1}^+,m_i^+}^+\}, \tag{2.18}$$

$$\sigma_i^+ = \inf\{s > m_i^+,\ S_{m_i^+,s}^+ - S_s \ge \Gamma_t\}, \tag{2.19}$$

and in the same way on the left hand side of the origin.

We would like to make remarks about this construction:

Remark 2.6. Notice that for all $i \ge 1$ the $\{M_i^+, m_i^+, M_{i+1}^+\}$ are valleys of depth larger than Γ_t such that we can not construct a sub valley (with the operation of refinement, see Definition 2.4) of depth larger than Γ_t, the same remark is true if we replace the $+$ by the $-$. However we do not know how

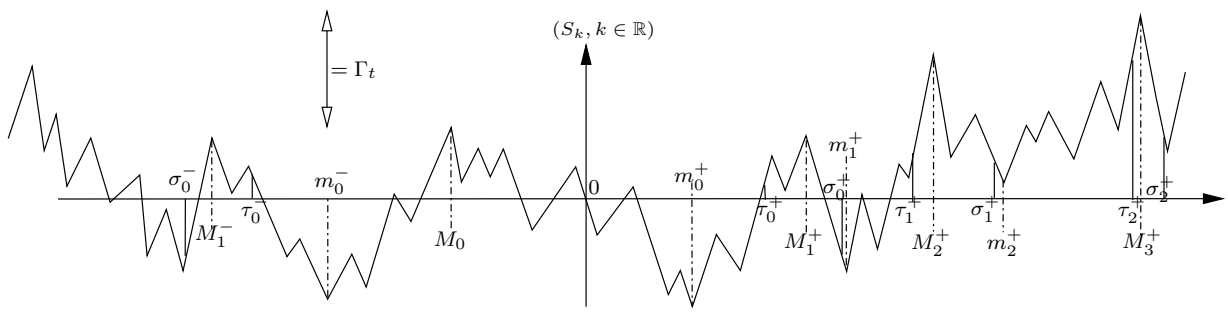

Figure 2: Construction of the valleys 2/2

deep the valleys $\{M_1^-, m_0^-, M_0\}$ and $\{M_0, m_0^+, M_1^+\}$ are, what we know from our construction is that we must have one of these two cases:

$$\min\left(d\left([M_1^-, M_0]\right), d\left([M_0, M_1^+]\right)\right) \geq \Gamma_t \text{ or} \tag{2.20}$$

$$\min\left(d\left([M_1^-, M_0]\right), d\left([M_0, M_1^+]\right)\right) < \Gamma_t \text{ and } d\left([M_1^-, M_1^+]\right) \geq \Gamma_t \tag{2.21}$$

In our drawings it is the case 2.20 that occurs.

Now let us describe the link between the construction we have exposed above and the support of f. For simplicity, we will assume that $supp(f) = [-K, K]$ with $K > 0$. Let n^+ be the largest integer such that $m_{n^+} < K(\log t)^2$ and symmetrically n^- the largest integer such that $m_{n^-} > -K(\log t)^2$. Note that both n^+ and n^- can be equal to zero. It is also important to notice that, we can have 2 different cases in both on the right and left hand-side of the origin. We will only discuss about the right hand-side: first case is when $m_{n^+} < K(\log t)^2 < M_{n^+ +1}$ in this case we will have to consider the particles that belong to the interval $(K(\log t)^2, M_{n^+ +1})$. Indeed most of them will tend to reach the bottom the valley m_{n^+} that belongs to $[-K(\log t)^2, K(\log t)^2]$. To the contrary in the case where $M_{n(f)+1} > K(\log t)^2 > m_{n^+ +1}$, we will not have to care about the particles in the interval $(M_{n(f)+1}, K(\log t)^2)$ because they will all tend to reach $m_{n^+ +1}$ that do not belongs to $[-K(\log t)^2, K(\log t)^2]$.

To simplify notations we re-numerate all the set of minima and maxima in the following way : we suppress the "+" and the "-", so denoting $n(f) \equiv n(K) = (n^+ + n^- - \mathbb{I}_{A_t^c}) \vee 0$ where $A_t = \{\min\left(d\left([M_1^-, M_0]\right), d\left([M_0, M_1^+]\right)\right) \geq \Gamma_t\}$, we denote: $M_{n^-}^- = M_1$, $m_{n^-}^- = m_1$, $M_{n^- -1}^- = M_2$, ..., $M_{n^+}^+ = M_{n(f)+1}$. Note that the $-\mathbb{I}_{A_t^c}$ is needed because we have two cases (2.20 and 2.21). In our drawing (see Figure 3) $n^+ = 2$, $n^- = 1$ and we are in the case 2.20 so $n(f) = 3$.

2.3 Indeterminate points and Set of good environments

Let $x \in [-K(\log t)^2, K(\log t)^2]$, due to the preceding construction we know that there exists $i \in \mathbb{N}$ such that $x \in [M_i, M_{i+1}]$, let $\bar{i} \equiv \bar{i}(t, f, \alpha) = \{y \in \mathbb{Z},\ y \in [M_i, M_{i+1}]\}$, $i \leq n(f)$. Now let us define the set of *indeterminate points*. Let $x \in \bar{l}$, $1 \leq l \leq n(f) + 1$, we will say that x is indeterminate if $x \in U_l \equiv [M_l - (\log_2 t)^2, M_l + (\log_2 t)^2]$. We call these sets "indeterminate", because at each top of the valley, actually a small interval on the top, we can not determine which valley the particles starting from there will choose. Notice, that the number of indeterminate points, belonging to V_f, are negligible comparing to a typical fluctuation of Sinai's random walk.

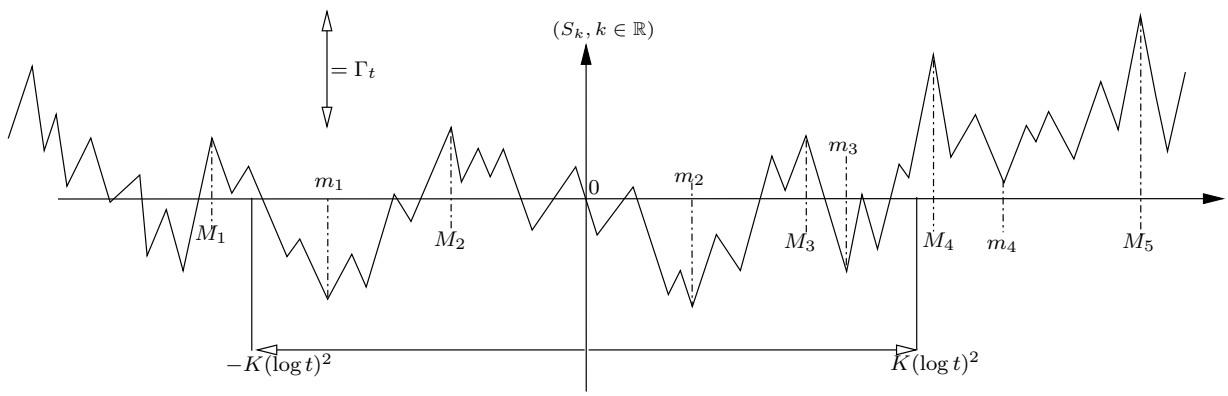

Figure 3: Support of f and the valleys

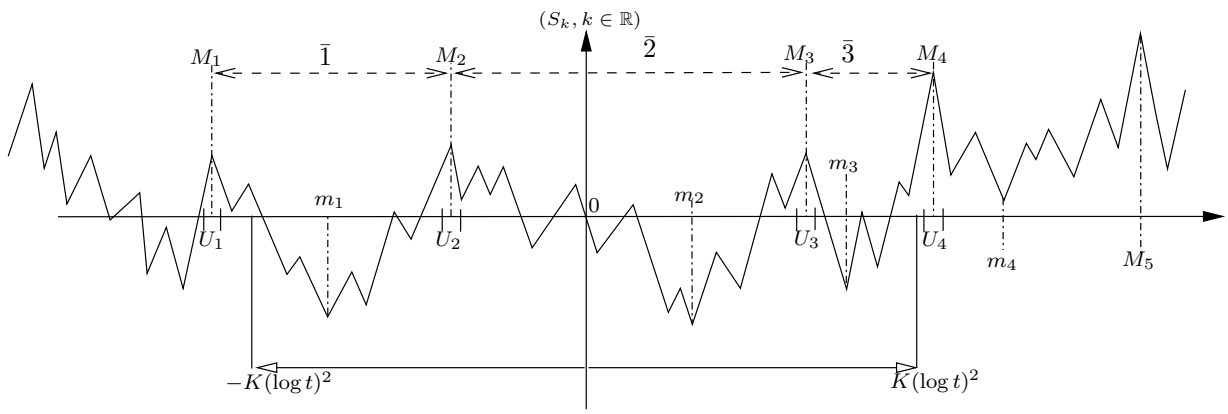

Figure 4: Random cover and sets of indeterminate points with $n(f) = 3$.

Now we give the main properties of the random environment, recall that $f \in C^1_\kappa$, $c_1 > 0$ and $c_2 > 0$:

(2.22) $n(f)$ is finite,

(2.23) $|V_f| \leq c_1(\log n)^2$,

(2.24) $\forall 1 \leq i \leq n(f),\ |\bar{i}| \equiv |M_i - M_{i+1}| \approx c_2(\log n)^2$,

(2.25) $\forall 1 \leq i \leq n(f)$ the valley$\{M_i, m_i, M_{i+1}\}$ have Sinai's basic properties.

Remark 2.7. (about 2.25) As we said before to get our result we need to prove the result of localization of Sinai for independent particles. For that we need some properties on the random environment we do not want to repeat here (see for example Andreoletti [2005] Definition 3.4, page 894), we have called these properties "Sinai's basic properties".

Let us define the set of good properties G_t, $G_t = \{\alpha \in \Omega_1, \alpha \text{ satisfies } 2.22 - 2.25\}$. The following Proposition, give a probability result for G_t:

Proposition 2.8. *There exists $c_1 > 0$, $c_2 > 0$ such that for all t large enough, $Q[G_t] = 1 - o(1)$.*

Proof.
We shortly discuss the proof of this Proposition at the end of the paper, Paragraph 5. ■

3 Results of localization for the particles

In this section we first prove two results (Proposition 3.1), about the no-migration of the particles from the valley where they start to another (different) one, then we state the key result which generalizes Sinai's result for a system of independent particles, let $f \in C^1_\kappa$ we denote $V_f = \{\bar{1}, \cdots, \bar{n}(f)\}$ the random cover of $[-K(\log t)^2, K(\log t)^2]$ (see the construction in the previous paragraph), we have

Proposition 3.1. *For all $f \in C^1_\kappa$, $\gamma > 12$, and all t large enough, $Q[G_t] = 1 - o(1)$ and*

$$\inf_{\alpha \in G_t} \left\{ \mathbb{P}^\alpha \left[\sum_{x \notin V_f} \sum_{k=1}^{\eta(x)} \mathbb{I}_{X_t^{x,k} \in V_f} = o(\log^2 t) \right] \right\} = 1 - o(1), \tag{3.1}$$

conversly, we have:

$$\inf_{\alpha \in G_t} \left\{ \mathbb{P}^\alpha \left[\bigcap_{i=1}^{n(f)} \bigcap_{x \in \bar{i} \setminus U_i} \bigcap_{k=1}^{\eta(x)} \left\{ X_t^{x,k} \in \bar{i} \right\} \right] \right\} = 1 - o(1). \tag{3.2}$$

3.1 shows that with an overwhelming probability a negligible number of particles, coming from the complementary of $V(f)$ in $\mathbb{Z}$, will reach V_f before the instant t. 3.2 shows that, within the interval of time t, there is no exchange of particles between two valleys except, may be, for the particles that belong to the sets of indeterminate points.

Proof.

First let us give the proof of 3.2, it is based on the two following facts,

$$\mathbb{P}^\alpha \left[\bigcap_{i=1}^{n(f)} \bigcap_{x \in \bar{i} \setminus U_i} \bigcap_{k=1}^{\eta(x)} \left\{ X_t^{x,k} \in \bar{i} \right\} \right] \geq \exp \left(\lambda \left[a - 1 \right] \sum_{k=1}^{n(f)} |\bar{k} \setminus U_k| \right), \forall \alpha, \tag{3.3}$$

$$a \geq 1 - \frac{cte}{(\log t)^{\gamma - 2}}, \ \forall \alpha \in G_t. \tag{3.4}$$

where $a = \inf_{1 \leq i \leq n(f)} \inf_{x \in \bar{i} \setminus U_j} \mathbb{P}^\alpha_x \left[\left\{ X_t^{i,1} \in \bar{i} \right\} \right]$. Fact 3.4 is basic, to get it we use a similar method that Andreoletti [2005] use to get the Proposition 2.8 page 888. We now give a proof of Fact 3.3,

$$\begin{aligned} \mathbb{P}^\alpha \left[\bigcap_{i=1}^{n(f)} \bigcap_{x \in \bar{i} \setminus U_i} \bigcap_{k=1}^{\eta(x)} \left\{ X_t^{x,k} \in \bar{i} \right\} \right] &= \mathbb{P}^\alpha \left[\bigcap_{i=1}^{n(f)} \bigcap_{x \in \bar{i} \setminus U_i} \bigcup_{k_x=0}^{+\infty} \left\{ \eta(x) = k_x, \bigcap_{k=1}^{k_x} \left\{ X_t^{x,k} \in \bar{i} \right\} \right\} \right], \\ &= \prod_{i=1}^{n(f)} \prod_{x \in \bar{i} \setminus U_i} \mathbb{P}^\alpha_x \left[\bigcup_{k_x=0}^{+\infty} \left\{ \eta(x) = k_x, \bigcap_{k=1}^{k_x} \left\{ X_t^{x,k} \in \bar{i} \right\} \right\} \right], \\ &= \prod_{i=1}^{n(f)} \prod_{x \in \bar{i} \setminus U_i} \exp \left(\lambda (\mathbb{P}^{\alpha,1}_x \left[\left\{ X_t^{x,1} \in \bar{i} \right\} \right] - 1) \right). \end{aligned} \tag{3.5}$$

In the first equality we have introduced a partition over the values of the $(\eta(x), x \in \mathbb{Z})$, the second equality comes from the independence of the particles starting from distinct points of the lattice, the last one from the fact that $(X_\cdot^{\cdot,s}, s \in \mathbb{N})$ are i.i.d. Finally

$$\mathbb{P}^\alpha \left[\bigcap_{i=1}^{n(f)} \bigcap_{x \in \bar{i} \setminus U_i} \bigcap_{k=1}^{\eta(x)} \left\{ X_t^{x,k} \in \bar{i} \right\} \right] \geq \prod_{i=1}^{n(f)} \prod_{y \in \bar{i} \setminus U_i} \exp \left(\lambda (a - 1) \right), \forall \alpha, \tag{3.6}$$

so we get 3.3. To end the proof we use 2.23 and 2.24. Notice that Fact 1 is true for all environments α whereas Fact 2 is only true for all good environments.
To get 3.1 we will consider three different distances from the starting point of the particles to the boundary of V_f: M_1 or $M_{n(f)+1}$ (Cases 1-3 below). We will only discuss the case when the boundary is $M_{n(f)+1}$, the other case can be treated in the same way.
Case 1 (very long distance) notice that $\forall x > M_{n(f)+1} + t$, $X_t^{x,\cdot} \notin V_f$,
Case 2 (long distance), let $x \in J_t \equiv [M_{n(f)+1} + \log_2 t(\log t)^2, M_{n(f)+1} + t]$. With the same computations we have obtained 3.3 we can prove that

$$\mathbb{P}^\alpha\left[\bigcap_{x\in J_t}\bigcap_{k=1}^{\eta(x)}\{X_t^{x,k}\notin V_f\}\right] \equiv \mathbb{P}^\alpha\left[\bigcap_{x\in J_t}\bigcap_{k=1}^{\eta(x)}\{X_t^{x,k} > M_{n(f)+1}\}\right] \geq \exp\left(t\lambda(b-1)\right), \forall\alpha \tag{3.7}$$

where $b = \inf_{x\in J_t}\mathbb{P}_x^\alpha\left[\left\{X_t > M_{n(f)+1}\right\}\right]$, moreover it is a basic fact (see Andreoletti [2005] Proposition 2.8 page 888) that for all $\alpha \in G_t$, $b \geq 1 - cte/(t(\log t)^{\gamma-2})$.
Case 3 (short distance), let $x \in K_t \equiv [M_{n(f)+1} + (\log_2 t)^2, M_{n(f)+1} + (\log_2 t)(\log t)^2]$, we have

$$\mathbb{P}^\alpha\left[\bigcap_{x\in K_t}\bigcap_{k=1}^{\eta(x)}\{X_t^{x,k}\notin V_f\}\right] \geq \exp\left((\log_2 t)(\log t)^2\lambda(c-1)\right), \forall\alpha, \tag{3.8}$$

where $c = \inf_{x\in K_t}\mathbb{P}_x^{\alpha,1}\left[\left\{X_t > M_{n(f)+1}\right\}\right]$, moreover it is also a basic fact, see the reference above, that for all $\alpha \in G_t$, $c \geq 1 - cte/(\log t)^{\gamma-2}$.
From this three cases we deduce that the only particles, starting from a point inside $(M_{n(f)} + 1, +\infty)$, that can reach $M_{n(f)} + 1$, are the particles that belong to the interval $(M_{n(f)+1}, M_{n(f)+1} + (\log_2 t)^2)$, which is a subset of the set of indeterminate points $U_{M_n(f)+1}$, of size $(\log_2 t)^2$, negligible comparing to $(\log n)^2$. ■

Proposition 3.2. *For all $f \in C_\kappa^1$, $\gamma > 12$, and all t large enough, $Q[G_t] = 1 - o(1)$ and*

$$\inf_{\alpha\in G_t}\left\{\mathbb{P}^\alpha\left[\bigcap_{i=1}^{n(f)}\bigcap_{x\in\bar{i}\setminus U_i}\bigcap_{k=1}^{\eta(x)}\left\{\left|\frac{X_t^{x,k}}{(\log t)^2} - m_i(t)\right| \leq \frac{cte}{(\log t)^{1/2}}\right\}\right]\right\} = 1 - o(1). \tag{3.9}$$

The Proposition above generalizes Sinai's localization result for independent particles. It shows that each particle from a given valley will be located at time t in a small neighborhood of the coordinate of the bottom of this same valley.

Proof.
The proof of this Proposition is based on the two following facts, let $I_{t,i} = (m_i(t) - cte(\log t)^{3/2}, m_i(t) + cte(\log t)^{3/2})$, we have

$$\mathbb{P}^\alpha\left[\bigcap_{i=1}^{n(f)}\bigcap_{x\in\bar{i}\setminus U_i}\bigcap_{k=1}^{\eta(x)}\left\{X_t^{x,k}\in I_{t,i}\right\}\right] \geq \exp\left(\lambda\left[d-1\right]\sum_{k=1}^{n(f)}|\bar{k}\setminus U_k|\right)$$

where $d = \inf_{1\leq i\leq n(f)}\inf_{x\in\bar{i}\setminus U_i}\mathbb{P}_i^{\alpha,1}\left[X_t^{x,1}\in I_{t,i}\right]$, which is obtained with the same computation we did to get the previous Proposition, and

$$d \geq 1 - \frac{cte}{(\log n)^{\gamma-12}} \tag{3.10}$$

which is obtained by using a similar method of Andreoletti [2005] (Theorem 2.11 page 889). ■

4 Proof of Theorem 1.1

In all this section we will always assume that $\alpha \in G_t$. The two Propositions 3.1 and 3.2 will be used frequently, as these results are true in $\mathbb{P}^\alpha$ probability we will mention it by using "$\doteq$" instead of the common "$=$". First using the definition of $\eta(x,t)$ and 3.1, we get

$$\sum_{y\in\mathbb{Z}} \eta(y,t) f(y/(\log t)^2) \equiv \sum_{x\in\mathbb{Z}} \sum_{i=1}^{\eta(x)} \sum_{y\in\mathbb{Z}} \mathbb{I}_{\{X_t^{x,i}=y\}} f(y/(\log t)^2) \tag{4.1}$$

$$\doteq \sum_{l=1}^{n(f)} \sum_{x\in\bar{l}} \sum_{k=1}^{\eta(x)} \sum_{y\in\mathbb{Z}} \mathbb{I}_{\{X_t^{x,k}=y\}} f(y/(\log t)^2) + o(\log^2 t). \tag{4.2}$$

Extracting the indeterminate particles and using 3.2, we have

$$\begin{aligned} \sum_{l=1}^{n(f)} \sum_{x\in\bar{l}} \sum_{k=1}^{\eta(x)} \sum_{y\in\mathbb{Z}} \mathbb{I}_{\{X_t^{x,k}=y\}} f(y/(\log t)^2) &\equiv \sum_{l=1}^{n(f)} \sum_{x\in\bar{l}-U_l} \sum_{k=1}^{\eta(x)} \sum_{y\in\mathbb{Z}} \mathbb{I}_{\{X_t^{x,k}=y\}} f(y/(\log t)^2) + \epsilon_n^{(1)} \\ &\doteq \sum_{l=1}^{n(f)} \sum_{x\in\bar{l}-U_l} \sum_{k=1}^{\eta(x)} \sum_{y\in\bar{l}} \mathbb{I}_{\{X_t^{x,k}=y\}} f(y/(\log t)^2) + \epsilon_n^{(1)}, \end{aligned} \tag{4.3}$$

where $\epsilon_n^{(1)} = \sum_{l=1}^{n(f)} \sum_{x\in U_l} \sum_{k=1}^{\eta(x)} \sum_{y\in\mathbb{Z}} \mathbb{I}_{\{X_t^{x,k}=y\}} f(y/(\log t)^2)$. Thanks to the fact that $f \in C_\kappa^1$, the law of large number for the sum of $\eta(x)$ and that by definition $|U_.| = 2(\log_2 t)^2$, it is easy to see that in P_1 probability $\epsilon_n^{(1)} = O(n(f)(\log_2 t)^2)$. Proposition 3.2 yields

$$\sum_{l=1}^{n(f)} \sum_{x\in\bar{l}-U_l} \sum_{k=1}^{\eta(x)} \sum_{y\in\bar{l}} \mathbb{I}_{\{X_t^{x,k}=y\}} f(y/(\log t)^2) \doteq \sum_{l=1}^{n(f)} \sum_{x\in\bar{l}-U_l} \sum_{k=1}^{\eta(x)} \sum_{y\in I(t,l)} \mathbb{I}_{\{X_t^{x,k}=y\}} f(y/(\log t)^2) \tag{4.4}$$

where $I(t,l) = \{m_l(t) - cte(\log t)^{3/2}, ..., m_l(t) + cte(\log t)^{3/2}\}$. Now using the fact that $f \in C_\kappa^1$, we get that

$$\begin{aligned} &\sum_{l=1}^{n(f)} \sum_{x\in\bar{l}-U_l} \sum_{k=1}^{\eta(x)} \sum_{y\in I(t,l)} \mathbb{I}_{\{X_t^{x,k}=y\}} f(y/(\log t)^2) \\ &= \sum_{l=1}^{n(f)} \left(f(m_l/(\log t)^2) + O\left(1/(\log t)^{1/2}\right)\right) \sum_{x\in\bar{l}-U_l} \sum_{k=1}^{\eta(x)} \sum_{y\in I(t,f)} \mathbb{I}_{\{X_t^{x,k}=y\}}. \end{aligned} \tag{4.5}$$

Using once again Proposition 3.2, we get

$$\sum_{l=1}^{n(f)} \sum_{x\in\bar{l}-U_l} \sum_{k=1}^{\eta(x)} \sum_{y\in I(t,l)} \mathbb{I}_{\{X_t^{x,k}=y\}} f(y/(\log t)^2) \doteq \sum_{l=1}^{n(f)} \left(f(m_l/(\log t)^2) + O\left(1/(\log t)^{1/2}\right)\right) \sum_{x\in\bar{l}-U_l} \eta(x),$$

collecting what we did above and using once again the law of large number, we get

$$\sum_{y\in\mathbb{Z}} \eta(y,t) f(y/(\log t)^2) \doteq \lambda \sum_{l=1}^{n(f)} \left(f(m_l/(\log t)^2) + O\left(1/(\log t)^{1/2}\right)\right) \left|\bar{l} - U_l\right| + o(n(f)\log^2 t).$$

To end the proof we use 2.22, the fact that by definition $|U_.| = 2(\log_2 t)^2$, and we divide what we get by $(\log t)^2$.

5 Proof of Propositions 1.2 and 2.8

Proof of Proposition 1.2 Using the vocabulary of Neveu and Pitman [1989] and [1989b], the points M_i, and m_i are Γ_t-extrema for a random walk. Moreover thanks to the theorem of Donsker (see for example Durrett [1996], page 406) $(S_{tl}/\sqrt{l}, l)$ converge in distribution to a two-sided Brownian motion, therefore we get that the points L_i are 1-minima and S_i 1-maxima for the two sided Brownian motion. The first part of our Proposition 1.2 follows from the Proposition page 241 of Neveu and Pitman [1989], indeed the 1-extrema built a stationary renewal process, such that the difference of two consecutive extrema form a i.i.d. sequence with Laplace transform equal to $1/\cosh(\sqrt{(}2\lambda))$. Moreover $n(f)$ is the number of renewal within the support of f. It is easy to check that $\mathbb{P}[n(f) \leq k] = \mathbb{P}[R_k \leq |supp(f)|]$, where $R_k = \sum_{l=1}^{k} x_l$ with $x_l = L_{l+1} - L_l$ so we get that if $supp(f)$ is large the distribution of $n(f)$ is given by a normal law with mean $\sigma^2 supp(f)/2$ and variance $\sigma^4 supp(f) * 3/4$.
Sketch of the proof of Proposition 1.2 The first 3 properties can easily be deduced from the proof above. About Sinai's basic properties the proof can be found in the Appendix A of Andreoletti [2005].

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